ON THE NUMBER OF SINGLE-PEAK SOLUTIONS
OF THE NONLINEAR SCHRÖDINGER EQUATION ☆

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ABSTRACT. – In this paper we consider uniqueness and multiplicity results for single-peak solutions for the nonlinear Schrödinger equation. For a suitable class of potentials V and critical points P we are able to establish exact multiplicity results. In particular we show that for any nondegenerate critical point of V there are only one solution concentrating at P. Moreover some examples of single-peak solutions concentrating at the same point P are provided.
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RÉSUMÉ. – Dans ce papier nous considérons les résultats d'unicité et de multiplicité pour les solutions de seul-pic pour l'équation de Schrödinger non linéaire. Pour une classe convenable de potentiel V et critiques points P nous pouvons établir une multiplicité exacte.
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1. Introduction

In this paper we study standing wave solutions of the nonlinear Schrödinger equation

$$i h \frac{\partial \psi}{\partial t} = -\frac{h^2}{2m} \Delta \psi + V(x)\psi - |\psi|^{p-1}\psi, \quad |V(x)| \leqslant C, \ x \in \mathcal{R}^N, \qquad (1.1)$$

i.e., solutions of the form

$$\psi(x, t) = e^{i\frac{Et}{h}} u(x), \quad u : \mathcal{R}^N \to \mathcal{R}^+. \qquad (1.2)$$

Substituting (1.2) in (1.1) and assuming that $m = \frac{1}{2}$ one has

$$\begin{cases} -h^2 \Delta u + (V(x) - E)u = u^p, \\ u > 0. \end{cases} \qquad (1.3)$$

A suitable choice of E makes V bounded from below by a positive constant. Hence, without loss of generality, it is possible to assume that $E = 0$ and $V \geqslant V_0 > 0$. If we set $h = \varepsilon$, (1.3) becomes

$$\begin{cases} -\varepsilon^2 \Delta u + V(x) u = u^p & \text{in } \mathcal{R}^N, \\ u > 0. \end{cases} \tag{1.4}$$

In [7], Floer and Weinstein considered the case $N = 1$, $p = 3$ and they proved that for small ε there exists a family of standing wave solutions which concentrates at each given nondegenerate critical point of the potential V. This result was generalized by Oh [16] of higher dimensions. The arguments in those papers are based on a Liapunov–Schmidt reduction.

The existence of solutions of (1.4) in the possibly degenerate setting was studied by many authors. In this context the first results seem due to Rabinowitz (see [17]) and Ding and Ni (see [6]). In [17] it was shown that if $\inf_{\mathcal{R}^N} V < \liminf_{|x| \to \infty} V(x)$ then the mountain pass theorem provides a solution for small ε. This solution concentrates around a global minimum of V as $\varepsilon \to 0$, as shown later by Wang (see [18]).

Later Ambrosetti, Badiale and Cingolani (see [1]) obtained existence of standing wave solutions by assuming that the potential V has a local minimum or maximum with nondegenerate mth derivative, for some integer m.

Another result concerning degenerate critical points of V is due to Li (see [12]), where a degeneracy of any order of the derivative is allowed. In [12] the author proves the existence of a solution for (1.4) by only assuming that the critical points of V are "stable" with respect to a small C^1-perturbation of V. In [5] Del Pino and Felmer consider a more general nonlinearity $f(t)$ and obtained a solution of (1.4) by considering a "topologically nontrivial" critical value of the energy functional associated. In such a this way they are able to menage a wide class of critical points of V. We point out that all these results are obtained by suitable perturbation of the solution U_y of

$$\begin{cases} -\Delta u + V(y) u = u^p & \text{in } R^N, \\ u > 0 & \text{in } R^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.5}$$

This type of solutions are sometimes called single-peak solutions since they have only one local maximum point (see Definition 2.3 for a more precise statement).

Moreover (see [18] or Theorem 2.7) concentration of single-peak solution may only occur at critical point of V. So the problem of the existence of at least one single-peak solution concentrating at P seems to be completely described.

At this stage a question arises:

How many solutions are generated by a stable critical point P of V?

In this paper we try to give an answer to this question for a suitable class of potential V. Now we briefly describe the class of potentials V considered. Roughly speaking we take in account $V \in C^1$ such that

$$\frac{\partial V}{\partial x_i}(x) = h_i(x) + R_i(x) \quad \text{in a neighborhood of } P, \tag{1.6}$$

where h_i is an homogenous function of degree α_i and R_i is a higher order term (see Section 3 for the precise definition). In this setting the functions h_i play an important role: indeed if we consider the following vector field

$$\mathcal{L}_P(y) = \left(\int\limits_{R^N} h_i(x + y + P) U_P^2(x)\, dx \right)_{i=1,\dots,N}, \tag{1.7}$$

where U_P is the solution of (1.5) and

$$Z = \left\{ y \in \mathcal{R}^N \text{ such that } y \text{ is a stable zero of } \mathcal{L} \right\}, \tag{1.8}$$

(see Definition 3.6 for the definition of stable zero) then we get the following theorem.

THEOREM 1.1. – *Let V be an admissible potential* (*see Section* 3 *for the definition*) *and suppose that* $\#Z < \infty$. *If, for any* $y \in Z$, *it holds*

$$\det \operatorname{Jac} \mathcal{L}_P(y) \neq 0, \tag{1.9}$$

then there exists $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *we have*

$$\#\{single\text{-}peak\ solution\ of\ (1.4)\ concentrating\ at\ x = P\} = \#Z. \tag{1.10}$$

In particular it is possible to show that, if P is a nondegenerate critical point of V then there is only one solution which concentrates at P (see Corollary 6.4). However, we think that the number of the solution of (1.4) concentrating at P is influenced by the shape of V in a neighborhood of P instead of its nondegeneracy (see Proposition 6.3) and example at the end of Section 6). Finally, again in the example at the end of Section 6, we construct an example of a potential V such that there exist exactly two single-peak solutions concentrating at the same point P. In the same way it is possible to give an example where a concentration of k single-peak solutions at P occurs. We do not know any example of such a phenomenon in literature, even for similar elliptic perturbed problems defined in bounded or unbounded domains.

Our results can be extended to the case of more general nonlinearities $f(u)$, using for example the assumption of [8]. However we state and prove our results only for the case $f(s) = s^p$.

The paper is organized as follows: in Section 2 we recall some known facts and introduce the main definitions. In Section 3 we introduce the class of admissible potentials V. In Section 4 we prove that stable zero of $\mathcal{L}_P$ generates solutions to (1.4). In Section 5 we establish an important estimate which allows, in Section 6, to deduce the exact multiplicity result.

2. Known facts and main definitions

In this section we recall some known facts.

