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A priori bounds and renormalized Morse indices of solutions of an elliptic system

by

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ABSTRACT. – We define a "renormalized" Morse index, and prove a Bahri–Lions type result for critical points of $E(u,v) = \int_\Omega \{\nabla u \cdot \nabla v - H(x,u,v)\}\,\mathrm{d}x$; i.e., we establish an a priori bound for critical points with bounded Morse index.

RÉSUMÉ. – Nous définissons un indice de Morse généralisé pour les points critiques de la fonction $E(u,v) = \int_\Omega \{\nabla u \cdot \nabla v - H(x,u,v)\}\,\mathrm{d}x$ défini sur $H_0^1(\Omega) \times H_0^1(\Omega)$.

Le but principal de ce travail est la démonstration d'une estimation de type Bahri–Lions [2] pour les points critiques. Nous montrons pour chaque entier $m \in \mathbb{N}$ que l'ensemble des points critiques dont l'indice renormalisé μ satisfait $\mu \leqslant m$ est borné dans $L^\infty(\Omega) \times L^\infty(\Omega)$.

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1. INTRODUCTION AND MAIN THEOREM

In [1] we obtained existence and multiplicity results for critical points in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ of the functional

$$f_H(u,v) = \int_\Omega \{\nabla u \cdot \nabla v - H(x,u,v)\}\,\mathrm{d}x \tag{1}$$

whose Euler–Lagrange equations are the following semilinear elliptic system

$$-\Delta u = H_v(x,u,v), \qquad -\Delta v = H_u(x,u,v), \tag{2}$$

with Dirichlet boundary conditions. If one chooses

$$H(x,u,v) = a(x)\frac{|u|^{p+1}}{p+1} + b(x)\frac{|v|^{q+1}}{q+1} \tag{3}$$

then (2) becomes

$$-\Delta u = b(x)v^q, \qquad -\Delta v = a(x)u^p. \tag{4}$$

Our method in [1] is to use Floer's version of Morse theory. In fact our motivation for this work and [1] was to see how well Floer's approach adapts to PDE problems involving indefinite functionals like f_H. In Floer's approach one defines a "renormalized Morse index" for critical points, and then defines homology groups which allow one to estimate how many critical points with a given index f_H should have. It turned out in [1] that Floer's method can indeed be used in a straightforward way, provided one can establish enough compactness, both for the critical points, and for the orbits of the gradient flow which connect the critical points. To our surprise we found that the flow which gives the best compactness properties for the connecting orbits is the gradient flow in $H_0^1(\Omega) \times H_0^1(\Omega)$, or more generally, $H^s \times H^{2-s}$. This flow is well posed, in contrast with the L^2 gradient flows that are usually chosen to define Floer homology. For the L^2 gradient flows ill-posedness of the initial value problem caused by ellipticity of the gradient flow PDE is largely responsible for compactness of the set of connecting orbits.

After establishing the Morse relations, and from there existence and multiplicity results for critical points of f_H in [1] it was natural to ask what could be said about the renormalized Morse index of critical points.

To do this we needed a compactness theorem for critical points with bounded index. Our main result in this paper is precisely such a theorem:

THEOREM 1A. – *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We assume $n \geqslant 3$, and we assume the system* (4) *is superlinear, i.e., $p, q > 1$, and subcritical*

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n}. \tag{5}$$

For any $m \in \mathbb{N}$ there is a constant C_m depending on a, b and Ω, such that any critical point of f_H with lower index $\mu_-(z_0) \leqslant m$ satisfies

$$\sup_\Omega\{|u|, |v|\} \leqslant C_m.$$

We recall the definition of the renormalized index in Section 2 below.

This theorem is similar to a theorem of Bahri and Lions [2] (see also Yang [8]) who show that boundedness of the Morse index of solutions of the scalar equation $\Delta u + u^p = 0$ imply *a priori* L^∞ estimates for the solutions. We cannot imitate their proof however, since they use the minimax characterization of eigenvalues of $-\Delta + V(x)$ in terms of the quotients

$$\frac{\int |\nabla\phi|^2 + V(x)\phi(x)^2\,\mathrm{d}x}{\int \phi(x)^2\,\mathrm{d}x}.$$

(See [3, Chapter 6].) This description always deals with "the first n eigenvalues" which makes no sense in our setting, since the second variation $\mathrm{d}^2 f_H(z)$ at a critical point $z \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ always has infinitely many positive and negative eigenvalues. In Section 3 we overcome this problem by giving an alternative description of the index of a critical point z in terms of the spectrum of an integral operator associated with the matrix

$$P(x) = \begin{pmatrix} H_{uu}(x.z(x)) & H_{vu}(\cdots) \\ H_{uv}(\cdots) & H_{vv}(\cdots) \end{pmatrix}.$$

In Section 4 we begin the compactness proof along the same lines as Bahri and Lions. Assuming compactness fails, we use a blow-up argument to reduce the problem to that of computing the index of entire solutions to the "constant coefficient version" of (4), i.e., (4) with $a(x)$ and $b(x)$ independent of x. This prompts us to study entire solutions, which we do in Sections 5–7, where we prove two Liouville type theorems. In Sections 8–10 we then complete the compactness proof.

The proof we give actually applies to more general functions H. To state the more general result we consider a sequence of functions $H^{(k)} \in C^2(\overline{\Omega} \times \mathbb{R}^2)$. We say this sequence satisfies condition $(*)$ if

$$\begin{cases} \text{For any sequence of points } P_k \in \overline{\Omega} \text{ and numbers } \lambda_k \uparrow \infty \\ \text{there is a sequence } k_i \uparrow \infty \text{ such that } \hat{H}^{(k_i)} \text{ defined by} \\ \widehat{H}^{(k)}(y, U, V) = \lambda_k^{-(p+1)(q+1)} H^{(k)}\big(P_k + \varepsilon_k y, \lambda_k^{q+1} U, \lambda_k^{p+1} V\big), \\ \varepsilon_k = \lambda_k^{-(pq-1)/2}, \\ \text{converges in } C^2_{\mathrm{loc}} \text{ to } a|U|^{p+1} + b|V|^{q+1} \text{ for some } a, b > 0. \end{cases} \tag{$*$}$$

This hypothesis is satisfied by "lower order perturbations" of (3), i.e., functions of the form

$$H(x, u, v) = a(x)\frac{|u|^{p+1}}{p+1} + b(x)\frac{|v|^{q+1}}{q+1} + h(x, u, v),$$

with

$$h(x, u, v) = \mathrm{o}\big(|u|^{p+1} + |v|^{q+1}\big),$$

and similar growth conditions for the first and second derivatives of h.

THEOREM 1B. – *Let $H^{(k)}$ be a sequence of functions satisfying $(*)$, as well as*

$$\frac{\partial^2 H^{(k)}}{\partial u \partial v} \leqslant 0, \qquad \frac{\partial^2 H^{(k)}}{\partial^2 u}, \frac{\partial^2 H^{(k)}}{\partial^2 v} \geqslant 0.$$

Then any sequence z_k of critical points of $f_{H^{(k)}}$ with uniformly bounded renormalized Morse indices is uniformly bounded in $L^\infty(\Omega; \mathbb{R}^2)$.

The method used in this paper appears to give an optimal result with respect to the exponents p and q. On the other hand our method does impose restrictions on $\mathrm{d}^2 H(z)$ and on the dimension of the domain. The method used by Bahri and Lions and by Yang for semi-definite elliptic equations does not share these restriction due to the direct variational characterization of the eigenvalues which is possible in the semidefinite case. We believe that the imposed restrictions on $\mathrm{d}^2 H(z)$ and n are of technical nature and that the result (Theorem 1A) should hold true under milder hypotheses on $\mathrm{d}^2 H(z)$ and for all $n \geqslant 1$.

2. THE RENORMALIZED INDEX

The second variation of f_H at a critical point $z = \binom{u}{v}$ is given by

$$d^2 f_H(z) \cdot (\phi, \phi) = (\phi, \mathcal{E}\phi)_{L^2(\Omega;\mathbb{R}^2)}, \qquad \phi \in H^2 \cap H_0^1(\Omega; \mathbb{R}^2).$$

where $\mathcal{E}$ is the elliptic operator given by $\mathcal{E} = -\partial_\Delta - P(x)$, with

$$-\partial_\Delta = \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix},$$

$$P(x) = \begin{pmatrix} H_{uu}(x, u(x), v(x)) & H_{vu}(\cdots) \\ H_{uv}(\cdots) & H_{vv}(\cdots) \end{pmatrix}.$$

The operator $\mathcal{E}$ is elliptic and self-adjoint. It is also a bounded perturbation of the operator $-\partial_\Delta$ whose spectrum consists of the bi-infinite sequence of eigenvalues $\{\pm\sqrt{\lambda_k} \mid k = 1, 2, \ldots\}$, where λ_k are the eigenvalues of $-\Delta$ on Ω with Dirichlet boundary conditions. Thus $\mathcal{E}$ also has a bi-infinite sequence of eigenvalues and the ordinary Morse index of the critical point z is infinite.

Let $\mathbb{S}_2$ be the 3-dimensional space of symmetric 2×2 matrices, and let $\mathfrak{P} = L^\infty(\Omega, \mathbb{S}_2)$ be the space of "potentials". We will first define the index of $\mathcal{E} = -\partial_\Delta - P(x)$ if $\mathcal{E}$ is nondegenerate (invertible), so let $\mathfrak{P}_0$ be the set of $P \in \mathfrak{P}$ for which $-\partial_\Delta - P(x)$ is invertible. $\mathfrak{P}_0$ is an open subset of $\mathfrak{P}$, and its complement can be written as the union $\bigcup_{i=1}^\infty \mathfrak{P}_i$, where $\mathfrak{P}_i$ consists of those potentials P for which $-\partial_\Delta - P(x)$ has i-dimensional kernel. Each $\mathfrak{P}_i$ is a smooth submanifold of $\mathfrak{P}$ with codimension $i(i+1)/2$ (see [4]).

