Vortices for a variational problem
related to superconductivity

by

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ABSTRACT. – We study minimizers of Ginzburg-Landau functionals, which depend on a parameter ϵ. These functionals appear in superconductivity and two dimensional abelian Higgs models. We study the asymptotic limit, as $\epsilon \to 0$, of minimizers and show that the limiting configuration has vortices, which have topological degree one.

RÉSUMÉ. – Nous étudions les applications minimisantes de la fonctionnelle de Ginzburg-Landau dépendant d'un paramètre ϵ. Cette fonctionnelle intervient dans les problèmes de supraconductivité ainsi que dans le modèle abélien de Higgs en dimension deux. Plus particulièrement, nous étudions le comportement asymptotique de ces applications minimisantes lorsque ϵ tend vers 0 et nous montrons que la configuration limite a des tourbillons de degré 1.

INTRODUCTION

Let Ω be a simply connected bounded domain in $\mathbb{R}^2$. Let $\epsilon > 0$ be a parameter and $u_0 : \partial\Omega \to S^1$ be a smooth map. In a recent book,

F. Bethuel, H. Brezis and F. Hélein [2] (see also [1], [3], and [4]) have studied the functional

$$F_\epsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\epsilon^2} \int_\Omega \left(1 - |u|^2\right)^2 \qquad (1)$$

for $u \in H^1_{u_0}\left(\Omega, I\!R^2\right)$. Let u_ϵ be a minimizer of $F_\epsilon(u)$ on $H^1_{u_0}\left(\Omega, I\!R^2\right)$. They have studied the asymptotic limit $\epsilon \to 0$ of u_ϵ and proved that u_ϵ converges to a limiting map u_*, which has exactly $|d|$ singularities (vortices), where $d = deg(u, \partial\Omega)$. Moreover, the distribution of the singularities was characterized, and the map u_* identified. They also proved that u_ϵ converges to u_* in C^1_{loc}, away from the singularities.

Functionals of the type of F_ϵ arises in many problems in low temperature physics, for instance superfluidity. Our purpose in this paper is to extend the methods and results of [2] (see also H. Brezis, F. Merle and T. Riviere [5] and [6]) to related functionals arising in superconductivity and two-dimensional abelian Higgs models. More precisely, consider the functional

$$G_\epsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla u - iA\, u|^2 + \frac{1}{4\epsilon^2} \int_\Omega \left(1 - |u|^2\right)^2 + \frac{1}{2} \int_\Omega |dA|^2, \quad (2)$$

where $u \in H^1\left(\Omega, I\!R^2\right)$, and A (the vector potential), is a real valued 1-form, that is

$$A = A_1\, dx_1 + A_2\, dx_2.$$

The main characteristic of $G_\epsilon(u, A)$ is its invariance under gauge transformations. More precisely let ϕ be a function in $H^2(\Omega, I\!R)$ and consider

$$\begin{cases} u_\phi = e^{i\phi}\, u \\ A_\phi = A + d\phi. \end{cases} \qquad (3)$$

We have

$$G_\epsilon(u_\phi, A_\phi) = G_\epsilon(u, A). \qquad (4)$$

The functional G was introduced by Ginzburg and Landau in 1950, in their study of phase transitions, for superconductors. The gauge invariance of the functional allows to account for electromagnetic effects. In particular

$$h = \star dA \qquad (5)$$

is the induced magnetic field. Moreover physically relevant quantities, are those which are, as h, gauge invariant, for instance $|u|$, the density of

supraconductor), $J = (iu, d_A u)$ (the current), where $d_A u = du - iA\,u$, and the degree of u on the boundary if we assume for instance $|u| = 1$ on the boundary (see below).

In order to have a well posed minimization problem, we need to supplement the functional with a boundary condition. Clearly a Dirichlet type boundary condition (as for F_ϵ) is not consistent with the gauge invariance. We adopt instead the following approach.

Let $d > 0$, be an integer, and $g : \partial\Omega \to I\!\!R$ be a smooth given function. We assume throughout the paper that Ω is simply connected. Consider the space

$$V = \left\{ \begin{array}{c} (u, A) \in H^1(\Omega, I\!\!R^2) \times H^1(\Omega, I\!\!R^2), \text{ such that} \\ |u| = 1 \quad \text{on} \quad \partial\Omega \\ J.\tau \equiv (iu, \tau.(\nabla_A u)) = g \\ deg(u, \partial\Omega) = d. \end{array} \right\} \quad (6)$$

Here τ denotes the unit tangent vector to $\partial\Omega$, such that (ν, τ) is direct, ν denoting the exterior normal to $\partial\Omega$, and $\nabla_A u = \nabla u - iA\,u$. We recall that $\partial\Omega$ is connected, since Ω is assumed simply connected. The boundary condition $J.\tau \equiv (iu, \tau.\nabla_A u)$ can be written as

$$\partial_\tau u = (g + A.\tau)\,iu \qquad \text{on } \partial\Omega. \quad (7)$$

Since the r.h.s. of (7) is clearly in L^2 (by the trace theorem, and the assumption $g \in C^1(\partial\Omega)$), we see that if (7) is satisfied $u \in H^1(\partial\Omega, S^1)$, for $u \in V$. In particular

$$deg(u, \partial\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} u \times u_\tau \, d\tau$$

is well defined (one could also invoke instead a result in [7], which shows that the degree is well defined in $H^{\frac{1}{2}}(\partial\Omega, S^1)$).

THEOREM 1. – *We have*

$$\begin{array}{c} Inf \quad G_\epsilon(u, A) \\ (u,A)\in V \end{array} \quad (8)$$

is achieved. ∎

In view of the gauge invariance, once we have a minimizer we have actually a whole family (u_ϕ, A_ϕ) of minimizers. It is convenient to reduce this degeneracy by imposing the condition

$$\left\{ \begin{array}{ll} d^* A = 0 & \text{in } \Omega \\ A \cdot \nu = 0 & \text{on } \partial\Omega. \end{array} \right.$$

We have

THEOREM 2. – *There is a minimizer (u_ϵ, A_ϵ) of (8) satisfying*

$$\begin{cases} d^* A_\epsilon = 0 & in \quad \Omega \\ A_\epsilon.\nu = 0 & on \quad \partial\Omega. \end{cases} \qquad (9)$$

■

Carrying out an asymptotic analysis as in [2], we are able to prove the following, which is our main result

THEOREM 3. – *Let (u_ϵ, A_ϵ) be minimizers of (8), satifying (9). There is a subsequence $\epsilon_n \to 0$ and exactly d points $a_1, a_2, ..., a_d$ in Ω and (u_*, A_*) smooth except at the points $a_1, a_2, ..., a_d$ such that*

$$\begin{cases} d^* A_* = 0 & in \quad \Omega \\ A_*.\nu = 0 & on \quad \partial\Omega, \end{cases} \qquad (10)$$

$$u_{\epsilon_n} \to u_* \ strongly \ in \ H^1_{loc}\left(\Omega \backslash \bigcup_{i=1}^d \{a_i\}\right) \ and \ W^{1,p}(\Omega) \ for \ p < 2. \quad (11)$$

Set $h_{\epsilon_n} = \star dA_{\epsilon_n}$ and $h_* = \star dA_*$. We have

$$h_{\epsilon_n} \to h_* \ strongly \ in \ H^1_{loc}\left(\Omega \backslash \bigcup_{i=1}^d \{a_i\}\right) \ and \ W^{1,p}(\Omega) \ for \ p < 2, \quad (12)$$

and h_* satified the equation

$$\begin{cases} -\Delta h_* + h_* = 2\pi \sum_{i=1}^n \delta_{a_i} & in \quad \Omega \\ \dfrac{\partial h_*}{\partial \nu} = -g & on \quad \partial\Omega. \end{cases} \qquad (13)$$

■

Once the points a_i are determined, we are able to recover all informations from the equation (13) which is referred to in the physical litterature as the London equation. Let ξ_* be the solution of

$$\begin{cases} \Delta \xi_* = h_* & on \quad \Omega \\ \xi_* = 0 & on \quad \partial\Omega. \end{cases} \qquad (14)$$

We have

$$\begin{aligned} A_* &= \star d\xi_* \\ J_* &= (iu_*, d_{A_*} u_*) = -\star dh. \end{aligned} \qquad (15)$$

Finally u_* is determined (up to a rotation) by

$$u_* = \prod_{k=1}^{d} \frac{z - a_k}{|z - a_k|} \, e^{i\phi}, \tag{16}$$

where ϕ is an harmonic function satisfying

$$\begin{cases} \Delta\phi = 0 & \text{in} \quad \Omega \\ \phi = \phi_0 & \text{on} \quad \partial\Omega \end{cases} \tag{17}$$

where ϕ_0 satisfies on $\partial\Omega$

$$\frac{\partial\phi_0}{\partial\tau} = - \left(i \prod_{k=1}^{d} \frac{z - a_k}{|z - a_k|} \, , \, \frac{\partial}{\partial\tau} \left(\prod_{k=1}^{d} \frac{z - a_k}{|z - a_k|} \right) \right) + \frac{\partial\xi}{\partial\nu} + g \tag{18}$$

(ϕ_0 is uniquely determined up to a constant, this is also the case for ϕ).