THEOREM 2.1. – *Let V a continuous function verifying $0 < V_0 \leqslant V \leqslant V_1$ in $\mathcal{R}^N$. Then for any $y \in \mathcal{R}^N$ the problem*

$$\begin{cases} -\Delta u + V(y)u = u^p & in \ \mathcal{R}^N, \\ u > 0 & in \ \mathcal{R}^N, \\ u(x) \to 0 & as \ |x| \to \infty, \end{cases} \tag{2.1}$$

with $1 < p < \frac{N+2}{N-2}$, admits only one solution $U_y \in H^1(\mathcal{R}^N)$ which satisfies

(i) *U_y is spherically symmetric with respect to some point of $\mathcal{R}^N$,*
(ii) *U_y is in $H_r^2(\mathcal{R}^N) = \{u \in H^2(\mathcal{R}^N): u = u(|x|)\}$,*
(iii) *$\lim_{|x|\to\infty} e^{|x|}|x|^{\frac{N-1}{2}} u(x) = a > 0$,*
(iv) *the kernel of the linearized operator $L = -\Delta + (V(y) - U_y^{p-1})\mathrm{Id}$*

        is spanned by the function $\partial U / \partial x_i$ for $i = 1, \ldots, N$. $\qquad$ (2.2)

Proof. – See [11] for the uniqueness of the solution and the proof of (ii), [9] for (i) and (iii), [14] for (iv). $\quad \square$

It is easily seen that (1.4) is equivalent to

$$\begin{cases} -\Delta u + V(\varepsilon x + P)u = u^p & in \ \mathcal{R}^N, \\ u > 0 \end{cases} \tag{2.3}$$

for $P \in \mathcal{R}^N$.

Now we give the following definitions.

DEFINITION 2.2. – *We say that u_ε is a family of bound state solutions of (1.4) if there is a constant C which does not depend on ε such that*

$$\varepsilon^N \int_{\mathcal{R}^N} \left(\varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2\right) \mathrm{d}x \leqslant C. \tag{2.4}$$

So if we set

$$v_\varepsilon(x) = u_\varepsilon(\varepsilon x + P),$$

where u_ε is a bound state solution of (1.4), we get that v_ε is a solution of (2.3) and

$$\int_{\mathcal{R}^N} \left(|\nabla v_\varepsilon|^2 + v_\varepsilon^2\right) \mathrm{d}x \leqslant C. \tag{2.5}$$

DEFINITION 2.3. – *We say that a family of bound state solution u_ε of (1.4) is single-peaked if it has only one local maximum point. The point $P_\varepsilon \in \mathcal{R}^N$ where this maximum is achieved is called the peak of the solution. If $P_\varepsilon \to P$ we say that u_ε concentrates at P.*

This paper deals with single-peak solution of (1.4). By blow-up arguments ([4]) we get the following

PROPOSITION 2.4. – *Let u_ε be a family of single-peak solution of* (1.4) *concentrating at P and let P_ε be its peak. If U_P is the unique solution of* (2.1) *symmetric with respect to the origin we have*

$$u_\varepsilon(x) = U_P\left(\frac{x - P_\varepsilon}{\varepsilon}\right) + \omega_\varepsilon(x), \tag{2.6}$$

and

$$u_\varepsilon(x) \leqslant C e^{-\left|\frac{x - P_\varepsilon}{\varepsilon}\right|}, \tag{2.7}$$

where $\omega_\varepsilon \to 0$ in $L^\infty(\mathcal{R}^N)$.

Proof. – This is a known result (see [4]) but we repeat the proof for reader's convenience. Set $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + P_\varepsilon)$. Since u_ε is a bound state solution of (1.4), by standard boot-strap arguments we get that v_ε is uniformly bounded on the compact set of $\mathcal{R}^N$. Hence, passing to the limit in (2.3) we get that $v_\varepsilon \to U_P$ where $U_P \in H^1(\mathcal{R}^N)$ solves (1.5) with $y = P$.

Since positive solutions of (1.5) decay exponentially (see [9]) and $U_P \in H^1(\mathcal{R}^N)$ is uniformly bounded we obtain the existence of $R > 0$ such that $U_P(x) < \eta$ for all $|x| = R$. Let us choose $\eta^{p-1} < V_0$. But also $u_\varepsilon(x) < \eta$ if $|x| = R$ and n large and then, since v_ε has only one local maximum we obtain that $v_\varepsilon(x) < \eta$ for all $|x| > R$. Then we can use the maximum principle to conclude that $v_\varepsilon(x) < w_0(x)$ for all $|x| > R$, where w_0 is a suitable multiple of the fundamental solution of $\Delta w - bw = 0$ with $b = V_0 - \eta^{p-1} > 0$. Since w_0 decays exponentially we deduce (2.7) and that

$$v_\varepsilon \to U_P \quad \text{uniformly in } \mathcal{R}^N. \tag{2.8}$$

This ends the proof. □

Concerning single-peak solutions Wang (see [18]) proved this useful identity.

PROPOSITION 2.5. – *Let us suppose that u_ε is a family of single-peak solutions of* (1.4) *and*

$$|\nabla V(x)| = O(e^{\gamma|x|}) \quad \text{at infinity for some } \gamma > 0. \tag{2.9}$$

Then

$$\int_{\mathcal{R}^N} \frac{\partial V}{\partial x_i}(\varepsilon x + P)u_\varepsilon^2(\varepsilon x + P)\,dx = 0 \quad \text{for any } i = 1, \ldots, N. \tag{2.10}$$

From the previous identity Wang deduced the following result.

THEOREM 2.6. – *Let us suppose that V satisfies* (2.9) *and it is nondecreasing in one direction in $\mathcal{R}^N$. Then there is no solution to* (1.4).

Proof. – It follows by (2.10). □

THEOREM 2.7. – *Let u_ε be a family of single-peak solutions to* (1.4) *and P_ε the point of $\mathcal{R}^N$ where u_ε achieves its maximum. Suppose* (2.9) *holds, $\|u_\varepsilon - U_P\|_{H^1(\mathcal{R}^N)} \to 0$ as $\varepsilon \to 0$. If $V \in C^1(\mathcal{R}^N)$ we have that $P_\varepsilon \to P$ with $\nabla V(P) = 0$.*

Proof. – It follows by (2.10) (see [18]). □

3. Assumptions on the potential *V*

This section is devoted to the assumptions on the potential V. For $Q \in \mathcal{R}^N$ and $\rho \in \mathcal{R}$ let us denote by $B_{Q,\rho} = \{x \in \mathcal{R}^N$ such that $|x - Q| \leqslant \rho\}$. We give the following definition.

DEFINITION 3.1. – *We say that a function* $h : \mathcal{R}^N \mapsto \mathcal{R}$ *is homogenous of degree* $\alpha \in \mathcal{R}^+$ *with respect to* $P \in \mathcal{R}^N$ *if*

$$h(tx + P) = t^\alpha h(x + P) \quad \text{for any } t \in \mathcal{R}^+ \text{ and } x \in \mathcal{R}^N. \tag{3.1}$$

Remark 3.2. – If $h \in C(\mathcal{R}^N)$ is homogenous of degree α from the previous definition it follows that

$$|h(x)| \leqslant C|x - P|^\alpha \quad \text{for any } x \in \mathcal{R}^N, \tag{3.2}$$

where C is the maximum of h on $\partial B_{P,1}$.

