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# The nonrelativistic limit of the nonlinear Dirac equation

by

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ABSTRACT. – It is proved that there exist solutions of the nonlinear Dirac equation, smooth in time, on a time interval which is independent of $c$. Moreover after multiplication by a phase factor (dependent on $c$) these solutions converge to the solution of a coupled system of nonlinear Schrödinger type equations.

*Key words :* Nonlinear Dirac equations, nonrelativistic limit, Schrödinger equation.

RÉSUMÉ. – *La limite non relativistique de l'équation de Dirac non linéaire.* – On montre qu'existent des solutions de l'équation de Dirac non linéaire, régulières en temps, sur un intervalle en temps indépendant de $c$. En plus, après la multiplication avec un facteur de phase ($c$-dépendant) ces solutions convergent vers la solution d'un système couplé d'équations du type Schrödinger non linéaire.

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## INTRODUCTION

We consider the initial value problem for the nonlinear Dirac equation

$$\left.\begin{aligned} i\frac{\partial\Psi}{\partial t} &= -ic\,\alpha\nabla\Psi + c^2\beta\Psi + 2\lambda(\beta\Psi\,|\,\Psi)\beta\Psi \\ \Psi(0) &= \Psi_{0c}, \end{aligned}\right\} \tag{1}$$

where $\lambda$ and $c$ are positive constants, $\Psi$ is a function from $\mathbb{R}^3$ into $\mathbb{C}^4$, $\alpha\nabla$ stands for $\sum_{j=1}^{3}\alpha_j\partial_j$ and $\alpha_j$ and $\beta$ are $4\times 4$ matrices satisfying $\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}\mathrm{I}$, $\alpha_j\beta+\beta\alpha_j=0$, $\beta$ is diagonal and $\beta^2=\mathrm{I}$.

This equation has first been investigated by the physicists; see the references in [1], [2], [3]. L. Vazquez, T. Cazenave *et al.* ([3], [2], [1]) recently started the investigation of the existence of localized (or stationary) solutions of (1).

We are interested in the solutions of the equation with arbitrary (*i. e.* not necessarily radially symmetric as in the above references) initial conditions and their convergence as the speed of light $c$ increases to $\infty$.

Introducing $\varepsilon=\dfrac{1}{2c^2}$, we write (1) in the equivalent form

$$\left.\begin{aligned} i\frac{\partial\Psi}{\partial t} &= -i\sqrt{\frac{1}{2\varepsilon}}\,\alpha\nabla\Psi + \frac{1}{2\varepsilon}\beta\Psi + 2\lambda(\beta\Psi\,|\,\Psi)\beta\Psi \\ \Psi(0) &= \Psi_{0\varepsilon}. \end{aligned}\right\} \tag{2}$$

We first prove the existence of a classical solution of (2) on an interval independent of $\varepsilon$.

In the following theorem as well as in the rest of this paper $\mathrm{H}^{\mathrm{S}}$ stands for the Sobolev space $\mathrm{H}^{\mathrm{S}}(\mathbb{R}^3)^4$ and $\mathrm{L}^2$ stands for $\mathrm{L}^2(\mathbb{R}^3)^4$.

THEOREM 1. – *Let* $\varepsilon_0>0$. *Assume that* $\Psi_{0\varepsilon}\in\mathrm{H}^2$ $(0<\varepsilon\leqq\varepsilon_0)$ *and*

$$\sup_{\varepsilon\leqq\varepsilon_0}\|\Psi_{0\varepsilon}\|_{\mathrm{H}^2}<\infty.$$

*Then there exists an interval* $\mathrm{J}=[-\mathrm{T},\mathrm{T}]$ *such that for every* $\varepsilon$ *with* $0<\varepsilon\leqq\varepsilon_0$ *there exists a unique solution*

$$\Psi_\varepsilon\in\mathrm{C}^2(\mathrm{J},\mathrm{L}^2)\cap\mathrm{C}^1(\mathrm{J},\mathrm{H}^1)\cap\mathrm{C}(\mathrm{J},\mathrm{H}^2)$$

*of the initial value problem* (2).