Finally the configuration $\{a_i\}$ is governed by a renormalized energy, which is very similar to the one introduced in [2]. Consider a configuration $b = (b_1, ..., b_d)$ of d distinct points of Ω and the function

$$\begin{aligned} W(b) = &- \pi \sum_{i \neq j} log|b_i - b_j| - \pi \sum_{i=1}^{d} R(b_i) + \pi \sum_{i=1}^{d} \xi_*(b_i) \\ &+ \frac{1}{2} \int_{\partial\Omega} \Phi \left(g + \frac{\partial\xi}{\partial\nu} \right) + \frac{1}{2} \int_{\partial\Omega} \Delta\xi_* \frac{\partial\xi_*}{\partial\nu}, \end{aligned} \tag{19}$$

where ξ_* is the solution of the linear problem

$$\begin{cases} -\Delta^2\xi_* + \Delta\xi_* = 2\pi \sum_{i=1}^{d} \delta_{b_i} & \text{in} \quad \Omega \\ \xi_* = 0 & \text{on} \quad \partial\Omega \\ \dfrac{\partial}{\partial\nu}\Delta\xi_* = -g & \text{on} \quad \partial\Omega, \end{cases} \tag{20}$$

and the function R is given by

$$R(x) = \Phi(x) - \sum_{i=1}^{d} log|x - b_i|, \tag{21}$$

where Φ is the solution of

$$\begin{cases} \Delta\Phi = 2\pi \sum_{i=1}^{d} \delta_{b_i} & \text{in} \quad \Omega \\ \dfrac{\partial\Phi}{\partial\nu} = g + \dfrac{\partial\xi}{\partial\nu} & \text{on} \quad \partial\Omega. \end{cases} \tag{22}$$

We prove

THEOREM 4. – *Let a_i be as in theorem 3. The configuration (a_i) minimizes W.* ■

The arguments in the proofs of theorems 3 and 4 follow closely the strategy of [2]. Let us however emphasize two important differences. First, we have to use a local version of Pohozaev's identity (by local we mean on balls of radius ϵ^α, for $\alpha < 1$), thereas this identity yielded directly an important estimate in [2]. Second, most of our estimates can be derived from the equation verified by h, the magnetic field.

This paper is organized as follows. The next section is devoted to the proof of theorem 1 and 2, and to some general remarks. Section II is devoted to some basic properties of (u_ϵ, A_ϵ), in particular the Ginzburg-Landau equations, the maximum principle for $|u_\epsilon|$, and an uniform bound for $\nabla_A u$. Section III is devoted to local estimates. Section IV contains a localization of the singularities (in the spirit of [2]). In Section V we deduce global L^p estimates for $p < 2$. Section VI is devoted to the proof of the convergence. In section VII we derive the London equation verified by h_* and we characterize u_*. Finally Section VIII is devoted to the proof of theorem 4.

Remarks on Physics. – In the theory of superconductivity, conducting electrons are described as a fluid existing in two phases, the superconducting one and the normal one. In the superconducting state the material has an infinite electric conductivity, and repeals magnetic fields (the London effect). On a microscopic scale, the superconducting state corresponds to a pairing of the electrons. The pairs of electrons are described on a macroscopic scale by a complex-valued function u, often called condensate wave function. Roughly speaking, one could think of $|u(x)|$ as representing the density of pairs of superconducting electrons. The Ginzburg-Landau functional then governs the interaction between u and the magnetic field. The parameter $\kappa = \epsilon^{-1}$ is called the Ginzburg-Landau parameter, and depends on the material. Material with different values of κ have very different properties.

If $\kappa < \dfrac{1}{\sqrt{2}}$, the material is called a type I superconductor. If one applies an exterior magnetic field to the sample, then for a critical value of the applied field, the sample passes from the superconducting state to the normal state. For $\kappa \geqslant \dfrac{1}{\sqrt{2}}$, the situation is quite different and the sample passes gradually from the superconducting to the normal state. This corresponds to the coexistence of the normal and superconducting state: the normal

phase is confined in vortices or filaments which are quite similar to the ones studied in this paper. for a detailed description see [7], [9], [12] and [16]. We will turn to a mathematical description of the physical situation in a forthcoming paper.

For abelian Higgs models we refer to [11]. See also related problems studied in [8], [13] and [14].

Remarks on notation. – d and $\star$ denote respectively the exterior derivative and the linear operator on the $I\!\!R$-valued forms of $I\!\!R^2$ such that

$$\star dx_1 = dx_2, \quad \star dx_2 = -dx_1, \quad \star 1 = dx_1 \wedge dx_2 \quad \text{and} \quad \star dx_1 \wedge dx_2 = 1.$$

By d^* we mean the operator $\star^{-1} d\star$, where $\star^{-1}$ is the inverse of $\star$.

The scalar product on $\mathbb{C}$ is denoted by $(\ ,\ )$, *i.e.* $(a, b) = \dfrac{1}{2}(a\,\overline{b} + \overline{a}\,b)$. More generally, if $f = f_I\,dx^I$ and $g = g_J\,dx^J$ are two forms with coefficients in $\mathbb{C}$, $(f \wedge g)$ denotes the 2l-form with real coefficients $\sum (f_I, g_J)\,dx^I \wedge dx^J$.

$\times$ represents the wedge product between two vectors of $I\!\!R^2$, it is considered as a real number.

We often identify a 1-form and the associated vector by the scalar product. We also often identify a vector of $I\!\!R^2$ and the corresponding complex number. For instance, as a result of our identifications, $u \times \dfrac{\partial u}{\partial \tau}$ stands for $\left(iu, \dfrac{\partial u}{\partial \tau} \right)$.

1. THE VARIATIONAL PROBLEM

We begin this section by some definitions and remarks.

DEFINITION I.1. – *We say that* $(u, A) \in H^1(\Omega, I\!\!R^2) \times H^1(\Omega, I\!\!R^2)$ *is gauge equivalent to* $(v, B) \in H^1(\Omega, I\!\!R^2) \times H^1(\Omega, I\!\!R^2)$, *if there exists a function* $\phi \in H^2(\Omega)$ *such that*

$$\begin{cases} v = e^{i\phi}\, u \\ B = A + d\phi. \end{cases}$$

$\blacksquare$

As we have mentionned in the introduction, it is often very useful to reduce the gauge invariance of the problem by imposing a condition on A. We have the standard result:

PROPOSITION I.1. – *Let* $(u, A) \in H^1(\Omega) \times H^1(\Omega)$. *There is some* $(\tilde{u}, \tilde{A}) \in H^1(\Omega) \times H^1(\Omega)$, *gauge equivalent to* (u, A) *such that*

$$\begin{cases} d^* \tilde{A} = 0 & in \quad \Omega \\ \tilde{A}.\nu = 0 & on \quad \partial\Omega. \end{cases} \tag{I.1}$$

■

Remark. – (I.1) is often termed a Coulomb or Lorentz gauge.

Proof. – Consider the linear problem

$$\begin{cases} \Delta\phi = -d^* A & in \quad \Omega \\ \dfrac{\partial\phi}{\partial\nu} = -A.\nu & on \quad \partial\Omega. \end{cases} \tag{I.2}$$

We easily verify that

$$\begin{cases} \tilde{u} = e^{i\phi} u \\ \tilde{A} = A + d\phi \end{cases}$$

satisfies (I.1). ■

In the course of our proof, we will often need local versions of proposition I.1. We have

PROPOSITION I.2. – *Let G be some smooth subdomain of Ω, and let (u, A) be smooth. There is some smooth $(\tilde{u}, \tilde{A})$ gauge equivalent to (u, A) such that*

$$\begin{cases} d^* \tilde{A} = 0 & in \quad G \\ \tilde{A}.\nu = 0 & on \quad \partial G. \end{cases} \tag{I.3}$$

■

Note that, in proposition I.2, we impose only a condition for $\tilde{A}$ in G and on ∂G. The proof is the same as the proof of proposition I.1 (performed in G), except that we have to extend ϕ to Ω in a smooth way. ■

Proof of theorem 1 and 2. – Let (u_n, A_n) be a minimizing sequence for (8). In view of proposition I.1 we may assume that

$$\begin{cases} d^* A_n = 0 & in \quad \Omega \\ A_n.\nu = 0 & on \quad \partial\Omega. \end{cases} \tag{I.4}$$

We deduce from (I.7) that there is some $\xi_n \in H^2(\Omega)$ such that $(A_n^1, A_n^2) = (-\xi_{n_{x_2}}, \xi_{n_{x_1}})$, where (x_1, x_2) are cartesian coordinates on $\mathbb{R}^2$. Hence

$$\begin{cases} \Delta\xi_n = \star dA_n & in \quad \Omega \\ \xi_n = 0 & on \quad \partial\Omega. \end{cases}$$

It follows, by standard elliptic estimates, that

$$\int_\Omega |A_n|^2 + \int_\Omega |\nabla A_n|^2 = \int_\Omega |\nabla \xi_n|^2 + \int_\Omega |\nabla^2 \xi_n|^2$$

$$\leqslant |\xi_n|_{W^{2,2}} \leqslant \int_\Omega |dA_n|^2$$

$$\leqslant C\, G_\epsilon(u_n, A_n) < C.$$

Hence A_n is uniformly bounded in $H^1(\Omega)$. Since

$$\int_\Omega |\nabla u_n - i A_n\, u_n|^2 \leqslant 2\, G_\epsilon(u_n, A_n) < C,$$

we obtain

$$\int_\Omega |\nabla u_n|^2 \leqslant C + 2 \int_\Omega A_n^2 |u_n|^2 \leqslant C + 2 \int_\Omega A_n^2 (1 - |u_n|^2) + 2 \int_\Omega A_n^2$$

$$\leqslant C \left(\int_\Omega A_n^4 \right)^{\frac{1}{2}} \leqslant C.$$

We deduce that u_n is also a bounded sequence in $H^1(\Omega, \mathbb{R}^2)$. Passing to a subsequence if necessary we have

$$u_n \rightharpoonup u \quad \text{in} \quad H^1(\Omega) \text{ weakly} \tag{I.5}$$

$$A_n \rightharpoonup A \quad \text{in} \quad H^1(\Omega) \text{ weakly.} \tag{I.6}$$

Passing to the limit in (I.7) we are led to

$$\begin{cases} d^*A = 0 & \text{in} \quad \Omega \\ A.\nu = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{I.7}$$

Moreover by lower semicontinuity

$$G_\epsilon(u, A) \leqslant \liminf_{n \to +\infty} G_\epsilon(u_n, A_n)$$

$$\leqslant \operatorname*{Inf}_{(v,B) \in V} G_\epsilon(v, B). \tag{I.8}$$

It remains to prove that $(u, A) \in V$, that is that the boundary conditions are preserved.