At this stage we can state our crucial condition on the potential V.

DEFINITION OF ADMISSIBLE POTENTIAL. – *Let us assume that* $V \in C^1(R^N)$ *satisfies*

$$|\nabla V(x)| \leqslant C e^{\gamma |x|} \quad \text{at infinity} \tag{3.3}$$

and

$$0 < V_0 \leqslant V(x) \leqslant V_1 \tag{3.4}$$

for some $\gamma > 0$. *We say that* V *is an admissible potential at* $P \in R^N$ *if there exist continuous functions* $h_i : \mathcal{R}^N \mapsto \mathcal{R}$, $R_i : B_{P,r} \mapsto \mathcal{R}$ *and real numbers* $\alpha_i \geqslant 1$, $i = i, \ldots, N$, *such that*
  (i) $\frac{\partial V}{\partial x_i}(x) = h_i(x) + R_i(x)$ *in* $B_{P,r}$,
  (ii) $R_i(x) \leqslant C|x - P|^{\beta_i}$ *in* $B_{P,r}$ *with* $\beta_1 > \alpha_i$ *for any* $i = 1, \ldots, N$,
  (iii) $h_i(x) = 0$ *if and only if* $x = P$,
  (iv) h_i *is homogeneous of degree* α_i. $\tag{3.5}$

Remark 3.3. – We point out that if P an isolated critical point of V, (iv) holds and $R_i(x) \equiv 0$ then (iii) holds. Indeed if by contradiction there exists $y \in \mathcal{R}^N$ such that $h_i(y) = 0$ then we have that $h_i(t(y - P) + P) = 0$ for any $t \in \mathcal{R}^+$. Hence from (i) of (3.5) we get $\frac{\partial V}{\partial x_i}(t(y - P) + P) = 0$ for any positive t. So P is not an isolated critical point and this gives a contradiction.

Remark 3.4. – It is easy to see that the function h_i is determined in an unique way.

Remark 3.5. – The assumptions (i) and (ii) of the previous definition are satisfied if $V \in C^k(B_{P,r})$ and admits a nonzero derivative with respect to x_i for any index $i \in \{1, \ldots, N\}$. In this case $\alpha_i + 1$ is the order of the first nonzero derivative and the function h_i are given by Taylor's formula. In order to state a sufficient condition which implies (iii) we need some notations. Let $\mathbf{k} = (k_1, k_2, \ldots, k_N)$ be a multi-index and set $|\mathbf{k}| = k_1 + k_2 + \cdots + k_N$, $\mathbf{k}! = k_1! k_2! \cdots k_N!$, and for $w = (w_1, w_2, \ldots w_N) \in R^N$,

$w^{\mathbf{k}} = w_1^{k_1} w_2^{k_2} \cdots w_N^{k_N}$. So if

$$D^{\mathbf{k}} f = \frac{\partial^{|\mathbf{k}|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_N^{k_N}}$$

we have that

$$h_i(x) = \sum_{|\mathbf{k}|=\alpha_i+1} \frac{D^{\mathbf{k}} V(P)}{\mathbf{k}!} (x - P)^{\mathbf{k}}, \tag{3.6}$$

and condition (iii) becomes that the nonlinear system

$$\sum_{|\mathbf{k}|=\alpha_i} \frac{D^{\mathbf{k}} V(P)}{\mathbf{k}!} (x - P)^{\mathbf{k}} = 0 \tag{3.7}$$

admits only the solution $x = P$.

DEFINITION 3.6. – *Let $G \in C(\mathcal{R}^N; \mathcal{R}^N)$ be a vector field. We say that y is a stable zero for G if*
  (i) *$G(y) = 0$,*
 (ii) *y is isolated,*
 (iii) *if G_n is a sequence of vector fields such that $\|G_n - G\|_{C(B_{y,\rho})} \to 0$ for some $\rho > 0$ then there exists y_n such that $G_n(y_n) = 0$ and $y_n \to y$.*

A sufficient condition on G and y which implies that y is a stable zero is that the index $i(G, y, 0) \neq 0$, where the index of y at zero is given by

$$i(G, y, 0) = \lim_{\rho \to 0} \deg(G, B_{y,\rho}, 0),$$

with $\deg(G, B_{y,\rho}, 0)$ denoting the Brouwer degree.

Now let us set

$$\mathcal{L}_P(y) = \left(\int_{\mathcal{R}^N} h_i(x + y + P) U_P^2(x) \, dx \right)_{i=1,\dots,N}, \tag{3.8}$$

where U_P is the solution of (2.1). By Remark 3.2 and the exponential decay of U_P we have that (3.8) is well-posed. Finally set

$$Z = \{ y \in \mathcal{R}^N \text{ such that } y \text{ is a stable zero of } \mathcal{L} \}. \tag{3.9}$$

Remark 3.7. – Condition (1.9) implies that any zero of $\mathcal{L}_P$ is stable. Indeed if $\mathcal{L}_P(y) = 0$ by (1.9) we have that the Brower degree $\deg(\mathcal{L}_P, B_{P,1}, 0) \neq 0$ (see [13]) and this gives that $y \in Z$.

4. The existence result

We start this section by recalling the following result.

THEOREM 4.1. – *Let us assume that V satisfies* (3.4). *Then there exist positive constant δ and ε_0 such that for every y and ε with $|y| < \delta$ and $0 < \varepsilon < \varepsilon_0$ there exists $u_{y,\varepsilon} \in H^2(\mathcal{R}^N)$ such that*

$$-\Delta u_{y,\varepsilon} + V(\varepsilon x + y)u_{y,\varepsilon} - |u_{y,\varepsilon}|^{p-1}u_{y,\varepsilon} = \sum_{i=1}^{N} c_{i,y,\varepsilon} \frac{\partial U_y}{\partial x_i}, \qquad (4.1)$$

with

$$\|u_{y,\varepsilon} - U_y\|_{H^2(\mathcal{R}^N)} \to 0 \quad as\ \varepsilon \to 0, \qquad (4.2)$$

for some constant $c_{i,y,\varepsilon} \in \mathcal{R}$. Here U_y is the unique solution of (2.1).

Proof. – The result is classical (see [16] for example).   □

In the next lemma we give some useful estimates:

LEMMA 4.2. – *Let us consider the function $u_{y,\varepsilon}$ given by Theorem* 4.1. *Then*
  (i) *$u_{y,\varepsilon}$ decays exponentially at the infinity,*
 (ii) *$u_{y,\varepsilon} \to U_y$ uniformly on $\mathcal{R}^N$.*

Proof. – First of all we show that, for $|y| < \delta$, $c_{i,y,\varepsilon} \to 0$ as $\varepsilon \to 0$ for any $i = \{1, \ldots, N\}$.
Indeed, multiplying (4.1) by $\frac{\partial U_y}{\partial x_j}$ and integrating we get

$$c_{j,y,\varepsilon} \int_{\mathcal{R}^N} \left(\frac{\partial U}{\partial x_j}\right)^2 = \sum_{i=0}^{N} c_{i,y,\varepsilon} \int_{\mathcal{R}^N} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j}$$

$$= -\int_{\mathcal{R}^N} \left(\Delta u_{y,\varepsilon} + V(\varepsilon x + y)u_{y,\varepsilon} - |u_{y,\varepsilon}|^{p-1}u_{y,\varepsilon}\right)\frac{\partial U}{\partial x_j}\,\mathrm{d}x \to 0. \quad (4.3)$$

by (4.2). Then, by using the weak Harnack inequality (see [16], or [8,18]) it is possible to prove that

$$u_{y,\varepsilon}(x) \to 0 \quad \text{uniformly as } |x| \to \infty. \qquad (4.4)$$

So there exists $R > 0$ such that $V(\varepsilon x + y) - |u|^{p-1}u \geqslant \frac{V_0}{2} > 0$ for $|x| > R$. At this stage we can repeat the same proof of Lemma 6.4 of [15] which proves (i).