Next we investigate the nonrelativistic limit $\varepsilon\to 0$ [*i.e.* $c\to\infty$ in (1)]. Since $\Psi_\varepsilon$ cannot be expected to converge, motivated by the linear theory (*see* [5], [6]), we introduce the new function $\Phi=2e^{i\beta t/\varepsilon}\beta\Psi$. Inserting this into (2) we obtain an equivalent initial value problem

$$\left.\begin{aligned}\frac{\partial\Phi}{\partial t}=\sqrt{\frac{1}{2\varepsilon}}e^{i\beta t/\varepsilon}\alpha\nabla\Phi-\frac{i\lambda}{2}(\beta\Phi\,|\,\Phi)\,\beta\Phi\\ \Phi(0)=\Phi_{0\,\varepsilon},\end{aligned}\right\}\tag{3}$$

with $\Phi_{0\,\varepsilon}=2\,\beta\Psi_{0\,\varepsilon}$.

Theorem 1 evidently holds verbatim for $\Phi_\varepsilon$. Differentiating (3) with respect to $t$ we conclude that the smooth solutions of (3) (their existence is shown in Theorem 1) is also the solution of the initial value problem

$$\left.\begin{aligned}\frac{\partial^2\Phi}{\partial t^2}-\frac{1}{2\varepsilon}\left[\Delta\Phi+2\,i\,\beta\frac{\partial\Phi}{\partial t}-\lambda(\beta\Phi\,|\,\Phi)\,\Phi\right]=\mathrm{F}_\varepsilon(\Phi)\\ \Phi(0)=\Phi_{0\,\varepsilon},\qquad\frac{\partial\Phi}{\partial t}(0)=\Phi_{1\,\varepsilon},\end{aligned}\right\}\tag{4}$$

where $\Phi_{1\,\varepsilon}=\sqrt{\frac{1}{2\varepsilon}}\alpha\nabla\Phi_{0\,\varepsilon}-\frac{i\lambda}{2}(\beta\Phi_{0\,\varepsilon}\,|\,\Phi_{0\,\varepsilon})\,\beta\Phi_{0\,\varepsilon}$ and

$$\begin{aligned}\mathrm{F}_\varepsilon(t,\,\Phi)=&-\frac{\lambda^2}{2}(\beta\Phi\,|\,\Phi)^2\,\Phi\\ &-\frac{i\lambda}{2}\varepsilon^{-1/2}\sum_j\{e^{i\beta t/\varepsilon}[\partial_j(\beta\Phi\,|\,\Phi)]\,\alpha_j\,\beta\Phi+2\,\mathrm{Re}\,(\beta\Phi\,|\,e^{i\beta t/\varepsilon}\alpha_j\partial_j\Phi)\,\beta\Phi\}.\end{aligned}$$

Conversely, as the initial value problem (3) is equivalent to the initial value problem (2) which has a skew-adjoint linear part, standard arguments show that a solution of (4) which is $\mathrm{C}^2$ in time is also a solution of the first order system (3).

This suggests that the solutions $\Phi_\varepsilon$ of (3) converge to the solution $\Phi_0$ of the nonlinear Schrödinger type equation

$$\left.\begin{aligned}\frac{\partial\Phi}{\partial t}=\frac{1}{2}i\,\beta\Delta\Phi-\frac{i\lambda}{2}(\beta\Phi\,|\,\Phi)\,\beta\Phi\\ \Phi(0)=\Phi_{00}.\end{aligned}\right\}\tag{5}$$

We shall prove an existence result for $\Phi_0$ and the convergence of $\Phi_\varepsilon$ to $\Phi_0$:

THEOREM 2. — *Assume that* $\Phi_{00}\in\mathrm{H}^2$. *Then there exists an interval* $\mathrm{J}=[-\mathrm{T},\,\mathrm{T}]$ *such that the initial value problem* (5) *has a unique solution*

$$\Phi_0\in\mathrm{C}^1(\mathrm{J},\,\mathrm{L}^2)\cap\mathrm{C}(\mathrm{J},\,\mathrm{H}^2).$$

THEOREM 3. — *Let* $\varepsilon_0>0$. *Assume that* $\Phi_{0\,\varepsilon}\in\mathrm{H}^2\,(0\leqq\varepsilon\leqq\varepsilon_0)$ *and moreover*

$$\sup_{\varepsilon\leqq\varepsilon_0}\|\Phi_{0\,\varepsilon}\|_{\mathrm{H}^2}<\infty.\tag{6}$$