We clearly have

$$A_n.\tau \rightharpoonup A.\tau \quad \text{in} \quad H^{1/2}(\partial\Omega). \tag{I.9}$$

On the other hand, we deduce from the condition $J_n.\tau = g$ that

$$\frac{\partial u_n}{\partial \tau} = iu_n \, (g + A_n.\tau) \qquad \text{on} \quad \partial\Omega. \tag{I.10}$$

Since by (I.9) $u_n \rightharpoonup u$ in $H^{1/2}(\partial\Omega)$ weakly, we see that $u_n \to u$ strongly in $L^p(\partial\Omega)$ for any $1 < p < +\infty$. Similarly we may infer that $g + A_n.\tau \to g + A.\tau$ strongly in $L^p(\partial\Omega)$. It follows that

$$\frac{\partial u_n}{\partial \tau} \to iu \, (g + A.\tau) \quad \text{strongly in } L^p(\partial\Omega) \text{ for any } 1 < p < +\infty. \tag{I.11}$$

Passing to the limit (in the sense of distributions) in (I.15) we obtain

$$\frac{\partial u}{\partial \tau} = iu \, (g + A.\tau) \qquad \text{on} \quad \partial\Omega. \tag{I.12}$$

This leads to

$$J.\tau = (iu, \tau.\nabla_A u) = g. \tag{I.13}$$

(where $\nabla_A u = \nabla u - iA \, u$).

Finally it remains to prove that

$$deg(u, \partial\Omega) = d. \tag{I.14}$$

We have

$$deg(u, \partial\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} u \times \frac{\partial u}{\partial \tau} = \frac{1}{2\pi} \int_{\partial\Omega} (g + A.\tau)$$

and

$$d = deg(u_n, \partial\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} (g + A_n.\tau) \longrightarrow \frac{1}{2\pi} \int_{\partial\Omega} (g + A.\tau),$$

by (I.14), and (I.19) follows. This completes the proof of theorem 1. Since A satisfies (I.11), we also have proved theorem 2. ∎

2. BASIC PROPERTIES OF (u_ϵ, A_ϵ)

2.1. The Ginzburg-Landau equation

PROPOSITION II.1. – *Any minimizer* (u_ϵ, A_ϵ) *of* G_ϵ *on* V *satisfies the Ginzburg-Landau equations on* Ω

$$-\nabla^2_{A_\epsilon} u_\epsilon = \frac{1}{\epsilon^2} u_\epsilon (1 - |u_\epsilon|^2) \tag{II.1}$$

$$- \star dh_\epsilon = J_\epsilon \tag{II.2}$$

where $J_\epsilon = (iu_\epsilon, d_{A_\epsilon} u_\epsilon)$, *and* $d_{A_\epsilon} u_\epsilon = du_\epsilon - iA_\epsilon u_\epsilon$. ∎

For a proof see for instance [11], [9]. Remark that equations (II.1) and (II.2) are independent of the gauge (in particular they hold even if we do not assume (9)). We recall that ∇^2_A is the operator defined by

$$\nabla^2_A = \sum_{j=1}^{2} \left(\frac{\partial}{\partial x_j} - iA_j \right) \left(\frac{\partial}{\partial x_j} - iA_j \right).$$

Recall also that, in view on our variational problem, equations (II.1) and (II.2) are completed with the boundary conditions

$$J_\epsilon . \tau = g \quad \text{on} \quad \partial\Omega \tag{II.3}$$

$$|u_\epsilon| = 1 \quad \text{on} \quad \partial\Omega. \tag{II.4}$$

Moreover we have chosen (u_ϵ, A_ϵ) so that (I.1) is satisfied.

PROPOSITION II.2. – *Any minimizer* (u_ϵ, A_ϵ) *to (8) in* V *satisfies*

$$|u_\epsilon| \leqslant 1 \quad in \ \Omega. \tag{II.5}$$

Moreover if A_ϵ *satisfies (1.1), then* (u_ϵ, A_ϵ) *are smooth.* ∎

Proof. – See [11], [7] for the proof of regularity of (u_ϵ, A_ϵ). Inequality (II.5) is a consequence of the maximum principle, since we have

$$\frac{1}{2}\Delta |u_\epsilon|^2 = -\frac{1}{\epsilon^2} |u_\epsilon|^2 (1 - |u_\epsilon|^2) + |\nabla_{A_\epsilon} u_\epsilon|^2. \tag{II.6}$$

∎

A very useful consequence of propositions II.1 and II.2 is the following.

PROPOSITION II.3. – *We have*

$$\frac{\partial h_\epsilon}{\partial \nu} = -g \quad on \quad \partial\Omega \tag{II.7}$$

∎

Proof. – Since h_ϵ is smooth up to $\partial\Omega$, (II.2) holds on $\partial\Omega$ and yields (II.7). ∎

The next equality plays the same role as Theorem III.3 in [2] and is a Pohozaev type equality.

PROPOSITION II.4. – *Let G be a subdomain of Ω. We have*

$$\int_G \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)^2 - h_\epsilon^2 = \int_{\partial G} (x.\nu)\left[(\tau.\nabla_{A_\epsilon} u_\epsilon)^2 - (\nu.\nabla_{A_\epsilon} u_\epsilon)^2\right]$$
$$+ \int_{\partial G} \frac{1}{2}(x.\nu)\left(\frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)^2 - h_\epsilon^2\right)$$
$$- \int_{\partial G} (x.\tau)(\tau.\nabla_{A_\epsilon} u_\epsilon, \nu.\nabla_{A_\epsilon} u_\epsilon). \qquad \text{(II.8)}$$

∎

Proof. – Use the stress - energy tensor as defined in [11] and integrate by parts on G. ∎

2.2. An upper bound for the energy of minimizers

As in [2] (theorem III.1), we may derive the following upper bound.

PROPOSITION II.5. – *We have, for $\epsilon < 1$*

$$G_\epsilon(u_\epsilon, A_\epsilon) \leqslant \pi\, d\log\frac{1}{\epsilon} + C, \qquad \text{(II.9)}$$

where C depends only on Ω, g and d. ∎

Remark. – The proposition below, as written, could have been a direct consequence of Theorem III.1 of [2], but here we establish a specific dependence of C on Ω, g and d: for a given degree d we prove that C is independent of the scale of Ω, by changing Ω into $\Omega_\lambda = \frac{1}{\lambda}\Omega$, and replacing g by $g_\lambda(x) = \lambda\, g(\lambda x)$ for $\lambda < 1$. This scale invariance, that we will use during the proof of proposition II.6, represents one of the major difference between the gauge invariant functional, G, and the non gauge invariant one, F. One of its consequence is illustrated for instance by the fact that we may consider the functional G on all of $\mathbb{R}^2$. Even if we prescribe some (non-zero) degree at infinity the functional may remain bounded (see [11]). This is not the case for the functional F (see [5]).

Proof. – It suffices to construct an explicit test function $(v_\epsilon, B_\epsilon) \in V$ such that

$$G_\epsilon(v_\epsilon, B_\epsilon) \leqslant \pi \, d \log \frac{1}{\epsilon} + C.$$

To that aim, we proceed as in [2] and fix d distinct points $a_1, ..., a_d$ in Ω, and $R > 0$ so small that

$$B(a_i, R) \subset \Omega,$$

and

$$B(a_i, R) \cap B(a_j, R) = \emptyset \quad \text{if } i \neq j.$$

Set

$$\tilde{\Omega} = \Omega \setminus \bigcup_{i=1}^{d} B(a_i, R),$$

and let w_0 be a smooth map from $\tilde{\Omega}$ to S^1 verifying for all j

$$w_0 = e^{i\theta} = \frac{z - a_j}{|z - a_j|} \quad \text{on} \quad \partial B(a_j, R).$$

Since w_0 is a smooth map from $\tilde{\Omega}$ to S^1 we have

$$2 \frac{\partial w_0}{\partial x_1} \times \frac{\partial w_0}{\partial x_2} = \frac{\partial}{\partial x_1} \left(w_0 \times \frac{\partial w_0}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left(w_0 \times \frac{\partial w_0}{\partial x_1} \right) = 0.$$

We set

$$B_1^0 = w_0 \times \frac{\partial w_0}{\partial x_1} = \left(i w_0, \frac{\partial w_0}{\partial x_1} \right)$$

$$B_2^0 = w_0 \times \frac{\partial w_0}{\partial x_2} = \left(i w_0, \frac{\partial w_0}{\partial x_2} \right).$$

We verify easily that

$$(\nabla - iB^0) w_0 = 0 \quad \text{in} \quad \tilde{\Omega} \tag{II.10}$$

and

$$dB^0 = 0 \quad \text{in} \quad \tilde{\Omega}. \tag{II.11}$$

Let B^1 be the solution of

$$\begin{cases} \Delta B^1 = 0 & \text{in} \quad \Omega \\ B^1 . \tau = g, \quad B^1 . \nu = 0 & \text{on} \quad \partial \Omega \end{cases} \tag{II.12}$$

We deduce from elliptic estimates that

$$\int_{\Omega} |B^1|^2 + |\nabla B^1|^2 \; < \; C. \tag{II.13}$$

On $\tilde{\Omega}$ we set

$$B_\epsilon = B^0 + B^1 \quad \text{and} \quad v_\epsilon = w_0;$$

we verify that

$$|v_\epsilon| = 1 \quad \text{in} \quad \tilde{\Omega}$$

$$(\nabla - iB_\epsilon)\, v_\epsilon = -i\, B^1\, v_\epsilon \quad \text{in} \quad \tilde{\Omega}$$

and that

$$dB_\epsilon = dB^1 \quad \text{in} \quad \tilde{\Omega}.$$

Hence we compute

$$\frac{1}{2} \int_{\tilde{\Omega}} |(\nabla - iB_\epsilon)\, v_\epsilon|^2 \; + \; |dB_\epsilon|^2 \; + \; \frac{1}{2\epsilon^2}(1 - |v_\epsilon|^2)^2$$

$$= \frac{1}{2} \int_{\tilde{\Omega}} |B^1|^2 + |dB^1|^2 \; < \; C. \tag{II.14}$$

In view of (II.13).

