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A system of non-linear functional differential equations arising in an equilibrium model of an economy with borrowing constraints

by

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ABSTRACT. — We study the existence of solutions to a nonlinear functional differential system that describes the equilibrium of a dynamic stochastic economy with heterogeneous agents facing borowing constraints. Our model is a generalization of the one studied in Scheinkman and Weiss [3] to display the effect of borrowing constraints in aggregate economic activity, and the mathematical techniques that we developp can be useful to deal with more complex models.

The system has aspects of a free boundary value problem in which there are different equations for different domains, with the domains themselves

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and their boundary determined by the solutions. It also has some of the characteristics of hyperbolic systems that one would obtain by differentiating Hamilton-Jacobi equations. These aspects combine to require, at least apparently, a very specific new approach to the existence problem. One payoff of this technique is that the existence proof is also an algorithm for computation.

RÉSUMÉ. – On étudie l'existence de solutions d'un système différentiel fonctionnel non linéaire décrivant l'équilibre d'une économie stochastique dynamique avec des agents hétérogènes soumis à des contraintes d'endettement. Notre modèle est une généralisation de celui étudié par Scheinkman et Weiss [3] qui mettait en valeur les effets de contraintes d'endettement sur l'activité économique agrégée, et sa résolution introduit des techniques mathématiques utilies pour la résolution de modèles plus complexes.

Le système possède certains aspects d'un problème avec frontière libre pour lequel il y a des équations différentes pour différents domaines, les domaines et leur frontière étant déterminés par les solutions. Il possède également certaines des caractéristiques des systèmes hyperboliques obtenu par différentiation d'équations de Hamilton-Jacobi. La combinaison de ces aspects nécessite une approche spécifique du problème d'existence. L'un des avantages de cette technique est que la preuve d'existence fournit un algorithme de calcul des solutions.

1. INTRODUCTION

Our purpose is to study a system of non-linear functional differential equations [*see* equations (3.1) to (3.6) below] that arise in the study of a dynamic model in economics that generalizes the model developed in Scheinkman and Weiss [3]. The system comes from a dynamic model in which each agent has to solve a stochastic optimal control problem.

This paper should be of interest to mathematicians: since it is very non classical, system (3.1) to (3.6) requires mathematical developments that are, at least to us, novel. The system at first glance looks like a system of non linear delay equations. Actually, as the sign of the delay is reversed, one can not consider these equations as delay equations. Moreover, as the coefficients of the equations are discontinuous functions of the solutions, this system has a flavor of a free boundary problem, *i.e.* one in which there are different equations for distinct domains, with the domains themselves and their "free" boundary determined by the solution. Finally this system has also some of the characteristics of the hyperbolic system that one would obtain by differentiating an Hamilton-Jacobi equation. These aspects combined to require, at least apparently, a very specific new approach to the existence problem.

The paper should also be of interest to economists: the proof of the existence of at least one solution to system (3.1) to (3.6) provides an entirely constructive algorithm and makes easy to produce simulations of the model.

Similarly to what often occurs when deriving the Hamilton-Jacobi-Belman equation for an optimal control problem, we first make an heuristic derivation of system (3.1) to (3.6), and then show that any solution to this system provides an equilibrium of the model.

The paper is organized as follows: in section 2, we introduce the model and write it as an equilibrium problem in which to every agent is associated an optimal control problem. We then expose heuristic considerations that provide system (3.1) to (3.6).

In section 3, we prove that every solution of this system is an equilibrium solution.

Section 4 presents the existence of a least one solution to system (3.1) to (3.6). We first introduce a reduced system that allows us to construct a map from a functional space to itself. We show that solutions of system (3.1) to (3.6) are fixed points of this map. We then expose the main theorem that claims the existence of at least one fixed point.

Sections 5, 6 and 7 provide a collection of results that allows us to prove the main theorem. Section 5 contains results concerning the reduced system. In section 6, we construct a supersolution and a subsolution to the fixed point problem. In section 7, we show the convergence of the algorithm that provides a fixed point.

Finally, in section 8 we present a complementary results in a particular case of the model.

2. THE MODEL

Consider an economy with a large number (infinite) of each of two types of infinitely long lived individuals, indexed by $i=1, 2$. Each "agent" consumes a single good and also works in the production of that good. His labor productivity is however random and determined by the "state" of the economy. At any time the state j is an element of a finite set J with 2N elements. In state j one unit of labor of type 1 individual produces α_j units of the consumption good while one unit of labor of type 2 individual produces α_{-j} units of the consumption good. If state j prevails a state change will occur with an exponential probability distribution with mean duration $1/\rho_j$. If a state change occurs, the probability that the next state is $k \in J$ is given by $\nu_{j,k}$, where for convenience we choose $\nu_{j,j}=0$. We write $\lambda_{j,k}=\rho_j \nu_{j,k}$, and the process $\{s_t\}$ represents the state at time t. ([1]) In order to make the two agents symmetric from an ex-ante point of view, we require that associated to each state j there is another state $\varphi(j)$ such that $\alpha_{\varphi(j)}=\alpha_{-j}$, $\alpha_{-\varphi(j)}=\alpha_j$, $\rho_{\varphi(j)}=\rho_j$ and $\lambda_{\varphi(j),-k}=\lambda_{j,k}$ for all $k \in J$. This hypothesis allows us to write without ambiguity $-j$ in place of $\varphi(j)$.

At each point in time an individual must choose how much to work and how much to consume. There is also an asset that the individuals can use to save. We choose units in such that when any one type holds a unit of the asset per capita, the other must hold zero. The initial distribution of money is assumed to be the same within each type. At each time t, at each event $\omega \in \Omega$, let $p(t, \omega)$ denote the price of a unit of the consumption good. Agents of type i, $i=1, 2$, take $p(t, \omega)$ as given and solve the problem (P^i) [equations (2.1) to (2.4)]:

$$\max_y E \int_0^{+\infty} e^{-\mu t} [U(c^i(t, \omega)) - l^i(t, \omega)]\, dt \tag{2.1}$$

subject to

$$y^i(0, \omega) = y_0^i > 0 \tag{2.2}$$

$$\dot{y}^i = p(t, \omega)[\alpha_j l^i(t, \omega) - c^i(t, \omega)] \tag{2.3}$$

with $j=s(t, \omega)$ if $i=1$ and $j=-s(t, \omega)$ if $i=2$, and

$$c^i(t, \omega) \geqq 0, \qquad l^i(t, \omega) \geqq 0, \qquad y^i(t, \omega) \geqq 0 \tag{2.4}$$

Notice that any solution to (P^i) must satisfy

$$\alpha_j U'(c^i(t, \omega)) = 1 \qquad \text{if} \quad l^i(t, \omega) > 0. \tag{2.5}$$

with $j=s(t, \omega)$ if $i=1$ and $j=-s(t, \omega)$ if $i=2$.

In (P^i) the individual chooses any consumption and working plan that satisfies the constraint that no borrowing is allowed. An equilibrium is a stochastic process $p(t, \omega)$ such that if $y^i(t, \omega)$, $c^i(t, \omega)$ and $l^i(t, \omega)$ solve (P^i) for $i=1, 2$, then

$$y^1(t, \omega) + y^2(t, \omega) = 1 \tag{2.6}$$

(demand for money equals supply), and

$$c^1(t, \omega) + c^2(t, \omega) = \alpha_{s(t, \omega)} l^1(t, \omega) + \alpha_{-s(t, \omega)} l^2(t, \omega) \tag{2.7}$$

(demand for goods equals supply). Notice that since constraint (2.3) appears in (P^i), equation (2.7) actually follows from equation (2.6).

In principle one could find an equilibrium by considering for each process $p(t, \omega)$ the process $y^i(t, \omega)$ that solves (P^i) and checking whether

([1]) Formally, $\{s_t\}$ is a Markov process on a probability space $(\Omega, \mathscr{F}, P)$ with $P(s_t=j, s_{t+\tau}=k) = \lambda_{j,k}\tau + o(\tau)$.

(2.6) holds. Economic considerations suggest, however, another route to compute at least one equilibrium. We will give here an heuristic argument that an equilibrium should satisfy a certain set of non-linear equations and later proceed to show precisely that a solution to these equations is in fact an equilibrium and that these equations have a solution as well as exhibiting an algorithm to compute it.

Let $z(t, \omega)$ denote the average amount of asset held by type 1 at time t and in event ω. Notice that due to the no-borrowing constraints, we may assume $0 \leqq z(t, \omega) \leqq 1$. We will look at an equilibrium where

$$p(t, \omega) = p(z(t, \omega), \mathrm{s}(t, \omega)) > 0,$$

where p is a C^1 function of its first argument. In reality it is more convenient to work with the price of money in terms of goods, so we write $q_j(z) = 1/p(z, j)$. In order to treat the two types symmetrically, we assume that

$$q_j(z) = q_{-j}(1-z). \tag{2.8}$$

Thus, one may think that consumers take $z(t, \omega)$ and the functions $q_j : [0, 1] \to \mathbf{R}$ for all $j \in \mathrm{J}$ as given, and solve (P^i). Assume further that consumers forecast that z will be an absolutely continuous functions such that, where the derivative exists,

$$\dot{z}(t, \omega) = \varphi_j(z(t, \omega)), \qquad \text{with} \quad \varphi_j(z) + \varphi_{-j}(1-z) = 0 \tag{2.9}$$

where φ_j is function for each $j \in \mathrm{J}$.

With this structure, we may redefine an equilibrium as a set of functions $q_j : [0, 1] \to \mathbf{R}$, $j \in \mathrm{J}$, with $q_j(z) = q_{-j}(1-z)$, and a stochastic process $z(t, \omega)$ with values in [0, 1], such that if consumer i solves (P^i) with

$$p(t, \omega) = q_{s(t,\,\omega)}(z(t, \omega)) \qquad \text{and} \qquad y_0^1 = z(0, \omega),\ y_0^2 = 1 - z(0, \omega),$$

then $\dot{y}^1(t, \omega) = \dot{z} + t, \omega)$ and $\dot{y}^2(t, \omega) = 1 - \dot{z}(t, \omega)$, *i.e.* type i holds the predicted amount of money. Notice that the usual dynamic programming arguments imply that in such an equilibria, $c^i(t, \omega) = c^i(z(t, \omega))$ and similarly, $l^i(t, \omega) = l^i(z(t, \omega))$.

Other economic considerations can be used to further characterize the equilibria. First, notice that since $\mathrm{U}'(0) = +\infty$, for any z, at least one of the two types must work. Since each type has an unlimited potential supply of labor and has a linear disutility for it is natural to conjecture that at each state j there exist a z_j^* such that if $z \leqq z_j^*$, type 1 works, whereas if $z \geqq z_j^*$ type 2 works. The intuition here is that the richest type will demand most in order to sacrifice its leisure.

Let us define

$$a_j(z) = \mathrm{U}'(c^1(z))\, q_j(z). \tag{2.10}$$

Since U is strictly concave, $g=(U')^{-1}$ is well defined, and hence if $z>z_j^*$ [using (2.5) for $i=2$, (2.8) and (2.10)], we obtain

$$\varphi_j(z)=-\frac{c^1(z)}{q_j(z)}=-g\left(\frac{a_j(z)}{q_j(z)}\right)\frac{1}{q_j(z)}=-g\left(\frac{a_j(z)}{\alpha_{-j}a_{-j}(1-z)}\right)\frac{1}{\alpha_{-j}a_{-j}(1-z)}.$$

Similarly, if $z<z_j^*$, then

$$\varphi_j(z)=\frac{c^2(z)}{q_j(z)}=g\left(\frac{a_{-j}(1-z)}{\alpha_j a_j(z)}\right)\frac{1}{\alpha_j a_j(z)}.$$

At $z=z_j^*$, we are free to define $\varphi_j(z_j^*)$ and we choose $\varphi_j(z_j^*)=0$.

The function a_j can be interpreted as the marginal value of a unit of money to consumer of type 1 when the average holdings of his type is z and he holds z. As such it is natural to conjecture that given that the utility function for consumption is strictly concave, $z\mapsto a_j(z)$ will be strictly decreasing. Since both types have the same marginal disutility per unit of labor the value z_j^* should be determined by

$$\alpha_j a_j(z_j^*)=\alpha_{-j}a_{-j}(1-z_j^*) \tag{2.11}$$

Clearly such a z_j^* does not need to exist. We will show however that one can always find an equilibrium where (2.11) has a solution in [0, 1].