Concerning the proof of (ii) this follows from (i) and the standard boot-strap (here we are using that $1 < p < \frac{N+2}{N-2}$).   □

Now we are able to prove the main result of this section.

THEOREM 4.3. – *Let us suppose that $V(x)$ is an admissible potential and $\#Z < \infty$. Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have that $\#\{$single-peak solution of* (1.4) *concentrating at $P\} \geqslant \#Z$.*

Proof. – Let us multiply (4.1) by $\frac{\partial u_{y,\varepsilon}}{\partial x_j}$ and integrate. We have that

$$\int_{\mathcal{R}^N} \left(-\Delta u_{y,\varepsilon} - |u_{y,\varepsilon}|^{p-1}u_{y,\varepsilon}\right)\frac{\partial u_{y,\varepsilon}}{\partial x_j}\,\mathrm{d}x = 0, \qquad (4.5)$$

(see [18] for a detailed proof) and

$$\sum_{i=0}^{N} c_{i,y,\varepsilon} \frac{\partial U}{\partial x_i} \frac{\partial u_{y,\varepsilon}}{\partial x_j} \, dx = \int_{\mathcal{R}^N} V(\varepsilon x + y) u_{y,\varepsilon} \frac{\partial u_{y,\varepsilon}}{\partial x_j} \, dx$$

$$= -\frac{\varepsilon}{2} \int_{\mathcal{R}^N} \frac{\partial V}{\partial x_j} (\varepsilon x + y) u_{y,\varepsilon}^2 \, dx. \tag{4.6}$$

Let $y_0 \in Z$ and set $y = P + \varepsilon \tilde{y}$ with $\tilde{y} \in B_{y_0,1}$. By the assumption on V we get

$$\int_{\mathcal{R}^N} \frac{\partial V}{\partial x_j} (\varepsilon x + y) u_{y,\varepsilon}^2 \, dx = \int_{|\varepsilon x + y - P| \leqslant r} \left(\varepsilon^{\alpha_i} h_j(x + \tilde{y} + P) + R_j(\varepsilon x + y) u_{y,\varepsilon}^2 \, dx \right)$$

$$+ \int_{|\varepsilon x + y - P| > r} \frac{\partial V}{\partial x_j} (\varepsilon x + y) u_{y,\varepsilon}^2 \, dx. \tag{4.7}$$

So by (ii) of (3.5) we deduce

$$\left| \int_{|\varepsilon x + y - P| \leqslant r} R_j(\varepsilon x + y) \right| u_{y,\varepsilon}^2 \, dx$$

$$\leqslant C \int_{|\varepsilon x + y - P| \leqslant r} |\varepsilon x + y - P|^{\beta_i} u_{y,\varepsilon}^2 \, dx \leqslant C \varepsilon^{\beta_i} \int_{\mathcal{R}^N} |x + \tilde{y}|^{\beta} u_{y,\varepsilon}^2 \, dx$$

$$\leqslant C \varepsilon^{\beta_i} \int_{\mathcal{R}^N} \left(|x| + 1 \right)^{\beta} u_{y,\varepsilon}^2 \, dx = O(\varepsilon^{\beta_i}) \quad \text{uniformly with respect to } y. \tag{4.8}$$

Finally since $\frac{\partial V}{\partial x_i}(x) = O(e^{\gamma|x|})$ at infinity and Lemma 4.2 we have

$$\int_{|\varepsilon x + y - P| > r} \frac{\partial V}{\partial x_i} (\varepsilon x + y - P) u_{y,\varepsilon}^2 \, dx \leqslant C \int_{|\varepsilon x + y - P| > r} e^{\gamma|\varepsilon x + y|} e^{-2|x|} \, dx$$

$$\leqslant C \int_{|x| > \frac{r}{2\varepsilon}} e^{-|x|} = O(e^{-\frac{r}{\varepsilon}}) \quad \text{uniformly with respect to } y. \tag{4.9}$$

Then by (4.7), (4.8) and Lemma 4.2 we get

$$\sum_{i=0}^{N} c_{i,y,\varepsilon} \int_{\mathcal{R}^N} \frac{\partial U}{\partial x_i} \frac{\partial u_{y,\varepsilon}}{\partial x_j} \, dx = \varepsilon^{\alpha_i} \left(\int_{\mathcal{R}^N} h_j(x + y + P) u_{y,\varepsilon}^2 \, dx + O(\varepsilon^{\beta_i - \alpha_i}) \right). \tag{4.10}$$

Again by (ii) of Lemma 4.2 and Remark 3.2 we obtain that

$$\int_{|\varepsilon x + y - P| \leqslant r} h_j(x + \tilde{y} + P) u_{y,\varepsilon}^2 \, dx \to \int_{\mathcal{R}^N} h_j(x + \tilde{y} + P) U_y^2 \, dx \tag{4.11}$$

uniformly in $B_{y_0,1}$. So by the definition of stable zero we get that there exists $y_\varepsilon \to y_0$ such that

$$\int_{\mathcal{R}^N} \frac{\partial V}{\partial x_j}(\varepsilon x + y_\varepsilon) u^2_{y_\varepsilon,\varepsilon}\, dx = 0. \tag{4.12}$$

Thus

$$\sum_{i=0}^N c_{i,y_\varepsilon,\varepsilon} \frac{\partial U_{P+\varepsilon y_\varepsilon}}{\partial x_i} \frac{\partial u_{P+\varepsilon y_\varepsilon}}{\partial x_j}\, dx = 0. \tag{4.13}$$

From the uniqueness of the solution of (2.1) and the implicit function theorem we get that $U_{P+\varepsilon y_\varepsilon} \to U_P$ in $H^2(\mathcal{R}^N)$ (see Lemma 4.1 of [8] for a detailed proof). Then

$$\int_{\mathcal{R}^N} \frac{\partial U_{P+\varepsilon y_\varepsilon}}{\partial x_i} \frac{\partial u_{P+\varepsilon y_\varepsilon}}{\partial x_j}\, dx \to \delta_i^j \int_{\mathcal{R}^N} \left(\frac{\partial U_P}{\partial x_i}\right)^2 \neq 0 \tag{4.14}$$

and this implies that, for ε small, the linear system (4.13) admits only the trivial solution $c_{i,y_\varepsilon,\varepsilon} = 0$ for any $i = \{1, \dots, N\}$. This proves that, for any stable zero $y_0 \in Z$ there exists a solution u_ε of (2.3) such that

$$\|u_\varepsilon - U_P\|_{H^2(\mathcal{R}^N) \cap L^\infty(\mathcal{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.15}$$

By standard arguments (see [8] or [18] for example), we get that $u_\varepsilon > 0$ in $\mathcal{R}^N$. In order to prove our claim we have to show that two different stable zeroes y_1 and y_2 generates two different solutions.