It remains to define (v_ϵ, B_ϵ) on $\bigcup_{i=1}^{d} B(a_i, R)$. Let ζ be a smooth function from $I\!\!R$ to $I\!\!R^+$ such that

$$\zeta(t) = 1 \quad \text{if } t \geqslant 1$$

$$\zeta(t) = 0 \quad \text{if } t \leqslant \frac{1}{2}.$$

We set, on $B(a_i, R)$

$$v_\epsilon(x) = \frac{x - a_i}{|x - a_i|} \, \zeta\left(\frac{x - a_i}{\epsilon}\right), \quad \text{as soon as} \quad \epsilon < R,$$

and

$$B_\epsilon(x) = B_\epsilon\left(\frac{x - a_i}{|x - a_i|} R\right) \zeta\left(\frac{x - a_i}{R}\right).$$

A simple computation yields

$$\frac{1}{2} \int_{B(a_i, R)} |(\nabla - i B_\epsilon) v_\epsilon|^2 + |d B_\epsilon|^2$$
$$+ \frac{1}{2\epsilon^2} (1 - |v_\epsilon|^2)^2 < \pi \, log \left(\frac{1}{\epsilon} \right) + C, \tag{II.15}$$

where C is a constant depending only on R. Clearly we have $(v_\epsilon, B_\epsilon) \in H^1(\Omega) \times H^1(\Omega)$ and combining (II.14) and (II.15) we obtain

$$G_\epsilon(v_\epsilon, B_\epsilon) \leqslant \pi \, d \, log \frac{1}{\epsilon} + C. \tag{II.16}$$

On the other hand we verify that

$$deg(v_\epsilon, \partial\Omega) = deg(w_0, \partial\Omega) = d$$

$$|v_\epsilon| = |w_0| = 1 \quad \text{on} \quad \partial\Omega$$

and

$$J_\epsilon.\tau = B^1.\tau = g \quad \text{on} \quad \partial\Omega.$$

Hence $(v_\epsilon, B_\epsilon) \in V$, and (II.9) follows from (II.16). ∎

2.3. An estimate for $|\nabla_{A_\epsilon} u_\epsilon|$

The equivalent of lemma A2 of [1] is the following.

PROPOSITION II.6. – *We have*

$$|\nabla_{A_\epsilon} u_\epsilon| \leqslant C/\epsilon, \tag{II.17}$$

and hence

$$|\nabla h_\epsilon| \leqslant |\nabla_{A_\epsilon} u_\epsilon| \leqslant C/\epsilon, \tag{II.18}$$

where C is some constant depending only on g, Ω and d. ∎

Proof. – We deduce from proposition II.5 that

$$\int_\Omega |h_\epsilon|^2 \leqslant C \, log \frac{1}{\epsilon}. \tag{II.19}$$

It will be convenient to make the following change of scale. Set

$$\overline{x} = \frac{x}{\epsilon} \quad \text{and} \quad \overline{\Omega} = \frac{1}{\epsilon} \Omega, \tag{II.20}$$

$$\overline{u}(\overline{x}) = u_\epsilon(\epsilon\overline{x}) = u_\epsilon(x) \qquad \text{i.e} \quad u_\epsilon(x) = \overline{u}(x/\epsilon) \tag{II.21}$$

and

$$\overline{A}(\overline{x}) = \epsilon \, A_\epsilon(\epsilon\,\overline{x}). \tag{II.22}$$

Hence

$$\nabla_{\overline{A}}\overline{u}(\overline{x}) = \epsilon \, \nabla_{A_\epsilon} u_\epsilon(\epsilon\,\overline{x}) = \epsilon \, \nabla_{A_\epsilon} u_\epsilon(x), \tag{II.23}$$

and

$$\overline{h}(\overline{x}) = \star d\overline{A}(\overline{x}) = \epsilon^2 \, h_\epsilon(\epsilon\,\overline{x}) = \epsilon^2 \, h_\epsilon(x). \tag{II.24}$$

It follows that $(\overline{u}, \overline{A})$ is a minimizer of the functional

$$\overline{G}(v, B) = \int_{\overline{\Omega}} \frac{1}{2}|(\nabla - iB)\,v|^2 + \frac{1}{2\epsilon^2}|dB|^2 + \frac{1}{4}(1 - |v|^2)^2$$

on the space

$$\overline{V} = \left\{ \begin{array}{c} (v, B) \in H^1(\overline{\Omega}) \times H^1(\overline{\Omega}) \text{ such that} \\ |v| = 1 \text{ on } \partial\overline{\Omega} \\ J.\tau = \overline{g} \qquad deg(v, \partial\overline{\Omega}) = d \end{array} \right\}$$

where $\overline{g}(\overline{x}) = \epsilon \, g(\epsilon\,\overline{x})$.

The Ginzburg-Landau equations for $(\overline{u}, \overline{A})$ are therefore

$$\begin{aligned} -\nabla^2_{\overline{A}}\overline{u} &= \overline{u}(1 - |\overline{u}|^2) \qquad \text{in } \overline{\Omega} \\ \frac{1}{\epsilon^2}\, d^* \, d\overline{A} &= (i\overline{u}, d_{\overline{A}}\overline{u}). \end{aligned} \tag{II.25}$$

since $d^*\overline{A} = 0$, (II.25) implies

$$-\frac{1}{\epsilon^2}\Delta\overline{A} + |\overline{u}|^2\,\overline{A} = (-i\overline{u}, d\overline{u}) \quad \text{in } \overline{\Omega}. \tag{II.26}$$

Let $\overline{x}_0 \in \overline{\Omega}$, and consider the ball $B(\overline{x}_0, 2)$. We are going to show that

$$|\nabla_{\overline{A}}\overline{u}| \leqslant C \qquad \text{in } B(\overline{x}_0, 1). \tag{II.27}$$

To that aim we distinguish two cases.

Case 1. $-dist(\overline{x}_0, \partial\Omega) > 2$. It follows that $B(\overline{x}_0, 2) \cap \partial\overline{\Omega} = \emptyset$. In view of proposition I.2 we may choose $(\tilde{u}, \tilde{A})$ gauge equivalent to $(\overline{u}, \overline{A})$ such that

$$\begin{cases} d^*\tilde{A} = 0 & \text{in } B(\overline{x}_0, 2) \\ \tilde{A}.\nu = 0 & \text{on } \partial B(\overline{x}_0, 2). \end{cases}$$

Hence there is some $\tilde{\xi} \in H^2(B(\overline{x}_0, 2))$ such that

$$\tilde{A} = (-\tilde{\xi}_{x_2}, \tilde{\xi}_{x_1}), \tag{II.28}$$

where $\tilde{\xi}$ satisfies

$$\begin{cases} \Delta \tilde{\xi} = \overline{h} & \text{in} \quad B(\overline{x}_0, 2) \\ \tilde{\xi} = 0 & \text{on} \quad \partial B(\overline{x}_0, 2). \end{cases}$$

By standard elliptic estimates, we obtain

$$||\nabla \xi||_{L^2(B(\overline{x}_0, 2))} + ||\nabla^2 \xi||_{L^2(B(\overline{x}_0, 2))} \leqslant C\, ||\overline{h}||_{L^2(\overline{\Omega})} \leqslant C\,\epsilon\,|log\epsilon|^{1/2},$$

where we have used (II.19) and (II.24). It follows that

$$||\tilde{A}||^2_{H^1(B(\overline{x}_0, 2))} \leqslant C\,\epsilon^2\,log\epsilon. \tag{II.29}$$

In the gauge $\tilde{A}$, (II.25) becomes

$$-\Delta \tilde{u} = \tilde{u}\,(1 - |\tilde{u}|^2) - \tilde{A}^2\,\tilde{u} - 2i\tilde{A}.d\tilde{u}. \tag{II.30}$$

Where $\tilde{A}.d\tilde{u} = \tilde{A}_1\,\tilde{u}_{x_1} + \tilde{A}_2\,\tilde{u}_{x_2}$. We write

$$\tilde{u} = \tilde{u}_0 + \tilde{u}_1 \qquad \text{on} \quad B(\overline{x}_0, 2),$$

where $\tilde{u}_0$ satisfies

$$\begin{cases} \Delta \tilde{u}_0 = 0 & \text{in} \quad B(\overline{x}_0, 2) \\ \tilde{u}_0 = \tilde{u} & \text{on} \quad \partial B(\overline{x}_0, 2), \end{cases}$$

and $\tilde{u}_1$ satisfies

$$\begin{cases} -\Delta \tilde{u}_1 = \tilde{u}\,(1 - |\tilde{u}|^2) - \tilde{A}^2\,\tilde{u} - 2i\tilde{A}.d\tilde{u} & \text{in} \quad B(\overline{x}_0, 2) \\ \tilde{u}_1 = 0 & \text{on} \quad \partial B(\overline{x}_0, 2). \end{cases} \tag{II.31}$$

From elliptic estimates, we obtain

$$|\nabla \tilde{u}_0| < C \qquad \text{on} \quad B(\overline{x}_0, 3/2). \tag{II.32}$$

We have for any $p < +\infty$, by (II.29)

$$||\tilde{A}^2\tilde{u}||_{L^p} \leqslant ||\tilde{A}^2||_{L^p} = ||\tilde{A}||^2_{L^{2p}} \leqslant ||\tilde{A}||^2_{H^1} \leqslant C\,\epsilon^2\,|log\epsilon|, \tag{II.33}$$

and for $p < 2$,

$$||\tilde{A}.\nabla\tilde{u}||_{L^p} \leqslant ||\tilde{A}.\nabla_{\tilde{A}}\tilde{u}||_{L^p} + ||\tilde{A}^2\tilde{u}||_{L^p}$$
$$\leqslant ||\nabla_{\tilde{A}}\tilde{u}||_{L^2} ||\tilde{A}||_{L^{pq}} + ||\tilde{A}^2||_{L^p}, \qquad (\text{II}.34)$$

where $\frac{1}{q} + \frac{p}{2} = 1$. Hence, since $||\nabla_{\tilde{A}}\tilde{u}||_{L^2(\overline{\Omega})} \leqslant C|log\epsilon|$ and $||\tilde{A}||_{L^{pq}} \leqslant ||\tilde{A}||_{H^1}$, we deduce, by (II.29) that

$$||\tilde{A}.\nabla\tilde{u}||_{L^p} \leqslant C\epsilon|log\epsilon|^{3/2}, \qquad (\text{II}.35)$$

and from (II.31), (II.33), and (II.35) that

$$||\tilde{u}_1||_{W^{2,p}(B(\overline{x}_0,2))} \leqslant C_p\epsilon|log\epsilon|^{3/2} \qquad \forall p < 2.$$