We will define the index μ of the operator $-\partial_\Delta - P(x)$ by requiring that it be locally constant on $\mathfrak{P}_0$, and by specifying how $\mu(-\partial_\Delta - P(x))$ changes when P crosses from one component of $\mathfrak{P}_0$ to another. The following lemma makes this possible.

LEMMA 2A. – *$\mathfrak{P}_1$ has a natural co-orientation.*

Proof. – Let $P_0 \in \mathfrak{P}_1$. A co-orientation of $\mathfrak{P}_1$ at P_0 is an orientation of $T_{P_0}\mathfrak{P}/T_{P_0}\mathfrak{P}_1$.

By definition 0 is a simple eigenvalue of $-\partial_\Delta - P_0(x)$. Standard perturbation theory implies that the operator $-\partial_\Delta - P(x)$ has a simple eigenvalue $\lambda(P)$ near 0 for all $P \in \mathfrak{P}$ near P_0. The function $P \mapsto \lambda(P)$ is smooth, and its derivative is given by

$$d\lambda(P) \cdot \delta P = -(\phi_P, \delta P \cdot \phi_P)_{L^2}, \tag{6}$$

where ϕ_P is a unit eigenvector of $-\partial_\Delta - P(x)$ for the eigenvalue $\lambda(P)$. If $\delta P \in T_{P_0}\mathfrak{P}$ is not tangent to $\mathfrak{P}_1$, then by the implicit function theorem $d\lambda(P_0)\cdot\delta P \neq 0$, and the sign of this expression provides us with a co-orientation. □

The proof actually provides us with *two* co-orientations: we will call δP positive if $d\lambda(P_0)\cdot\delta P$ is negative(!)

Given $\mathcal{E} = -\partial_\Delta - P$, with $P \in \mathfrak{P}_0$, we choose a generic path $\{P_\theta \mid 0 \leqslant \theta \leqslant 1\}$ connecting $P_0 = 0$ to $P_1 = P$. A generic path will not intersect any of the $\mathfrak{P}_i$ with $i \geqslant 2$ since they have codimension 3 or more. A generic path can intersect $\mathfrak{P}_1$, but we may assume that it does so tranversally. The co-orientation then assigns a sign to each intersection of the path with $\mathfrak{P}_1$. We define the sum of these signs to be $\mu(\mathcal{E})$. A generic homotopy of paths will also miss all the $\mathfrak{P}_i$ with $i \geqslant 2$, and will also be transversal to $\mathfrak{P}_1$. Therefore the number of intersections (counted with orientation) of the path with $\mathfrak{P}_1$ does not depend on the path.

Briefly, $\mu(\mathcal{E})$ is the number of positive eigenvalues of $\mathcal{E}_\theta = -\partial_\Delta - P_\theta$ which become negative as θ increases from 0 to 1 minus the number of negative eigenvalues of $\mathcal{E}_\theta = -\partial_\Delta - P_\theta$ which become positive as θ increases from 0 to 1 (cf. the "spectral flow formula" in [6]).

LEMMA 2B. – *If P_0, $P_1 \in \mathfrak{P}_0$ and $P_1(x) \geqslant P_0(x)$ pointwise, then $\mu(-\partial_\Delta - P_1) \geqslant \mu(-\partial_\Delta - P_0)$.*

Proof. – Since $\mathfrak{P}_0$ is open we may assume that $P_1 \geqslant P_0 + \varepsilon \mathrm{I}$ for some small $\varepsilon > 0$. Now let $P_\theta = \theta P_1 + (1-\theta)P_0$. One has $\partial P_\theta/\partial\theta \geqslant \varepsilon\mathrm{I}$, and any sufficiently small C^1 perturbation of this path will also have $\partial P_\theta/\partial\theta > 0$. For a generic perturbation (6) tells us that every intersection of the perturbed path with $\mathfrak{P}_1$ is positive. □

It is relatively straightforward to compute the index of operators with constant coefficient potentials. Let

$$P_{A,B,C}(x) = \begin{pmatrix} A & -C \\ -C & B \end{pmatrix},$$

where A, B, C are constants.

LEMMA 2C. – *If $-C \pm \sqrt{(AB)}$ is not an eigenvalue of $-\Delta$, in particular if $AB < 0$, then $P \in \mathfrak{P}_0$. In this case the index of $-\partial_\Delta - P_{A,B,C}$ is determined as follows*:

(a) *If $AB \leqslant 0$ then $\mu(-\partial_\Delta - P_{A,B,C}) = 0$;*

(b) *If* $AB > 0$ *and* $A, B > 0$ *then* $\mu(-\partial_\Delta - P_{A,B,C})$ *is the number of eigenvalues of* $-\Delta$ *lying in the interval* $-C - \sqrt{(AB)} < \lambda < -C + \sqrt{(AB)}$;

(c) *If* $AB > 0$ *and* $A, B < 0$ *then* $\mu(-\partial_\Delta - P_{A,B,C})$ *is* minus *the number of eigenvalues of* $-\Delta$ *lying in the interval* $-C - \sqrt{(AB)} < \lambda < -C + \sqrt{(AB)}$;

Proof. – For any ϕ, ψ one has $\binom{\phi}{\psi} \in \operatorname{kern}(-\partial_\Delta - P_{A,B,C})$ iff

$$-A\phi + (C - \Delta)\psi = 0,$$
$$(C - \Delta)\phi - B\psi = 0.$$

Add B times the first equation to $(C - \Delta)$ times the second to find that

$$\{(C - \Delta)^2 - AB\}\phi = 0.$$

A similar manipulation shows that ψ also satisfies this equation. If $-C \pm \sqrt{(AB)}$ are not eigenvalues of $-\Delta$ then this equation forces both ϕ and ψ to vanish, so that $P_{A,B,C} \in \mathfrak{P}_0$.

Assume $AB \leqslant 0$ and $P_{A,B,C} \in \mathfrak{P}_0$. Since $\mathfrak{P}_0$ is open, we can perturb A and B slightly to cause $AB < 0$. Then, keeping A and B fixed, we can vary C without ever causing $-\partial_\Delta - P_{A,B,C}$ to become singular; we move C to $C = 0$. Finally we let A and B move linearly to $A = 0$ and $B = 0$, and again our operator $-\partial_\Delta - P_{A,B,C}$ remains nonsingular, while the potential P at the end of these deformations has become the zero potential. Hence the original operator $-\partial_\Delta - P_{A,B,C}$ had index zero.

Let $A > 0$ and $B > 0$, and assume again $P_{A,B,C} \in \mathfrak{P}_0$. After perturbing C slightly we can assume that $-C$ is not an eigenvalue of $-\Delta$. We now deform A linearly to 0, i.e., we consider the operators $-\partial_\Delta - P_{\theta A,B,C}$ with $0 \leqslant \theta \leqslant 1$. This operator has a monotonically decreasing potential, so its index drops at each θ for which it becomes singular. Thus for each θ for which $-C \pm \sqrt{(\theta AB)}$ is an eigenvalue of $-\Delta$ the index of $-\partial_\Delta - P_{\theta A,B,C}$ jumps by the multiplicity of the eigenvalue in question. The end result of this deformation is a nondegenerate operator with $AB = 0$. We have just seen that such an operator has index zero, and hence the index of our original $-\partial_\Delta - P_{A,B,C}$ must equal the number of eigenvalues of $-\Delta$ counted with multiplicity in the interval $|\lambda + C| < \sqrt{(AB)}$.

In the remaining case, $A < 0$, $B < 0$, one can apply the same argument. The only difference is now that the deformation $-\partial_\Delta - P_{\theta A,B,C}$ has monotonically *increasing* potential, so one arrives at the same numerical value, but the opposite sign for the index. □

In [1] we also introduced an upper and lower index $\mu_+(-\partial_\Delta - P(x))$ and $\mu_-(-\partial_\Delta - P(x))$ for degenerate critical points, which are defined by

$$\mu_+\big(-\partial_\Delta - P(x)\big) = \lim_{\varepsilon\downarrow 0} \sup\big\{\mu(-\partial_\Delta - P') \mid \|P' - P\|_{L^\infty} < \varepsilon,\ P' \in \mathfrak{P}_0\big\},$$

$$\mu_-\big(-\partial_\Delta - P(x)\big) = \lim_{\varepsilon\downarrow 0} \inf\big\{\mu(-\partial_\Delta - P') \mid \|P' - P\|_{L^\infty} < \varepsilon,\ P' \in \mathfrak{P}_0\big\}.$$

They satisfy

$$\mu_+\big(-\partial_\Delta - P(x)\big) - \mu_-\big(-\partial_\Delta - P(x)\big) = \dim \operatorname{kern}\big(-\partial_\Delta - P(x)\big).$$

3. A VARIATIONAL PRINCIPLE FOR THE INDEX

Let $\mathcal{E} = -\partial_\Delta - P(x)$ with

$$P(x) = \begin{pmatrix} A(x) & -C(x) \\ -C(x) & B(x) \end{pmatrix},$$

and assume that $A(x)$, $B(x)$ and $C(x)$ are pointwise nonnegative. This implies that the operator $-\Delta + C(x)$ is invertible. We define the bounded compact operator

$$T_P f = \sqrt{A}(-\Delta + C)^{-1} B(-\Delta + C)^{-1}\sqrt{A} f$$

on $L^2(\Omega)$. One can write T_P as $(S_P)^* S_P$, where

$$S_P = \sqrt{B}(-\Delta + C)^{-1}\sqrt{A},$$

from which one sees that T_P is selfadjoint and nonnegative.