Set $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ the solutions of (2.3) generates by y_1 and y_2 and $y_\varepsilon^1 \to y_1$, $y_\varepsilon^2 \to y_2$ with $y_1 \neq y_2$. We recall that $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ solve

$$-\Delta u_{1,\varepsilon} + V(\varepsilon x + P + \varepsilon y_\varepsilon^1) u_{1,\varepsilon} = u_{1,\varepsilon}^p,$$

and

$$-\Delta u_{2,\varepsilon} + V(\varepsilon x + P + \varepsilon y_\varepsilon^2) u_{2,\varepsilon} = u_{2,\varepsilon}^p.$$

Let

$$z_{1,\varepsilon}(x) = u_{1,\varepsilon}\left(\frac{x - (P + \varepsilon y_\varepsilon^1)}{\varepsilon}\right)$$

and

$$z_{2,\varepsilon}(x) = u_{2,\varepsilon}\left(\frac{x - (P + \varepsilon y_\varepsilon^2)}{\varepsilon}\right)$$

the corresponding solutions of (1.4). By (4.15) we deduce that

$$z_{1,\varepsilon}(P + \varepsilon y_\varepsilon^1) = u_{1,\varepsilon}(0) \to U_P(0),$$
$$z_{2,\varepsilon}(P + \varepsilon y_\varepsilon^1) = u_{1,\varepsilon}(y_\varepsilon^1 - y_\varepsilon^2) \to U_P(y_1 - y_2) \neq U_P(0).$$

This proves that $u_{1,\varepsilon} \neq u_{2,\varepsilon}$. $\square$

5. A crucial estimate

The aim of this section is to compute asymptotics for the ratio

$$\frac{P - P_\varepsilon}{\varepsilon}, \tag{5.1}$$

where P_ε is the peak of u_ε. We know that $P_\varepsilon \to P$ as $\varepsilon \to 0$. Next proposition gives more information.

PROPOSITION 5.1. – *Assume that $V(x)$ is an admissible potential and u_ε is a solution of* (1.4). *Then there exists a positive constant C such that*

$$\left| \frac{P - P_\varepsilon}{\varepsilon} \right| \leqslant C. \tag{5.2}$$

Proof. – By contradiction let us suppose that there exists a sequence $\varepsilon_n \to 0$ such that

$$\frac{|P_{\varepsilon_n} - P|}{\varepsilon_n} \to \infty. \tag{5.3}$$

Set $P_n = P_{\varepsilon_n}$ and $v_n(x) = u_{\varepsilon_n}(\varepsilon_n x + P_n)$. By Proposition 2.4 $v_n \to U_P$ in $L^2(\mathcal{R}^N)$ and then

$$\int\limits_{R^N} \frac{\partial V}{\partial x_i} (\varepsilon_n x + P_n) v_n^2(x) \, dx = 0. \tag{5.4}$$

Hence

$$\int\limits_{R^N} \frac{\partial V}{\partial x_i} (\varepsilon_n x + P_n) v_n^2(x) \, dx$$

$$= \int\limits_{|\varepsilon_n x + P_n - P| \leqslant r} h_i (\varepsilon_n x + P_n) v_n^2(x) \, dx + \int\limits_{|\varepsilon_n x + P_n - P| \leqslant r} R_i (\varepsilon_n x + P_n) v_n^2(x) \, dx$$

$$+ \int\limits_{|\varepsilon_n x + P_n - P| > r} \frac{\partial V}{\partial x_i} (\varepsilon_n x + P_n) v_n^2(x) \, dx. \tag{5.5}$$

Concerning the first integral we have

$$\int\limits_{|\varepsilon_n x + P_n - P| \leqslant r} h_i (\varepsilon_n x + P_n) v_n^2(x) \, dx$$

$$= |P_n - P|^{\alpha_i} \int\limits_{|\varepsilon_n x + P_n - P| \leqslant r} h_i \left(\frac{\varepsilon_n x}{|P_n - P|} + \frac{P_n - P}{|P_n - P|} + P \right) v_n^2(x) \, dx. \tag{5.6}$$

Up to a subsequence $\frac{P_n - P}{|P_n - P|} \to \zeta$ with $|\zeta| = 1$. Then we get

$$I_{1,n} = |P_n - P|^{\alpha_i} h_i (\zeta + P) \int\limits_{\mathcal{R}^N} U^2(x) \, dx + o\big(|P_n - P|^{\alpha_i}\big). \tag{5.7}$$

Moreover, from the definition of R_i,

$$|I_{2,n}| \leqslant C \int_{|\varepsilon_n x + P_n - P| \leqslant r} |\varepsilon_n x + P_n - P|^{\beta_i} v_n^2(x)\, dx$$

$$= C|P_n - P|^{\beta_i} \int_{\mathcal{R}^N} U^2(x)\, dx + o(|P_n - P|^{\beta_i}). \tag{5.8}$$

Finally since $\frac{\partial V}{\partial x_i}(x) = O(e^{\gamma x})$ at infinity and (iii) of Theorem 2.1 we have

$$I_{3,n} = \int_{|\varepsilon_n x + P_n - P| > r} \frac{\partial V}{\partial x_i}(\varepsilon_n x + P_n) v_n^2(x)\, dx \leqslant C \int_{|\varepsilon_n x + P_n - P| > r} e^{\gamma|\varepsilon_n x + P_n|} e^{-2|x|}\, dx$$

$$\leqslant \int_{|x| > r/2\varepsilon_n} e^{-|x|} = O(e^{-\frac{r}{\varepsilon_n}}). \tag{5.9}$$

Thus (5.5) becomes

$$h_i(\zeta + P) \int_{\mathcal{R}^2} U^2(x)\, dx = o(|P_n - P|^{\beta_i - \alpha_i}) + \frac{e^{-\frac{\gamma}{\varepsilon_n}}}{|P_n - P|^{\alpha_i}}. \tag{5.10}$$

This clearly implies $h_i(\zeta + P) = 0$ and since $|\zeta| = 1$ we have a contradiction with (iii) of (3.5). $\quad\square$

PROPOSITION 5.2. – *Let ε_n be a sequence which goes to zero and $u_{\varepsilon_n} = u_n$ the corresponding solutions of* (1.4). *Then, up to a subsequence, if P_n denotes the peak of the solution u_n we have*

$$\frac{P_n - P}{\varepsilon_n} \to y, \tag{5.11}$$

where y satisfies $\mathcal{L}(y) = 0$.