Finally, we notice that "money" yields no direct utility and is only held to finance future consumption. As of time t the marginal utility of money, in equilibrium, at time τ is given by the random variable $e^{-\mu\tau}a_{s(\tau,\omega)}(z(\tau,\omega))$. Since money yields no direct utility, this random variable should be an $\mathcal{F}$-martingale. Using a first-order expansion, we get,

$$\begin{aligned}\mathrm{E}\,[e^{-\mu(t+dt)}a_{s(t+dt,\omega)}(z(t+dt,\omega))\,|\,z(t,\omega)=z,\ s(t,\omega)=j]\\ =e^{-\mu t}a_j(z)+e^{-\beta t}[a_j'(z)\,\varphi_j(z)\\ +\sum_{k\in J}\lambda_{j,k}[a_k(z)-a_j(z)]-\mu\,a_j(z)]+o(dt)\end{aligned}$$

Making $dt\to 0$, we get

$$a_j'(z)\,\varphi_j(z)+\sum_{k\in J}\lambda_{j,k}[a_k(z)-a_j(z)]-\mu\,a_j(z)=0.$$

Finally, if $z_j^*=1$, then, in equilibrium, $a_j'(1)=0$, and we get

$$\sum_{k\in J}\lambda_{j,k}[a_k(1)-a_j(1)]-\mu\,a_j(1)=0.$$

3. THE MAIN DIFFERENTIAL SYSTEM

Putting together the results that have been heuristically derived in the previous section, we get the set of non linear functional differential equations which we call system (S) (equations (3.1) to (3.6)): for all $j \in \mathrm{J}$,

$$a_j \in \mathrm{C}^1([0, 1]), \qquad a_j > 0, \quad a_j \text{ strictly decreasing} \tag{3.1}$$

$$a_j'(z)\,\varphi_j(z) + \sum_{k \in \mathrm{J}} \lambda_{j,k}[a_k(z) - a_j(z)] - \mu\, a_j(z) = 0, \qquad \forall z \in]0, 1[\tag{3.2}$$

with

$$\left.\begin{array}{l} \varphi_j(z) = -g\left(\dfrac{a_j(z)}{\alpha_{-j}a_{-j}(1-z)}\right)\dfrac{1}{\alpha_{-j}a_{-j}(1-z)} \\ \text{if} \\ \qquad \alpha_j a_j(z) < \alpha_{-j}a_{-j}(1-z) \end{array}\right\} \tag{3.3}$$

$$\varphi_j(z) = g\left(\frac{a_{-j}(1-z)}{\alpha_j a_j(z)}\right)\frac{1}{\alpha_j a_j(z)} \qquad \text{if} \quad \alpha_j a_j(z) > \alpha_{-j}a_{-j}(1-z) \tag{3.4}$$

and

$$\alpha_j a_j(1) \leqq \alpha_{-j}a_{-j}(0). \tag{3.5}$$

$$\text{If} \quad \alpha_j a_j(1) = \alpha_{-j}a_{-j}(0), \quad \text{then} \quad \sum_{k \in \mathrm{J}} \lambda_{j,k}[a_k(1) - a_j(1)] - \mu\, a_j(1) = 0. \tag{3.6}$$

Assuming now that we have a solution to system (S), we want to show that the corresponding process of wealth z is such that $z(t, \omega)$ [resp. $1 - z(t, \omega)$] is a solution for problem (P^1) (resp. (P^2)). This is provided by the following theorem:

THEOREM 1. – *Let* (a_j) *be a solution of system* (S). *Let* $z(t, \omega)$ *be the process defined by* $\dot{z} = \varphi_j(z)$ *with* $j = s(t, \omega)$ *if* $\alpha_j a_j(z) \neq \alpha_{-j}a_{-j}(1-z)$, *and* $\dot{z} = 0$ *otherwise. Then the process* z *is a solution of problem* (P):

$$\max_y \mathrm{E}\int_0^{+\infty} e^{-\mu t}[\mathrm{U}(c_t) - l_t]\,dt$$

subject to

$$y(0) = z_0 > 0$$

$$\dot{y} = \frac{\alpha_j l_t - c_t}{q_j(z)} \qquad \textit{with} \quad j = s(t) \quad \textit{if} \ \alpha_j a_j(z) \neq \alpha_{-j}a_{-j}(1-z)$$

$$\dot{y} = 0 \qquad \textit{otherwise.}$$

$$c_t \geqq 0, \qquad l_t \geqq 0, \qquad y(t) \geqq 0$$

with

$$q_j(z) = \alpha_j a_j(z) \qquad \textit{if} \quad \alpha_j a_j(z) > \alpha_{-j}a_{-j}(1-z)$$

$$q_j(z) = \alpha_{-j}a_{-j}(1-z) \qquad \text{if} \quad \alpha_j a_j(z) < \alpha_{-j}a_{-j}(1-z).$$

Since a_j and a_{-j} are strictly positive and C^1 on [0, 1], it is obvious that $\dot{z}$ is essentially bounded. We will only consider solutions to (P) such that ess sup $|\dot{y}+t, \omega)|<+\infty$. From the previous remark, z satisfies this requirement. The proof will use the following result:

LEMMA 1. – *$e^{-\mu t} a_{s(t)}(z(t))$ is a $\mathscr{F}$-martingale.*

Proof of the lemma. – We first show that for all $t \geqq 0$ and $\Delta t>0$,

$$\mathrm{E}[e^{-\mu(t+\Delta t)} a_{s(t+\Delta t)}(z(t+\Delta t)) \mid \mathscr{F}_t]=e^{-\mu t} a_{s(t)}(z(t))+o(\Delta t). \quad (3.7)$$

We set $\Delta z=z(t+\Delta t)-z(t)$. Since $\dot{z}$ is essentially bounded, $|\Delta z| \leqq \mathrm{R}\,\Delta t$ for some $\mathrm{R}>0$. Using this and a first order expansion, we get that

$$\begin{aligned}\mathrm{E}[e^{-\mu(t+\Delta t)} & a_{s(t+\Delta t)}(z(t+\Delta t)) \mid z(t), s(t)] \\ &= e^{-\mu t} a_{s(t)}(z(t))+o(\Delta t) \\ &+\Delta t e^{-\mu t}\Big[\sum_{k \neq s(t)} \lambda_{s(t), k}\, a_k(z(t))-\mu a_{s(t)}(z)\Big] \\ &+e^{-\mu t} a'_{s(t)}(z(t))\, \mathrm{E}[\Delta z\, 1_{s(t+\Delta t)=s(t)} \mid z(t), s(t)]+o(\Delta t).\end{aligned}$$

Now how evaluate

$$\mathrm{E}[\Delta z\, 1_{s(t+\Delta t)=s(t)} \mid z(t), s(t)].$$

Consider first the case where $\alpha_{s(t)} a_{s(t)}(z(t)) \neq \alpha_{-s(t)} a_{s(t)}(1-z(t))$. For Δt small enough, since $|\Delta z|<\mathrm{R}\,\Delta t$, we get that for every $\tau>0$ such that $\tau<\Delta t$, $\alpha_{s(t)} a_{s(t)}(z(\tau)) \neq \alpha_{-s(t)} a_{-s(t)}(1-z(\tau))$. This implies that for all $\tau<\Delta t$, the process $z(t+\tau)$ belongs to an open set on which $z \mapsto \dot{z}(z, s(t))=\varphi_{s(t)}(z)$ is C^1. Hence

$$\mathrm{E}[\Delta z\, 1_{s(t+\Delta t)=s(t)} \mid z(t), s(t)]=\varphi_{s(t)}(z(t))\, \Delta t+o(\Delta t).$$

Using equations (3.1) and (3.2), We have (3.7).

Now consider the case where $\alpha_{s(t)} a_{s(t)}(z(t))=\alpha_{-s(t)} a_{-s(t)}(1-z(t))$. Conditionally on $s(t+\tau)=s(t)$ for every $\tau \leqq \Delta t$, we have $z(t+\tau)=z(t)$, and hence

$$\mathrm{E}[\Delta z\, 1_{s(t+\Delta t)=s(t)} \mid z(t), s(t)]=o(\Delta t),$$

and by continuity of the a_k and a'_k, we get

$$\sum_{k \neq s(t)} \lambda_{s(t), k}\, a_k(z(t))-\mu a_{s(t)}(z)=0$$

and (3.7) is established.

The law of iterated expectations then holds for all $\tau>0$, *i.e.*

$$\mathrm{E}[e^{-\mu(t+r)} a_{s(t+\tau)}(z(t+\tau)) \mid \mathscr{F}_t]=e^{-\mu t} a_{s(t)}(z(t)).$$

Q.E.D.

We now give the proof of theorem 1.

Proof of theorem 1: Let

$$\mathrm{L}(y, \dot{y}, z, j)=\sup \left\{\mathrm{U}(c)-l / \dot{y}=\frac{\alpha_j l-c}{q_j(z)},\ c \geqq 0,\ l \geqq 0\right\}.$$

Then it is easy to verify that if $-q_j(z)\dot{y} \leqq g(1/\alpha_j)$, then

$$L(y, \dot{y}, z, j) = U\left(g\left(\frac{1}{\alpha_j}\right)\right) - \frac{q_j(z)\dot{y} + g(1/\alpha_j)}{\alpha_j}$$
$$\nabla_{y, \dot{y}} L(y, \dot{y}, z, j) = \left(0, -\frac{q_j(z)}{\alpha_j}\right)$$

while if $-q_j(z)\dot{y} \geqq g(1/\alpha_j)$, then

$$L(y, \dot{y}, z, j) = U(-q_j(z)\dot{y})$$
$$\nabla_{y, \dot{y}} L(y, \dot{y}, z, j) = (0, q_j(z) U'(-q_j(z)\dot{y})).$$

In particular, when $y = z$ and $\dot{y} = \dot{z}$, we get using the expression giving $\dot{z}$ that if $\alpha_j a_j(z) > \alpha_j a_j(1-z)$, then

$$\nabla_{y, \dot{y}} L(z, \dot{z}, z, j) = (0, -q_j(z)/\alpha_j) = (0, -a_j(z)),$$

while if $\alpha_j a_j(z) < \alpha_{-j} a_{-j}(1-z)$, then

$$\nabla_{y, \dot{y}} L(z, \dot{z}, z, j) = (0, -q_j(z) U'(-q_j(z)\dot{z})) = (0, -a_j(z)).$$

By concavity of L, we obtain for all $T > 0$.

$$E \int_0^T e^{-\mu t} [L(y, \dot{y}, z, s(t)) - L(z, \dot{z}, z, s(t))] \, dt \leqq E \int_0^T e^{-\mu t} a_{s(t)}(z(t)) [\dot{z} - \dot{y}] \, dt.$$

Since $a_{s(t)}(z(t))$ has bounded variation we get

$$E \int_0^T e^{-\mu t} a_{s(t)}(z(t)) [\dot{z} - \dot{y}] dt = e^{-\mu T} a_{s(T)}(z T)) [z(T) - y(T)] + \int_0^T (y - z) \, d(e^{-\mu t} a_{s(t)}(z(t))).$$

Since ess sup $|\dot{y}| < +\infty$, we have that $|y|$ is bounded on each [0, T] with $T > 0$. Also z is in [0, 1] and thus is bounded. Hence, since $e^{-\mu t} a_{s(t)}(z(t))$ is a bounded martingale, we get by applying the results on stochastic integral against a square integrable martingale as in [1] that the expectation of the second integral is zero. We also have, since $z(t)$ and $a_{s(t)}(z(t))$ are bounded that $\limsup_{T \to +\infty} |e^{-\mu T} a_{s(T)}(z(T))\, z(T)| = 0$, and thus

$$\limsup_{T \to +\infty} E \int_0^T e^{-\mu t} [L(y, \dot{y}, z, s(t)) - L(z, \dot{z}, z, s(t))]\, dt \geqq 0.$$

Q.E.D.

4. EXISTENCE RESULTS

This section provides a necessary and sufficient condition for the existence of a solution to system (S). The proof of this result is quite long, but entirely constructive. Hence, the mathematics which follows also provide an algorithm for the numerical construction of a solution.

The methodology that is used here is first to show that a solution of system (S) is a fixed point of a certain map. The monotony of this map, and the characterization of a sub and a supersolution allows us to construct two sequences of functions that converge to fixed points, *i.e.* solutions of system (S).

We denote by E the space of strictly decreasing, strictly positive continuous functions on [0, 1]. Since $\# J = 2N$, we have $E^J \equiv E^{2N}$.

4.1. A reduced system

Let ν, α, β be three given constants with $\nu > 0$ and $0 < \alpha, \beta < 1$.

DEFINITION 1. – *For all pair* $(u, v) \in E \times E$, *we define the switch point* z^* *by:*

$$\begin{aligned} z^* &= 0 \qquad \text{if} \quad \alpha u(0) < \beta v(1) \\ z^* &= 1 \qquad \text{if} \quad \alpha u(1) > \beta v(0) \\ \alpha u(z^*) &= \beta v(1 - z^*) \qquad \text{otherwise.} \end{aligned}$$

Since $z \mapsto \alpha u(z) - \beta v(z)$ is strictly decreasing, this defines a unique point $z^* \in [0, 1]$.

Given (u, v) in $E \times E$, we consider the functional differential system (R) (equations (4.1) to (4.8) below) in two unknown functions a and b:

$$(a, b) \in E \times E \tag{4.1}$$

$$a'(z)\, \varphi(z) + u(z) - \nu a(z) = 0, \qquad \forall z \in]0, 1[\tag{4.2}$$

$$-b'(z)\, \varphi(1 - z) + v(z) - \nu b(z) = 0, \qquad \forall z \in]0, 1[\tag{4.3}$$

with

$$\varphi(z) = -g\left(\frac{a(z)}{\beta b(1 - z)}\right) \frac{1}{\beta b(1 - z)} \qquad \text{if} \quad z > z^* \tag{4.4}$$

$$\varphi(z) = g\left(\frac{b(1-z)}{\alpha a(z)}\right)\frac{1}{\alpha a(z)} \quad \text{if} \quad z < z^* \tag{4.5}$$

and

$$\alpha a(z^*) = \beta b(1-z^*) \tag{4.6}$$

$$\nu a(z^*) = u(z^*) \quad \text{if} \quad z^* > 0 \tag{4.7}$$

$$\nu b(1-z^*) = v(1-z^*) \quad \text{if} \quad 1-z^* > 0 \tag{4.8}$$

Note that only two of the three conditions of (4.6) to (4.8) hold when $z^*=0$ or $z^*=1$, while the three conditions hold when $z^* \in]0, 1[$. But in this last case, only two of the three conditions are independent.

THEOREM 2. – *Assume that g is C^1, decreasing from $]0, +\infty[$ into itself, and the map $b \mapsto g(a/b)/b$ is increasing. For any pair $(u, v) \in E \times E$, there exists a unique solution (a, b) to system* (R). *Moreover, if two pairs (u_1, v_1) and (u_2, v_2) in $E \times E$ satisfy $u_1 \geqq u_2$ and $v_1 \geqq v_2$, and if (a_1, b_1) and (a_2, b_2) are the solutions to system* (R) *corresponding to (u_1, v_1)* and *(u_2, v_2) respectively, then one has $a_1 \geqq a_2$ and $b_1 \geqq b_2$.*

The proof of this theorem will be given later in section 5.

Note that since $g = U'^{-1}$, the condition that $b \mapsto g(a/b)/b$ is increasing is equivalent to the condition ([2])

$$\frac{c\,U''(c)}{U'(c)} \geqq -1, \qquad \forall c > 0. \tag{4.9}$$

This may be easily seen by writting that (set $c = g(a/b)$, *i. e.* $a/b = U'(c)$)

$$\frac{d}{db}\,\frac{g(a/b)}{b} = -\frac{1}{b^2}g(a/b) - \frac{a}{b^3}g'(a/b) = -\frac{a}{b^3}\left[\frac{c}{U'(c)} + \frac{1}{U''(c)}\right].$$

Also since U' is decreasing, g is decreasing.