LEMMA 3A. – *The operator $\mathcal{E}$ is nondegenerate iff* 1 *is not an eigenvalue of T_P. The Morse index of $\mathcal{E}$ equals the number of eigenvalues λ of T_P with $\lambda > 1$. If $\mathcal{E}$ is degenerate, then* dim kern $\mathcal{E}$ *coincides with* dim kern $(T_P - 1)$. *The lower index of $\mathcal{E}$ is the number of eigenvlaues of T_P exceeding* 1; *the upper index is the number of eigenvalues λ of T_P with $\lambda \geqslant 1$.*

Proof. – $\mathcal{E}$ is degenerate iff there are $\binom{\phi}{\psi}$ such that

$$-A(x)\phi + (C(x) - \Delta)\psi = 0,$$
$$(C(x) - \Delta)\phi - B(x)\psi = 0.$$

Since $C(x) \geqslant 0$ the operator $C(x) - \Delta$ has a bounded inverse $(C(x) - \Delta)^{-1}$ on $L^2(\Omega)$. This allows us to eliminate ψ, after which we find that $\mathcal{E}$ is degenerate exactly when there is some ϕ with

$$\phi = (C(x) - \Delta)^{-1}\bigl[B(x)\bigl(C(x) - \Delta\bigr)^{-1}\{A(x)\phi\}\bigr],$$

i.e., whenever 1 is an eigenvalue of the operator $T_P' = (C - \Delta)^{-1}B(C - \Delta)^{-1}A$. This operator is formally conjugate with T_P, namely $T_P[\sqrt{A}f] = \sqrt{A}T_P'[f]$.

To compute the index of $\mathcal{E}$, let A and B vary monotonically to 0, and count the number of times $\mathcal{E}$ and $T_P - 1$ become degenerate: both operators vary monotonically, so this number gives both the change in index of $\mathcal{E}$ and the number of positive eigenvalues of $T_P - 1$. Since operators of the form $-\partial_\Delta - \begin{pmatrix} 0 & -C \\ -C & 0 \end{pmatrix}$ are always nondegenerate (provided $C(x) \geqslant 0$ of course) they all have the same index: this index must be the index of $-\partial_\Delta$ itself, i.e., zero. □

COROLLARY 3B. – *If either* $A(x) \equiv 0$ *or* $B(x) \equiv 0$, *then* $\mathcal{E}$ *has index* 0.

Proof. – The operator T_P vanishes, and hence has no eigenvalues exceeding 1. □

COROLLARY 3C. – *If* $A(x) \equiv B(x)$, *then the index of* $\mathcal{E}$ *equals the number of negative eigenvalues of the Schrödinger operator* $-\Delta - A(x) + C(x)$.

Proof. – The operator T_P is the square of $\sqrt{A}(C - \Delta)^{-1}\sqrt{A}$, which has eigenvalue 1 exactly when the Schrödinger operator $-\Delta + A(x) + C(x)$ is singular. Replace $A(x)$ with $\theta A(x)$, and let θ vary monotonically from 1 to 0. All negative eigenvalues of $-\Delta + A(x) + C(x)$ then move to the positive real axis, since $-\Delta + C(x)$ is positive definite. Every time an eigenvalue of $-\Delta + \theta A(x) + C(x)$ crosses 0, an eigenvalue of T_{P_θ} crosses 1. Hence T_P has as many eigenvalues with $\lambda > 1$ as $-\Delta + A(x) + C(x)$ has negative eigenvalues. □

The following corollary sheds some light on our hypothesis concerning the signs of H_{uu}, etc. in Theorem 1B.

COROLLARY 3D. – *If* H *satisfies* $H_{uu} \geqslant 0$, $H_{vv} \geqslant 0$ *and* $H_{uv} \leqslant 0$, *then* f_H *has no critical points with negative renormalized Morse index.*

4. THE BLOW-UP ARGUMENT

Let $H^{(k)}$ be a sequence of functions in $C^2(\overline{\Omega};\mathbb{R}^2)$ satisfying the conditions of Theorem 1B. Assume that there is a sequence of critical points $z_k=(u_k,v_k)$ of $f_{H^{(k)}}$, with

$$\lim_{k\to\infty}\|u_k\|_{L^\infty}+\|v_k\|_{L^\infty}=\infty.$$

Assume also that the renormalized index of the z_k is $\leqslant m-1$. Then we define

$$\lambda_k=\sup_{x\in\Omega}\bigl\{\max\bigl(|u_k(x)|^{1/(q+1)},|v_k(x)|^{1/(p+1)}\bigr)\bigr\},\qquad \varepsilon_k=\lambda_k^{-(pq-1)/2}.$$

We assume that the supremum is attained in $P_k\in\Omega$ and define

$$U_k(y)=\alpha\lambda_k^{-(q+1)}u_k(P_k+\varepsilon_k y),$$
$$V_k(y)=\beta\lambda_k^{-(p+1)}v_k(P_k+\varepsilon_k y)$$

with $\alpha,\beta>0$ to be specified in a moment. We also define the rescaled domains

$$\Omega_k=\frac{\Omega-P_k}{\varepsilon_k}.$$

A short calculation shows that (U_k,V_k) is a critical point of $f_{\widetilde{H}^{(k)}}$ on $C_0^1(\overline{\Omega}_k;\mathbb{R}^2)$, where

$$\widetilde{H}^{(k)}(y,U,V)=\frac{1}{\alpha\beta}\widehat{H}^{(k)}(y,\alpha U,\beta V),$$

and $\hat{H}^{(k)}$ is as described in the condition $(*)$. By our assumption $(*)$ we can extract a subsequence for which $\widehat{H}^{(k)}(y,U,V)$ converges in C^2_{loc} to

$$\widehat{H}(U,V)=a|U|^{p+1}+b|V|^{q+1},$$

for certain positive constants a,b. The $\widetilde{H}^{(k)}(y,U,V)$ then converge in C^2_{loc} to

$$H(U,V)=\frac{a\alpha^p}{\beta}|U|^{p+1}+\frac{b\beta^q}{\alpha}|V|^{q+1}.$$

By choosing α and β appropriately we can arrange that H is given by

$$H(U,V)=\frac{|U|^{p+1}}{p+1}+\frac{|V|^{q+1}}{q+1}. \tag{7}$$

The (U_k, V_k) are uniformly bounded in L^∞, and satisfy the Euler–Lagrange equations,

$$-\Delta U_k = \frac{\partial \widetilde{H}^{(k)}}{\partial V}(y, U, V), \qquad -\Delta V_k = \frac{\partial \widetilde{H}^{(k)}}{\partial U}(y, U, V). \tag{8}$$

Elliptic regularity implies that the (U_k, V_k) are uniformly bounded in $C^{2,\alpha}$ for any $\alpha < 1$. Hence there is some subsequence for which the (U_k, V_k) converge in $C^{2,\alpha}_{\mathrm{loc}}$. The limits (U_*, V_*) are then bounded solutions of

$$-\Delta U = V^q, \qquad -\Delta V = U^p. \tag{9}$$

The domain of U and V is $\Omega_* = \lim_{k\to\infty} \Omega_k$. If

$$\limsup_{k\to\infty} \frac{\mathrm{dist}(P_k, \partial\Omega)}{\varepsilon_k} = \infty,$$

then we can extract a subsequence along which Ω_k converges to $\Omega_* = \mathbb{R}^n$. Otherwise we recall that $\partial\Omega$ was assumed to be smooth, so that along some subsequence the Ω_k converge to a half space Ω_* containing the origin in its interior.

For now we shall assume that Ω_* is all of $\mathbb{R}^n$, and at the end of this section we indicate which changes must be made if Ω_* is a half space.

We consider the index of the solutions $z_k = (u_k, v_k)$. By Lemma 3A the index of z_k equals the number of eigenvalues above 1 of the operator $T_k = (S_k)^* S_k$, with

$$S_k\phi(x) = \sqrt{H^{(k)}_{vv}\big(x, z_k(x)\big)}\big(-\Delta - H^{(k)}_{uv}\big(x, z_k(x)\big)\big)^{-1} \times \Big[\sqrt{H^{(k)}_{uu}\big(x, z_k(x)\big)}\phi(x)\Big].$$

We have

$$H^{(k)}_{uu}\big(x, z_k(x)\big) = \frac{\alpha}{\beta}\lambda_k^{(p-1)(q+1)}\widetilde{H}^{(k)}_{UU}\big(y, U_k(y), V_k(y)\big),$$
$$H^{(k)}_{vv}\big(x, z_k(x)\big) = \frac{\beta}{\alpha}\lambda_k^{(p+1)(q-1)}\widetilde{H}^{(k)}_{VV}\big(y, U_k(y), V_k(y)\big),$$
$$H^{(k)}_{uv}\big(x, z_k(x)\big) = \varepsilon_k^{-2}\widetilde{H}^{(k)}_{UV}\big(y, U_k(y), V_k(y)\big).$$

Using these relations one then easily finds how S_k changes under rescaling.

LEMMA 4A. – *Given* $\phi \in C_c^\infty(\mathbb{R}^n)$ *let* $\phi_k(x) = \phi((x - P_k)/\varepsilon_k)$. *Then*

$$(S_k\phi_k)(P_k + \varepsilon_k y) = \tilde{S}_k\phi(y),$$

where $\tilde{S}_k$ *is the operator given by*

$$\tilde{S}_k\phi(y) = \sqrt{\widetilde{H}^{(k)}_{VV}\big(y, z_k(y)\big)}\big(-\Delta - \widetilde{H}^{(k)}_{UV}\big(y, z_k(y)\big)\big)^{-1} \times \Big[\sqrt{\widetilde{H}^{(k)}_{UU}\big(y, z_k(y)\big)}\phi(y)\Big].$$

We now let k tend to infinity.