Proof. – By the previous proposition we have that $\frac{P - P_n}{\varepsilon_n}$ is bounded and so we can assume that (up to a subsequence) $\frac{P - P_n}{\varepsilon_n} \to y$.

Again as in (5.5) we get

$$0 = \int_{R^N} \frac{\partial V}{\partial x_i}(\varepsilon_n x + P_n) v_{\varepsilon_n}^2(x)\, dx = I_{1,n} + I_{2,n} + I_{3,n}. \tag{5.12}$$

Now

$$I_{1,n} = \varepsilon_n^{\alpha_i} \int_{|\varepsilon_n x + P_n - P| \leqslant r} h_i\left(x + \frac{P_n - P}{\varepsilon_n} + P\right) v_{\varepsilon_n}^2(x)\, dx$$

$$= \varepsilon_n^{\alpha_i} \int_{\mathcal{R}^N} h_i(x + y + P) U^2(x)\, dx + o(1). \tag{5.13}$$

Proceeding as in Proposition 5.2 we get

$$I_{2,n} = O(\varepsilon_n^{\beta_i}), \tag{5.14}$$

and

$$I_{3,n} = \mathrm{O}\big(\mathrm{e}^{-\gamma/\varepsilon_n}\big). \tag{5.15}$$

Then (5.12) becomes

$$\int_{\mathcal{R}^N} h_i(x + y + P)U^2(x)\,\mathrm{d}x = \mathrm{O}\big(\varepsilon_n^{\beta_i - \alpha_i}\big). \tag{5.16}$$

Passing to the limit in (5.16) we get the claim. □

From the previous proposition we get the following "local" nonexistence result

COROLLARY 5.3. – *Let us suppose that V is an admissible potential and $\mathcal{L}(y) \neq 0$ for any y. Then there is no single peak solution of (1.4) concentrating at P.*

In the next lemmas we show some cases where Z is nonempty.

LEMMA 5.4. – *Let P be a nondegenerate critical point of $V \in C^3(B_{P,r})$. Then $Z = \{0\}$.*

Proof. – By Taylor's formula we have that

$$V(x) = V(P) + \frac{1}{2}\sum_{j,k=1}^{N} \frac{\partial^2 V}{\partial x_j \partial x_k}(P)(x_j - P_j)(x_k - P_k) + R(x). \tag{5.17}$$

So in this case

$$h_i(x) = \sum_{j=1}^{N} \frac{\partial^2 V}{\partial x_i \partial x_j}(P)(x_j - P_j) \quad \text{and} \quad R_i(x) = \frac{\partial R(x)}{\partial x_i}.$$

Then

$$0 = \sum_{j=1}^{N} \frac{\partial^2 V}{\partial x_i \partial x_j}(P) \int_{R^N} (x_i + y_i)U^2(x)\,\mathrm{d}x = \sum_{i=1}^{N} \frac{\partial^2 V}{\partial x_i \partial x_j}(P) \int_{R^N} y_i U^2(x)\,\mathrm{d}x. \tag{5.18}$$

Since the matrix $\frac{\partial^2 V}{\partial x_i \partial x_j}(P)$ is invertible we have that the linear system

$$\sum_{i=1}^{N} \frac{\partial^2 V}{\partial x_i \partial x_j}(P)y_i = 0 \tag{5.19}$$

admits only the solution $y = 0$. Moreover, again by the invertibility of the matrix $\frac{\partial^2 V}{\partial x_i \partial x_j}(P)$ we have $\det \mathrm{Jac}\, \mathcal{L}_P(0) \neq 0$ and so y is a zero stable of h_i. □

More in general we have the following

LEMMA 5.5. – *Let us suppose that $V(x) = c_0 + \sum_{i=1}^{m} a_i x_i^{\alpha_i + 1}$ in a neighborhood of $P = 0$. Here α_i are positive even integers, $a_i \in R$ with $a_i \neq 0$ and $c_0 > 0$. Then $Z = \{0\}$.*

Proof. – We have that

$$h_i(x) = (\alpha_i + 1)a_i \int\limits_{R^N} x_i^{\alpha_i} U^2(x)\, dx.$$

Let us look for the zeroes of h_i.

$$\int\limits_{R^N} (x_i + y_i)^{\alpha_i} U^2(x)\, dx = 0 \quad \text{for any } i = 1, \ldots, N. \tag{5.20}$$

Since α_i are even numbers and U is an even function we get

$$\int\limits_{R^N} (x_i + y_i)^{\alpha_i} U^2(x)\, dx = \sum_{k=0}^{\alpha_i} \binom{\alpha_i}{k} \int\limits_{R^N} x_i^k y_i^{\alpha_i - k} U^2(x)\, dx \tag{5.21}$$

$$= \sum_{k=0}^{\frac{\alpha_i}{2}} \binom{\alpha_i/2}{2k} \int\limits_{R^N} x_i^{2k} y_i^{\alpha_i - 2k} U^2(x)\, dx. \tag{5.22}$$

Because of the function $\sum \gamma_i z^{\delta_i}$ is strictly increasing if γ_i is positive and δ_i is odd we get that (5.22) admit only the solution $y = 0$. Moreover it is easy to verify that $\det \text{Jac}\, \mathcal{L}_P(0) \neq 0$ and so $Z = \{0\}$. $\quad \square$

As application of Corollary 5.3 we obtain the following nonexistence result.

PROPOSITION 5.6. – *Let us suppose that $V(x) = c_0 + \sum_{j=1}^m a_j x^{\alpha_j}$ in a neighborhood of $x = 0$. Assume that at least one of the integers α_j is odd, $a_j \in R$ with $a_j \neq 0$ and $c_0 > 0$. Then there is no solution u_ε concentrating at $P = 0$.*

6. An exact multiplicity result

In this section we prove Theorem 1.1. Let us start with the following

PROPOSITION 6.1. – *Let ε_n be a sequence which goes to zero and u_n the corresponding single-peak solution of (1.4). Let P_n the peaks of u_n and assume that (5.11) holds. Then*

$$u_n(x) - U_P\left(\frac{x - P}{\varepsilon_n} - y\right) \to 0 \quad \text{uniformly in } R^N. \tag{6.1}$$

Proof. – Let us consider $v_n(x) = u_n(\varepsilon_n x + P_n)$. So v_n is a positive solutions of

$$\Delta v_n + V(\varepsilon_n x + P)v_n = v_n^p \quad \text{in } R^N. \tag{6.2}$$

Proceeding as in the proof of Proposition 2.7 we get

$$v_n \to U_P \quad \text{uniformly in } R^N. \tag{6.3}$$

Moreover $\|v_n\|_\infty \geqslant V_0^{1/(p-1)}$ and then we get that $U_P \not\equiv 0$. Since

$$v_n\left(\frac{P_n - P}{\varepsilon_n}\right) = u_n(P_n) = \max_{x \in \mathcal{R}^N} u_n(x)$$

we derive

$$U_P(y) = \max_{x \in \mathcal{R}^N} U_P(x).$$

But U_P is radially symmetric with respect to some point (see [9]) so that $U_P(x) = U_P(|x - y|)$. This proves the claim. $\quad\square$