*Further assume that there exists some* $\alpha \in [0, 1]$ *such that*

$$\lim_{\varepsilon \to 0} \Phi_{0\,\varepsilon} = \Phi_{00} \quad \text{in } H^{\alpha}. \tag{7}$$

*Let* $J = [-T, T]$ *be such that there exists a unique solution* $\Phi_\varepsilon \in C^2(J, L^2) \cap C^1(J, H^1) \cap C(J, H^2)$ *of the initial value problem* (3) *for* $\varepsilon > 0$ *and a unique solution* $\Phi_0 \in C^1(J, L^2) \cap C(J, H^2)$ *of the initial value problem* (5). *Then*

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon = \Phi_0 \quad \text{in } C(J, H^{\alpha}). \tag{8}$$

*Proofs*

LEMMA 4. — *There exists* $K > 0$ *such that for all* $u \in H^2$ *following inequalities hold*:

$$\| u \|_{L^\infty} \leqq K \| u \|_{L^2}^{1/4} \| \Delta u \|_{L^2}^{3/4}, \tag{9}$$

$$\| \nabla u \|_{L^4} \leqq K \| u \|_{L^2}^{1/8} \| \Delta u \|_{L^2}^{7/8}. \tag{10}$$

*Proof.* — Let $\hat{u}$ be the Fourier transform of $u$. The Plancherel Theorem implies that

$$\| u \|_{L^\infty}^2 \leqq \| \hat{u} \|_{L^1}^2 \leqq \left( \int_{|x| \leqq R} |\hat{u}|^2 \right) \left( \int_{|x| < R} dx \right) + \left( \int_{|x| \geqq R} |x^2 \hat{u}|^2 \right) \left( \int_{|x| \geqq R} \frac{dx}{|x|^4} \right) \leqq C \left( R^3 \| u \|_{L^2}^2 + \frac{1}{R} \| \Delta u \|_{L^2}^2 \right),$$

$$\| \nabla u \|_{L^4}^4 \leqq \| x \hat{u} \|_{L^{4/3}}^4 \leqq \left( \int |\hat{u}|^2 \right)^2 \left( \int_{|x| < R} |x|^4 \, dx \right) + \left( \int_{|x| \geqq R} |x|^4 |\hat{u}|^2 \right)^2 \left( \int_{|x| \geqq R} \frac{dx}{|x|^4} \right) \leqq C \left( R^7 \| u \|_{L^2}^4 + \frac{1}{R} \| \Delta u \|_{L^2}^4 \right).$$

Setting $R = \left( \frac{\| \Delta u \|_{L^2}}{\| u \|_{L^2}} \right)^{1/2}$ we obtain (9) and (10).

LEMMA 5. — *Assume* $\varepsilon_0 > 0$ *and*

$$\sup_{0 < \varepsilon \leqq \varepsilon_0} \| \Psi_{0\,\varepsilon} \|_{H^2} < \infty.$$

*Then there exists an interval* $J = [-T, T]$ *such that if for each* $\varepsilon \in (0, \varepsilon_0)$ *the initial value problem* (2) *has a solution* $\Psi_\varepsilon \in C^1(J, L^2) \cap C(J, H^2)$ *then*

$$\sup_{0 < \varepsilon \leqq \varepsilon_0} \| \Psi_\varepsilon \|_{C(J, H^2)} < \infty, \tag{11}$$

$$\sup \{ \| \Psi_\varepsilon(t) \|_{L^\infty} : 0 < \varepsilon \leqq \varepsilon_0, \, t \in J \} < \infty. \tag{12}$$

*Proof.* – It is sufficient to prove (11) since (12) follows from (11) and the embedding theorem. In fact it is sufficient to prove

$$\sup_{0<\varepsilon\leqq\varepsilon_0} \|\Psi_\varepsilon\|_{C([0,T],H^2)} < \infty, \tag{13}$$

since $t\in[-T, 0]$ is treated similarly.

In the proof of (13) we need the conservation of $L^2$ norm

$$\|\Psi_\varepsilon(t)\|_{L^2} = \|\Psi_{0\,\varepsilon}\|_{L^2} \qquad (t\in J) \tag{14}$$

which follows easily from (2) by scalar multiplication in $L^2$ by $\Psi_\varepsilon$.