LEMMA 4B. – *If* $n \geqslant 3$ *then*

$$\lim_{k\to\infty} \tilde{S}_k\phi(y) = S\phi(y) \tag{10}$$

uniformly on compact sets in $\mathbb{R}^n$. *Here* $S\phi$ *is defined by*

$$S\phi(y) = \sqrt{pq}\,\big|V(y)\big|^{(q-1)/2}(-\Delta)^{-1}\big|U(y)\big|^{(p-1)/2}\phi(y).$$

Here $(-\Delta)^{-1}$ is the Newton potential.

Proof. – It follows from C^2 convergence of $\widetilde{H}^{(k)}$ to $\frac{|U|^{p+1}}{p+1} + \frac{|V|^{q+1}}{q+1}$ that

$$\widetilde{H}^{(k)}_{UU} \to p|U(y)|^{p-1}, \quad \widetilde{H}^{(k)}_{VV} \to q|V(y)|^{q-1} \quad \text{and} \quad \widetilde{H}^{(k)}_{UV} \to 0$$

uniformly in compact subsets of $\mathbb{R}^n$.

For $n \geqslant 3$ the Newton potential $|x|^{2-n}/(n-2)\omega_n$ is positive (ω_n is the surface “area” of the unit sphere S^{n-1} in $\mathbb{R}^n$). Together with $\widetilde{H}^{(k)}_{UV} \leqslant 0$ and the maximum principle this implies that

$$\big|\big(-\Delta - \widetilde{H}^{(k)}_{UV}\big(x, z_k(x)\big)\big)^{-1} f(y)\big| \leqslant \frac{1}{(n-2)\omega_n}\int_{\mathbb{R}^n} \frac{|f(y')|}{|y-y'|^{n-2}}\,\mathrm{d}y'$$

for any $f \in C_c^\infty(\Omega_k)$. This uniform bound allows one to pass to the limit and conclude that

$$\lim_{k\to\infty}\big(-\Delta - \widetilde{H}^{(k)}_{UV}\big(x, z_k(x)\big)\big)^{-1} f = (-\Delta)^{-1} f. \quad \square$$

In Section 8, Theorem 8A, we will prove for arbitrary $m \in \mathbb{N}$ and $\varepsilon > 0$ that there exist $\phi_i \in C_c^\infty(\mathbb{R}^n)$, $i = 1, \ldots, m$, for which $\|S\phi\|_{L^2} \geqslant (\sqrt{pq} -$

$\varepsilon)\|\phi\|_{L^2}$ for any linear combination $\phi = c_1\phi_1 + \cdots + c_m\phi_m$. Choose ε so small that $\sqrt{pq} - \varepsilon > 1$. Then we find that for sufficiently large k there is an m-dimensional space on which $\|S_k\phi\|^2 = (\psi, T_k\psi) > \|\psi\|^2$, and hence that T_k must have at least m eigenvalues larger than 1. This contradicts our assumption that the indices of the z_k were all less than m, so that our main theorem is proved as soon as we establish Theorem 8A.

We now briefly consider the situation in which $\operatorname{dist}(P_k, \partial\Omega) \leqslant C\varepsilon_k$. In this case we may assume after passing to a subsequence that P_k tends to some point P_* on the boundary. One now "flattens the boundary", i.e., one chooses coordinates $\xi_1, \ldots, \xi_n$ near P_* such that P_* becomes the origin, and Ω gets mapped to the half space $\mathbb{H}^n = \{\xi \mid \xi_1 > 0\}$.

Then we define

$$U_k(\eta) = \lambda_k^{-(q+1)} u_k\big(X(\varepsilon_k\eta)\big), \qquad V_k(\eta) = \lambda_k^{-(p+1)} v_k\big(X(\varepsilon_k\eta)\big),$$

where $\xi \mapsto X(\xi)$ is the inverse to the chart $x \mapsto (\xi_1(x), \ldots, \xi_n(x))$.

Then the U_k and V_k are defined on $B_{R_k} \cap \mathbb{H}^n$, with $R_k \sim \varepsilon_k^{-1}$, and they satisfy (8), provided one interprets Δ as $\varepsilon_k^{-2}\times$ the Euclidean Laplacian in η coordinates. In the limit $k \to \infty$ this equation ends to (9), and one can extract a subsequence for which the U_k and V_k converge to bounded nontrivial solutions U and V of (9) on $\mathbb{H}^n$ which vanish on $\partial\mathbb{H}^n = \{0\} \times \mathbb{R}^{n-1}$. By odd reflection in $\partial\mathbb{H}^n$ one can extend such solutions to entire solutions of (9), and all results in the following sections therefore apply.

For $n \geqslant 3$ the operator $\tilde{S}_k$ also converges to

$$\tilde{S}_* = \sqrt{pq}|V|^{(q-1)/2}(\Delta_{\mathbb{H}^n})^{-1}|U|^{(p+1)/2},$$

for the same reasons as in the case where $\Omega_* = \mathbb{R}^n$. Rather than considering the action of $\tilde{S}_*$ on functions on $\mathbb{H}^n$, one can consider the associated operator

$$\bar{S}_* = \sqrt{pq}|V|^{(q-1)/2}(\Delta_{\mathbb{R}^n})^{-1}|U|^{(p+1)/2}$$

acting on odd functions on $\mathbb{R}^n$ (odd meaning $\phi(-\eta_1, \eta_2, \ldots, \eta_n) = -\phi(\eta_1, \eta_2, \ldots, \eta_n)$). All arguments in the following section apply to this operator without modification, and thus one can again show that $\|\tilde{S}_k\phi\| > \|\phi\|$ holds on some m dimensional subspace of $L^2(\Omega)$ for large enough k.

5. THE BLOWN-UP EQUATION

THEOREM 5A. – *Let u, v be solutions of* (9) *on the ball with radius R. Then one has for large enough m and arbitrary small $\varepsilon > 0$*

$$\left|\int_{B_R} \rho\left(\frac{x}{R}\right)^m \{|u|^{p+1} - |v|^{q+1}\}\,\mathrm{d}x\right| \leqslant \varepsilon \int_{B_R} \rho\left(\frac{x}{R}\right)^m \left\{|u|^{p+1} + |v|^{q+1}\right\}\mathrm{d}x + \frac{C}{R^n},$$

where $\rho(x) = 1 - |x|^2$. The constant C depends on ε, m, p, and q but not on R or the solutions u, v. Here m is large enough if it exceeds $2(p+1)(q+1)/(pq-1)$.

COROLLARY 5B. – *If (u, v) are bounded entire solutions of* (9), *with $\int_{\mathbb{R}^n} |u|^{p+1}$ finite, then*

$$\int_{\mathbb{R}^n} |v|^{q+1} = \int_{\mathbb{R}^n} |u|^{p+1}.$$

In particular the $(q+1)$-norm of v is also finite.

We will show later on that u and v must actually vanish.

Proof. – Theorem 5A implies that

$$I_R = \int \rho(x/R)^m |u|^{p+1}, \qquad J_R = \int \rho(x/R)^m |v|^{q+1}$$

satisfy

$$(1-\varepsilon)I_R - \frac{C_\varepsilon}{R} \leqslant J_R \leqslant (1+\varepsilon)I_R + \frac{C_\varepsilon}{R}.$$

Letting $R \uparrow \infty$ one concludes that $J = \int |v|^{q+1}$ converges, and that $(1-\varepsilon)I \leqslant J \leqslant (1+\varepsilon)I$ for any $\varepsilon > 0$, where $I = \int |u|^{p+1}$. Thus $I = J$, as claimed. □

We now prove Theorem 5A. We may assume that $R = 1$, since the general case then follows by rescaling. Put $\zeta = \rho^m$ and compute:

$$\int \zeta \{|u|^{p+1} - |v|^{q+1}\}\,\mathrm{d}x = \int \zeta (v\Delta u - u\Delta v)\,\mathrm{d}x = \int u\nabla\zeta \cdot \nabla v - v\nabla\zeta \cdot \nabla u.$$

Hence, using $|\nabla\zeta| = m\rho^{m-1}|\nabla\rho| \leqslant C\rho^{m-1}$ we arrive at

$$\begin{aligned}
&\left|\int \zeta\left\{|u|^{p+1} - |v|^{q+1}\right\}\mathrm{d}x\right| \\
&\quad\leqslant C\int \rho^{m-1}\left\{|u||\nabla v| + |v||\nabla u|\right\}\mathrm{d}x \\
&\quad\leqslant \sigma\int \rho^{m}\left\{|u|^{p+1} + |v|^{q+1}\right\}\mathrm{d}x + C_\sigma\int \rho^{m-1-1/p}|\nabla v|^{(p+1)/p}\,\mathrm{d}x \\
&\qquad + C_\sigma\int \rho^{m-1-1/q}|\nabla u|^{(q+1)/q}\,\mathrm{d}x \qquad (11)
\end{aligned}$$

for arbitrary $\sigma > 0$.