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. – By Theorem 4.3 we know that #{single-peak solution of (1.4) concentrating at $x = P$} $\geqslant$ #Z. By contradiction let us suppose that #{single-peak solution of (1.4) concentrating at $x = P$} > #Z. Since #$Z < \infty$ and Proposition 5.2 there exist $y \in Z$, $\varepsilon_n \to 0$ and two distinct positive solutions $u_{1,n}$ and $u_{2,n}$ of (1.4) with $\varepsilon = \varepsilon_n$ such that if $P_{1,n}$ and $P_{2,n}$ are their peaks we have

$$\lim_{n\to\infty} \frac{P_{1,n} - P}{\varepsilon_n} = \lim_{n\to\infty} \frac{P_{2,n} - P}{\varepsilon_n} = y. \tag{6.4}$$

From Proposition 6.1 the functions $v_{1,n}(x) = u_{1,n}(\varepsilon_n x + P)$ and $v_{2,n}(x) = u_{2,n}(\varepsilon_n x + P)$ satisfy $v_{1,n}(x), v_{2,n}(x) \to U_P(|x - y|)$ uniformly on $\mathcal{R}^N$.

Since $u_{1,n} \not\equiv u_{2,n}$ we can consider

$$w_n(x) = \frac{v_{1,n} - v_{2,n}}{\|v_{1,n} - v_{2,n}\|_\infty} \quad \text{in } \mathcal{R}^N, \tag{6.5}$$

and so w_n satisfies

$$-\Delta w_n + V(\varepsilon_n x + P)w_n = c_n(x)w_n \quad \text{in } \mathcal{R}^N \tag{6.6}$$

with

$$c_n(x) = p \int_0^1 \left(t v_{1,n}(x) + (1 - t)v_{2,n}(x)\right)^{p-1} \mathrm{d}t, \tag{6.7}$$

and $c_n(x) \to U_P^{p-1}(|x - y|)$ in $C^1_{\mathrm{loc}}(\mathcal{R}^N) \cap L^\infty(\mathcal{R}^N)$.

Passing to the limit in (6.6) we get that $w_n \to w$ in $C^1_{\mathrm{loc}}(\mathcal{R}^N)$ where w solves

$$-\Delta w + V(P)w = pU_P(|x - y|)^{p-1}w \quad \text{in } \mathcal{R}^N. \tag{6.8}$$

By (iv) of Theorem 2.1 we obtain that

$$w(x) = \sum_{j=1}^N a_j \frac{\partial U_P(|x - y|)}{\partial x_j}$$

with $a_j \in \mathcal{R}$.

Now the proof will be divided in two steps.

Step 1: we have that $a_j = 0$ for any $j = 1, \ldots, N$.

From (2.10) we get

$$\int_{\mathcal{R}^N} \frac{\partial V}{\partial x_i}(\varepsilon_n x + P) v_{1,n}^2(x)\, dx = \int_{\mathcal{R}^N} \frac{\partial V}{\partial x_i}(\varepsilon_n x + P) v_{2,n}^2(x)\, dx = 0. \qquad (6.9)$$

Then we have

$$\int_{\mathcal{R}^N} \frac{\partial V}{\partial x_i}(\varepsilon_n x + P) w_n(x)\big(v_{1,n}(x) + v_{2,n}(x)\big)\, dx = 0. \qquad (6.10)$$

Since $|w_n| \leqslant 1$ and $v_{1,n}$ and $v_{2,n}$ decay exponentially (see Lemma 4.2) we can proceed as in the previous sections. So by (6.10) we obtain

$$\sum_{j=1}^{N} a_j \int_{\mathcal{R}^N} h_j(x + P) \frac{\partial U_P(|x - y|)}{\partial x_i} U_P(|x - y|)\, dx = 0. \qquad (6.11)$$

Recalling the definition of $\mathcal{L}_P$ we have

$$\text{Jac}\, \mathcal{L}_P(y) = \left(2 \int_{\mathcal{R}^N} h_j(x + P) \frac{\partial U_P(|x - y|)}{\partial x_i} U_P(|x - y|)\, dx \right)_{i,j=1,\ldots,N}. \qquad (6.12)$$

Hence (1.9) implies that the linear system (6.11) admits only the trivial solution $a_j = 0$ for any $j = 1, \ldots, N$. So $w \equiv 0$.

Step 2: $w \equiv 0$ cannot occur.

Let x_n be such that $w_n(x_n) = \|w_n\|_\infty = 1$ (the same proof applies if $w_n(x_n) = -1$). If $|x_n| \leqslant C$ a contradiction arises since by the previous step $w_n \to 0$ in $C^1_{\text{loc}}(\mathcal{R}^N)$. On the other hand if $|x_n| \to \infty$ we have $\Delta w_n(x_n) \leqslant 0$ and by (6.6) $V_0 \leqslant V(\varepsilon_n x + P) = c_n(x_n)$. This gives a contradiction since $c_n(x_n) \to 0$ and so the claim of the theorem follows. $\quad\square$

THEOREM 6.2. – *Let V be an admissible potential and suppose that* (1.9) *holds. Then for ε small enough any single-peak solution u_ε of* (1.4) *is nondegenerate, i.e., the linear problem*

$$\begin{cases} -\varepsilon^2 \Delta \phi + V(x)\phi = u_\varepsilon^{p-1}\phi & \text{in } \mathcal{R}^N, \\ v \in H^1(\mathcal{R}^N). \end{cases} \qquad (6.13)$$

admits only the trivial solution $\phi \equiv 0$.

Proof. – By contradiction let us suppose that there exists a sequence $\varepsilon_n \to 0$ and a solution ϕ_n of (6.13) with $u_\varepsilon = u_{\varepsilon_n}$ such that

$$\begin{cases} -\varepsilon_n^2 \Delta \phi_n + V(x)\phi_n = u_{\varepsilon_n}^{p-1}\phi_n & \text{in } \mathcal{R}^N, \\ \phi \in H^1(\mathcal{R}^N). \end{cases}$$

Set $u_{\varepsilon_n} = u_n$ and let us denote by P_n the peak of the solution u_n. Consider $\widetilde{\phi}_n(x) = \phi_n(\varepsilon_n x + P)$. Since V is admissible $\frac{P_n - P}{\varepsilon_n} \to y$ with $y \in Z$ and $\widetilde{\phi}_n(x)$ satisfies

$$
\begin{cases}
-\Delta\widetilde{\phi}_n + V(\varepsilon_n x + P_n)\widetilde{\phi}_n = u_n^{p-1}(\varepsilon_n x + P_n)\widetilde{\phi}_n & \text{in } \mathcal{R}^N, \\
\widetilde{\phi}_n \in H^1(R^N).
\end{cases} \tag{6.14}
$$

Finally set

$$
v_n = \frac{\widetilde{\phi}_n}{\|\widetilde{\phi}_n\|_{L^\infty(\mathcal{R}^N)}}.
$$