Now we turn to the proof of (13). Apply the operator $\Delta$ to both sides of (2). It follows that $\omega=\Delta\Psi$ is the solution of the initial value problem.

$$\left.\begin{aligned} i\frac{\partial\omega}{\partial t} = -\sqrt{\frac{1}{2\varepsilon}}\alpha\nabla\omega + \frac{1}{2\varepsilon}\beta\omega + 2\lambda(\beta\Psi\,|\,\Psi)\,\beta\omega \\ +2\lambda[\Delta(\beta\Psi\,|\,\Psi)]\,\beta\Psi + 2\lambda\sum_j[\partial_j(\beta\Psi\,|\,\Psi)]\,\beta\,\partial_j\Psi \\ \omega(0)=\omega_{0\,\varepsilon}:\;=\Delta\Psi_{0\,\varepsilon}. \end{aligned}\right\} \tag{15}$$

Multiplying by $\omega$ and taking the imaginary part we find

$$\begin{aligned}\frac{d}{dt}\|\omega_\varepsilon(t)\|_{L^2}^2 &\leqq K\int(|\Psi_\varepsilon|^2|\omega_\varepsilon|^2 + |\Psi_\varepsilon||\omega_\varepsilon||\nabla\Psi_\varepsilon|^2) \\ &\leqq K(\|\Psi_\varepsilon\|_{L^\infty}^2\|\omega_\varepsilon\|_{L^2}^2 + \|\Psi_\varepsilon\|_{L^\infty}\|\nabla\Psi_\varepsilon\|_{L^4}^2\|\omega_\varepsilon\|_{L^2}).\end{aligned}$$

From (9), (10) and (14) it follows that

$$\frac{d}{dt}\|\omega_\varepsilon(t)\|_{L^2} \leqq K\|\omega_\varepsilon(t)\|_{L^2}^{5/2}.$$

From this it follows that if $KT\|\Psi_{0\,\varepsilon}\|_{H^2}^{3/2}<1$ then

$$\sup_{0\leqq t\leqq T}\|\Delta\Psi_\varepsilon(t)\| \leqq \left(\frac{\|\Psi_{0\,\varepsilon}\|_{H^2}^{3/2}}{1-KT\|\Psi_{0\,\varepsilon}\|_{H^2}^{3/2}}\right)^{2/3}.$$

Together with (14) this implies (13).

Let $\gamma$ be a matrix and $j, k\in\{1, 2, 3\}$. Define the functions

$$f_1(u):\;=(\gamma u\,|\,u)\,u,\qquad f_2(u):\;=(\gamma u\,|\,u)^2\,u,$$
$$f_3:\;=\partial_j f_1,\qquad f_4:\;=\partial_{jk}^2 f_1,\qquad f_5:\;=\partial_j f_2,\qquad f_6:\;=j_{jk}^2 f_2.$$

LEMMA 6. – *The functions $f_i$, $i=1, \ldots, 6$, are locally Lipschitz continuous functions from $H^2$ to $L^2$: for every $K>0$ there exists $C_K>0$ such that $\|u\|_{H^2}\leqq K$, $\|u\|_{H^2}\leqq K$ imply*

$$\|f_j(u)-f_j(v)\|_{L^2}\leqq C_K\|u-v\|_{H^2} \tag{16}$$

*Moreover for $\alpha=0$ and 1*

$$\|f_1(u)-f_1(u)\|_{H^\alpha}\leqq C_K\|u-v\|_{H^\alpha}. \tag{17}$$

We omit the easy proof (which repeatedly uses Lemma 4 and the Embedding Theorem).

Define the mappings $N_\varepsilon (\varepsilon \geqq 0)$ by

$$N_0(\Phi) := -\frac{\lambda}{2}(\beta\Phi \mid \Phi)\Phi$$
$$N_\varepsilon(\Phi) := \varepsilon F_\varepsilon(\Phi) + N_0(\Phi)\ (\varepsilon > 0).$$

LEMMA 7. — (*a*) *For* $\varepsilon > 0$ *the mapping* $N_\varepsilon$ *is a locally Lipschitz continuous map from* $H^2$ *to* $H^1$: *for every* $K > 0$ *there exists* $c_K > 0$ *such that* $\|\Phi\|_{H^2} \leqq K$, $\|\Psi\|_{H^2} \leqq K$ *imply*

$$\| N_\varepsilon(\Phi) - N_\varepsilon(\Psi) \|_{H^1} \leqq C_K \| \Phi - \Psi \|_{H^2}.$$

(*b*) *The mapping* $N_0$ *is a locally Lipschitz continuous map on* $H^2$.