LEMMA 5C. – *For arbitrary $u \in C^2(B)$, $1 < p < \infty$ and $\delta > 0$, there is a $C_{\delta,p} < \infty$ such that*

$$\int_B \rho^m|\nabla u|^p\,\mathrm{d}x \leqslant C_\delta\int_B \rho^{m-p}|u|^p\,\mathrm{d}x + \delta\int_B \rho^{m+p}|\Delta u|^p\,\mathrm{d}x.$$

Proof. – This follows from L^p interior estimates for the Laplacian, and a covering argument. □

We apply the lemma to (11). For the third term in (11) we find, using $\Delta v = -u^p$,

$$\begin{aligned}
&\int \rho^{m-1-1/p}|\nabla v|^{(p+1)/p}\,\mathrm{d}x \\
&\quad\leqslant C_\delta\int \rho^{m-2-2/p}|v|^{(p+1)/p}\,\mathrm{d}x + \delta\int \rho^{m}|\Delta v|^{(p+1)/p}\,\mathrm{d}x \\
&\quad\leqslant C_\delta\int \rho^{m-2-2/p}|v|^{(p+1)/p}\,\mathrm{d}x + \delta\int \rho^{m}|u|^{p+1}\,\mathrm{d}x. \qquad (12)
\end{aligned}$$

We now observe that $(p+1)/p < 2 < q+1$, so that

$$r = \frac{p}{p+1}(q+1) > 1,$$

and so that one has $x^r \leqslant \tau^{-r'/r} + \tau x^r$, $r' = r/(r-1)$, for any $x \geqslant 0$ and $\tau > 0$. Thus

$$\begin{aligned}
\rho^{m-2-2/p}|v|^{(p+1)/p} &\leqslant \tau^{-r'/r} + \tau\left(\rho^{m-2-2/p}|v|^{(p+1)/p}\right)^r \\
&= \tau^{-r'/r} + \tau\rho^{mr-2(q+1)}|v|^{q+1} \\
&\leqslant \tau^{-r'/r} + \tau\rho^{m}|v|^{q+1}
\end{aligned}$$

provided $m > 2(q+1)/(r-1) = 2(p+1)(q+1)/(pq-1)$. Apply this inequality to (12), and you get

$$\begin{aligned}&\int \rho^{m-1-1/p}|\nabla v|^{(p+1)/p}\,\mathrm{d}x\\&\quad\leqslant C+\tau\int \rho^m|v|^{q+1}\mathrm{d}x+\delta\int \rho^m|u|^{p+1}\,\mathrm{d}x \qquad (13)\end{aligned}$$

which implies the theorem.

6. A LIOUVILLE THEOREM

In this section we will prove:

THEOREM 6A. – *Let u and v be bounded entire solutions of* (9). *If $\int_{\mathbb{R}^n}|u|^{p+1}$ is finite, then both u and v vanish.*

We have shown that $\int_{\mathbb{R}^n}|u|^{p+1}<\infty$ implies that $\int|v|^{q+1}<\infty$, and that both integrals are in fact equal.

The idea of the proof is as follows: first we show that the action of the solution (u, v)

$$E(u,v)=\int_{\mathbb{R}^n}\left\{\nabla u\cdot\nabla v-\frac{|u|^{p+1}}{p+1}-\frac{|v|^{q+1}}{q+1}\right\}\mathrm{d}x$$

is finite. Then we observe that for any $\lambda>0$ the functions

$$u^\lambda(x)=\lambda^{q+1}u(x/\varepsilon),\qquad v^\lambda(x)=\lambda^{p+1}v(x/\varepsilon),$$

with $\varepsilon=\lambda^{-(pq-1)/2}$, are also solutions of our system. Moreover, these solutions also have finite action. Direct substitution shows that the action of (u^λ, v^λ) is

$$E\left(u^\lambda,v^\lambda\right)=\lambda^\alpha E(u,v), \qquad (14)$$

with

$$\alpha=\frac{n}{2}(p+1)(q+1)\left(1-\frac{2}{n}-\frac{1}{p+1}-\frac{1}{q+1}\right)\neq 0.$$

On the other hand the (u^λ, v^λ) are critical points of the action, so $E(u^\lambda, v^\lambda)$ should not depend on λ. This can only happen if u and v both vanish.

We now go through the details of the argument.

LEMMA 6B. – *If* $u \in L^{p+1}(\mathbb{R}^n)$ *then* $|\nabla u| \in L^r$ *and* $|\nabla v| \in L^s$ *where*

$$\frac{1}{r} = \frac{1}{2}\left(1 + \frac{1}{p+1} - \frac{1}{q+1}\right), \qquad \frac{1}{s} = \frac{1}{2}\left(1 - \frac{1}{p+1} + \frac{1}{q+1}\right). \quad (15)$$

In particular, $|\nabla u| \cdot |\nabla v| \in L^1$*, and* $|u\nabla v| + |v\nabla u| \in L^t$ *for some* $1 < t < n/(n-1)$.

Proof. – We have $\Delta u \in L^{1+1/q}$ and $u \in L^{p+1}$, so that, by interpolation, $|\nabla u| \in L^r$, where

$$\frac{1}{r} = \frac{1}{2}\left(\frac{1}{p+1} + \frac{q}{q+1}\right)$$

which implies the first part of (15). The second part follows in the same way.

Using the subcriticality of p and q, one finds

$$\frac{1}{r} > \frac{n-1}{n} - \frac{1}{q+1}.$$

Hölder's inequality and $v \in L^{q+1}$ then imply $|v\nabla u| \in L^t$, where

$$\frac{1}{t} = \frac{1}{r} + \frac{1}{q+1} > \frac{n-1}{n}. \quad \square$$

This lemma implies the action is well defined, and moreover that, by dominated convergence

$$E(u,v) = \lim_{R\to\infty} E^R(u,v),$$
$$E^R(u,v) = \int \eta\left(\frac{x}{R}\right)\left\{\nabla u \cdot \nabla v - \frac{|u|^{p+1}}{p+1} - \frac{|v|^{q+1}}{q+1}\right\} \mathrm{d}x,$$

for any smooth compactly supported function with $\eta(0) = 1$. We shall assume that

$$\eta(x) \begin{cases} \equiv 1 & \text{for } |x| \leqslant 1/2, \\ \equiv 0 & \text{for } |x| \geqslant 1. \end{cases}$$

LEMMA 6C. –

$$E(u,v) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int |u|^{p+1}\,\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{q+1}\right)\int |v|^{q+1}\,\mathrm{d}x$$
$$= \left(1 - \frac{1}{p+1} - \frac{1}{q+1}\right)\int |u|^{p+1}\,\mathrm{d}x.$$

Proof. – Formally we integrate by parts and use the Euler–Lagrange equations. To deal with the infinite domain, we work with E^R:

$$E^R(u,v) = \int \eta^R \left\{ \left(\frac{1}{2} - \frac{1}{p+1}\right)|u|^{p+1} + \left(\frac{1}{2} - \frac{1}{q+1}\right)|v|^{q+1} \right\} \mathrm{d}x \\ + \frac{1}{2}\int \nabla\eta^R \cdot \{u\nabla v + v\nabla u\}\,\mathrm{d}x$$

where $\eta^R(x) = \eta(x/R)$. Combining $u\nabla v \in L^t$ with $t < n/(n-1)$ and $|\nabla\eta^R| \leqslant C/R$, one shows that the last integral vanishes as $R \to \infty$. □

This lemma directly implies that the action scales as stated in (14).

LEMMA 6D. – $\frac{\mathrm{d}}{\mathrm{d}\lambda}E(u^\lambda, v^\lambda) = 0$.

Proof. – Again we deal with E^R first. Let $h = \frac{\partial u^\lambda}{\partial\lambda}|_{\lambda=1}$ and $k = \frac{\partial v^\lambda}{\partial\lambda}|_{\lambda=1}$. Then

$$\frac{\mathrm{d}E^R}{\mathrm{d}\lambda} = -\int \nabla\eta \cdot \{h\nabla v + k\nabla u\}\,\mathrm{d}x.$$

On substituting

$$h = -\frac{1}{2}(pq-1)x\cdot\nabla u + (q+1)u$$

and

$$k = -\frac{1}{2}(pq-1)x\cdot\nabla v + (p+1)v,$$

one ends up with four integrals. Two of these are bounded by

$$\int |\nabla\eta| \big\{ |u\nabla v| + |v\nabla u| \big\}\,\mathrm{d}x.$$

As in the previous lemma one shows that this is o(1) for $R \to \infty$.

The other two integrals are of the form $\int (x\cdot\nabla u)(\nabla\eta\cdot\nabla v)\,\mathrm{d}x$. We now recall that $\eta(x) \equiv 1$ for $|x| \leqslant 1/2$, so that $\nabla(\eta(x/R))$ is supported on $B_R \setminus B_{R/2}$, and is bounded by C/R on this annulus. By Hölder's inequality we then get that

$$\left| \int (x\cdot\nabla u)(\nabla\eta\cdot\nabla v)\,\mathrm{d}x \right| \leqslant C \int_{B_R\setminus B_{R/2}} |\nabla u||\nabla v|\,\mathrm{d}x,$$

which, since $|\nabla u||\nabla v| \in L^1(\mathbb{R}^n)$, is o(1) for large R. Thus we see that $\lim_{R\to\infty} \frac{\mathrm{d}}{\mathrm{d}\lambda}E^R = 0$. □

The Liouville theorem now follows immediately, since $E(u^\lambda, v^\lambda) = \lambda^\alpha E(u, v)$ is found to be constant, and hence must vanish. The explicit formula for $E(u, v)$ then implies $u \equiv v \equiv 0$.

7. A SECOND LIOUVILLE THEOREM

THEOREM 7A. – *Let u, v be bounded entire solutions of* (9)*. Then, if $(a + b_1x_1 + \cdots + b_nx_n)|u(x)|^{(p-1)/2}$ belongs to $L^2(\mathbb{R}^n)$, both u and v must vanish.*

Proof. – If $b_i = 0$ for all i, then our hypothesis is $\int |u(x)|^{p-1}\,\mathrm{d}x < \infty$, which by boundedness of u implies $\int |u(x)|^{p+1}\,\mathrm{d}x < \infty$. The first Liouville theorem now forces u and v to vanish.

Assume henceforth that some $b_i \neq 0$.