Theorem 2 allows us to define a family of maps $F(\alpha, \beta, \nu; .,.)$ from $E \times E$ into itself given by

$$F(\alpha, \beta, \nu; u, v) = (a, b)$$

where (a, b) is the unique solution of system (R).

Remark 1. – *One can check that simultaneous interchange of* α, β *and* u, v *leads to interchange* a *and* b, *that is:* $F(\alpha, \beta, \nu; u, v) = (a, b)$ *if and only if* $F(\beta, \alpha, \nu; v, u) = (b, a)$.

([2]) The right hand side of (4.9) is the coefficient of relative risk aversion of the utility function U.

4.2. The complete system as a fixed point problem

This section will show how to construct a solution to system (S) by using the map F defined above.

For all $j \in J$, we set

$$\lambda_j = \sum_{k \in J} \lambda_{j,k}, \qquad \mu_j = \lambda_j + \mu.$$

Since $\lambda_{j,k} = \lambda_{-j,-k}$, we have $\mu_j = \mu_{-j}$.

Since $\mu_j = \mu_{-j}$, remark 1 implies that

$$(a, b) = F(\alpha_j, \alpha_{-j}, \mu_j; u_j, u_{-j})$$
$$\text{if and only if } (b, a) = F(\alpha_{-j}, \alpha_j, \mu_{-j}; u_{-j}, u_j).$$

Hence, for any family $u = (u_j)_{j \in J}$ of functions in E, we can define another family $H(u) = (a_j)_{j \in J}$ of functions in E by:

$$(a_j, a_{-j}) = F(\alpha_j, \alpha_{-j}, \mu_j; u_j, u_{-j}), \qquad \forall j \in J. \tag{4.10}$$

The map H is by theorem 2 increasing, *i.e.* for any u, $\tilde{u}$ in E^J such that $\forall j \in J$, $u_j \geqq \tilde{u}_j$, we have $\forall j \in J$, $H(u)_j \geqq H(\tilde{u})_j$.

Let $G : E^J \to E^J$ be given by

$$\forall j \in J, \quad (G(a))_j = \sum_{j \in J} \lambda_{j,k} a_k.$$

Note that since the $\lambda_{j,k}$ are positive, the map G is also increasing. Hence, the map $H \circ G$ is increasing from E^j into itself.

One has the following result:

THEOREM 3. – *A family of functions* $a = (a_j)_{j \in J} \in E^J$ *is a solution of system* (S) *if and only if its is a fixed point for the map* $H \circ G$.

Proof. – Suppose $a \in E^J$ is a fixed point of $H \circ G$. Fix $j \in J$, and set $u = \sum_{k \in J} \lambda_{j,k} a_k$, $v = \sum_{k \in J} \lambda_{-j,k} a_k$. By equation (4.6), the switch point z_j^* of the pair (u, v) is also the switch point of the pair (a_j, a_{-j}). Hence, equations (4.1) to (4.5) imply that equations (3.1) to (3.4) are satisfied. Also (4.6) together with $z_j^* \in [0, 1]$ implies that $\alpha_j a_j(1) \leqq \alpha_{-j} a_{-j}(0)$, *i.e.* (3.5). Moreover, if the equality holds, then $z_j^* = 1$ and (4.7) implies equation (3.6). Hence, a fixed point of $H \circ G$ is a solution of system (S).

Conversely, let a be a solution of system (S). Then, (4.1) implies that $a \in E^J$, and that the switch point z_j^* of (a_j, a_{-j}) is well defined. If $z_j^* \in]0, 1[$, then we have

$$\lim_{z \to z_j^*,\, z < z_j^*} \varphi_j(z) < 0$$

and

$$\lim_{z \to z_j^*,\, z > z_j^*} \varphi_j(z) > 0.$$

By continuity, (3.2) implies that $u(z_j^*)=(\lambda_j+\mu)\,a_j(z_j^*)$. Exchanging j by $-j$ gives $v(1-z_j^*)=(\lambda_j+\mu)\,a_{-j}(1-z_j^*)$. If $z_j^*=1$, then (3.6) gives $u(1)=(\lambda_j+\mu)\,a_j(1)$. Exchanging j by $-j$ gives from (3.2) that for all $z\in]0, 1[$, $v(z)<(\lambda_j+\mu)\,a_{-j}(z)$. Hence, since $\alpha_j a_j(1)=\alpha_{-j}a_{-j}(0)$, we have $\alpha_j u(1)\geqq\alpha_{-j}v(0)$. If $z_j^*=0$, then exchanging j in $-j$ gives $\alpha_j u(0)\leqq\alpha_{-j}v(1)$. Hence, z_j^* is also the switch point of the pair (u, v). From (3.1) to (3.4), we have that (4.1) to (4.5) are satisfied. From the above arguments showing that z_j^* is the switch point of (u, v), we get equations (4.6) to (4.8).

Q.E.D.

Showing that system (S) has a solution is now reduced to demonstrate the existence of fixed point for $H\circ G$. As this map is increasing, the main part of the proof consists in showing the existence of a subsolution $\underline{a}$ and a supersolution $\bar{a}$. This will be done in section 6. Because of the lack of compactness of E^J, we need to complete the proof with a convergence result for a^n defined by $a^{n+1}=H(G(a^n))$ and $a^0=\underline{a}$ or $a^0=\bar{a}$, which is in section 7.

4.3. The existence result (main theorem)

DEFINITION 2 (Undecomposability of the model). – *For i and j in* J, *we say that the pairs $(i, -i)$ and $(j, -j)$ are connected if there exists a sequence $i_0, \ldots, i_m$ in* J *such that $i_0=i$ and $i_m=\pm j$* (i.e. *$i_m=j$ or $i_m=-j$*) *such that $\lambda_{i_{k-1}, i_k}\neq 0$ for all $k=1, \ldots, m$.*

Recall that $\lambda_{-m,-p}=\lambda_{m,p}$ for all $(m, p)\in J\times J$. Hence, the preceding definition of a path just depend on the sets $(i, -i)$ and $(j, -j)$ (and not on the ordered pairs) and this connection relation is an equivalence relation.

We will suppose that each pair $(i, -i)$ is connected to any pair $(j, -j)$. If this is not the case, one should break J in its connected components and study them independently.

DEFINITION 3. – *We call w-subset, or working states subset, any set* $K\subset J$ *such that*

$$K\cap(-K)=\varnothing,\ K\cup(-K)=J.$$

Hence the cardinality of K is equal to N for any w-subset.

To each w-subset, we associated the $N\times N$ square matrix $\theta=\theta(K)$ with positive coefficients $(\theta_{i,j},\ i, j\in K\times K)$ defined by:

$$\theta_{i,j}=\left(\lambda_{i,j}+\lambda_{i,-j}\frac{\alpha_j}{\alpha_{-j}}\right)\frac{1}{(\lambda_i+\mu)}.$$

Using the hypothesis that $\lambda_{i,j}=\lambda_{-i,-j}$ and $\alpha_k>0$ for all i, j and k in J, one verifies that $\lambda_{m,n}>0$ implies $\theta_{i,j}\neq 0$ where (i, j) is the unique pair such that $|i|=|m|$, $|j|=|n|$, $(i, j)\in K\times K$. From this, one deduces that the connectivity hypothesis made previously implies the classical notion of undecomposability for the matrix θ, *i.e.* for any $(i, j)\in K\times K$, there exists a path $i_0, \ldots, i_m$ such that $i_0=i$, $i_m=j$ and $\theta_{i_{k-1}, i_k}\neq 0$ for all $k=1, \ldots, m$.

We now define $\rho(K)$ the largest eingenvalue of $\theta(K)$, and a number $\rho>0$ which plays a crucial role for our system:

$$\rho=\sup\{\rho(K)/K\ w\text{-subset of } J\}.$$

As the set of w-subset is finite, there exists at least one w-subset K_0 such that $\rho=\rho(K_0)$.

We now state the main theorem:

THEOREM 4. — *If the relative risk aversion is less than or equals to one, that is*

$$\frac{c\,U''(c)}{U'(c)}\geqq -1, \qquad \forall c>0. \tag{4.11}$$

then a necessary and sufficient condition for the existence of at least one solution to the system (S) *is that* ρ *satisfies* $\rho>1$.

Let us recall that condition (4.11) is equivalent to the hypothesis on the function $g=U'^{-1}$ made in theorem 2, namely g is decreasing and $b\mapsto g(a/b)/b$ is increasing.

One can easily give sufficient conditions which imply $\rho>1$. For instance, assume that there exist $k\in J$ such that

$$\alpha_k(\lambda_k+\mu)<\alpha_{-k}\lambda_{k,-k}. \tag{4.12}$$

Consider now a w-subset K such that $-k\in K$. From the definition of $\theta_{i,j}$, *one gets*

$$\theta_{-k,-k}\geqq\left(\lambda_{-k,k}\frac{\alpha_{-k}}{\alpha_k}\right)\frac{1}{(\lambda_{-k}+\mu)}.$$

But, as $\lambda_{k,-k}=\lambda_{-k,k}$ and $\lambda_k=\lambda_{-k}$, this gives us $\theta_{-k,-k}>1$. Considering the vector $x=(x_j, j\in K)$ with $x_j=0$ if $j\neq -k$ and $x_{-k}=1$, it is easy to check, since θ has positive coefficients, that for all j in K, we have $(\theta x)_j\geqq\theta_{-k,-k}x_j$. Applying the Perron-Frobenius theorem as stated in Nikaido [2], we get $\rho(K)>1$. Hence, we have the

COROLLARY 1. — *If relative risk aversion is less than or equals to one, and if condition* (4.12) *holds, then the system* (S) *has at least one solution.*

In the case $N=1$, $\theta(K)$ reduces to a scalar for every K. We set $\alpha=\alpha_1$, $\beta=\alpha_{-1}$ and $\lambda=\lambda_{1,-1}=\lambda_{-1,1}$, and we get the:

COROLLARY 2. — *If* $N=1$ *and the relative risk aversion is less than or equals to one, then a necessary and sufficient condition for the existence of at least one solution to system* (S) *is that* $\inf\{\alpha/\beta, \beta/\alpha\}<\lambda/\mu$.

4.4. Decomposition of the proof of the main theorem

Theorem 4 follows from following propositions 1 to 4 bellow. Propositions 2 and 3 consist in constructing super and sub solutions to the fixed point problem of the map $H \circ G$. Proposition 4 consists in showing that starting either from the super or sub solution, the sequence of functions defined in section 4.2 converges to a fixed point of $H \circ G$.

PROPOSITION 1. — *When* $\rho \leqq 1$, *the system* (S) *has no solution.*

Proof. — The proof is by contradiction. We assume that $\rho \leqq 1$ and that there exist a solution $a \in E^J$ to system (S). Let K be the w-subset defined by $j \in K$ if $z_j^* > 1/2$ and by choosing any of the two integers j and $-j$ if $z_j^* = 1/2$, where z_j^* denotes the switch point associated to the pair (a_j, a_{-j}). Such a w-subset exists and is unique since for all j, $z_j^* = 1 - z_{-j}^*$. For all $j \in K$, we have $a_j'(z) \leqq 0$ and $\varphi_j(z) > 0$ if $z < 1/2$. Hence, equation (3.2) and continuity of the a_k imply

$$\forall j \in K, \quad (\lambda_j + \mu)\, a_j(1/2) \leqq \sum_{k \in J} \lambda_{j,k}\, a_k(1/2).$$

For all $j \notin K$, we have $z_j^* = 1 - z_{-j}^*$ and $-j \in K$. Hence, $z_j^* \leqq 1/2$ implies

$$\alpha_j a_j(1/2) \leqq \alpha_{-j} a_{-j}(1/2).$$

From these two equations, we get

$$\forall j \in K, \quad (\lambda_j + \mu)\, a_j(1/2) \leqq \sum_{k \in K} \left[\lambda_{j,k} + \lambda_{j,-k} \frac{\alpha_k}{\alpha_{-k}} \right] a_k(1/2).$$

Hence, with $\theta = \theta(K)$ and $A_k = a_k(1/2)$ for all $k \in K$, we get $A \leqq \theta A$.

Let us first suppose that $\rho < 1$. From $\rho(K) \leqq \rho < 1$, one deduces that θ^n is a contraction for n sufficiently large (*see* Nikaido [2]). As θ has positive coefficients, $A \leqq \theta A$ implies by induction that $A \leqq \theta^n A$. hence, $A = 0$, which contradicts $a_k(1/2) > 0$.

We now turn to the case where $\rho = 1$. If $A \neq \theta A$, then one can prove that $(1 + \varepsilon) A \leqq \theta^m A$ for some $\varepsilon > 0$ and $m \in N$. Also, $(\theta^m/(1 + \varepsilon))^n$ is a contraction for n sufficiently large. Hence $A = 0$. If $A = \theta A$, then

$$(\lambda_j + \mu)\, a_j\left(\frac{1}{2}\right) = \sum_{k \in J} \lambda_{j,k}\, a_k\left(\frac{1}{2}\right), \qquad \forall j \in K$$

and

$$\alpha_j a_j\left(\frac{1}{2}\right) = \alpha_{-j} a_{-j}\left(\frac{1}{2}\right), \qquad \forall j \notin K.$$

This last equality implies that $z_j^* = 1/2$ for $j \notin K$. Hence

$$(\lambda_j + \mu)\, a_j\left(\frac{1}{2}\right) = \sum_{k \in J} \lambda_{j,k}\, a_k\left(\frac{1}{2}\right), \qquad \forall j \notin K.$$

This establishes that $B=MB$ where $B=(a_j(1/2), j\in J)\in \mathbf{R}^J$ and $M=(m_{i,j}. i\in J, j\in J)$ with $m_{i,j}=\lambda_{i,j}/(\lambda_i+\mu)$. From $m_{i,j}>0$ and $\sum_{j\in J} m_{i,j}=\lambda_i/(\lambda_i+\mu)<1$, we deduce that M is a contraction. Hence, $B=0$ which contradicts $a_j(1/2)>0$.

Q.E.D.