(*c*) *There exists* $C > 0$ *such that for all* $\Phi, \Psi \in H^2$ *and* $\alpha \in [0, 1]$ *the estimate*

$$\| N_0(\Phi) - N_0(\Psi) \|_{H^\alpha} \leqq C(\|\Phi\|_{H^2} + \|\Psi\|_{H^2})^2 \|\Phi - \Psi\|_{H^\alpha}. \tag{18}$$

*Proof.* — (*a*) and (*b*) are straightforward consequences of Lemma 6. To prove (*c*), fix $\Phi, \Psi$ and define the linear map $R_{\Phi, \Psi}$ by $R_{\Phi, \Phi}(\theta) = (\beta\Phi \mid \Phi)\theta + (\beta\Phi \mid \theta)\Psi + (\theta \mid \beta\Psi)\Psi$. Then

$$N_0(\Phi) - N_0(\Psi) = R_{\Phi, \Psi}(\Phi - \Psi).$$

It is easy to show that $\| R_{\Phi, \Psi}(\theta) \|_{H^\alpha} \leqq C(\|\Phi\|_{H^2} + \|\Psi\|_{H^2})^2 \|\theta\|_{H^\alpha}$ for $\alpha = 0$ and $\alpha = 1$ (in fact such estimate appears in the proof of Lemma 6). Since $R_{\Phi, \Psi}$ is linear, the same estimate holds for all $\alpha \in [0, 1]$.

*Proof of Theorem* 1. — As mentioned in the Introduction, it is sufficient to prove the statement of Theorem 1 for the solution of the initial value problem (4), since it follows by standard methods that this solution is also solution of (2).

The substitution $U = e^{-i\beta t/2\varepsilon}\Phi$ transforms the initial value problem (4) into

$$\left.\begin{aligned} &\varepsilon \frac{\partial^2 U}{\partial t^2} - \frac{1}{2}\Delta U + \frac{1}{4\varepsilon} U = N_\varepsilon(U) \\ &U(0) = U_{0\varepsilon} := \Phi_{0\varepsilon}, \qquad \frac{\partial U}{\partial t}(0) = U_{1\varepsilon} := \Phi_{1\varepsilon} - \frac{i\beta}{2\varepsilon}\Phi_{0\varepsilon}. \end{aligned}\right\} \tag{19}$$

Converting this equation into an integral equation (*cf.* [4]) we find that any smooth solution of (4) is also a solution of

$$\Phi_\varepsilon(t) = I_\varepsilon(t)\Phi_{0\varepsilon} + J_\varepsilon(t)\Phi_{1\varepsilon} + \frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s) N_\varepsilon(\Phi_\varepsilon(s))\, ds \tag{20}$$

with

$$I_\varepsilon(t)=e^{i\beta t/2\varepsilon}\left(\cos t\,A_\varepsilon-\frac{i\beta}{2\varepsilon}A_\varepsilon^{-1}\sin t\,A_\varepsilon\right), \tag{21}$$

$$J_\varepsilon(t)=e^{i\beta t/2\varepsilon}A_\varepsilon^{-1}\sin t\,A_\varepsilon \tag{22}$$

and $A_\varepsilon$ the positive square root of $\frac{1}{\varepsilon^2}\left(\varepsilon A+\frac{1}{4}\right)$, $A=-\frac{1}{2}\Delta$, naturally defined in $L^2$.

We shall use the following well known result: let $\mathscr{A}$ be a $m$-dissipative operator in a Hilbert space X and let $D(\mathscr{A})$ be its domain endowed with the graph norm. Let F be a locally Lipschitz continuous mapping on $D(\mathscr{A})$. Then for every $x_0\in D(\mathscr{A})$ the initial value problem

$$\frac{dx}{dt}=\mathscr{A}x+F(x)$$
$$x(0)=x_0$$

has a maximal classical solution

$$x\in C^1([0, T_{max}), X)\cap C([0, T_{max}), D(\mathscr{A})).$$

Moreover if $T_{max}<\infty$ then

$$\lim_{t\nearrow T_{max}}(\|\mathscr{A}x(t)\|_X+\|x(t)\|_X)=\infty.$$

Fix $\varepsilon>0$. We apply the above result to the space $X=D(A_\varepsilon)\times L^2=H^1\times L^2$ endowed with the norm