Since u and v are bounded solutions, their gradients are also bounded. Let $M = \sup|\nabla u|$. For any $x \in \mathbb{R}^n$ we define

$$r_x = \frac{|u(x)|}{10M}, \qquad B_x = B(x, r_x), \qquad 5B_x = B(x, 5r_x).$$

On the larger ball $5B_x$ it follows from $|\nabla u| \leqslant M$ that we have

$$\frac{1}{2}\bigl|u(x)\bigr| \leqslant \bigl|u(y)\bigr| \leqslant \frac{3}{2}\bigl|u(x)\bigr|, \quad \forall y \in 5B_x,$$

and hence

$$\int_{5B_x} \bigl|u(y)\bigr|^{p+1}\,\mathrm{d}y \leqslant C\bigl|u(x)\bigr|^{p+1} r_x^n \leqslant C'\bigl|u(x)\bigr|^{p+n+1}. \tag{16}$$

On B_x we have

$$\int_{B_x} (a + b_1y_1 + \cdots + b_ny_n)^2\,\mathrm{d}y \geqslant cr_x^{n+2} \tag{17}$$

for some constant which only depends on $a, b_1, \ldots, b_n$. Hence

$$\int_{B_x} (a + b_1y_1 + \cdots + b_ny_n)^2\bigl|u(y)\bigr|^{p-1}\,\mathrm{d}y \geqslant \left(\frac{|u(x)|}{2}\right)^{p-1} cr_x^{n+2}$$
$$= c'\bigl|u(x)\bigr|^{p+n+1}. \tag{18}$$

Putting (18) and (16) together we get

$$\int_{5B_x} |u(y)|^{p+1}\,\mathrm{d}x \leqslant \frac{C'}{c'}\int_{B_x}(a+b_1y_1+\cdots+b_ny_n)^2|u(y)|^{p-1}\,\mathrm{d}y.$$

We can now choose $x_1, x_2, \ldots$, such that the B_{x_i} are pairwise disjoint, and such that the $5B_{x_i}$ cover $\mathbb{R}^n$ (see [7, Section I.1.6]). One then has

$$\begin{aligned}\int_{\mathbb{R}^n} |u(y)|^{p+1}\,\mathrm{d}y &\leqslant \sum_i \int_{5B_{x_i}} |u(y)|^{p+1}\,\mathrm{d}y \\ &\leqslant \frac{C'}{c'}\sum_i \int_{B_{x_i}}(a+b_1y_1+\cdots+b_ny_n)^2|u(y)|^{p-1}\mathrm{d}y \\ &\leqslant \frac{C'}{c'}\int_{\mathbb{R}^n}(a+b_1y_1+\cdots+b_ny_n)^2|u(y)|^{p-1}\,\mathrm{d}y\end{aligned}$$

and we find again that $\int |u|^{p+1} < \infty$, which implies that u and v vanish, as claimed. □

By slightly modifying the proof we get the following stronger version of this theorem.

THEOREM 7B. – *Let $\psi \in C^1(\mathbb{R}^n)$, with $|\psi(x)| + |\nabla\psi(x)| \to 0$ for $|x| \to \infty$. Then $(a+b_1x_1+\cdots+b_nx_n+\psi(x))v(x)^{(q-1)/2}$ can only be in $L^2(\mathbb{R}^n)$ if $a = b_1 = \cdots = b_n = 0$.*

Proof. – The proof proceeds exactly as before, the only difference being that (17) no longer holds. However, outside some large enough ball B_{R_1} one has $|\psi| << a$ and $|\nabla\psi| << |b|$, so that (17) does hold for all balls B_x outside B_{R_1}. □

8. INDEX OF ENTIRE SOLUTIONS

Let u and v be entire solutions to (9). We will essentially show in this section that the generalized Morse index of such a solution is infinite. Thus we would like to consider the operator

$$T = pq\Psi \circ (-\Delta)^{-1} \circ \Phi^2 \circ (-\Delta)^{-1} \circ \Psi,$$

where Ψ is multiplication with $|v(x)|^{(q-1)/2}$ and Φ is mutiplication with $|u(x)|^{(p-1)/2}$. Unfortunately this operator is not necessarily well defined on $L^2(\mathbb{R}^n)$, even if we restrict its domain to, say $C_c^\infty(\mathbb{R}^n)$. Thus instead of studying T we consider

$$S = \sqrt{pq}\,\Phi \circ (-\Delta)^{-1} \circ \Psi,$$

i.e., for $\phi \in C_c^\infty$ we define

$$S\phi(x) = \sqrt{pq} \int_{\mathbb{R}^n} \frac{|v(x)|^{(q-1)/2} |u(y)|^{(p-1)/2}}{(n-2)\omega_n |x-y|^{n-2}} \phi(y)\,\mathrm{d}y. \tag{19}$$

This way we have a continuous integral operator from C_c^∞ to L^∞.

Formally, $T\phi = S^* S\phi$, where S^* is the "L^2-adjoint" of S. To make this precise, we choose a domain for S which makes it a possibly unbounded operator in L^2. Let $\mathfrak{D} \subset C_c^\infty$ be the subspace of all testfunctions which satisfy

$$\int_{\mathbb{R}^n} \left|u(x)\right|^{(p-1)/2} \phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} x_i \left|u(x)\right|^{(p-1)/2} \phi(x)\,\mathrm{d}x = 0, \quad 1 \leqslant i \leqslant n.$$

To motivate this definition recall that the Newton potential of a compactly supported function ψ has an asymptotic expansion of the form

$$\frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{\psi(y)}{|x-y|^{n-2}}\,\mathrm{d}y = \frac{M_0}{r^{n-2}} + \frac{M_1(\hat{x})}{r^{n-1}} + \frac{M_2(\hat{x})}{r^n} + \cdots,$$

where $r = |x|$, $\hat{x} = x/r$ and the $M_k(\hat{x})$ are spherical harmonics of order k. The M_k depend linearly on the kth order moments of ψ. If the moments of order $\leqslant k-1$ of ψ vanish, then the Newton potential $(-\Delta)^{-1}\psi$ decays like $\mathcal{O}(r^{-n-k+2})$.

Consequently, since $\mathfrak{D}$ consists of those ϕ for which the moments of order 0 and 1 of $|u|^{(p-1)/2}\phi$ vanish, $S\phi(x)$ is bounded by $C(\phi)/r^n$ when $\phi \in \mathfrak{D}$. For $n \geqslant 3$ this implies $S\phi \in L^2(\mathbb{R}^n)$, i.e., our definition of $\mathfrak{D}$ makes $S : \mathfrak{D} \to L^2$ a well defined (perhaps unbounded) operator.

The main result in this section is

THEOREM 8A. – *For any* $\varepsilon > 0$ *and any integer* m *there exist* $\phi_1, \ldots, \phi_m \in \mathfrak{D}$ *such that*

$$\|S\phi\|_{L^2} \geqslant (\sqrt{pq} - \varepsilon)\|\phi\|_{L^2}$$

holds for all $\phi = c_1\phi_1 + \cdots + c_m\phi_m$.

Since formally we have $(\phi, T\phi) = \|S\phi\|^2$, we find by choosing $\varepsilon > \sqrt{pq} - 1$ that $(\phi, T\phi) > \|\phi\|^2$ on some m dimensional subspace of L^2, which, in view of our characterization of the generalized Morse index, we interpret as index $(u, v) \geqslant m$. Since m is arbitrary we say (u, v) has infinite index.

The main technical tools in proving the theorem are the following two lemmas.

LEMMA 8B. – *The domain $\mathfrak{D}$ is dense in L^2. The operator $S : \mathfrak{D} \to L^2(\mathbb{R}^n)$ has a closed extension.*

Proof. – If $\mathfrak{D}$ were not dense, then some $g \in L^2$ would be perpendicular to $\mathfrak{D}$. By linear algebra this g must be a linear combination of $|u|^{(p-1)/2}$, and the $x_i|u|^{(p-1)/2}$ $(1 \leqslant i \leqslant n)$. Thus for some a, b_i we find that $(a + b_1x_1 + \cdots + b_nx_n)|u|^{(p-1)/2}$ is an L^2 function. Our second Liouville type theorem excludes this.

We now prove that $S|_{\mathfrak{D}}$ is closeable, i.e., we show that for any sequence $f_i \in \mathfrak{D}$ with $\|f_i\|_{L^2} \to 0$ and $Sf_i \to g$ in L^2 one must have $g = 0$.

Let

$$h_k = (-\Delta)^{-1}\big(|u|^{(p-1)/2} f_k\big).$$

Since $u \in L^\infty$, and since $f_k \to 0$ in L^2, it follows that $\nabla^2 h_k \to 0$ in L^2, for $\nabla^2 h_k$ is the Riesz transform of $|u|^{(p-1)/2} f_k$, and the Riesz transform is bounded on L^2.

The set $\mathcal{O} = \{x \in \mathbb{R}^n \mid v(x) \neq 0\}$ is open and nonempty, and since $Sf_k = |v|^{(q-1)/2}h_k$, the h_k converge in $L^2_{\mathrm{loc}}(\mathcal{O})$ to $g|v|^{-(q-1)/2}$.

It follows that h_k actually converges in $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^n)$ to some function h, whose second derivatives vanish, i.e., $h(x) = a + b_1x_1 + \cdots + b_nx_n$.

Thus we find that $g = (a + b_1x_1 + \cdots + b_nx_n)|v|^{(q-1)/2}$ belongs to $L^2(\mathbb{R}^n)$. The second Liouville theorem now forces $u \equiv v \equiv 0$. □

LEMMA 8C. – *Let $\varphi^R(x) = \eta(x/R)u(x)^{(p+1)/2}$. Then*

$$\liminf_{R\to\infty} \frac{\|S\varphi^R\|_{L^2}}{\|\varphi^R\|_{L^2}} \geqslant \sqrt{pq}.$$

This lemma does not claim that $\varphi^R \in \mathfrak{D}$, in fact one expects this not to be the case in general. Thus $\|S\varphi^R\|$ is defined by the integral (19), and may be infinite.