PROPOSITION 2. — *For any* $\gamma>0$, *there exist a function* $A\in C^1([0, 1])$ *such that*

$$A'(z)=-\mu A(z)\frac{A(0)}{g(A(z)/A(0))}$$

and

$$A(0)=\gamma A(1)>0.$$

If the constant γ *satisfies* $\gamma\geqq \sup\{\alpha_j/j\in J\}/\inf\{\alpha_j/j\in J\}$, *then* $\bar{a}=(\bar{a}_j)\in E^J$ *where* $\bar{a}_j=A$ *for all* $j\in J$ *satisfies* $\bar{a}\geqq H(G(\bar{a}))$, i. e. $\bar{a}$ *is a supersolution.*

Proof. — The proof is in section 6.

PROPOSITION 3. — *Suppose* $\rho>1$. *Let* K *be a w-subset such that* $\rho=\rho(K)$. *Let* $B=(B_j, j\in K)$ *be a positive eigenvector of the matrix* $\theta=\theta(K)$ *associated to the eigenvalue* ρ. *For* $j\in(-K)$, *define* B_j *by* $B_j=B_{-j}\alpha_{-j}/\alpha_j$. *For* $\eta>0$, *define* B^η *by* $B^\eta_j(z)=B_j-\eta z$. *For* η *and* ε *sufficiently small,* $\underline{a}=\varepsilon B^\eta$ *is a subsolution* [i. e. *one has* $\underline{a}\in E^J$ *and* $\underline{a}\leqq H(G(\underline{a}))$].

Proof. — The proof stays in section 6.

PROPOSITION 4. — *If there exists* $\underline{a}$ *and* $\bar{a}$ *in* E^J *such that*

$$\underline{a}\leqq\bar{a}$$
$$\underline{a}\leqq H(G(\underline{a}))$$
$$\bar{a}\geqq H(G(\bar{a}))$$

then the map $H\circ G$ *has at least one fixed point. More precisely, the sequences* $(\underline{a}^n)$ *and* $(\bar{a}^n)$ *defined by*

$$\underline{a}^0=\underline{a},\qquad \underline{a}^{n+1}=H(G(\underline{a}^n))$$
$$\bar{a}^0=\bar{a},\qquad \bar{a}^{n+1}=H(G(\bar{a}^n))$$

are respectively increasing and decreasing and converge in E^J *to respectively* $\underline{a}^*$ *and* $\bar{a}^*$ *which are fixed points of* $H\circ G$.

The proof of proposition 4 is in section 7. From proposition 3 we may choose ε small enough to obtain $\underline{a}\leqq\bar{a}$. Hence this proposition gives a constructive algorithm to find two solutions $\underline{a}^*$ and $\bar{a}^*$ of our equations. In all simulations we did find $\underline{a}^*=\bar{a}^*$. But at the time there is no theoretical evidence that this equality always holds.

5. PROOF OF THE RESULTS CONCERNING THE REDUCED SYSTEM

The proof of theorem 2 will be first established in a particular case, which we call balanced case, then in the general case by an approximation procedure. We set $f(a, b)=g(a/b)/b$. The assumptions made on g imply that $a \mapsto f(a, b)$ is decreasing and that $b \mapsto f(a, b)$ is increasing.

5.1. The balanced case

DEFINITION 4. — *$(u, v) \in \mathrm{E} \times \mathrm{E}$ is said to be balanced if* $\alpha u(0)>\beta v(1)$, $\alpha u(1)<\beta v(0)$, i.e. *the corresponding switch point of definition* 1 *is in* $]0, 1[$.

PROPOSITION 5 (proof of theorem 2 in the balanced case). — *Let* α, β *and* ν *be given constants such that* $\nu>0$ *and* $0<\alpha$, $\beta<1$. *Let* f *be a* C^1 *function from* $]0, +\infty[^2$ *into* $]0, +\infty[$. *Let* $(u, v) \in \mathrm{E}+\mathrm{E}$ *and let* $z^* \in]0, 1[$ *be the associated switch point. Then there exists a unique pair* (a, b) *solution to* (5.1) *to* (5.7):

$$(a, b) \in \mathrm{E} \times \mathrm{E} \tag{5.1}$$
$$a'(z)\,\varphi(z)+u(z)-\nu a(z)=0, \qquad \forall z \in]0, 1[\tag{5.2}$$
$$-b'(z)\,\varphi(1-z)+v(z)-\nu b(z)=0, \qquad \forall z \in]0, 1[\tag{5.3}$$

with

$$\varphi(z)=-f(a(z), \beta b(1-z)) \qquad \textit{if} \quad z>z^* \tag{5.4}$$
$$\varphi(z)=f(b(1-z), \alpha a(z)) \qquad \textit{if} \quad z<z^* \tag{5.5}$$

and

$$\nu a(z^*)=u(z^*) \tag{5.6}$$
$$\nu b(1-z^*)=v(l-z^*). \tag{5.7}$$

Moreover, a and b satisfy

$$\left.\begin{array}{ll} u(z^*)<\nu a(z)<u(z) & \textit{if} \quad z<z^* \\ u(z)<\nu a(z)<u(z^*) & \textit{if} \quad z>z^* \\ v(1-z)<\nu b(1-z)<v(1-z^*) & \textit{if} \quad z<z^* \\ v(1-z^*)<\nu b(1-z)<v(1-z) & \textit{if} \quad z>z^* \end{array}\right\} \tag{5.8}$$

Proof of proposition 5. — Set $w(z)=v(1-z)$ and $c(z)=b(1-z)$. From (5.1) to (5.7) we obtain

$$a'(z)\,\varphi(z)+u(z)-\nu a(z)=0, \qquad \forall z \in]0, 1[$$
$$c'(z)\,\varphi(z)+w(z)-\nu c(z)=0, \qquad \forall z \in]0, 1[$$

with

$$\varphi(z)=-f(a(z), \beta c(z)) \qquad \text{if} \quad z>z^*$$
$$\varphi(z)=f(c(z), \alpha a(z)) \qquad \text{if} \quad z<z^*$$

and

$$\nu a(z^*) = u(z^*)$$
$$\nu c(z^*) = w(z^*)$$

This new system is equivalent to a set of two Cauchy problems. The first one is a forward Cauchy problem consisting in two non linear O.D.E. on $[z^*, 1]$:

$$\left.\begin{array}{l} a' = (u - \nu a)/f(a, \beta c) \\ c' = (w - \nu c)/f(a, \beta c) \\ \nu a(z^*) = u(z^*) \\ \nu c(z^*) = w(z^*). \end{array}\right\} \quad (5.9)$$

The second one is a backward Cauchy problem consisting in two non linear O.D.E. on $[0, z^*]$:

$$\left.\begin{array}{l} a' = -(u - \nu a)/f(c, \alpha a) \\ c' = -(w - \nu c)/f(c, \alpha a) \\ \nu a(z^*) = u(z^*) \\ \nu c(z^*) = w(z^*). \end{array}\right\} \quad (5.10)$$

Note that replacing (β, z, a, c, u, w) by $(\alpha, 1-z, c, a, w, u)$ in (5.9) gives (5.10). This is again the symmetry of our problem. Hence it suffices to prove existence and unicity for the Cauchy problem (5.9).

As the function f is C^1 on the open set $\Omega =]0, +\infty[\times]0, +\infty[$, it is a standard result on Cauchy problems that (5.9) has a unique solution (a, c) defined on a maximal interval I^+ which is either $I^+ = [z^*, 1]$ or $I^+ = [z^*, z^+]$ for some $z^+ \in]z^*, 1]$. In the first case, we are done. In the second case, it is a property of maximal solutions that $(a(z), c(z)) \to \partial\Omega$ when $z \to z^+$, which means that for any compact C in Ω, there exist $\varepsilon > 0$ such that $z^+ - \varepsilon < z < z^+$ implies $(a(z), c(z)) \notin C$. This case will be eliminated by a corollary to the lemma below:

LEMMA 2. — *Let z_1 and z_2 be two points in I^+. Let y and φ be two functions defined on interval $I_{1,2} = [z_1, z_2[$. Assume that φ is continuous and y is C^1, and that*

$$y'(z)\varphi(z) + u(z) - \nu y(z) = 0 \quad \text{in } I_{1,2}$$
$$u(z_1) - \nu y(z_1) = 0$$

Then, if $\varphi(z) < 0$ for all $z \in I_{1,2}$ and $z_2 > z_1$, one has

$$u(z) < \nu y(z) < u(z_1), \qquad \forall z \in]z_1, z_2[.$$

If $\varphi(z) > 0$ for all $z \in I_{1,2}$ and $z_2 < z_1$, one has

$$u(z_1) < \nu y(z) < u(z), \qquad \forall z \in]z_1, z_2[.$$

In any of these two cases, one has that y is strictly decreasing on $]z_1, z_2[$.

COROLLARY 3. — *A solution* (a, c) *of* (5.9) *defined on* $[z^*, z^+[$ *cannot satisfy* $(a(z), c(z)) \to \partial\Omega$ *when* $z \to z^+$. *Hence*, $I^+ = [z^*, 1]$.

Proof. — It is clear using the lemma that for all z, $(a(z), c(z))$ remains in the set $C = [u(z^+)/\nu, u(z^*)/\nu] \times [w(z^*)/\nu, w(z^+)/\nu]$ which is a compact set in Ω.

Q.E.D.

COROLLARY 4. — *The unique solution of equations* (5.2) *to* (5.7) *satisfies* (5.1) *and* (5.8).

Proof. — It suffices to use the inequalities provided by the lemma.

Q.E.D.

Proof of the lemma. — We assume $z_2 > z_1$ (the proof in the other case is exactly the same).

Let us first suppose that u is C^1. Define $\gamma \in C^1(I_{1,2})$ by $\gamma(z_1) = 0$ and $\gamma'(z) = -\nu/\varphi(z)$, and $x \in C^1(I_{1,2})$ by $x(z) = (y(z) - u(z)/\nu)\, e^{\gamma(z)}$. Using the differential equation satisfied by y, we get $x(z_1) = 0$ and

$$x'(z) = -\frac{1}{\nu} u'(z)\, e^{\gamma(z)} \tag{5.11}$$

As $u' < 0$, this gives $x' > 0$ which implies $x(z) > x(z_1) = 0$ for $z > z_1$. We also have

$$y'(z) = e^{\gamma(z)} \frac{x(z)}{\varphi(z)} < 0. \tag{5.12}$$

Hence y is strictly decreasing and $y(z) < y(z_1) = u(z_1)/\nu$ for $z > z_1$.

If u is not C^1 but only continuous and strictly decreasing, we can do the same computation using the weak derivative on $I_{1,2}$, and we get (5.11) in the weak sense. Since u is strictly decreasing, we get $\operatorname{Supp} u' = I_{1,2}$ and because of (5.11), we have for the positive measure x that $\operatorname{Supp} x' = I_{1,2}$. This implies $x(z) > x(z_1) > 0$. Because of (5.12), we have $y' < 0$ on $]z_1, z_2[$.

Q.E.D.

This achieves the proof of proposition 5.

We now give a monotony result in the balanced case.

PROPOSITION 6. — *Let* (u_1, v_1) *and* (u_2, v_2) *be two balanced pairs such that* $u_1 \geqq u_2$ *and* $v_1 \geqq v_2$. *Let* (a_1, b_1) *and* (a_2, b_2) *be the unique solutions of* (5.1) *to* (5.7) *with respectively* $(u, v) = (u_1, v_1)$ *and* $(u, v) = (u_2, v_2)$. *Then* $a_1 \geqq a_2$ *and* $b_1 \geqq b_2$.

Proof of the proposition. — It is sufficient to prove the result in the two special cases $(u_1 = u_2, v_1 \geqq v_2)$ and $(u_1 \geqq u_2, v_1 = v_2)$. Let $n > 0$ be an integer and $(u^k, v^k)_{k\in\{0,\ldots,2n\}}$ the finite sequence of functions

$$u^0=u_1$$
$$u^{2k-1}=u^{2k}=\frac{n-k}{n}u_1+\frac{k}{n}u_2, \qquad k\in\{1,\ldots,n\}$$
$$v^{2k+1}=v^{2k}=\frac{n-k}{n}v_1+\frac{k}{n}v_2, \qquad k\in\{0,\ldots,n-1\}$$
$$v^{2n}=v_2.$$

As $(\alpha, \beta, u_1, v_1)$ and $(\alpha, \beta, u_2, v_2)$ are balanced, there exists $\varepsilon>0$ such that

$$\alpha u_1(0)-\beta v_1(1)>\varepsilon, \qquad \alpha u_1(1)-\beta v_1(0)<-\varepsilon,$$
$$\alpha u_2(0)-\beta v_2(1)>\varepsilon, \qquad \alpha u_2(1)-\beta v_2(0)<-\varepsilon.$$

Hence, for all $k\in\{0,\ldots,n-1\}$,

$$\begin{aligned}\alpha u^{2k+1}(0)-\beta v^{2k+1}(1)\\ =\frac{n-k-1}{n}[\alpha u_1(0)-\beta v_1(1)]+\frac{k}{n}[\alpha u_2(0)-\beta v_2(1)]+\frac{1}{n}[\alpha u_2(0)-\beta v_1(1)]\\ >\frac{1}{n}[(n-1)\varepsilon+\alpha u_2(0)-\beta v_1(1)]\end{aligned}$$

which in turn leads to $\alpha u^{2k+1}(0)>\beta v^{2k+1}(1)$ if

$$n>1-\frac{\alpha u_2(0)-\beta v_1(1)}{\varepsilon}.$$

Also

$$\begin{aligned}\alpha u^{2k+1}(1)-\beta v^{2k+1}(0)\\ =\frac{n-k-1}{n}[\alpha u_1(1)-\beta v_1(0)]+\frac{k}{n}[\alpha u_2(1)-\beta v_2(0)]+\frac{1}{n}[\alpha u_2(1)-\beta v_1(0)]\\ <\frac{1}{n}[-(n-1)\varepsilon+\alpha u_2(1)-\beta v_1(0)]\end{aligned}$$

and $\alpha u^{2k+1}(1)<\beta v^{2k+1}(0)$ if

$$n>1+\frac{\alpha u_2(1)-\beta v_1(0)}{\varepsilon}.$$

Moreover,

$$\alpha u^{2k}(0)-\beta v^{2k}(1)=\frac{n-k}{n}[\alpha u_1(0)-\beta v_1(1)]+\frac{k}{n}[\alpha u_2(0)-\beta v_2(1)]>0,$$
$$\alpha u^{2k}(1)-\beta v^{2k}(0)=\frac{n-k}{n}[\alpha u_1(1)-\beta v_1(0)]+\frac{k}{n}[\alpha u_2(1)-\beta v_2(0)]<0.$$

Hence $(\alpha, \beta, u^k, v^k)$ is balanced for all $k\in\{0,\ldots,2n\}$ if n is big enough. It suffices to compare the solution to (R) for $(u^k.v^k)$ et (u^{k+1}, v^{k+1}) with $k=0,\ldots,2n-1$.