$$\|(u, v)\|_X=(\|A_\varepsilon u\|_{L^2}^2+\|v\|_{L^2}^2)^{1/2},$$

to the operator A in X defined on $D(\mathscr{A})=D(A_\varepsilon^2)\times D(A_\varepsilon)=H^2\times H^1$ by

$$\mathscr{A}(u, v)=(v, -A_\varepsilon^2 u),$$

and to the mapping F defined by

$$F(u, v)=(0, N_\varepsilon(u)).$$

Then $\mathscr{A}$ is skew-adjoint and F is locally Lipschitz continuous on $D(\mathscr{A})$ by lemma 7. We conclude that there exists $T_\varepsilon>0$ and a unique classical solution $U_\varepsilon$ of (19) on $[0, T_\varepsilon)$ and moreover that

$$\lim_{t\nearrow T_\varepsilon}\left(\|U_\varepsilon(t)\|_{H^2}+\left\|\frac{\partial U_\varepsilon}{\partial t}\right\|_{H^1}\right)=\infty$$

if $T_\varepsilon<\infty$.

Since (19) and (4) are equivalent, it follows that there exists a unique classical solution $\Phi_\varepsilon$ of (4) on $[0, T_\varepsilon)$ and moreover that if $T_\varepsilon<\infty$ then

$$\lim_{t\nearrow T_\varepsilon}\left(\|\Phi_\varepsilon(t)\|_{H^2}+\left\|\frac{\partial\Phi_\varepsilon(t)}{\partial t}-\frac{i\beta}{2\varepsilon}\Phi_\varepsilon(t)\right\|_{H^1}\right)=\infty.$$

Since $\Phi_\varepsilon$ is the solution of (3) and since (3) is equivalent to (2), we can apply lemma 5 and conclude that if $T_\varepsilon < T$ (T is the number in that lemma) then $\lim_{t \nearrow T_\varepsilon} \| \Phi_\varepsilon(t) \|_{H^2} < \infty$, consequently $\lim_{t \nearrow T_\varepsilon} \left\| \frac{\partial \Phi_\varepsilon(t)}{\partial t} \right\|_{H^1} = \infty$. However the equation (2) implies $\left\| \frac{\partial \Phi_\varepsilon(t)}{\partial t} \right\|_{H^1} \leqq C \| \Phi_\varepsilon(t) \|_{H^2}$ which leads to a contradiction.

This implies $T_\varepsilon \geqq T$. From the symmetry of the equation (19) with respect to $t$ we conclude that the maximal existence for $\Phi_\varepsilon$ (hence also for $\Psi_\varepsilon$) is in fact $(-T_\varepsilon, T_\varepsilon)$.

*Proof of Theorem* 2. – Using the notation from the proof of Theorem 1, the initial value problem (5) can be written as

$$\left.\begin{aligned} \frac{d\Phi}{dt} &= -i\beta A \Phi + i\beta N_0(\Phi) \\ \Phi(0) &= \Phi_{00}, \end{aligned}\right\} \tag{23}$$

Since $-i\beta A$ is a skew adjoint operator in $L^2$ and $N_0$ is a locally Lipschitz map on $D(A) = H^2$ by Lemma 7, the conclusions of Theorem 2 are a straightforward application of the classical existence result that was used in the proof of Theorem 1.

*Proof of Theorem* 3. – Denote

$$I_0(t) = e^{-i\beta A t}.$$

Then the solution of the initial value problem (23) [and consequently (5)] satisfies also the integral equation

$$\Phi_0(t) = I_0(t)\Phi_{00} + i \int_0^t \beta I_0(t-s) N_0(\Phi_0(s))\, ds; \tag{24}$$

note that $\beta$, A and $I_0(t)$ are diagonal commuting.