Hence, by Sobolev imbedding

$$||\nabla\tilde{u}_1||_{L^p(B(\overline{x}_0,2))} \leqslant C_p\epsilon|log\epsilon|^{3/2} \qquad \forall\, 1 < p < +\infty. \qquad (\text{II}.36)$$

Combining (II.36) and (II.32) we are led to

$$||\nabla\tilde{u}||_{L^p(B(\overline{x}_0,3/2))} \leqslant C_p \qquad \forall\, 1 < p < +\infty. \qquad (\text{II}.37)$$

Working now on the ball $B(\overline{x}_0, 3/2)$ instead of the ball $B(\overline{x}_0, 2)$, we see that equation (II.31) leads to

$$||\tilde{u}_1||_{W^{2,p}(B(\overline{x}_0,3/2))} \leqslant C_p \qquad \forall\, 1 < p < +\infty.$$

This yields in particular, by embedding

$$|\nabla\tilde{u}_1| \leqslant C \qquad \text{on} \quad B(\overline{x}_0, 3/2),$$

which shows, combining with (II.32) that

$$|\nabla\tilde{u}| \leqslant C \qquad \text{on} \quad B(\overline{x}_0, 3/2). \qquad (\text{II}.38)$$

Using next equation (II.26) on $B(\overline{x}_0, 3/2)$ we deduce that

$$||\tilde{A}||_{L^\infty(B(\overline{x}_0,1))} \leqslant C. \qquad (\text{II}.39)$$

Combining (II.38) and (II.39) we deduce (II.27) in case 1.

Case 2. $- \overline{x}_0 \in \partial\overline{\Omega}$. We follow essentially the same arguments. we choose $(\tilde{u}, \tilde{A})$ gauge equivalent to $(\overline{u}, \overline{A})$ such that

$$\begin{cases} d^* \tilde{A} = 0 & \text{in} \quad \overline{\Omega} \cap B(\overline{x}_0, 2) \\ \tilde{A}.\nu = 0 & \text{on} \quad \partial\big(\overline{\Omega} \cap B(\overline{x}_0, 2)\big). \end{cases}$$

As above, we write

$$\tilde{u} = \tilde{u}_0 + \tilde{u}_1,$$

where $\tilde{u}_0$ verifies

$$\begin{cases} \Delta\tilde{u}_0 = 0 & \text{on} \quad \overline{\Omega} \cap B(\overline{x}_0, 2) \\ \tilde{u}_0 = \tilde{u} & \text{on} \quad \partial\big(\overline{\Omega} \cap B(\overline{x}_0, 2)\big). \end{cases}$$

We have $\tilde{A} = (-\tilde{\xi}_{x_2}, \tilde{\xi}_{x_1})$ where $\tilde{\xi}$ satifies

$$\begin{cases} \Delta\tilde{\xi} = \overline{h} & \text{on} \quad \overline{\Omega} \cap B(\overline{x}_0, 2) \\ \tilde{\xi} = 0 & \text{on} \quad \partial\big(\overline{\Omega} \cap B(\overline{x}_0, 2)\big). \end{cases} \tag{II.40}$$

We have

$$\tilde{A}.\tau = \frac{\partial\tilde{\xi}}{\partial\nu}, \tag{II.41}$$

and

$$\frac{\partial\tilde{u}}{\partial\tau} = i\left(\frac{\partial\tilde{\xi}}{\partial\nu} + g\right)\tilde{u} \qquad \text{on} \quad \partial\overline{\Omega} \cap B(\overline{x}_0, 2). \tag{II.42}$$

From (II.40) and (II.42) we deduce, by classical linear estimates, that

$$\left\|\frac{\partial\tilde{u}}{\partial\tau}\right\|_{H^{1/2}(\partial\overline{\Omega}\cap B(\overline{x}_0, 3/2))} \leqslant C.$$

Hence since $\tilde{u}_0$ is harmonic we deduce that

$$\|\tilde{u}_0\|_{H^2(B(\overline{x}_0, 3/2)\cap\Omega)} \leqslant C. \tag{II.43}$$

As for (II.29) we have

$$\|\tilde{A}\|_{H^1(B(\overline{x}_0, 3/2)\cap\Omega)} \leqslant C. \tag{II.44}$$

Similarly arguing as in (II.31)-(II.38), we derive

$$\|\tilde{u}\|_{H^1(B(\overline{x}_0, 3/2)\cap\overline{\Omega})} \leqslant C. \tag{II.45}$$

In particular, we obtain

$$\|\nabla_{\tilde{A}}\tilde{u}\|_{H^1(B(\overline{x}_0, 3/2) \cap \overline{\Omega})} \leqslant C. \qquad (\text{II.46})$$

On the other hand, we have by (II.2)

$$- \star d\overline{h} = \epsilon^2 (i\tilde{u}, d_{\tilde{A}}\tilde{u}).$$

Hence, we deduce from (II.44), (II.45) and (II.46)

$$|\nabla \overline{h}|_{L^2(B(\overline{x_0}, 3/2) \cap \overline{\Omega})} \leqslant \epsilon^2 \, |\nabla_{\tilde{A}}\tilde{u}|_{L^2(B(\overline{x_0}, 3/2) \cap \Omega)} \leqslant C\epsilon^2.$$

Hence, since $|\overline{h}|_{L^2(\overline{\Omega})} \leqslant C\epsilon \, log(1/\epsilon)$, we obtain by Sobolev embedding, for any $p > 2$

$$|\overline{h}|_{L^p(B(\overline{x_0}, 3/2) \cap \Omega)} \leqslant C\,\epsilon^2 \, log\left(\frac{1}{\epsilon}\right) \quad \longrightarrow 0.$$

Iterating the argument we deduce that

$$\|\tilde{u}_0\|_{W^{2,p}(B(\overline{x}_0, 5/4) \cap \overline{\Omega})} \leqslant C,$$

and hence

$$|\nabla \tilde{u}_0| \leqslant C.$$

We complete the proof of (II.27) as in case 1. ∎

3. LOCAL ESTIMATES

As we may see from proposition II.6, ϵ is the characteristic length of variations of (u_ϵ, A_ϵ). The aim of this section is to derive estimates on domains of a larger scale, of order ϵ^α for $\alpha < 1$. These estimates will be an important step for deriving global estimates on Ω (see section V).

Let $0 < \alpha < 1$ be fixed. If not otherwise necessary, we drop the indice ϵ for u_ϵ, h_ϵ, A_ϵ and simply denote u, h, A.

PROPOSITION III.1. – *Let* $0 < \alpha < 1$ *and let* $x_0 \in \Omega$. *We have*

$$\int_{B(x_0, \epsilon^\alpha) \cap \Omega} |h|^2 \leqslant C\,\epsilon^\alpha \, |log\epsilon|, \qquad (\text{III.1})$$

and

$$\frac{1}{\epsilon^2} \int_{B(x_0,\epsilon)\cap\Omega} (1-|u|^2)^2 \leqslant C, \tag{III.2}$$

where C depends only on g, d, α and Ω.   ∎

Proof. Step 1. Proof of (III.1). – We have, by (II.2),

$$|\nabla h| \leqslant |\nabla_A u|, \tag{III.3}$$

and hence by proposition II.5.

$$\int_\Omega |\nabla h|^2 \leqslant C |log\epsilon|.$$

On the other hand, we have also

$$\int_\Omega |h|^2 \leqslant C |log\epsilon|.$$

Hence, by Sobolev embedding, we deduce that for $p > 2$

$$\left(\int_\Omega |h|^p \right)^{\frac{1}{p}} \leqslant C_p |log\epsilon|^{\frac{1}{2}}. \tag{III.4}$$

It follows that

$$\left(\int_{B(x_0,\epsilon^\alpha)\cap\Omega} |h|^2 \right)^{\frac{1}{2}} \leqslant \left(\int_\Omega |h|^p \right)^{\frac{1}{p}} [meas\ (B(x_0,\epsilon^\alpha)]^{\frac{1}{2}-\frac{1}{p}}$$
$$\leqslant C |log\epsilon|^{\frac{1}{2}}\ \epsilon^{\alpha\left(1-\frac{2}{p}\right)}, \tag{III.5}$$

which yields (III.1) with $p = 4$.

Step 2. Proof of (III.2). – Recall that, by Proposition II.5, we have

$$G_\epsilon(u, A) \leqslant C |log\epsilon|,$$

which yields

$$\frac{1}{2} \int_{B(x_0,\epsilon^{\alpha/2})\cap\Omega} |\nabla_A u|^2 + |dA|^2 + \frac{1}{2\epsilon^2}(1-|u|^2)^2 \leqslant C|log\epsilon|.$$

By Fubini's theorem, we may hence write

$$\int_{\epsilon^\alpha}^{\epsilon^{\alpha/2}} I_r(u, A)\, dr \leqslant C |log\epsilon| \tag{III.6}$$

where

$$I_r(u, A) = \frac{1}{2} \int_{\partial B(x_0, r) \cap \Omega} |\nabla_A u|^2 + |dA|^2 + \frac{1}{2\epsilon^2}(1 - |u|^2)^2.$$

We claim that there is some $\rho \in (\epsilon^\alpha, \epsilon^{\alpha/2})$ and a constant C_α depending on α, g and d such that

$$I_\rho(u, A) \leqslant \frac{C_\alpha}{\rho}. \qquad (III.7)$$

We argue as in [2] and assume by contradiction that (III.6) does not hold for any $\rho \in (\epsilon^\alpha, \epsilon^{\alpha/2})$ and any C_α. Integrating we obtain

$$\int_{\epsilon^\alpha}^{\epsilon^{\alpha/2}} I_r(u, A) \, dr \geqslant \int_{\epsilon^\alpha}^{\epsilon^{\alpha/2}} \frac{C_\alpha}{r} \, dr$$

$$\geqslant \frac{1}{2} \alpha \, C_\alpha \, |log\epsilon|.$$

This contradicts (III.6) for C_α choosen sufficiently large. We distinguish next two cases.