Also the symmetry of the system (5.1) to (5.7) implies that each of the special case reduces to the other one. So, in the proof, we will suppose that $u_1 \geqq u_2$ and $v_1 = v_2 = v$.

We now introduce the auxiliary functions $c_1(z) = b_1(1-z)$, $c_2(z) = b_2(1-z)$ and $w(z) = v(1-z)$. With these notations, the system becomes, with $i = 1,2$.

$$a_i'(z)\varphi_i(z) + u_i(z) - \nu a_i(z) = 0, \qquad \forall z \in]0, 1[$$
$$c_i'(z)\varphi_i(z) + w(z) - \nu c_i(z) = 0, \qquad \forall z \in]0, 1[$$

with

$$\varphi_i(z) = -f(a_i(z), c_i(z)) \quad \text{if } z > z_i^*$$
$$\varphi_i(z) = f(c_i(z), a_i(z)) \quad \text{if } z < z_i^*$$

and

$$\nu a_i(z^*) = u_i(z_i^*)$$
$$\nu c_i(z^*) = w(z_i^*).$$

Recall that z_i^* is the switch point related to the balanced pair (u_i, v), *i.e.* $\alpha u_i(z_i^*) = \beta w(z_i^*)$.

The following lemma will be very useful:

LEMMA 3. – *If $u_1 \geqq u_2$ and $v_1 = v_2$, then*

$$z_1^* \geqq z_2^* \tag{5.13}$$
$$u_1(z_1^*) \geqq u_2(z_2^*) \tag{5.14}$$
$$a_1(z_2^*) \geqq a_1(z_1^*) \geqq a_2(z_2^*) \geqq a_2(z_1^*) \tag{5.15}$$
$$c_1(z) \geqq c_2(z), \qquad \forall z \in [z_2^*, z_1^*] \tag{5.16}$$
$$c_1(z_2^*) \geqq c_2(z_2^*), \qquad c_1(z_1^*) \geqq c_2(z_1^*). \tag{5.17}$$

Proof. – Suppose $z_1^* < z_2^*$. Take $z \in]z_1^*, z_2^*[$. From the definition of the switch point, we get $\alpha u_2(z) > \beta w(z) > \alpha u_1(z)$, which contradicts $u_1 \geqq u_2$. Hence (5.13) is proved.

As w is increasing, we have $w(z_1^*) \geqq w(z_2^*)$. Hence we get

$$u_1(z_1^*) = \frac{\beta}{\alpha} w(z_1^*) \geqq \frac{\beta}{\alpha} w(z_2^*) = u_2(z_2^*)$$

which gives (5.14).

As a_1 and a_2 are decreasing, (5.13) implies

$$a_1(z_2^*) \geqq a_1(z_1^*) \qquad \text{and} \qquad a_2(z_2^*) \geqq a_2(z_1^*).$$

Also

$$a_1(z_1^*) = \frac{1}{\nu} u_1(z_1^*) \geqq \frac{1}{\nu} u_2(z_2^*) \geqq a_2(z_2^*).$$

Hence (5.15).

From (5.8) it follows that for all $z \leqq z_1$, $\nu c_1(z) \geqq w(z)$, and for all $z \geqq z_2$, $\nu c_2(z) \leqq w(z)$. Hence, we get (5.16) and (5.17).

Q.E.D.

We have to prove that inequalities $a_1 \geqq a_2$ and $c_1 \geqq c_2$ hold on [0, 1].

We will split the interval (and the proof) in the three pieces $[0, z_2^*]$, $[z_2^*, z_1^*]$, $[z_1^*, 1]$.

LEMMA 4. – *$a_1 \geqq a_2$ and $c_1 \geqq c_2$ on $[z_2^*, z_1^*]$.*

Proof. – Since a_1 and a_2 are decreasing, we have for all $z \in [z_2^*, z_1^*]$ that $a_1(z) \geqq a_1(z_1^*)$ and $a_2(z) \leqq a_2(z_2^*)$. This, together with $a_1(z_1^*) \geqq a_2(z_2^*)$ gives the result.

Q.E.D.

LEMMA 5. – *$a_1 \geqq a_2$ and $c_1 \geqq c_2$ on $[z_1^*, 1]$.*

Proof. – Since $z_1^* \geqq z_2^*$, the system becomes on $[z_1^*, 1]$:

$$a_i'(z) = g_i(z, a_i(z), c_i(z))$$
$$c_i'(z) = h(z, a_i(z), c_i(z))$$

with

$$g_i(z, A, C) = \frac{(u_i(z) - \nu A)}{f(A, \beta C)}$$
$$h(z, A, C) = \frac{(w(z) - \nu C)}{f(A, \beta C)}.$$

Hence the function $C \to g_2(z, A, C)$ is increasing for all $z \in [z_1^*, 1]$ and A such that $u_2(z) - \nu A < 0$, while the function $A \mapsto h(z, A, C)$ is increasing for all $z \in [z_1^*, 1]$ and C such that $w(z) - \nu C > 0$.

We let $(.)_+$ denote $\max(., 0)$, and set

$$x(z) = (a_2(z) - a_1(z), 0)_+^2 + (c_2(z) - c_1(z), 0)_+^2.$$

x is C^1, and from lemma 4, we have that $x(z_1^*) = 0$. We want to show that $x = 0$ on $[z_1^*, 1]$.

We define a and c by

$$a(z) = \max(a_1(z), a_2(z)) \qquad \text{and} \qquad c(z) = \max(c_1(z), c_2(z)).$$

Since $c \geqq c_2$, we get

$$g_2(z, a_2(z), c(z)) \geqq g_2(z, a_2(z), c_2(z)).$$

Hence, we obtain

$$\begin{aligned} a_2'(z) - a_1'(z) &= g_2(z, a_2(z), c_2(z)) - g_1(z, a_1(z), c_1(z)) \\ &\leqq g_2(z, a_2(z), c(z)) - g_1(z, a_1(z), c_1(z)) \\ &\leqq g_1(z, a_2(z), c(z)) - g_1(z, a_1(z), c_1(z)). \end{aligned}$$

We now choose ε small enough so that $\varepsilon \leqq a_i(z) \leqq 1/\varepsilon$ and $\varepsilon \leqq c_i(z) \leqq 1/\varepsilon$ for $i = 1,2$ and all $z \in [z_1^*, 1]$, and we define

$$R = \sup \left\{ \left| \frac{\partial g_1}{\partial A}(z, A, C) \right| + \left| \frac{\partial g_1}{\partial C}(z, A, C) \right|, \varepsilon \leqq A, C \leqq \frac{1}{\varepsilon} \right\}.$$

Hence, we have

$$|g_1(z, A, C) - g_1(z, A', C')| \leqq R(|A - A'| + |C - C'|),$$

and we get

$$a_2'(z) - a_1'(z) \leqq R(|a_2(z) - a_1(z)| + |c(z) - c_1(z)|).$$

This gives us

$$\begin{aligned}(a_2'(z) - a_1'(z))(a_2(z) - a_1(z))_+ &\\ \leqq R(|a_2(z) - a_1(z)| + |c(z) - c_1(z)|)(a_2(z) - a_1(z))_+ &\\ \leqq R((a_2(z) - a_1(z))_+ + (c_2(z) - c_1(z))_+)(a_2(z) - a_1(z))_+.&\end{aligned}$$

Doing the same computation on c_1, c_2 and g_2, we get

$$\begin{aligned}(c_2'(z) - c_1'(z))(c_2(z) - c_1(z))_+ \leqq R'((a_2(z) - a_1(z))_+ &\\ + (c_2(z) - c_1(z))_+)(c_2(z) - c_1(z))_+.&\end{aligned}$$

Hence, we have, with $R'' = \operatorname{Sup}\{R, R'\}$,

$$\begin{aligned}x'(z) &\leqq R''[(a_2(z) - a_1(z))_+ + (c_2(z) - c_1(z))_+]^2\\ &\leqq 2R''x(z).\end{aligned}$$

Since $x(z_1^*) = 0$, Gronwall's lemma gives us $x(z) = 0$ for all $z \in [z_1^*, 1]$.

Q.E.D.

LEMMA 6. — *$a_1 \geqq a_2$ and $c_1 \geqq c_2$ on $[0, z_2^*]$.*

Proof. — On $[0, z_2^*]$, the system becomes:

$$\begin{aligned}a_i'(z) &= g_i(z, a_i(z), c_i(z))\\ c_i'(z) &= h(z, a_i(z), c_i(z))\end{aligned}$$

with

$$\begin{aligned}g_i(z, A, C) &= -\frac{(u_i(z) - \nu A)}{f(C, \alpha A)}\\ h(z, A, C) &= -\frac{(w(z) - \nu C)}{f(C, \alpha A)}.\end{aligned}$$

The same arguments as in the proof of lemma 5 establish the result.

Q.E.D.

The proof of proposition 6 is now complete, and the map F is well defined in the balanced case.

5.2. The general case

Using propositions 5 and 6, we are able to make the proof of theorem 2 by an approximation procedure.

Given two constants $e>0$ and $d>1$, we define the function $\pi_{e,d}:[0, 1]\to \mathbf{R}_+$ by

$$\pi_{e,d}(x)=\begin{cases} e-dex & \text{if } x\leqq 1/d \\ 0 & \text{if } x\geqq 1/d. \end{cases} \tag{5.18}$$

For any u in E, we define $u_{e,d}=u+\pi_{e,d}$. From $e>0$ and $d>1$, we see that $u_{e,d}$ is in E.

LEMMA 7. — *For any pair* $(u, v)\in \mathrm{E}\times\mathrm{E}$, *and for d and e such that* $d>1$ *and* $e>\max\{u(1), v(1)\}\beta/\alpha$, *the pair* $(u_{e,d}, u_{e,d})\in \mathrm{E}\times\mathrm{E}$ *is balanced.*

Proof. — We clearly have that $\alpha u(0)>\beta v(1)$ and $\alpha u(1)<\beta v(0)$ which gives the result.

Q.E.D.

PROPOSITION 7. — *If* $e>\sup\{u(1), v(1)\}\beta/\alpha$ *and*

$$(a_n, b_n)=\mathrm{F}(\alpha, \beta, \nu, u_{e,n}, v_{e,n}). \tag{5.19}$$

then a_n *(resp.* b_n*) converges (at least) pointwise to* a *(resp.* b*) where* $(a, b)\in \mathrm{E}\times\mathrm{E}$ *is the unique solution of system* (R).

COROLLARY 5. — *The map* $(u, v)\mapsto \mathrm{F}(\alpha, \beta, \nu, u, v)$ *is increasing from* $\mathrm{E}\times\mathrm{E}$ *into itself.*

Proof. — This may be checked by using proposition 6 applied to $(u_{e,n}, v_{e,n})$ and by taking the limit when $n\to\infty$.

Q.E.D.

Proof of the proposition. — Lemma 7 implies that for all n, the pair $(u_{e,n}, v_{e,n})$ is balanced. It follows from the monotony of F, that since the sequence $n\mapsto(u_{e,n}, v_{e,n})$ is decreasing (this can be easily checked), the sequence $n\mapsto(a_n, b_n)$ is also decreasing. Also, for all n, the functions a_n and b_n are positive. Hence, the sequence (a_n, b_n) converges pointwise to a pair (a, b).

We set $w_{e,n}(z)=v_{e,n}(1-z)$, $w(z)=v(1-z)$, $c_n(z)=b_n(1-z)$. Recall that $z_n^*\in]0, 1[$ is the switch point associated to $(u_{e,n}, v_{e,n})$. For all $n\in\mathbf{N}$, (a_n, b_n) is the unique solution in $\mathrm{E}\times\mathrm{E}$ of:

$$\begin{cases} a_n'=(u_{e,n}-\nu a_n)/f(a_n, \beta c_n) \\ c_n'=(w_{e,n}-\nu c_n)/f(a_n, \beta c_n) \end{cases} \tag{5.20}$$

on $]z_n^*, 1]$,

$$\begin{cases} a_n'=-(u_{e,n}-\nu a_n)/f(c_n, \alpha a_n) \\ c_n'=-(w_{e,n}-\nu c_n)/f(c_n, \alpha a_n) \end{cases} \tag{5.21}$$

on $[0, z_n^*[$, and

$$\begin{cases} \nu a_n(z^*)=u_{e,n}(z^*) \\ \nu c_n(z^*)=w_{e,n}(z^*). \end{cases} \tag{5.22}$$

It follows from the definition of z_n^* that $z_n^* \to z^*$ when $n \to \infty$. Proposition 5 implies that

$$\min\{u_{e,n}(z), u_{e,n}(z_n^*)\}/\nu \leqq a_n(z) \leqq \max\{u_{e,n}(z), u_{e,n}(z_n^*)\}/\nu$$
$$\min\{w_{e,n}(z), w_{e,n}(z_n^*)\}/\nu \leqq a_n(z) \leqq \max\{w_{e,n}(z), w_{e,n}(z_n^*)\}/\nu.$$

Also.

$$u(z) \leqq u_{e,n}(z) \leqq u(z) + e$$
$$w(z) \leqq w_{e,n}(z) \leqq w(z) + e.$$

Since u and w are strictly positive and continuous, this implies the existence of $\varepsilon > 0$ such that for all n and $z \in [0, 1]$, $\varepsilon \leqq a_n(z) \leqq 1/\varepsilon$ and $\varepsilon \leqq c_n(z) \leqq 1/\varepsilon$. Moreover, f is bounded on $[\varepsilon, 1/\varepsilon] \times [\varepsilon, 1/\varepsilon]$ by strictly positive constants. Hence, from equations (5.21) and (5.22), we have that $|a_n'|$ and $|c_n'|$ are bounded on [0, 1] by constants that are independent of n, which gives that a_n and c_n are Lipschitz on $[z_0, z_1]$ uniformly in n. This provides the following lemma:

LEMMA 8. – *The convergence of* (a_n, c_n) *to* (a, c) *is uniform on* $[z_0, z_1]$.