Using the representations (20) and (24) of the solutions $\Phi_\varepsilon$ and $\Phi_0$, their difference can be written as

$$\Phi_\varepsilon(t) - \Phi_0(t) = \sum_{i=1}^{6} L_\varepsilon^{(i)}(t),$$

where

$$L_\varepsilon^{(1)}(t) = [I_\varepsilon(t) - I_0(t)]\Phi_{00}$$

$$L_\varepsilon^{(2)}(t) = I_\varepsilon(t)(\Phi_{0_\varepsilon} - \Phi_{00})$$

$$L_\varepsilon^{(3)}(t) = J_\varepsilon(t)\Phi_{1_\varepsilon}$$

$$L_\varepsilon^{(4)}(t) = \int_0^t \left[\frac{1}{\varepsilon} J_\varepsilon(t-s) - i\beta I_0(t-s)\right] N_0(\Phi_0(s))\, ds$$

$$L_\varepsilon^{(5)}(t)=\frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)\,[N_\varepsilon(\Phi_\varepsilon(s))-N_0(\Phi_\varepsilon(s))]\,ds$$

$$L_\varepsilon^{(6)}(t)=\frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)\,[N_0(\Phi_\varepsilon(s))-N_0(\Phi_0(s))]\,ds$$

We shall prove

$$\lim_{\varepsilon\to 0} L_\varepsilon^{(i)}=0 \quad \text{in } C([0, T]; H^\alpha) \tag{$25_i$}$$

for $i=1, \ldots, 5$.

Since $\Phi_{00}\in H^\alpha$ and $N_0(\Phi_0(\cdot))\in C([0, T]; H^2)$ by Theorem 2, the statements $(25_1)$ and $(25_4)$ follow directly from Lemma 2.2 in [5]. Noting the estimate

$$\sup_{\substack{\varepsilon>0\\ t\in\mathbb{R}}}\left(\|I_\varepsilon(t)\|_{\mathscr{L}(L^2)}+\left\|\frac{1}{\varepsilon}J_\varepsilon(t)\right\|_{\mathscr{L}(L^2)}\right)<\infty, \tag{26}$$

we see that $(25_2)$ is a consequence of (7).

Similarly, the estimate

$$\begin{aligned}\varepsilon\|\Phi_{1\varepsilon}\|_{H^1} &\leqq C(\varepsilon^{1/2}\|\Phi_{0\varepsilon}\|_{H^2}+\varepsilon\|N_0(\Phi_{0\varepsilon})\|_{H^1})\\ &\leqq C(\varepsilon^{1/2}\|\Phi_{0\varepsilon}\|_{H^2}+\varepsilon\|\Phi_{0\varepsilon}\|_{L^\infty}^2\|\Phi_{0\varepsilon}\|_{H^1})\end{aligned}$$

together with (6) and the Sobolev embedding theorem imply $\lim_{\varepsilon\to 0}\varepsilon\Phi_{1\varepsilon}=0$ in $H^1$, therefore $(25_3)$ follows from (26). Next

$$\|F_\varepsilon(\Phi)\|_{H^1}\leqq C(\varepsilon\|\Phi\|_{L^\infty}^4+\varepsilon^{1/2}\|\Phi\|_{L^\infty}^2\|\Phi\|_{H^2}+\varepsilon^{1/2}\|\Phi\|_{W^{1,4}}^2\|\Phi\|_{L^\infty}),$$

hence it follows from (26) and Lemma 4 that

$$\|L_\varepsilon^{(5)}(t)\|_{H^\alpha}\leqq C\varepsilon\int_0^t\|F_\varepsilon(\Phi_\varepsilon(s)\|_{H^\alpha}\,ds\leqq C\varepsilon\int_0^t\|F_\varepsilon(\Phi_\varepsilon(s))\|_{H^1}\,ds$$

Using Lemma 4 and Lemma 5 [recall that in (11) and (12) $\Psi_\varepsilon$ can be replaced by $\Phi_\varepsilon$] it follows that $\|L_\varepsilon^{(5)}(t)\|_{H^\alpha}\leqq C t\varepsilon^{1/2}$, hence the equality $(24_5)$ is also proved. Using (26) once again we see that

$$\|L_\varepsilon^{(6)}(t)\|_{H^\alpha}\leqq C\int_0^t\|\Phi_\varepsilon(s)-\Phi_0(s)\|_{H^\alpha}\,ds.$$

It follows that

$$\|\Phi_\varepsilon(t)-\Phi_0(t)\|_{H^\alpha}\leqq C\left(a_\varepsilon+\int_0^t\|\Phi_\varepsilon(s)-\Phi_0(s)\|_{H^\alpha}\,ds\right)\qquad (0\leqq t\leqq T)$$

with $\lim_{\varepsilon\to 0} a_\varepsilon=0$. This implies (8) on [0, T]; the proof for [$-$T, 0] is identical.

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