Proof. – We suppress the subscript R from our notation for the duration of this proof. Thus

$$S\varphi(x) = \left|v(x)\right|^{(q-1)/2}(-\Delta)^{-1}\left(\eta(x/R)u(x)^p\right).$$

Define ψ by

$$(-\Delta)^{-1}\left(\eta(x/R)u(x)^p\right) = \eta v + \psi.$$

Then, using $\Delta v + u^p = 0$ one computes that ψ satisfies

$$-\Delta\psi = 2\nabla\eta\cdot\nabla v + v\Delta\eta = 2\nabla\cdot(v\nabla\eta) - v\Delta\eta, \tag{20}$$

so that

$$\psi = 2\nabla(-\Delta)^{-1}(v\nabla\eta) - (-\Delta)^{-1}(v\Delta\eta). \tag{21}$$

As in Section 4 we define

$$I = I_R = \int \eta^2|u|^{p+1}\,\mathrm{d}x, \qquad J = J_R = \int \eta^2|v|^{q+1}\,\mathrm{d}x.$$

The Liouville theorem implies that $I, J \to \infty$ as $R \to \infty$, so by Theorem 4A we have

$$I_R = \left(1 + \mathrm{o}(1)\right)J_R \quad (R \to \infty).$$

We now compute the L^2 norm of $S\varphi$ on B_R:

$$\begin{aligned}
\int_{B_R} \left|S\varphi(x)\right|^2\mathrm{d}x &= \int_{B_R} \left|v(x)\right|^{q-1}\left\{\eta^2v^2 + 2\eta v\psi + \psi^2\right\}\mathrm{d}x \\
&\geqslant \int_{B_R} \left(\eta^2\left|v(x)\right|^{q+1} + 2\eta\psi v^q\right)\mathrm{d}x \\
&= \int_{B_R} \left(\eta^2\left|v(x)\right|^{q+1} - 2u\Delta(\eta\psi)\right)\mathrm{d}x \\
&= J_R - 2\int_{B_R}\left\{u\psi\Delta\eta + 2u\nabla\psi\cdot\nabla\eta + u\eta\Delta\psi\right\}\mathrm{d}x.
\end{aligned}$$

At this point we substitute the formulas (20) and (21) for ψ, which on expansion turns the last integral into one with six terms:

$$\int_{B_R} \left|S\varphi(x)\right|^2\mathrm{d}x \geqslant J_R + K_1 + \cdots + K_6,$$

where

$$K_1 = -4\int u\Delta\eta\nabla(-\Delta)^{-1}(v\nabla\eta)\,\mathrm{d}x,$$
$$K_2 = +2\int u\Delta\eta(-\Delta)^{-1}(v\Delta\eta)\,\mathrm{d}x,$$
$$K_3 = -8\int u\nabla\eta\cdot\nabla(-\Delta)^{-1}\nabla\cdot(v\nabla\eta)\,\mathrm{d}x,$$
$$K_4 = +4\int u\nabla\eta\cdot\nabla(-\Delta)^{-1}(v\Delta\eta)\,\mathrm{d}x,$$
$$K_5 = +2\int u\eta\nabla\eta\cdot\nabla v\,\mathrm{d}x,$$
$$K_6 = \int uv\eta\Delta\eta\,\mathrm{d}x.$$

We now proceed to estimate these terms one by one. It turns out that all terms except K_5 can be estimated following the same scheme. We show how to estimate K_4, and leave the other terms to the reader.

In doing such estimates it is convenient to have a slightly different notation for the L^r norms of functions on $\mathbb{R}^n$. We write

$$\|f\|_{L^r(\mathbb{R}^n)} = \left[f; \frac{1}{r}\right].$$

With this notation Hölder's inequality appears as

$$[fg; \alpha+\beta] \leqslant [f;\alpha]\cdot[g;\beta] \quad (0 \leqslant \alpha, \beta, \alpha+\beta \leqslant 1)$$

while one has the following estimates for the operator $(-\Delta)^{-1}$

$$\left[(-\Delta)^{-1}f;\alpha\right] \leqslant C(n,\alpha)\left[f;\alpha-\frac{2}{n}\right] \quad (2/n < \alpha < 1),$$
$$\left[\nabla(-\Delta)^{-1}f;\alpha\right] \leqslant C(n,\alpha)\left[f;\alpha-\frac{1}{n}\right] \quad (1/n < \alpha < 1),$$
$$\left[\nabla^2(-\Delta)^{-1}f;\alpha\right] \leqslant C(n,\alpha)[f;\alpha] \quad (0 < \alpha < 1).$$

Here the last estimate simply states L^p boundedness of the Riesz transforms for $1 < p < \infty$, while the first two are restatements of L^p/L^q mapping properties of the Riesz potentials (see [7, Section V.1]).

We shall also use that the specific form of our cutoff function, i.e., $\eta = \eta(x/R) = (1-|x/R|^2)^m$ implies

$$R|\nabla\eta| + R^2|\Delta\eta| \leqslant C\eta^{1-2/m}. \tag{22}$$

Moreover, we shall assume that m is "large".

We then have

$$
\begin{aligned}
|K_4| &\leqslant C\left[u|\nabla\eta|;\frac{1}{p+1}\right]\cdot\left[\chi_{B_R}\left|\nabla(-\Delta)^{-1}(v\Delta\eta)\right|;\frac{p}{p+1}\right]\\
&\leqslant \frac{C}{R}I_R^{\frac{1}{p+1}}[\chi;\theta]\cdot\left[\left|\nabla(-\Delta)^{-1}(v\Delta\eta)\right|;\frac{p}{p+1}-\theta\right]\\
&\leqslant CR^{-1+n\theta}I_R^{\frac{1}{p+1}}\cdot\left[|v\Delta\eta|;\frac{p}{p+1}-\theta+\frac{1}{n}\right]\\
&\leqslant CR^{-1+n\theta}I_R^{\frac{1}{p+1}}\cdot\left[|v\Delta\eta|;\frac{p}{p+1}-\theta+\frac{1}{n}\right]\\
&\leqslant CR^{-1+n\theta}I_R^{\frac{1}{p+1}}\cdot R^{-2}\left[\left|v\eta^{1-2/m}\right|;\frac{p}{p+1}-\theta+\frac{1}{n}\right]\\
&\leqslant CR^{-3+n\theta}I_R^{\frac{1}{p+1}}J_R^{\frac{1}{q+1}}\cdot\left[|\chi_{B_R}|;\frac{p}{p+1}-\theta+\frac{1}{n}-\frac{1}{q+1}\right]\\
&\leqslant CR^{-3+n\theta+n(\frac{p}{p+1}-\theta+\frac{1}{n}-\frac{1}{q+1})}I_R^{\frac{1}{p+1}}J_R^{\frac{1}{q+1}}\\
&= CR^{-n\alpha}I_R^{\frac{1}{p+1}}J_R^{\frac{1}{q+1}},
\end{aligned}
$$

where

$$\alpha=\frac{1}{p+1}+\frac{1}{q+1}-\frac{n-2}{n}$$

is positive because p and q are subcritical. In this calculation we have chosen $\theta\in(0,1)$ so that we can legitimately apply Hölder's inequality and the L^p mapping properties of $\nabla(-\Delta)^{-1}$. The constant θ must satisfy

$$\max\left(\frac{1}{n}-\frac{1}{p+1},0\right)<\theta<\min\left(1-\frac{1}{p+1},1-\frac{1}{p+1}-\frac{1}{q+1}+\frac{1}{n}\right).$$

Such θ exist.

As we mentioned before, a similar argument gives exactly the same estimate for the terms K_1, K_2, K_3, and K_6. To estimate K_5 we first observe the following

$$
\begin{aligned}
K_5=\int u\nabla\eta^2\cdot\nabla v\,\mathrm{d}x&=-\int\eta^2u\Delta v\,\mathrm{d}x-\int\eta^2\nabla u\cdot\nabla v\,\mathrm{d}x\\
&=-I_R-\int\eta^2\nabla u\cdot\nabla v\,\mathrm{d}x.
\end{aligned}
$$

By Theorem 4A we have $I_R = J_R + E_R$, where $E_R = \mathrm{o}(I_R)$ as $R \to \infty$. Hence

$$\begin{aligned} K_5 &= -J_R - \int \eta^2 \nabla u \cdot \nabla v \,\mathrm{d}x - E_R \\ &= -\int \eta^2 v \Delta u - \int \eta^2 \nabla u \cdot \nabla v \,\mathrm{d}x - E_R \\ &= -\int \eta^2 \nabla \cdot (v \nabla u)\mathrm{d}x - E_R = \int v \nabla u \cdot \eta^2 \,\mathrm{d}x - E_R \\ &= \frac{1}{2}\int \{ v\nabla u \cdot \eta^2 + u \nabla v \cdot \eta^2 \} \,\mathrm{d}x - \frac{E_R}{2} \\ &= \frac{1}{2}\int uv\Delta\eta^2 \,\mathrm{d}x - \frac{E_R}{2}. \end{aligned}$$

Here the remaining integral is of the same type as K_6 and can be estimated in the same way as K_6, with the same result. The last term $E_R/2$ was already known to be $\mathrm{o}(I_R)$, so the we can finally add all estimates together to obtain

$$\int_{B_R} |S\varphi(x)|^2 \,\mathrm{d}x \geqslant \big(pq + \mathrm{o}(1)\big) I_R.$$

Since $\|\varphi\|_{L^2} = I_R$ this completes the proof of Theorem 8C. □

9. PROOF OF THEOREM 8A WHEN S IS BOUNDED

If the operator $S : \mathfrak{D} \to L^2$ is bounded then it extends uniquely to a bounded operator $S_1 : L^2 \to L^2$. When $\phi \in C_c^\infty(\mathbb{R}^n)$ we have a formula that defines $S\phi$ as a function in $L^\infty(\mathbb{R}^n)$ (but not necessarily L^2). The following lemma addresses this ambiguity.