Proof. – a_n and c_n are uniformly Lipschitz and bounded on [0, 1], and converge pointwise to resp. a and c. From Ascoli's theorem, we have that the sequences a_n and c_n are compact with respect to the L^∞ norm. Hence, the convergence is uniform.

Q.E.D.

Similarly, the convergence is uniform on $[z_0, z_1]$ for every z_0 and z_1 such that $z_0 < z_1 < z^*$.

LEMMA 9. – *If* $z^* \in]0, 1[$, *then*

$$a(z^*) = u(z^*)/\nu$$
$$c(z^*) = w(z^*)/\nu$$
$$\alpha a(z^*) = \beta c(z^*).$$

If $z^* = 0$, *then*

$$c(z^*) = w(z^*)/\nu$$
$$\alpha a(z^*) = \beta c(z^*).$$

If $z^* = 1$, *then*

$$a(z^*) = u(z^*)/\nu$$
$$\alpha a(z^*) = \beta c(z^*).$$

Proof. – In the first case, since $z_n^* \to z^*$, we may take $\varepsilon > 0$ such that $z_n^* \in [\varepsilon, 1-\varepsilon]$ for n big enough. For n big enough, we have that $u_{e,n} = u$ and $w_{e,n} = w$ on $[\varepsilon, 1-\varepsilon]$. Hence $u_{e,n}(z_n^*) \to u(z^*)$ and $w_{e,n}(z_n^*) \to w(z^*)$. Because of the uniform convergence of a_n and c_n, we also have $a_n(z_n^*) \to a(z^*)$ and $c_n(z_n^*) \to c(z^*)$.

The result in the second and in the third case are obtained by the fact that $w_{e,n}=w$ on $[0, 1-\varepsilon]$ (resp. $u_{e,n}=u$ on $[\varepsilon, 1]$) for n big enough and by the same argument of uniform convergence.

Q.E.D.

We now take z_0 and z_1 such that $z^*<z_0<z_1$. For n big enough, we have $z_n^*<z_0$.

LEMMA 10. — *The functions a and c are* C^1 *on* $[z_0, z_1]$ *and satisfy*

$$\begin{cases} a'(z)=(u(z)-\nu\, a(z))/f(a(z), \beta\, c(z)) \\ c'(z)=-(w(z)-\nu\, c(z))/f(a(z), \beta\, c(z)) \end{cases}$$

on $[z_0, z_1]$.

Proof. — For n big enough and all $z\in[z_0, z_1]$, we have, since $z_n^*<z_0$,

$$\begin{cases} a_n(z)=a_n(z_0)+\displaystyle\int_{z_0}^{z}(u_n(s)-\nu\, a_n(s))/f(a_n(s), \beta\, c_n(s))\,ds \\ c_n(z)=c_n(z_0)+\displaystyle\int_{z_0}^{z}(w_n(s)-\nu\, c_n(s))/f(a_n(s), \beta\, c_n(s))\,ds \end{cases} \tag{5.23}$$

Since $\varepsilon\leqq a_n\leqq 1/\varepsilon$ and $\varepsilon\leqq c_n\leqq 1/\varepsilon$ on $[z_0, z_1]$ uniformly in n, we have that $\varepsilon\leqq a\leqq 1/\varepsilon$ and $\varepsilon\leqq c\leqq 1/\varepsilon$ on $[z_0, z_1]$. Since f is Lipschitz on $[\varepsilon, 1/\varepsilon]^2$, and because of the uniform convergence of (a_n, c_n) to (a, c) on $[z_0, z_1]$, we can take the limit in (5.23), and we obtain

$$\begin{cases} a(z)=a(z_0)+\displaystyle\int_{z_0}^{z}(u(s)-\nu\, a(s))/f(a(s), \beta\, c(s))\,ds \\ c(z)=c(z_0)+\displaystyle\int_{z_0}^{z}(w(s)-\nu\, c(s))/f(a(s), \beta\, c(s))\,ds. \end{cases} \tag{5.24}$$

The uniform convergence of the continuous functions a_n and c_n implies that a and c are continuous on $[z_0, z_1]$. Hence (5.24) implies that a and c are C^1 on $[z_0, z_1]$.

Q.E.D.

Using the same, we may extend the result for every z_0 and z_1 such that $z_0<z_1<z^*$. Hence, a and c are C^1 on $[0, z^*[\bigcup]z^*, 1]$. Lemma 9 implies that a and c are in fact continuous on $[0, 1]$. Equation (5.24) then implies that a and c are C^1 on $[0, 1]$.

The proof of the proposition will now be achieved by the following lemma:

LEMMA 11. — *a (resp. c) is strictly decreasing (resp. increasing) on* $[0, 1]$.

Proof. — We will establish the result concerning a. The proof of the result on c is exactly the same. Because of the convergence of a_n to a, and since for all n a_n is strictly decreasing, we have that a is decreasing.

Suppose that a is not strictly decreasing. Then there exist z_0 and z_1 with $z_0<z_1$ such that $a(z_0)=a(z_1)$. a is then constant on $[z_0, z_1]$. Hence, even if $z^*\in[z_0, z_1]$, there exist z_2 and z_3 such that for instance $z^*<z_2<z_3$ and $a(z_2)=a(z_3)$. We also have

$$a(z_3)=a(z_2)+\int_{z_2}^{z_3}(u(s)-\nu a(s))/f(a(s), \beta c(s))\,ds,$$

and since $f>0$, this implies $a(z)=u(z)/\nu$ for all $z\in[z_2, z_3]$. But u is strictly decreasing and this provides a contradiction.

Q.E.D.

Using the inequalities of (5.8) and the convergence, we get

$$\left\{\begin{array}{lll} u(z^*)\leqq \nu a(z)\leqq u(z) & \text{if} & z<z^* \\ u(z)\leqq \nu a(z)\leqq u(z^*) & \text{if} & z>z^* \\ v(1-z)\leqq \nu b(1-z)\leqq v(1-z^*) & \text{if} & z<z^* \\ v(1-z^*)\leqq \nu b(1-z)\leqq v(1-z) & \text{if} & z>z^* \end{array}\right. \tag{5.25}$$

These inequalities will be useful in section 6

6. SUB AND SUPERSOLUTIONS

This section provides the proofs of propositions 2 to 3.

6.1. A supersolution

The proof of proposition 2 will be provided by the two following lemmas:

LEMMA 12. – *For any $\gamma>0$, there exist a function $A\in C^1([0, 1])$ such that*

$$A'(z)=-\mu A(z)\frac{A(0)}{g(A(z)/A(0))}$$

and

$$A(0)=\gamma A(1)>0.$$

Proof. – Set $A(0)=x$ and $y(x)=A(z)/x$. The problem is equivalent to find a function $y_x\in E$ such that

$$y'_x(z)=-\mu x\frac{y_x(z)}{g(y_x(z))},\qquad y_x(0)=1,\quad x>0, \tag{6.1}$$

and $y_x(1)=1/\gamma$. Since (6.1) is a standard Cauchy Problem and since $y\mapsto y/g(y)$ is locally Lipschitz on $]0, +\infty[$, there is a unique solution on a maximal interval I_x^+ which is either [0, 1] or $[0, z_x^+[$ for some $z_x^+ < 1$. On $]0, +\infty[$, we can define a strictly increasing and C^1 function h by

$$h(y) = \int_1^y \frac{g(s)}{s} ds.$$

h has an inverse function h^{-1} which is also strictly increasing and C^1. On I_x^+, we have $y_x(z) = h^{-1}(-\mu x z)$. Since $s \mapsto g(s)$ is decreasing, we have that $\int_1^y g(s)/s\, ds$ is divergent in the neighborhood of 0. Hence $h(y) \to -\infty$ when $y \to 0$. Moreover $h(1) = 0$, and hence $]-\infty, 0] \subset \operatorname{Supp} h^{-1}$ which implies that for all $z \in [0, 1]$, $-\mu x z \in \operatorname{Supp} h^{-1}$. Hence $I_x^+ = [0, 1]$. Thus $y_x(1) = h^{-1}(-\mu x)$, and in particular $x \mapsto y_x(1)$ is continuous, strictly decreasing, and $\lim_{x \to +\infty} y_x(1) = \lim_{x \to +\infty} h^{-1}(\mu x) = 0$. This implies that there exist a unique $x_\gamma > 0$ for which we have $y_{x_\gamma}(1) = 1/\gamma$. Moreover, x_γ is characterized by $x_\gamma = -h(1/\gamma)/\mu$.

LEMMA 13. — *Take* A *defined as in lemma* 12 *with* $\gamma > \sup\{\alpha_j / j \in J\} / \inf\{\alpha_j / j \in J\}$ *and let* $\bar{a}_j = A$ *for all* $j \in J$. *Then* $\bar{a} = (\bar{a}_j)$ is a supersolution.

Proof. — Since $\bar{a} \in E^J$, we need to show that $\tilde{a} = H \circ G(\bar{a}) \leqq \bar{a}$. Fix $j \in J$. Then $(\tilde{a}_j, \tilde{a}_{-j}) = F(\alpha_j, \alpha_{-j}, \lambda_j + \mu, u_j, u_{-1})$ with

$$u_j = \sum_{k \in J} \lambda_{j,k} \bar{a}_k = \lambda_j A$$

and

$$u_{-j} = \sum_{k \in J} \lambda_{-j,k} \bar{a}_k = \sum_{k \in J} \lambda_{j,-k} \bar{a}_k = \sum_{k \in J} \lambda_{j,k} \bar{a}_{-k} = \lambda_j A = u_j.$$

Since $\gamma \geqq \alpha_{-j}/\alpha_j$ and $\gamma \geqq \alpha_j/\alpha_{-j}$,

$$\alpha_j u_j(0) = \lambda_j \alpha_j A(0) > \lambda_j \alpha_{-j} A(1) = \alpha_{-j} u_j(1)$$

and

$$\alpha_j u_j(1) = \lambda_j \alpha_j A(1) < \lambda_j \alpha_{-j} A(0) = \alpha_{-j} u_j(0).$$

Hence, the pair (u_j, u_j) is balanced. Set $u(z) = u_j(z)$, $w(z) = u_j(1-z)$, $a(z) = \tilde{a}_j(z)$, $c(z) = \tilde{a}_{-j}(1-z)$ and $\nu = \lambda_j + \mu$. Let z^* be the switch point associated to (u, u). Hence $z^* \in]0, 1[$. From (5.8), we have on $[0, z^*]$ that

$$a(z) \leqq \frac{u(z)}{\nu} = \frac{\lambda_j}{\lambda_j + \mu} A(z) < A(z) = \bar{a}_j(z).$$

On $[z^*, 1]$, (a, c) satisfies

$$\begin{cases} a' = (u - \nu a) c / g(a/\beta c) \\ c' = (\omega - \nu c) c / g(a/\beta c) \\ \quad \nu a(z^*) = u(z^*) \\ \quad \nu c(z^*) = \omega(z^*). \end{cases}$$

The inequalities of (5.8) give

$$c(z) \leqq \frac{w(z)}{\nu} = \frac{\lambda_j}{\lambda_j + \mu} A(1-z) < A(1-z) \leqq A(0).$$

Hence, since $c \mapsto g(a/c)/c$ is increasing, we get

$$a'(z) < (u - \nu a(z)) \frac{\beta A(0)}{g(a(z)/\beta A(0)} = (\lambda_j A(z) - (\lambda_j + \mu) a(z)) \frac{\beta A(0)}{g(a(z)/\beta A(0))}.$$

Moreover, we have

$$A'(z) = -\mu A(z) \frac{A(0)}{g(A(z)/A(0))}$$

and $a(z^*) = \lambda_j/(\lambda_j + \mu) A(z^*) < A(z^*)$. It is then easy to see that $a(z) \leqq A(z)$ on $[z^*, 1]$: by continuity, we have $a(z) \leqq A(z)$ in the neighborhood of z^*. We set $z^+ = \inf\{z, a(z) \geqq A(z)\}$. If $z^+ < 1$, we have $a(z^+) = A(z^+)$, and

$$a'(z^+) < -\mu A(z^+) \frac{\beta A(0)}{g(A(z^+)/\beta A(0))} \leqq -\mu A(z^+) \frac{A(0)}{g(A(z^+)/A(0))} = A'(z^+).$$

Hence, there exists $z > z^+$ such that $a(z) < A(z)$, which contradicts the definition of z^+. This implies $z^+ = 1$ and $a(z) \leqq A(z)$ on $[z^*, 1]$.

Q.E.D.

6.2. A Subsolution

Recall that $\rho > 1$, K is a w-subset such that $\rho = \rho(K)$ and B is a positive eigenvector of $\theta = \theta(K)$ associated to the eigenvalue ρ.

Because of the definition of B, and since $\rho > 1$, we have

$$\theta(B_j)_{j \in K} = \rho(B_j)_{j \in K} > (B_j)_{j \in K}$$

that is

$$\forall j \in K, \quad (\lambda_j + \mu) B_j < \sum_{k \in K} \left(\lambda_{j,k} + \lambda_{j,-k} \frac{\alpha_k}{\alpha_{-k}} \right) B_k \tag{6.2}$$

For $j \in (-K)$, B_j is defined by $B_j = B_{-j} \alpha_{-j}/\alpha_j$. For ε small enough, εB would be a subsolution, *i.e.* $\varepsilon B \leqq H \circ G(\varepsilon B)$. But for all j, B_j is constant and εB is not in E^J. Hence, we introduce $B^\eta = (B_j^\eta)$ with $B_j^\eta(z) = B_j - \eta z$ and show that there exist ε and η such that εB^η is a subsolution in E^J.

It is clear that for η small enough, $\beta^\eta \in E^J$.