As in the previous theorem it is easily seen that

$$
v_n(x) \to \sum_{j=1}^{N} a_j \frac{\partial U_P(|x - y|)}{\partial x_j} \quad \text{in } C_{\text{loc}}^1(\mathcal{R}^N).
$$

Let us write down the equation satisfied by $\frac{\partial v_n}{\partial x_i}$

$$
\begin{cases}
-\Delta\frac{\partial v_n}{\partial x_i} + V(\varepsilon_n x + P)\frac{\partial v_n}{\partial x_i} + \varepsilon_n \frac{V(\varepsilon_n x + P)}{\partial x_i}v_n = u_n^{p-1}\frac{\partial v_n}{\partial x_i} & \text{in } \mathcal{R}^N, \\
v \in H^1(R^N).
\end{cases} \tag{6.15}
$$

Multiplying (6.14) by $\frac{\partial u_n}{\partial x_i}$ and (6.15) by v_n we get

$$
\int_{\mathcal{R}^N} \frac{\partial V(\varepsilon_n x + P)}{\partial x_i} \frac{\partial u_n}{\partial x_i} v_n \, dx = 0. \tag{6.16}
$$

Passing to the limit in (6.15) we have

$$
\sum_{j=1}^{N} a_j \int_{\mathcal{R}^N} h_i(x + P)U_P(|x - y|)\frac{\partial U_P(|x - y|)}{\partial x_j} \, dx = 0. \tag{6.17}
$$

As in the previous theorem, by (1.9) we obtain $a_j = 0$ for any $j = 1, \ldots, N$. So $v_n \to 0$ in $C_{\text{loc}}^1(\mathcal{R}^N)$ and from this a contradiction follows (see Step 2 of the previous theorem). $\square$

Example. – Here we show an example of a potential V such that (1.4) admits exactly two single-peak solutions concentrating at P.

Let $V(x_1, x_2) \in C^1(\mathcal{R}^N) \cap L^\infty(\mathcal{R}^N)$ such that $V(x_1, x_2) = V_0 + x_1^3 - x_1 x_2^2$ in $B_{0,1}$, $V_0 > 0$. Of course 0 is an isolated critical point of V and V is admissible.

By direct computation we have that, for $y = (y_1, y_2)$,

$$
\mathcal{L}_0(y) = \left(2\int_{\mathcal{R}^N} x_1^2 U^2(x_1, x_2)\, dx_1\, dx_2 + (3y_1^2 - y_2^2)\int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2, \right.
$$

$$
\left. - 6y_1 y_2 \int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2 \right). \tag{6.18}
$$

Moreover

$$\mathcal{L}(z_1) = \mathcal{L}(z_2) = 0,$$

for

$$z_1 = \left(0, 2\frac{\int_{\mathcal{R}^N} x_1^2 U^2(x_1, x_2)\, dx_1\, dx_2}{\int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2}\right) \quad \text{and} \quad z_2 = \left(0, -2\frac{\int_{\mathcal{R}^N} x_1^2 U^2(x_1, x_2)\, dx_1\, dx_2}{\int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2}\right).$$

If we prove that $\operatorname{Jac} \mathcal{L}(z_1)$ and $\operatorname{Jac} \mathcal{L}(z_2)$ are invertible matrixes by Theorem 1.1 and Remark 3.7 we deduce the claim. Indeed

$$\operatorname{Jac} \mathcal{L}_0(y) = \begin{pmatrix} 6y_1 \displaystyle\int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2 & -2y_2 \displaystyle\int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2 \\ -6y_2 \displaystyle\int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2 & -6y_1 \displaystyle\int_{\mathcal{R}^N} U^2(x_1, x_2)\, dx_1\, dx_2 \end{pmatrix} \tag{6.19}$$

and $\det \operatorname{Jac} \mathcal{L}(z_1) = \det \operatorname{Jac} \mathcal{L}(z_2) \neq 0$.

We point out that with a little bit of computation it is possible to construct an example of V such that (1.4) admits k solution which "concentrate" in the same point P. We give a short proof of this.

Set $V(x_1, x_2) \in C^1(\mathcal{R}^N) \cap L^\infty(\mathcal{R}^N)$ such that

$$V(x_1, x_2) = V_0 + \sum_{i,j=1}^{L} a_{ij} x_1^i x_2^j \quad \text{with } a_{ij} \in \mathcal{R} \text{ and } L \in N.$$

We choose a_{ij} so that the following system

$$\begin{cases} \displaystyle\sum_{i,j=1}^{L} \sum_{m=0}^{i-1} \sum_{h=0}^{j} i a_{ij} \binom{h}{j} \binom{m}{i-1} y_1^{i-1-m} y_2^{j-h} \int_{\mathcal{R}^N} x_1^m x_2^h U^2(x_1, x_2)\, dx_1\, dx_2 = 0, \\ \displaystyle\sum_{i,j=1}^{L} \sum_{m=0}^{i} \sum_{h=0}^{j-1} j a_{ij} \binom{h}{j-1} \binom{m}{i} y_1^{i-m} y_2^{j-h-1} \int_{\mathcal{R}^N} x_1^m x_2^h U^2(x_1, x_2)\, dx_1\, dx_2 = 0 \end{cases} \tag{6.20}$$

admits k solutions $z_1, z_2, \ldots, z_k \in \mathcal{R}^2$. It is also possible to prove that we can choose a_{ij} in order to have that

$$\det \operatorname{Jac} \mathcal{L}(z_i) \neq 0 \quad \text{for any } i = 1, \ldots, k. \tag{6.21}$$

and this proves the claim. $\quad\square$

The shape of V near its critical point P plays an important role for multiplicity results of (2.3) of solutions concentrating at a point P. Next proposition tries to give partial answers in this direction.

PROPOSITION 6.3. – *Let us suppose that*

$$V(x) = c_0 + \sum_{i=1}^{m} a_i x_i^{\alpha_i + 1}$$

in a neighborhood of x = 0. Here α_i are positive even integers, $a_i \in R$ with $a_i \neq 0$ and $c_0 > 0$. Then either

(i) *at least one of the integers α_i is even and then there is no solution of* (2.3) *or*

(ii) *the integers α_i are odd and then there is only one nondegenerate solution of* (2.3) *which concentrates at x = P.*

Proof. – (i) follows by Corollary (5.3). Concerning (ii) we point out that in this case $\mathcal{L}_0(y) = (\alpha_i y_i^{\alpha_i - 1})_{i=1,\ldots,N}$ and so Jac $\mathcal{L}_0(0)$ is invertible. Hence the claim follows by Theorem 1.1 and Remark 3.7.   □

In the next corollary we deduced a uniqueness result for nongenerate critical points (see [2] for a similar result)

COROLLARY 6.4. – *Let us suppose that P is a nondegenerate critical point of V. Then there is only one single-peak solution of* (1.4) *concentrating at P*

Proof. – By Theorem 1.1 it is enough to prove that Z is a singleton. This follows by Lemma 5.4.   □

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