LEMMA 9A. – *If S is bounded then $S_1\phi = S\phi$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$.*

Proof. – Let $\phi_n \in \mathfrak{D}$ converge in L^2 to $\phi \in C_c^\infty(\mathbb{R}^n)$. Define

$$\psi_n = (-\Delta)^{-1}\big\{v^{(q-1)/2}\phi_n\big\}, \qquad \psi = (-\Delta)^{-1}\big\{v^{(q-1)/2}\phi\big\}.$$

Then, since the Riesz transforms $\nabla^2(-\Delta)^{-1}$ are bounded on L^2, $\nabla^2\psi_n$ converges in L^2 to $\nabla^2\psi$. Moreover, ψ_n converges to $\psi + a + b \cdot x$ in $W^{2,2}_{\mathrm{loc}}$, for some $a \in \mathbb{R}$, $b \in \mathbb{R}^n$. But then $S\phi_n = \sqrt{pq}|v|^{(q-1)/2}\psi_n$ converges to $\sqrt{pq}|v|^{(q-1)/2}(\psi + a + b\cdot x)$ so that $|v|^{(q-1)/2}(\psi + a + b \cdot x)$ belongs to L^2. Since ψ is the Newton potential of a compactly supported function, we may apply Theorem 7B to conclude that a and b vanish. Consequently, $S_1\phi = \lim S\phi_n = \sqrt{pq}|v|^{(q-1)/2}\psi = S\phi$. □

LEMMA 9B. – *For any $f \in L^2(\mathbb{R}^n)$ one has*

$$\lim_{R\to\infty} \frac{(\varphi^R, f)}{\|\varphi^R\|} = 0.$$

Proof. – First assume that f is compactly supported, i.e., $\operatorname{supp} f \subset B_{R_1}$ for some $R_1 > 0$. Since $\varphi^R = \eta(x/R)u(x)^{(p+1)/2}$ and $\eta(x) = 1$ for $|x| < 1/2$, the inner product (φ^R, f) is independent of R for $R \geqslant 2R_1$. But $\|\varphi^R\|^2 = \int \eta(x/R)^2|u(x)|^{p+1}\,\mathrm{d}x$ becomes infinite as $R \to \infty$, so for compactly supported f the lemma holds.

For general $f \in L^2$ we decompose $f = f_0 + f_1$, with f_0 compactly supoorted and $\|f_1\|_{L^2} \leqslant \varepsilon$. Then

$$\limsup_{R\to\infty}\left|\frac{(\varphi^R, f)}{\|\varphi^R\|}\right| = \limsup_{R\to\infty}\left|\frac{(\varphi^R, f_1)}{\|\varphi^R\|}\right| \leqslant \varepsilon.$$

This holds for arbitrary $\varepsilon > 0$ and thus the lemma holds for all $f \in L^2$. □

LEMMA 9C. – *Given any $\varepsilon > 0$ and m there exist $R_1 < R_2 < \cdots < R_m$ such that*

$$\left|(\phi_i, \phi_j)\right| + \left|(S\phi_i, S\phi_j)\right| < \varepsilon \quad (i \neq j), \tag{23}$$

and $\|S\phi_i\|^2 \geqslant pq - \varepsilon$ hold. Here $\phi_i = \varphi^{R_i}/\|\varphi^{R_i}\|$.

Proof. – By Theorem 8C we can assume that $\|S\phi^i\|^2 \geqslant pq - \varepsilon$, provided all R_i are chosen above some R_ε. We choose $R_1 = R_\varepsilon$ and proceed by induction. Let $R_1, \ldots, R_{m-1}$ be given. Since S is bounded its adjoint S^* is well defined, and we can write $(S\phi_m, S\phi_j) = (\phi_m, S^*S\phi_j)$. By Lemma 9B we can therefore make $|(S\phi_m, S\phi_j)| < \varepsilon$ for all $j < m$ by choosing R_m sufficiently large, while $R_m > R_1 = R_\varepsilon$ ensures that $\|S\phi_m\|^2 \geqslant pq - \varepsilon$. □

To conclude the proof of Theorem 8A, at least assuming boundedness of S, we note that $\phi = c_1\phi_1 + \cdots + c_m\phi_m$ satisfies

$$\|\phi\|^2 \leqslant \sum_1^m c_i^2 + \varepsilon\sum_{i\neq j} c_i c_j \leqslant \bigl(1 + (m-1)\varepsilon\bigr)\sum_1^m c_i^2,$$

and

$$\|S\phi\|^2 \geqslant \left(\sqrt{pq} - \varepsilon\right) \sum_1^m c_i^2 - \varepsilon \sum_{i \neq j} c_i c_j \geqslant \left(\sqrt{pq} - m\varepsilon\right) \sum_1^m c_i^2,$$

so that

$$\frac{\|S\phi\|}{\|\phi\|} \geqslant \frac{\sqrt{pq} - m\varepsilon}{1 + (m-1)\varepsilon} > \sqrt{pq} - \varepsilon'.$$

10. PROOF OF THEOREM 8A WHEN S IS NOT BOUNDED

We recall how one constructs the bounded self-adjoint operator $T = (\mathrm{I} + S^* S)^{-1}$ from the closed densely defined operator S (see [5, Section 118]).

Let $\mathfrak{V}$ be the Hilbert space completion of $\mathfrak{D}$ with inner product

$$(f, g)_{\mathfrak{V}} = (f, g) + (Sf, Sg).$$

It follows from the closedness of S that $\mathfrak{V}$ can be identified with a dense subspace of L^2 (the inclusion map $i : \mathfrak{D} \to L^2$ extends naturally to a bounded linear map $i' : \mathfrak{V} \to L^2$; closedness of S is needed to conclude the injectivity of i'.)

The Riesz representation theorem implies that for any $f \in L^2$ there exists a $g \in \mathfrak{V}$ with

$$(g, \phi)_{\mathfrak{V}} \equiv (f, \phi) \quad \forall \phi \in \mathfrak{V}.$$

We define $Tf = g$. One then easily shows that T is a bounded selfadjoint operator on L^2.

LEMMA 10A. – *Assume S is unbounded. Then T is injective, and $0 \in \sigma(T)$.*

Proof. – If $Tf = 0$ then $f \perp \mathfrak{V}$, so $f = 0$, which proves injectivity.

Assume 0 is not in the spectrum of T. Then T is invertible, and we have for arbitrary $\phi \in \mathfrak{D}$

$$\|\phi\|_{\mathfrak{V}}^2 = (\phi, \phi)_{\mathfrak{V}} = \left(T^{-1}\phi, \phi\right) \leqslant \|T^{-1}\| \cdot \|g\|_{L^2}^2.$$

This implies S is bounded, against our assumption. □

By the spectral theorem for bounded self adjoint operators we can write $T = \int_0^1 \lambda \, \mathrm{d}P_\lambda$, where $\{P_\lambda \mid 0 \leqslant \lambda \leqslant 1\}$ are the spectral projections of T. Assuming that S is not bounded we find that 0 is in the spectrum of T, while T is injective, i.e., 0 is not in the point spectrum of T. It follows that there exist $\lambda_n \downarrow 0$ such that the projections $\pi_n = P_{\lambda_n} - P_{\lambda_{n+1}}$ are non zero. Choose $\phi_n \in \mathrm{range}(\pi_n)$ with $\|\phi_n\|_{L^2} = 1$.

LEMMA 10B. – *The ϕ_n are mutually orthogonal. They belong to $\mathfrak{V}$ so that $S\phi_n$ is well defined, and they satisfy*

$$S\phi_n \perp S\phi_m \quad (n \neq m),$$
$$\|S\phi_n\|_{L^2}^2 \geqslant \frac{1}{\lambda_{n+1}} - 1.$$

Proof. – Let

$$\psi_n = \int_{\lambda_n}^{\lambda_{n+1}} \lambda^{-1} \, \mathrm{d}P_\lambda \phi_n, \tag{24}$$

then $T\psi_n = \phi_n$, so $\psi_n \in \mathrm{range}(T) \subset \mathfrak{V}$ and $\psi_n = T^{-1}\phi_n$. We have

$$\begin{aligned}\|S\phi_n\|_{L^2}^2 &= (S\phi_n, S\phi_n)_{L^2} = (\phi_n, \phi_n)_{\mathfrak{V}} - \|\phi_n\|_{L^2}^2 \\ &= (T\psi_n, \phi_n)_{\mathfrak{V}} - 1 = (\psi_n, \phi_n)_{L^2} - 1 \\ &= \int_{[\lambda_{n+1}, \lambda_n)} \frac{1}{\lambda} (\mathrm{d}P_\lambda \phi_n, \phi_n)_{L^2} - 1 \geqslant \frac{1}{\lambda_{n+1}} - 1.\end{aligned}$$

For $k \neq l$ we have $(\phi_k, \phi_l) = 0$ and also

$$\begin{aligned}(S\phi_k, S\phi_l) &= (\phi_k, \phi_l)_{\mathfrak{V}} - (\phi_k, \phi_l) = (\psi_k, \psi_l) \\ &= \int_0^\infty \frac{1}{\lambda} (\mathrm{d}P_\lambda \phi_k, \phi_l)_{L^2} = 0.\end{aligned}$$

□

If we assume that $\lambda_1 < 1/(1 + 2\sqrt{pq})$ then we have $\|S\phi_n\|^2 \geqslant 2pq$ for all n. Moreover, since the ϕ_n and $S\phi_n$ are pairwise orthogonal sets, any linear combination $\phi = c_1\phi_1 + \cdots + c_m\phi_m$ will also satisfy $\|S\phi\|^2 \geqslant 2pq\|\phi\|^2$. Since $\mathfrak{D} \subset \mathfrak{V}$ densely in the $\mathfrak{V}$ norm, we can approximate $\phi_1, \ldots, \phi_m$ in $\mathfrak{V}$ as closely as we like by $\phi'_j \in \mathfrak{D}$; in particular any linear combination $\phi' = c_1\phi'_1 + \cdots + c_m\phi'_m$ will satisfy $\|S\phi'\|^2 \geqslant pq\|\phi'\|^2$. This completes the proof of Theorem 8A, and hence of the main Theorem 1B.

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