Case 1. – $\partial B(x_0, \rho) \cap \Omega = \emptyset$. We apply proposition II.4 on the ball $B(x_0, \rho)$. This yields by (III.7)

$$\int_{B(x_0, \rho)} \frac{1}{\epsilon^2}(1 - |u|^2)^2 - |h|^2 \leqslant C \, \rho \, I_\rho(u, A) \leqslant C.$$

Combined with (III.1), this yields (III.2) in the case considered.

Case 2. – $\partial B(x_0, \rho) \cap \Omega \neq \emptyset$. We write

$$\partial[B(x_0, \rho) \cap \Omega] = (\partial B(x_0, \rho) \cap \Omega) \cup \Gamma_\rho$$

where

$$\Gamma_\rho = \partial\Omega \cap B(x_0, \rho).$$

Let x_1 be a point in Ω, take x_1 as the origin and apply proposition II.4 on $\Omega \cap B(x_0, \rho)$. This yield the estimate

$$\int_{B(x_0, \rho) \cap \Omega} \frac{1}{\epsilon^2}(1 - |u|^2)^2 - |h|^2 \leqslant C \, (\rho + |x_1 - x_0|) \, I_\rho(u, A)$$

$$+ \int_{\Gamma_\rho} (x - x_1).\nu \left[(\tau.\nabla_A u)^2 - (\nu.\nabla_A u)^2 \right]$$

$$- \frac{1}{2} \int_{\Gamma_\rho} (x - x_1).\nu \, h^2$$

$$+ \int_{\Gamma_\rho} (x - x_1).\tau(\tau.\nabla_A u, \nu.\nabla_A u). \qquad (III.8)$$

(We have used the fact that $|u| = 1$ on $\Gamma_\rho \subset \partial\Omega$.) We have

$$\int_{\Gamma_\rho} (x-x_1).\tau(\tau.\nabla_A u, \nu.\nabla_A u) \leqslant \frac{1}{2} \int_{\Gamma_\rho} |(x_1-x_0).\tau| \left[|\tau.\nabla_A u|^2 - |\nu.\nabla_A u|^2\right]$$

Assume that ϵ is sufficiently small, let πx_0 be the nearest point projection of x_0 on $\partial\Omega$ and choose x_1 such that $dist(x_1, \pi x_0) = 4\rho$ and $x_0 \in [\pi x_0, x_1]$. Then we have

$$(x - x_1).\nu \geqslant 0 \qquad \forall x \in \Gamma_\rho$$

and

$$\frac{1}{2}(x - x_1).\tau \leqslant (x - x_1).\nu \qquad \forall x \in \Gamma_\rho.$$

Then (III.8) yields

$$\int_{B(x_0,\rho)\cap\Omega} \frac{1}{\epsilon^2}(1 - |u|^2)^2 - |h|^2 \leqslant C\rho \left[I_\rho(u, A) + \int_{\Gamma_\rho} |\tau.\nabla_A u|^2\right]$$

$$\leqslant C\rho \left[I_\rho(u, A) + \int_{\Gamma_\rho} g^2\right] \leqslant C,$$

and this implies (III.2). ∎

4. LOCATING THE SINGULARITIES

The purpose of this section is to extend lemma IV.1 of [2] to our situation. More precisely we are going to give a first description of the vortices and their location. We will prove

THEOREM IV.1. – *There exist constants $N \in I\!N$, $\lambda_0 > 0$ depending only on g, d, Ω (and not on ϵ), and points $x_1^\epsilon, ..., x_i^\epsilon, ..., x_{N_\epsilon}^\epsilon$ in Ω such that*

$$N_\epsilon \leqslant N \tag{IV.1}$$

and

$$|u_\epsilon(x)| \geqslant \frac{1}{2} \quad for \quad x \in \Omega \setminus \left(\bigcup_{i=1}^{N_\epsilon} B(x_i^\epsilon, \lambda_0 \epsilon)\right). \tag{IV.2}$$

∎

In this section also, if not otherwise necessary, we drop the indice ϵ for u_ϵ, h_ϵ, A_ϵ and x_i^ϵ and simply denote u, h, A and x_i. We assume throughout that $\epsilon < 1$.

Our analysis here, follows closely the ideas of [2]. The main difference in our situation is that we only have the local estimate (III.2) (compare with Theorem III.3 of [2]).

The next result, as in [2], plays a crucial role:

PROPOSITION IV.1. – *There exists positive constants λ_0, μ_0 depending only on d, g and Ω such that if (u_ϵ, A_ϵ) is a minimizer of G_ϵ on V satisfying*

$$\int_{B(x_0, 2l) \cap \Omega} \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2)^2 \leqslant \mu_0 \qquad (IV.3)$$

with $x_0 \in \Omega$ and

$$\frac{l}{\epsilon} \geqslant \lambda_0 \quad , \qquad l \leqslant 1, \qquad (IV.4)$$

then

$$|u_\epsilon(x)| \geqslant \frac{1}{2} \quad on \quad \Omega \cap B(x_0, l). \qquad (IV.5)$$

∎

Proof. – The argument is exactly the same as the proof of the theorem III.4 of [2]. Indeed, observe that

$$|\nabla |u|| \leqslant |\nabla_A u|. \qquad (IV.6)$$

[If $|u| \neq 0$, we may write locally $u = |u| e^{i\phi}$, and we compute

$$\nabla_A u = \left(\nabla |u| + i|u| \left(\nabla \phi - A \right) \right) e^{i\phi}$$

and (IV.6) follows. Otherwise if $|u| = 0$

$$\nabla_A u = \nabla u,$$

and (IV.6) follows similarly]. By proposition II.6 we have therefore

$$|\nabla |u|| \leqslant C/\epsilon$$

and use the argument of [2]. ∎

Next we will (using proposition III.1) first give a local version of theorem IV.1.

PROPOSITION IV.2. – *Let $0 < \alpha < 1$, and $x_0 \in \Omega$ be given. There exists a constant $\overline{N}_\alpha$ depending only on α, g, and d, points $x_1^\epsilon, ..., x_{l_\epsilon}^\epsilon$ in $B(x_0, \epsilon^\alpha) \cap \Omega$ such that*

$$l_\epsilon \leqslant \overline{N}_\alpha, \qquad (IV.7)$$

and

$$|u_\epsilon(x)| \geqslant \frac{1}{2} \quad for \quad x \in B(x_0, \epsilon^\alpha) \cap \left(\Omega \setminus \left(\bigcup_{i=1}^{l_\epsilon} B(x_i^\epsilon, \lambda_0 \, \epsilon) \right) \right). \quad (IV.8)$$

■

Proof. – We follow the proof of lemma IV.4 of [2]. Consider a family of discs $(B(x_i, \lambda_0 \, \epsilon))_{i \in I}$, where λ_0 is the constant in proposition IV.1 such that

$$x_i \in \Omega \cap B(x_0, \epsilon^\alpha)$$

$$B\left(x_i, \frac{\lambda_0 \, \epsilon}{4} \right) \cap B\left(x_j, \frac{\lambda_0 \, \epsilon}{4} \right) = \emptyset \quad \forall \, i \neq j$$

$$\Omega \cap B(x_0, \epsilon^\alpha) \subset \bigcup_{i \in I} B(x_i, \lambda_0 \, \epsilon). \qquad (IV.9)$$

We say that $B(x_i, \lambda_0 \, \epsilon)$ is a bad disc if

$$\int_{B(x_i, 2\lambda_0 \, \epsilon) \cap \Omega} \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2)^2 \geqslant \mu_0. \qquad (IV.10)$$

Hence if x does not belong to a bad disc, we deduce, from proposition IV.1 that

$$|u(x)| \geqslant 1/2.$$

There is an absolute constant C such that

$$\sum_{i \in I} \int_{B(x_i, 2\lambda_0 \, \epsilon) \cap \Omega} (1 - |u_\epsilon|^2)^2 \leqslant C \int_{B(x_0, \epsilon^\alpha) \cap \Omega} (1 - |u_\epsilon|^2)^2. \quad (IV.11)$$

Therefore the number of bad discs is bounded by

$$\frac{C}{\epsilon^2} \int_{B(x_0, \epsilon^\alpha) \cap \Omega} (1 - |u_\epsilon|^2)^2$$

and the conclusion follows from proposition III.1. ■

The last ingredient in the proof of theorem IV.1 is to show that the contribution of a bad disc to the total energy is of the order of $|log\epsilon|$. Hence in view of proposition II.5 there cannot be many bad discs.

PROPOSITION IV.3. – *Let $0 < \alpha < 1$ and $x_0 \in \Omega$ be given. Assume that*

$$|u_\epsilon(x_0)| < \frac{1}{2}. \qquad (IV.12)$$

There exists a constant C_α depending only on α, g, and d such that

$$\int_{B(x_0,\epsilon^\alpha)\cap\Omega} |\nabla_{A_\epsilon} u_\epsilon|^2 \geqslant C_\alpha |log\epsilon|. \qquad \text{(IV.13)}$$

∎

Proof. – In order to prove proposition IV.3 we need the following lemma.

LEMMA IV.1. – *Let $D > 0$, $0 < \beta < 1$, $\gamma > 1$ be given constants, such that $\gamma\beta < 1$. Let $\rho < \epsilon^\beta$. Assume that $\rho^\gamma > \lambda_0\,\epsilon$, that*

$$I_\rho(u_\epsilon, A_\epsilon) < \frac{D}{\rho}, \qquad \text{(IV.14)}$$

$$|u_\epsilon| \geqslant \frac{1}{2} \qquad on \quad \partial B(x_0,\rho) \cap \Omega, \qquad \text{(IV.15)}$$

and

$$deg\left(\frac{u_\epsilon}{|u_\epsilon|}, \partial[B(x_0,\rho)\cap\Omega]\right) = 0. \qquad \text{(IV.16)}$$

Then we have

$$|u_\epsilon| \geqslant \frac{1}{2} \qquad on \quad B(x_0,\rho^\gamma) \cap \Omega, \qquad \text{(IV.17)}$$

provided $\epsilon < \epsilon_0$, for some constant ϵ_0 depending on D, β, γ, g, d and Ω. ∎

We postpone the proof of lemma IV.1 and complete the proof of proposition IV.3.