Define $u^\eta \in E^J$ by

$$u_j^\eta = \sum_{k \in J} \lambda_{j,k} B_k^\eta \quad \text{for each } j \in J.$$

We write $z_j^{*\eta}$ for the switch point associated to the pair (u_j^η, u_{-j}^η). Define $\tilde{A}^\eta$ by

$$\forall j\in J,\quad \tilde{A}_j^\eta(z)=\begin{cases}\dfrac{u_j^\eta(z)}{\lambda_j+\mu} & \text{if}\quad z<z_j^{*\eta}\\[2ex] \dfrac{u_{-j}^\eta(z)}{\lambda_j+\mu}\,\dfrac{\alpha_{-j}}{\alpha_j} & \text{if}\quad z>z_j^{*\eta}\end{cases}$$

and $\tilde{A}_j^\eta$ continuously prolonged in $z_j^{*\eta}$. Also set $\tilde{B}^{\varepsilon,\eta}=H\circ G(\varepsilon B^\eta)$.

LEMMA 14. – *There exists $\delta>0$ such for η sufficiently small,*

$$\tilde{A}_j^\eta(z)>\delta+B_j^\eta,\qquad \forall z\in[0,1],\quad \forall j\in J.$$

Proof. – The strict inequality (6.2) implies the existence of $\delta^*>0$ such that

$$\forall j\in K,\quad (\lambda_j+\mu)B_j+2\delta^*<\sum_{k\in K}\left(\lambda_{j,k}+\lambda_{j,-k}\frac{\alpha_k}{\alpha_{-k}}\right)B_k.$$

It is then clear that for $\eta>0$ sufficiently small, we have

$$\forall j\in K,\quad \forall z\in[0,1],\quad (\lambda_j+\mu)B_j^\eta(z)+\delta^*<\sum_{k\in K}\left(\lambda_{j,k}+\lambda_{j,-k}\frac{\alpha_k}{\alpha_{-k}}\right)B_k^\eta(z).$$

From the definition of u^η, this implies

$$\forall j\in K,\quad \forall z\in[0,1],\quad u_j^\eta(z)>\delta^*+(\lambda_j+\mu)B_j^\eta(z).$$

To prove lemma 14, take first $j\in K$. If $z<z_j^{*\eta}$, then

$$\tilde{A}_j^\eta(z)=\frac{u_j^\eta(z)}{\lambda_j+\mu}>B_j^\eta(z)+\frac{\delta^*}{\lambda_j+\mu}.$$

If $z>z_j^{*\eta}$, then

$$\tilde{A}_j^\eta(z)=\frac{u_{-j}^\eta(z)}{\lambda_j+\mu}\,\frac{\alpha_{-j}}{\alpha_j}\geqq\frac{u_j^\eta}{\lambda_j+\mu}>B_j^\eta(z)+\frac{\delta^*}{\lambda_j+\mu}.$$

If $z=z_j^{*\eta}$, we obtain the same inequality by continuity. Take now $j\in(-K)$. If $z<z_j^{*\eta}$, then

$$\tilde{A}_j^\eta(z)=\frac{u_j^\eta(z)}{\lambda_j+\mu}\geqq\frac{u_{-j}^\eta(z)}{\lambda_j+\mu}\,\frac{\alpha_{-j}}{\alpha_j}>B_{-j}^\eta(z)\frac{\alpha_{-j}}{\alpha_j}+\frac{\delta^*}{\lambda_j+\mu}\,\frac{\alpha_{-j}}{\alpha_j}=B_j^\eta(z)+\frac{\delta^*}{\lambda_j+\mu}.$$

If $z>z_j^{*\eta}$, then

$$\tilde{A}_j^\eta(z)=\frac{u_{-j}^\eta(z)}{\lambda_j+\mu}\,\frac{\alpha_{-j}}{\alpha_j}=\tilde{A}_{-j}^\eta(z)\frac{\alpha_{-j}}{\alpha_j}>B_{-j}^\eta(z)\frac{\alpha_{-j}}{\alpha_j}+\frac{\delta^*}{\lambda_j+\mu}\,\frac{\alpha_{-j}}{\alpha_j}=B_j^\eta(z)+\frac{\delta^*}{\lambda_j+\mu}.$$

If $z=z_j^{*\eta}$, we obtain the same inequality by continuity.

Since J is finite, we may take for δ

$$\delta=\inf\left\{\frac{\delta^*}{\lambda_j+\mu}, \frac{\delta^*}{\lambda_j+\mu}\frac{\alpha_{-j}}{\alpha_j} \quad j\in J\right\}>0.$$

Q.E.D.

LEMMA 15. — *For ε and η sufficiently small, we have*

$$\forall z\in[0, 1], \quad \forall j\in J, \qquad \frac{1}{\varepsilon}\tilde{B}_j^{\varepsilon, \eta}(z)\geqq\tilde{A}_j^\eta(z)-\frac{\delta}{2}.$$

Proof. — Fix η and ε. Take $j\in J$. The problem will be separated in the two cases $z\in[0, z_j^{*\eta}]$ (if $z_j^{*\eta}>0$) and $z\in[z_j^{*\eta}, 1]$ (if $z_j^{*\eta}<1$).

On $[0, z_j^{*\eta}]$, since $\tilde{B}_j^{\varepsilon, \eta}$ is a solution of system (S) for (u_j^η, u_j^η), we have

$$\tilde{B}_j^{\varepsilon, \eta}(z)\geqq\tilde{B}_j^{\varepsilon, \eta}(z_j^{*\eta})=\varepsilon\frac{u_j^\eta(z_j^{*\eta})}{\lambda_j+\mu}.$$

We also have $u_j^\eta(z)/(\lambda_j+\mu)=\tilde{A}_j^\eta(z)$ and

$$0\leqq u_j^\eta(z)-u_j^\eta(z_j^{*\eta}))\leqq\eta\sum_{k\in J}\lambda_{j, k}.$$

Hence,

$$\frac{1}{\varepsilon}\tilde{B}_j^{\varepsilon, \eta}(z)\geqq\tilde{A}_j^\eta(z)-\frac{\eta}{\varepsilon}\sum_{k\in J}\lambda_{j, k}.$$

On $[z_j^{*\eta}, 1]$, set $\nu=\lambda_j+\mu$. $\tilde{B}_j^{\varepsilon, \eta}$ satisfies

$$\begin{cases}\dfrac{d}{dz}\tilde{B}_j^{\varepsilon, \eta}(z)=(\varepsilon u_j^\eta(z)-\nu\tilde{B}_j^{\varepsilon, \eta}(z))/f(\tilde{B}_j^{\varepsilon, \eta}(z), \alpha_{-j}\tilde{B}_{-j}^{\varepsilon, \eta}(1-z))\\ \tilde{B}_j^{\varepsilon, \eta}(z_j^{*\eta})=\varepsilon\dfrac{\alpha_{-j}}{\alpha_j}\dfrac{u_{-j}^\eta(z_j^{*\eta})}{\nu}.\end{cases}$$

(since $u_{-j}^\eta(z_j^{*\eta})\alpha_{-j}/\alpha_j=u_j^\eta(z_j^{*\eta})$ if $z_j^{*\eta}>0$). The inequalities of (5.25) imply

$$\varepsilon u_j^\eta(z)\leqq\nu\tilde{B}_j^{\varepsilon, \eta}(z)\leqq\varepsilon u_j^\eta(z_j^{*\eta})$$
$$\varepsilon u_{-j}^\eta(1-z)\leqq\nu\tilde{B}_{-j}^{\varepsilon, \eta}(1-z)\leqq\varepsilon u_{-j}^\eta(1-z_j^{*\eta})$$

This implies (since $a\mapsto f(a, b)$ is decreasing and $b\mapsto f(a.b)$ is increasing) that

$$f(\tilde{B}_j^{\varepsilon, \eta}(z), \alpha_{-j}\tilde{B}_{-j}^{\varepsilon, \eta}(1-z))\geqq f\left(\varepsilon\frac{u_j^\eta(1)}{\nu}, \alpha_{-j}\varepsilon\frac{u_{-j}^\eta(0)}{\nu}\right)$$
$$=\varepsilon g\left(\frac{u_j^\eta(1)}{\alpha_{-j}u_{-j}^\eta(0)}\right)\frac{1}{\alpha_{-j}u_{-j}^\eta(0)}.$$

Hence, for all z in $[z_j^{*\eta}, 1]$,

$$0\leqq-\tilde{B}_j^{\varepsilon, \eta}(z)\leqq\varepsilon^2\frac{u_j^\eta(0)-u_j^\eta(1)}{g(u_j^\eta(1)/\alpha_{-j}u_{-j}^\eta(0))}\alpha_{-j}u_{-j}^\eta(0)$$

It is clear from the definition of u^η that

$$\frac{u_j^\eta(0)-u_j^\eta(1)}{g(u_j^\eta(1)/\alpha_{-j}u_{-j}^\eta(0))}\alpha_{-j}u_{-j}^\eta(0)\geqq R_j$$

with R_j a strictly positive constant independent of η. Hence, we get $0\leqq -\tilde{B}_j^{\varepsilon,\eta}(z)\leqq R_j\varepsilon^2$, which implies

$$\frac{1}{\varepsilon}\tilde{B}_j^{\varepsilon,\eta}(z)\geqq\frac{1}{\nu}\frac{\alpha_{-j}}{\alpha_j}u_{-j}^\eta(z_j^{*\eta})-R_j\varepsilon\geqq\frac{1}{\nu}\frac{\alpha_{-j}}{\alpha_j}u_{-j}^\eta(z)-R_j\varepsilon=\tilde{A}_j^\eta(z)-R_j\varepsilon.$$

To achieve the proof, we have to choose ε and η. Since J is finite, we may take ε small enough in order to get $R_j\varepsilon<\delta/2$ for all $j\in J$. We then take η small enough to get $(\eta/\varepsilon)\sum_{k\in J}\lambda_{j,k}<\delta/2$.

Q.E.D.

We then have $\tilde{B}^{\varepsilon,\eta}/\varepsilon\geqq B^\eta+\delta/2$ which establishes proposition 3.

7. THE CONVERGENCE OF THE FIXED POINT ALGORITHM

In this section, we give the proof of proposition 4.

We recall that the sequences $(\underline{a}^n)$ and $(\bar{a}_n)$ are defined by

$$\begin{cases}\underline{a}^0=\underline{a},\ \underline{a}^n=H\circ G(\underline{a}^{n-1})\\ \bar{a}^0=\bar{a},\ \bar{a}^n=H\circ G(\bar{a}^{n-1})\end{cases}$$

and that $\underline{a}$ and $\bar{a}$ are respectively a subsolution and a supersolution with $\underline{a}\leqq\bar{a}$.

Since $\underline{a}\leqq H\circ G(\underline{a})$, we have $\underline{a}^1\geqq\underline{a}^0$. By induction, we get for all $n\in N$, $\underline{a}^{n+1}\geqq\underline{a}$. Also by induction, we get $\bar{a}^{n+1}\leqq\bar{a}^n$ and $\underline{a}^n\leqq\bar{a}^n$. Hence, we have for all n that $\underline{a}^0\leqq\underline{a}^n\leqq\bar{a}^n\leqq\bar{a}^0$, and there exists $\delta>0$ such that for all $n\in N$, for all $j\in J$, for all $z\in[0, 1]$,

$$\delta\leqq\underline{a}_j^n(z)\leqq\bar{a}_j^n(z)\leqq\frac{1}{\delta}.$$

Since the sequences $(\underline{a}^n)$ and $(\bar{a}^n)$ are respectively increasing and decreasing, this implies that $\underline{a}^n$ and $\bar{a}^n$ converge pointwise to $\underline{a}^*$ and $\bar{a}^*$. From the bounds δ and $1/\delta$, we get that $\underline{a}^*$ and $\bar{a}^*$ are strictly positive. From $\underline{a}^n$ and $\bar{a}^n$ strictly decreasing for all n, we get that $\underline{a}^*$ and $\bar{a}^*$ are decreasing.

We want to show that $\underline{a}^*$ and $\bar{a}^*$ are in E and are fixed points of $H\circ G$. The proof will be done for $\underline{a}^*$, and is similar for $\bar{a}^*$.

LEMMA 16. — *$\underline{a}^n$ converges uniformly to $\underline{a}^*$ on* [0, 1], *and $\underline{a}^*$ is continuous on* [0, 1].

Proof. — Fix $j \in J$. We denote by z_j^{*n} the switch point associated to the pair $(\underline{a}_j^n, \underline{a}_{-j}^n)$. For all $n \in N$, the pair $(\underline{a}_j^n, \underline{a}_{-j}^n)$ is the solution of system (R) with $\alpha = \alpha_j$, $\beta = 1 - \alpha_j$, $\nu = \lambda_j + \mu$, $z^* = z_j^{*n}$, $u = \sum_{k \in J} \lambda_{j,k} \underline{a}_j^{n-1}$ and $v = \sum_{k \in J} \lambda_{-j,k} \underline{a}_{-j}^{n-1}$. For all n and j, $\underline{a}_j$ is bounded by δ and $1/\delta$. Since f is C^1, we have that $f(\underline{a}_j^n, \underline{a}_{-j}^n)$ is bounded by strictly positive constants. Hence, from equations (4.2) and (4.3), we get that $\underline{a}_j^{n\prime}$ and $\underline{a}_{-j}^{n\prime}$ are bounded independently of n. This implies that $\underline{a}^n$ is Lipschitz on [0, 1] uniformly in n. This and the pointwise convergence implies, from Ascoli's theorem, that the convergence of $\underline{a}^n$ to $\underline{a}^*$ is uniform on [0, 1]. Since for all $n \in N$, $\underline{a}^n$ is continuous, we have from the uniform convergence that $\underline{a}^*$ is continuous.

Q.E.D.

From the convergence of $\underline{a}^n$ to $\underline{a}^*$, we have that for all $j \in J$, $\underline{a}_j^*$ is decreasing. We need to prove that in fact $\underline{a}_j^*$ is strictly decreasing.