Proof of proposition IV.3 completed. – Set

$$\delta_\alpha = \frac{\alpha^{\frac{1}{2}} - \alpha}{2\,(\overline{N}_\alpha + 1)},$$

for $k = 1, ..., 2\,(\overline{N}_\alpha + 1)$, consider the intervals

$$I_k = \left(\epsilon^{\alpha^{1/2} - k\,\delta_\alpha}, \epsilon^{\alpha^{1/2} - (k+1)\,\delta_\alpha}\right),$$

and set

$$\mathcal{C}_k = [B(x_0, \epsilon^{\alpha^{1/2} - k\,\delta_\alpha}) \setminus B(x_0, \epsilon^{\alpha^{1/2} - (k+1)\,\delta_\alpha})] \cap \Omega.$$

We apply proposition IV.2 to the domain $B(x_0, \epsilon^\alpha)\cap\Omega$. Let $x_1, ..., x_i, ..., x_{l_\epsilon}$ be as in proposition IV.2. We claim that, if $\epsilon < \epsilon_0$ (depending on α, g, d, Ω) then there is some k_0 in $\{1, ..., 2\,(\overline{N}_\alpha + 1)\}$ such that

$$\mathcal{C}_{k_0} \cap \left(\bigcup_{i=1}^{l_\epsilon} B(x_i, \lambda_0\,\epsilon)\right) = \emptyset. \qquad \text{(IV.18)}$$

The proof of (IV.18) relies on a simple counting argument. Indeed, since $l_\epsilon \leqslant \overline{N}_\alpha$, the union of intervals

$$\bigcup_{i=1...l_\epsilon} (|x_i - x_0| - \lambda_0\,\epsilon, |x_i - x_0| + \lambda_0\,\epsilon)$$

cannot intersect all the I_k. It follows in particular from (IV.18) that

$$|u(x)| \geqslant \frac{1}{2} \qquad \forall x \in \mathcal{C}_{k_0}. \qquad (\text{IV.19})$$

Therefore, for any $\rho \in I_{k_0}$

$$d_{k_0} = deg\left(\frac{u_\epsilon}{|u_\epsilon|}, \partial[B(x_0, \rho) \cap \Omega]\right)$$

is well defined and does not depend on ρ. We complete the proof in two steps.

Step 1. – We have

$$d_{k_0} \neq 0. \qquad (\text{IV.20})$$

Indeed assume by contradiction that $d_{k_0} = 0$. Adapting the argument of the proof of proposition III.1, we would find some ρ_0 in I_{k_0} such that

$$I_{\rho_0}(u, A) < \frac{C}{\rho_0},$$

where C depends only on α, g, d, and Ω. Applying lemma IV.1 to $B(x_0, \rho_0) \cap \Omega$ we are led to $|u(x_0)| \geqslant \frac{1}{2}$, a contradiction.

Step 2. – Proof of (IV.13). Choose $(\tilde{u}, \tilde{A})$, gauge equivalent to (u, A) satifying

$$\begin{cases} d^*\tilde{A} = 0 & \text{in } B(x_0, \epsilon^\alpha) \cap \Omega \\ \tilde{A}.\nu = 0 & \text{on } \partial B(x_0, \epsilon^\alpha) \cap \Omega \\ \tilde{A}.\tau = 0 & \text{on } B(x_0, \epsilon^\alpha) \cap \partial\Omega. \end{cases} \qquad (\text{IV.21})$$

We may find $\tilde{A}$ by introducing the solution $\tilde{\xi}$ of

$$\begin{cases} \Delta\tilde{\xi} = h & \text{in } B(x_0, \epsilon^\alpha) \cap \Omega \\ \tilde{\xi} = 0 & \text{on } \partial B(x_0, \epsilon^\alpha) \cap \Omega \\ \dfrac{\partial\tilde{\xi}}{\partial\nu} = 0 & \text{on } B(x_0, \epsilon^\alpha) \cap \partial\Omega, \end{cases} \qquad (\text{IV.22})$$

so that $\tilde{A} = (-\tilde{\xi}_{x_2}, \tilde{\xi}_{x_1})$. Note that the domain $B(x_0, \epsilon^\alpha) \cap \Omega$ may have corners. However by a result of Grisvard [10], we may nevertheless assert that

$$||\nabla \xi||_{L^2} + ||\nabla^2 \xi||_{L^2(B(x_0,\epsilon^\alpha)\cap\Omega)} \leqslant C\, ||h||_{L^2(B(x_0,\epsilon^\alpha)\cap\Omega)},$$

and hence, we deduce that

$$\int_{B(x_0,\epsilon^\alpha)\cap\Omega} |\tilde{A}|^2 + |\nabla\tilde{A}|^2 \leqslant \int_{B(x_0,\epsilon^\alpha)\cap\Omega} |h|^2 \leqslant C. \qquad \text{(IV.23)}$$

We have

$$\forall \rho \in I_{k_0} \qquad |d_{k_0}| = \frac{1}{2\pi}\left|\int_{\partial[B(x_0,\rho)\cap\Omega]} \frac{1}{|u|^2}\left(\tilde{u} \times \frac{\partial\tilde{u}}{\partial\tau}\right)\right| \geqslant 1. \quad \text{(IV.24)}$$

On $B(x_0, \epsilon^\alpha) \cap \partial\Omega$, we have $\tilde{A}.\tau = 0$. Hence

$$\frac{\partial\tilde{u}}{\partial\tau} = i\, g\, \tilde{u} \qquad \text{on} \qquad B(x_0, \epsilon^\alpha) \cap \partial\Omega,$$

and

$$\left|\int_{B(x_0,\rho)\cap\partial\Omega}\left(\tilde{u} \times \frac{\partial\tilde{u}}{\partial\tau}\right)\right| = \left|\int_{B(x_0,\rho)\cap\partial\Omega} g\ \right| \leqslant C\,\epsilon^\alpha. \qquad \text{(IV.25)}$$

Combining (IV.24) and (IV.25) we are led to

$$\left|\int_{\partial B(x_0,\rho)\cap\Omega} \frac{1}{|u|^2}\left(\tilde{u} \times \frac{\partial\tilde{u}}{\partial\tau}\right)\right| \geqslant 2\pi\,(1 - C\,\epsilon^\alpha) \geqslant \pi.$$

By Cauchy-Schwarz inequality, we deduce that

$$\forall \rho \in I_{k_0} \qquad \int_{\partial B(x_0,\rho)\cap\Omega} |\nabla\tilde{u}|^2 \geqslant \frac{C}{\rho}, \qquad \text{(IV.26)}$$

and hence integrating on I_{k_0} we are led to

$$\int_{\mathcal{C}_{k_0}} |\nabla\tilde{u}|^2 \geqslant C\,|log\epsilon|.$$

Combining this with (IV.23) we deduce that

$$\int_{\mathcal{C}_{k_0}} |\nabla_{\tilde{A}}\tilde{u}|^2 = \int_{\mathcal{C}_{k_0}} |\nabla_A u|^2 \geqslant C_\alpha\,|log\epsilon|,$$

and this yields the conclusion.   ∎

We now turn to the proof of lemma IV.1.

Proof of lemma IV.1. – It is convenient to introduce the following notation. For any subset K of Ω, set

$$G(u, A, K) = \frac{1}{2} \int_K |\nabla_A u|^2 + |dA|^2 + \frac{1}{2\epsilon^2}(1 - |u|^2)^2.$$

We claim that

$$G(u, A, \Omega \cap B(x_0, \rho)) \leqslant C\left(\rho\left[I_\rho(u, A)\right] + o(1)\right), \qquad (IV.27)$$

where C is a constant depending only on g, d, Ω, and β and $o(1) \to 0$ uniformly as $\epsilon \to 0$.

Proof of (IV.27). – The idea is to construct a comparison function, and we adopt the construction of [1]. To that aim we are going to change the gauge, and consider $(\tilde{u}, \tilde{A})$ gauge equivalent to (u, A) such that

$$\begin{cases} d^* \tilde{A} = 0 & \text{on} \quad U \\ \tilde{A}.\tau = \dfrac{1}{|\Gamma|} \displaystyle\int_U h & \text{on} \quad \Gamma. \end{cases} \qquad (IV.28)$$

Here we have used the notation

$$U = B(x_0, \epsilon^\beta) \cap \Omega$$

and

$$\Gamma = \partial U = \partial\left[B(x_0, \epsilon^\beta) \cap \Omega\right].$$

This gauge $\tilde{A}$ can be found by introducing the solution $\tilde{\xi}$ to

$$\begin{cases} \Delta \tilde{\xi} = h & \text{on} \quad U \\ \dfrac{\partial \tilde{\xi}}{\partial \nu} = \dfrac{1}{|\Gamma|} \displaystyle\int_U h & \text{on} \quad \Gamma \\ \displaystyle\int_U \tilde{\xi} = 0, \end{cases} \qquad (IV.29)$$

so that $\tilde{A} = \left(-\tilde{\xi}_{x_2}, \tilde{\xi}_{x_1}\right)$ in U, $\tilde{A}$ is extended to Ω as in the proof of proposition I.2.

We easily verify that

$$\int_U |\nabla \tilde{\xi}|^2 \leqslant C \int_U |h|^2,$$

and hence by proposition III.1

$$\int_U |\tilde{A}|^2 \leqslant C\,\epsilon^\beta\,|log\epsilon| \longrightarrow 0. \qquad (IV.30)$$

Since $\displaystyle\int_\Gamma \tilde{A}.\tau = \int_U h \to 0$ as $\epsilon \to 0$, we deduce, from assumption (IV.14) that

$$\int_\Gamma \left|\frac{\partial \tilde{u}}{\partial \tau}\right|^2 \leqslant 2 \int_\Gamma |\nabla_A u|^2 + \int_\Gamma |\tilde{A}.\tau|^2. \qquad (IV.31)$$

We have by (IV.28)

$$\int_\Gamma |\tilde{A}.\tau|^2 \leqslant \frac{1}{|\Gamma|}\left(\int_U |h|\right)^2 \leqslant \frac{1}{|\Gamma|} \int_\Omega |h|^2 \, mes|U|$$
$$\leqslant C\epsilon^\beta \int_\Omega |h|^2 \leqslant C\,\epsilon^\beta\,|log\epsilon|.$$

Hence

$$\int_\Gamma \left|\frac{\partial \tilde{u}}{\partial \tau}\right|^2 \leqslant \frac{C}{\rho} + o(1),$$

where C is a constant depending on D. Since the degree of $\tilde{u}$ restricted to Γ is zero, we may write on Γ

$$\tilde{u} = |u|\,e^{i\,\tilde{\phi}}$$

where $\tilde{\phi}$ is a smooth map from Γ to $I\!R$.