LEMMA 17. — *For all $j \in J$, $\underline{a}_j^*$ is strictly decreasing.*

Proof. — The proof will be by contradiction. Suppose that for some $j \in J$, $\underline{a}_j^*$ is not strictly decreasing. Then there exists some z_1 and z_2 with $0 \leqq z_1 < z_2 \leqq 1$ such that $\underline{a}_j^*$ is constant on $[z_1, z_2]$. For all $n \in N$ and $z \in [0, 1]$, we have that $\underline{a}_j^{n\prime}(z) \leqq 0$. Hence,

$$\int_{z_1}^{z_2} |\underline{a}_j^{n\prime}(z)|\,dz = -\int_{z_1}^{z_2} \underline{a}_j^{n\prime}(z)\,dz$$

$$= \underline{a}_j^n(z_1) - \underline{a}_j^n(z_2) \to \underline{a}_j^*(z_1) - \underline{a}_j^*(z_2) \quad \text{when } n \to \infty,$$

which means that the restriction of $\underline{a}_j^n$ on $[z_1, z_2]$ converges (strongly) to 0 in $L^1([z_1, z_2])$. Also we have

$$0 = \underline{a}_j^{n\prime}(z)\,\varphi_j^n(z) + \sum_{k \in K} \lambda_{j,k}\,\underline{a}_k^n(z) - (\lambda_j + \mu)\,\underline{a}_j^n(z)$$

where φ_j^n is bounded by strictly positive constants. From the above convergence of $\underline{a}_j^{n\prime}$ to 0 and from the uniform convergence of $\underline{a}^n$ to $\underline{a}^*$, we obtain that for all $z \in [z_1, z_2]$,

$$\sum_{k \in J} \lambda_{j,k} \int_{z_1}^{z} \underline{a}_k^*(s)\,ds - (\lambda_j + \mu) \int_{z_1}^{z} \underline{a}_j^*(s)\,ds = 0$$

which implies (recall $\underline{a}^*$ is continuous) that

$$\sum_{k \in J} \lambda_{j,k}\,\underline{a}_k^*(z) = (\lambda_j + \mu)\,\underline{a}_j^*(z) = (\lambda_j + \mu)\,\underline{a}_j^*(z_1).$$

Hence, $\sum_{k \in J} \lambda_{j,k}\,\underline{a}_k^*(z)$ is continuous on $[z_1, z_2]$. Since for all $k \in J$, $\underline{a}_k^*$ is continuous and decreasing, this implies that $\underline{a}_k^*$ is constant on $[z_1, z_2]$ for all $k \in J$ such that $\lambda_{j,k} \neq 0$.

By induction, we get that $\underline{a}_k^*$ is constant on $[z_1, z_2]$ for all $k \in K$ the strictly connected component of j, *i.e.*

$$K = \{ j \in J / \exists\, i_0, \ldots, i_m, i_0 = j, i_m = k, \lambda_{i_{h-1}, i_h} \neq 0, \forall h = 1, \ldots, m \}.$$

Note that for all $k \in K$, K is also the strictly connected component of $-k$. Hence, by induction on the above equality, we get that for all $k \in K$,

$$\sum_{h \in K} \lambda_{k,h} A_h^* = (\lambda_k + \mu) A_k^*$$

where A_k^* denotes the value of $\underline{a}_k^*$ on $[z_1, z_2]$. This may be rewritten has $A^* = LA^*$ where $A^* = (A_k^*, k \in K) \in \mathbf{R}^K$ and $L = (l_{k,h}, k \in K, h \in K)$ is a $\# K \times \# K$ square matrix with $l_{k,h} = \lambda_{k,h}/(\lambda_k + \mu)$. This matrix has positive coefficient and $\sum_{h \in K} l_{k,h} < 1$ for all $k \in K$. Hence, it is a contraction, so that $A^* = LA^*$ implies $A^* = 0$ which contradicts $\underline{a}_k^* > 0$.

Q.E.D.

Since $\underline{a}_j^*$ is continuous and stricly decreasing for all $j \in J$, the switch point z_j^* associated to the pair $(\underline{a}_j^*, \underline{a}_{-j}^*)$ is well defined.

LEMMA 18. — *For all* $j \in J$, $z_j^{*n} \to z_j^*$ *when* $n \to \infty$.

Proof. — Choose $\varepsilon > 0$. Hence $\alpha_j \underline{a}_j^*(z_j^* - \varepsilon) > \alpha_{-j} \underline{a}_{-j}^*(z_j^* + \varepsilon)$. On the other hand as $n \to \infty$, $\underline{a}_j^n(z_j^* - \varepsilon) \to \underline{a}_j^*(z_j^* - \varepsilon)$ and $\underline{a}_{-j}^n(z_j^* + \varepsilon) \to \underline{a}_{-j}^*(z_{-j}^* + \varepsilon)$. Hence, for n big enough, we get $\alpha_j \underline{a}_j^n(z_j^* - \varepsilon) > \alpha_{-j} \underline{a}_{-j}^n(z_j^* + \varepsilon)$ which implies $z_j^{*n} \in]z_j^* - \varepsilon, z_j^* + \varepsilon[$.

Q.E.D.

LEMMA 19. — *For all* $j \in J$, *the pair* $(\underline{a}_j^*, \underline{a}_{-j}^*)$ *satisfies the boundary conditions* (4.6) *to* (4.8), *that is:*

$$\alpha_j \underline{a}_j^*(z_j^*) = \alpha_{-j} \underline{a}_{-j}^*(1 - z_j^*)$$
$$(\lambda_j + \mu) \underline{a}_j^*(z_j^*) = \sum_{k \in J} \lambda_{j,k} \underline{a}_k^*(z_j^*) \qquad \text{if} \quad z_j^* > 0$$
$$(\lambda_j + \mu) \underline{a}_{-j}^*(1 - z_j^*) = \sum_{k \in J} \lambda_{j,k} \underline{a}_{-k}^*(1 - z_j^*) \qquad \text{if} \quad z_j^* < 1.$$

Proof. — For all $n \in N$, we have

$$\alpha_j \underline{a}_j^n(z_j^{*n}) = \alpha_{-j} \underline{a}_{-j}^n(1 - z_j^{*n})$$
$$(\lambda_j + \mu) \underline{a}_j^n(z_j^{*n}) = \sum_{k \in J} \lambda_{j,k} \underline{a}_j^{n-1}(z_j^{*n}) \qquad \text{if} \quad z_j^{*n} > 0$$
$$(\lambda_j + \mu) \underline{a}_{-j}^n(1 - z_j^{*n}) = \sum_{k \in J} \lambda_{j,k} \underline{a}_{-k}^{n-1}(1 - z_j^{*n}) \qquad \text{if} \quad z_j^{*n} < 1.$$

Since $z_j^{*n} \to z_j^*$ and $\underline{a}^n$ converges uniformly to $\underline{a}^*$ when $n \to \infty$, we get the result.

Q.E.D.

It is only left to show that $\underline{a}^*$ is C^1 and satisfies the right differential equations.

LEMMA 20. — *For all* $j \in J$, $(\underline{a}_j^*, \underline{a}_{-j}^*)$ *is* C^1 *and satisfies on* [0, 1]:

$$\underline{a}_j^{*\prime}(z)\,\varphi_j(z) + \sum_{k \in J} \lambda_{j,k}[\underline{a}_k^*(z) - \underline{a}_j^*(z)] - \mu\,\underline{a}_j^*(z) = 0$$

$$-\underline{a}_{-j}^{*\prime}(z)\,\varphi_j(1-z) + \sum_{k \in J} \lambda_{-j,k}[\underline{a}_k^*(z) - \underline{a}_{-j}^*(z)] - \mu\,\underline{a}_{-j}^*(z) = 0$$

with

$$\varphi_j(z) = -g\left(\frac{\underline{a}_j^*(z)}{\alpha_{-j}\underline{a}_{-j}^*(1-z)}\right)\frac{1}{\alpha_{-j}\underline{a}_{-j}^*(1-z)} \qquad \textit{if} \quad \alpha_j \underline{a}_j^*(z) < \alpha_{-j}\underline{a}_{-j}^*(1-z)$$

$$\varphi_j(z) = g\left(\frac{\underline{a}_{-j}^{*n}(1-z)}{\alpha_j \underline{a}_j^*(z)}\right)\frac{1}{\alpha_j \underline{a}_j^*(z)} \qquad \textit{if} \quad \alpha_j \underline{a}_j^*(z) > \alpha_{-j}\underline{a}_{-j}^*(1-z)$$

Proof. — For all $n \in N$, we have

$$\underline{a}_j^{n\prime}(z)\,\varphi_j^n(z) + \sum_{k \in J} \lambda_{j,k}[\underline{a}_k^{n-1}(z) - \underline{a}_j^n(z)] - \mu\,\underline{a}_j^n(z) = 0$$

$$-\underline{a}_{-j}^{n\prime}(z)\,\varphi_j^n(1-z) + \sum_{k \in J} \lambda_{-j,k}[\underline{a}_k^{n-1}(z) - \underline{a}_{-j}^n(z)] - \mu\,\underline{a}_{-j}^n(z) = 0$$

with

$$\varphi_j^n(z) = -g\left(\frac{\underline{a}_j^n(z)}{\alpha_{-j}\underline{a}_{-j}^n(1-z)}\right)\frac{1}{\alpha_{-j}\underline{a}_{-j}^n(1-z)} \qquad \text{if} \quad \alpha_j \underline{a}_j^n(z) < \alpha_{-j}\underline{a}_{-j}^n(1-z)$$

$$\varphi_j^n(z) = g\left(\frac{\underline{a}_{-j}^n(1-z)}{\alpha_j \underline{a}_j^n(z)}\right)\frac{1}{\alpha_j \underline{a}_j^n(z)} \qquad \text{if} \quad \alpha_j \underline{a}_j^n(z) > \alpha_{-j}\underline{a}_{-j}^n(1-z)$$

From $z_j^{*n} \to z_j^*$ and from the uniform convergence of $\underline{a}^n$ to $\underline{a}^*$ when $n \to \infty$, with $\delta \leqq \underline{a}^* \leqq 1/\delta$ we get that φ_j^n converges uniformly to φ_j on [0, 1]. Writing the differential system on $\underline{a}^n$ on its integral form, we may take the limit because of the uniform convergence of all the functions to their limits. This gives us, on $\underline{a}_j^*$ for instance,

$$\underline{a}_j^*(z) = \underline{a}_j^*(0) + \int_0^z \frac{\sum_{k \in J} \lambda_{j,k}[\underline{a}_k^*(s) - \underline{a}_j^*(s)] - \mu\,\underline{a}_j^*(s)}{\varphi_j(s)}\,ds, \qquad \forall z \in [0, 1].$$

The continuity of φ_j on $[0, z_j^*[\bigcup] z_j^*, 1]$, and the continuity of $\underline{a}_j^*$ on [0, 1] imply that $\underline{a}_j^*$ is C^1 on $[0, z_j^*[\bigcup] z_j^*, 1]$. Hence, we may derive the above equation on $[0, z_j^*[\bigcup] z_j^*, 1]$, which gives us the desired system. The second one may be obtained in the same way. In the case $z_j^* \in]0, 1[$, from the boundary condition when $z = z_j^*$, the right member of the equation satisfied by $\underline{a}_j^*$ is continuous with value 0 at this point, and we get that $\underline{a}_j^{*\prime}$ is continuous with value 0 at this point.

Q.E.D.

Hence, $\underline{a}^*$ belongs to E^J and is a fixed point of $H \circ G$. This achieves the proof of proposition 4.

8. COMPLEMENTARY RESULTS IN THE CASE N = 1

This section provides complementary results in the case where $J = \{-1, 1\}$. We first introduce some notation adapted to this special case.

We set $\alpha = \alpha_{-1}$, $\beta = \alpha_1$, and we suppose (without loss of generality) that $\alpha < 1/2 \leqq \beta$. We also set $\lambda = \lambda_{-1,1} = \lambda_{1,-1}$, $\nu = \lambda + \mu$ and $z^* = z^*_{-1} = z^*_1$. We have $\nu > \lambda$.

Finally, we set $A = a_{-1}$ and $B = a_1$. We will establish in this two states case that the switch point z^* is always 0, which is the meaning of the following proposition:

PROPOSITION 8. — *In the two states case, the solution* (A, B) *of system* (S) *is the solution of the following problem:*

$$B'(z) = -[\lambda A(z) - \nu B(z)]\beta B(z)/g\left(\frac{A(1-z)}{\beta B(z)}\right) \tag{8.1}$$

$$A'(z) = [\lambda B(z) - \nu A(z)]\beta B(1-z)/g\left(\frac{A(z)}{\beta B(1-z)}\right) \tag{8.2}$$

and the two boundary relations

$$\alpha A(0) = \beta B(1) \tag{8.3}$$

$$\lambda A(1) = (\lambda + \mu) B(1) \tag{8.4}$$

Proof. — Set $z \in]0, 1[$. Since (A, B) are solutions of system (S), and since $A'(z) < 0$, $B'(z) < 0$, one of the following two sets of equations must hold:

$$\begin{cases} \alpha A(z) < \beta B(1-z) \\ \nu A(z) > \lambda B(z) \\ \nu B(z) < \lambda A(z) \end{cases} \tag{8.5}$$

or

$$\begin{cases} \alpha A(z) > \beta B(1-z) \\ \nu A(z) < \lambda B(z) \\ \nu B(z) > \lambda A(z). \end{cases} \tag{8.6}$$

Suppose (8.6) holds for z near 1. Then $B(z) > A(z)$, and since

$$\alpha A(z) < \alpha B(z) < \alpha B(1-z) < \beta B(1-z),$$

we would have a contradiction. Thus, for z near 1, (8.5) must hold.

Let $\Gamma = \{(A, B) \in \mathbf{R}^2_+ / \lambda B \geqq \nu A\}$ and $\tilde{\Gamma} = \{(A, B) \in \mathbf{R}^2_+ / \lambda A \geqq \nu B\}$. Γ and $\tilde{\Gamma}$ are closed with $\Gamma \cap \tilde{\Gamma} = (0, 0)$ and $(A(z), B(z)) \in \Gamma \cup \tilde{\Gamma}$ for each $z \in [0, 1]$. For z near 1, we have $(A(z), B(z)) \in \Gamma$, and for all $z \in [0, 1]$, we have $(A(z), B(z)) \neq (0, 0)$. Also A and B are continuous. Hence, we must have

$(A(z), B(z)) \in \Gamma$ for all $z \in [0, 1]$. From (3.5) and (3.6), we obtain the boundary conditions.

Q.E.D.

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