We are now able to introduce a comparison function (v, B). We first set

$$B = \tilde{A} \quad \text{in} \quad \Omega$$

and

$$\begin{cases} v = \eta\,e^{i\,\psi} & \text{in} \quad U \\ v = \tilde{u} & \text{in} \quad \Omega \setminus U. \end{cases}$$

Here ψ represents the solution of

$$\begin{cases} \Delta\psi = 0 & \text{in} \quad U \\ \psi = \tilde{\phi} & \text{on} \quad \Gamma = \partial U, \end{cases}$$

and η the solution of

$$\begin{cases} -\epsilon^2\,\Delta\eta + \eta = 1 & \text{in} \quad U \\ \eta = |u| & \text{on} \quad \Gamma = \partial U. \end{cases}$$

Arguing as in [1], (Theorem 2) we may assert by (IV.31) that

$$\int_U |\nabla \psi|^2 \leqslant C\left[\rho\, I_\rho(u, A)\right] + o(1),$$

$$\int_U |\nabla \eta|^2 \leqslant C\, \epsilon^{1/2}\left[\rho\, I_\rho(u, A)\right] + o(1),$$

$$\frac{1}{\epsilon^2}\int_U (1 - \eta)^2 \leqslant C\, \epsilon^{1/2}\left[\rho\, I_\rho(u, A)\right] + o(1),$$

and hence

$$\frac{1}{2}\int_U |\nabla v|^2 + \frac{1}{4\,\epsilon^2}\int_U (1 - |v|^2)^2 \leqslant C\left[\rho\, I_\rho(u, A)\right] + o(1) \leqslant C. \quad \text{(IV.32)}$$

Combining this with (IV.30) and proposition III.1 we deduce, from the definition of (v, B) that

$$G(u, A, U) \leqslant G(v, B, U) \leqslant C$$

This complete the proof of (IV.27).

We deduce from (IV.27) that

$$\int_{\rho^\gamma}^\rho I_r(u, A)\, dr \leqslant C$$

and hence, arguing as in the proof of proposition III.2, we deduce that there is some $r_0 \in [\rho^\gamma, \rho]$ such that

$$I_{r_0}(u, A) \leqslant \frac{C}{r_0\, |log\, r_0|^{\frac{1}{2}}},$$

where C does not depend on ϵ. Arguing as in the proof of (IV.27) we might now assert that

$$G(u, A, B(x_0, r_0) \cap \Omega) \leqslant C\left([r_0\, I_{r_0}(u, A)] + o(1)\right)$$
$$\leqslant \frac{C}{|log\, r_0|^{\frac{1}{2}}} + o(1) \leqslant \frac{C}{|log\, \epsilon|^{\frac{1}{2}}} + o(1).$$

In particular

$$\frac{1}{\epsilon^2}\int_{\Omega \cap B(x_0, \rho^\gamma)} (1 - |u|^2)^2 \leqslant \frac{C}{|log\, \epsilon|^{\frac{1}{2}}} + o(1) \quad \longrightarrow 0, \quad \text{as } \epsilon \to 0, \quad \text{(IV.33)}$$

and the conclusion of Lemma IV.1 follows from proposition IV.1. ∎

Proof of theorem IV.1. – Let $(x_i)_{i \in I}$ be a finite family of points in Ω, as in the proof of proposition IV.1, satisfying

$$x_i \in \Omega$$
$$B\left(x_i, \frac{\lambda_0 \, \epsilon}{4}\right) \bigcap B\left(x_j, \frac{\lambda_0 \, \epsilon}{4}\right) = \emptyset \qquad \forall \, i \neq j \tag{IV.34}$$
$$\Omega \subset \bigcup_{i \in I} B(x_i, \lambda_0 \, \epsilon).$$

We say that $B(x_i, \lambda_0 \, \epsilon)$ is a bad disc (in this proof) if there is some $y_i \in B(x_i, \lambda_0 \, \epsilon)$ such that

$$|u(y_i)| < \frac{1}{2}. \tag{IV.35}$$

We denote by J the set of indices for bad discs. In order to prove theorem IV.1, it suffices to show that

$$\# J < C, \tag{IV.36}$$

where C is some constant depending only on g, d, and Ω. To that aim, let $0 < \alpha < 1$ be fixed, and consider the balls $B(y_i, \epsilon^\alpha)$, for $i \in J$ (where y_i is one of the points verifying (IV.35) in the bad ball $B(x_i, \lambda_0\epsilon)$). Clearly, by (IV.35) and proposition IV.3, we have

$$\int_{B(y_i, \epsilon^\alpha) \cap \Omega} |\nabla_A u|^2 \geqslant C_\alpha \, |log \, \epsilon|. \tag{IV.37}$$

Set

$$W = \bigcup_{i \in J} B(x_i, 2 \, \epsilon^\alpha) \cap \Omega.$$

We claim that

$$\int_W |\nabla_A u|^2 \geqslant C \sum_{i \in J} \int_{B(x_i, 2 \, \epsilon^\alpha) \cap \Omega} |\nabla_A u|^2$$
$$\geqslant C \, (\# J) \, |log \, \epsilon| \tag{IV.38}$$

where C is a constant depending only on α, g, d and Ω.

Proof of (IV.38). – It suffices to bound (independently of ϵ) the number of balls $B(x_i, 2\epsilon^\alpha)$ to which any point y in W may belong. Let $y \in W$ and consider the subset K_y of indices of J such that

$$K_y = \{\, i \in J, \quad y \in B(y_i, 2\epsilon^\alpha)\,\}.$$

It follows that, for i in K_y we have (provided ϵ is sufficiently small)

$$x_i \in B(y, 2\epsilon^\alpha).$$

On the other hand, we deduce, from proposition IV.2, that the number of such x_i is bounded by a constant depending just on α, g, d and Ω, hence

$$\# K_y < C,$$

and this yields (IV.38). Combining (IV.38) with proposition II.5 we deduce (IV.36) and complete the proof of the theorem.   ∎

5. GLOBAL ESTIMATES

At this point we had only a local estimate for the magnetic field. Thanks to theorem IV.1, we will be able to derive an equation for h_ϵ which leads directly to global estimates. We are going to prove in this section.

THEOREM V.1. – *Given $1 < p < 2$ and $q \geqslant 1$ there are constants C_p (resp. C_q) depending only on d, g, Ω and p (resp. q) such that*

$$\int_\Omega |\nabla h_\epsilon|^p \leqslant C_p \qquad \forall p < 2, \tag{V.1}$$

$$\int_\Omega |h_\epsilon|^q \leqslant C_q \qquad \forall q < +\infty, \tag{V.2}$$

$$\int_\Omega |\nabla_{A_\epsilon} u_\epsilon|^p \leqslant C_p \qquad \forall p < 2. \tag{V.3}$$

∎

5.1. The equation for h

Recall that we have found (Theorem IV.1) N_ϵ points $x_1^\epsilon, ..., x_{N_\epsilon}^\epsilon$ in Ω such that

$$N_\epsilon \leqslant N$$

and

$$|u(x)| \geqslant \frac{1}{2} \qquad \text{on} \quad \Omega \setminus \bigcup_{i=1...N_\epsilon} B(x_i^\epsilon, \lambda_0 \epsilon).$$

Of course the limits as $\epsilon \to 0$ of the points x_i^ϵ are the good candidates for being the vortices of u_*. In theorem IV.1 we have not excluded the

possibility that some of the discs $B(x_i^\epsilon, \lambda_0\, \epsilon)$ may intersect. To avoid this unpleasant situation we will argue as in [2] (lemma X.3) and replace the discs $B(x_i^\epsilon, \lambda_0\, \epsilon)$ by slightly larger discs $B(x_i^\epsilon, \lambda\, \epsilon)$ for $i \in J^\epsilon$ a subset of $\{1, ..., l_\epsilon\}$ and where $\lambda > \lambda_0$ is another constant depending on g, Ω and such that

$$B(x_i^\epsilon, \lambda\, \epsilon) \cap B(x_i^\epsilon, \lambda\, \epsilon) = \emptyset \qquad \forall\, i \neq j, \qquad\qquad \text{(V.4)}$$

$$|u(x)| \geqslant \frac{1}{2} \qquad \text{if} \quad \bigcup_{i \in J^\epsilon} B(x_i^\epsilon, \lambda\, \epsilon). \qquad\qquad \text{(V.5)}$$

In this section also we drop the indice ϵ when no confusion is possible.

We set

$$\omega_i \,=\, B(x_i, \lambda\, \epsilon) \qquad \text{for} \quad i \in J$$

and

$$\Gamma_i = \partial B(x_i, \lambda\, \epsilon) \qquad \text{for} \quad i \notin J.$$

We consider the subset J_1 of J defined by

$$J_1 \,=\, \{\, i \in J,\, B(x_i, \lambda\, \epsilon) \cap \partial\Omega \,=\, \emptyset\,\}$$

and

$$J_2 \,=\, J \setminus J_1.$$

Set

$$\tilde{\Omega} \,=\, \Omega \setminus \bigcup_{i \in J} B(x_i, \lambda\, \epsilon),$$

$$\hat{\Omega} \,=\, \Omega \setminus \bigcup_{i \in J} B(x_i, 2\, \lambda\, \epsilon),$$

$$\Omega_1 \,=\, \Omega \setminus \bigcup_{i \in J_2} B(x_i, \lambda\, \epsilon),$$

and

$$\Gamma_0 \,=\, \partial\Omega_1.$$

PROPOSITION V.1. – *We have on* $\tilde{\Omega}$

$$\star d\left(i\frac{u_\epsilon}{|u_\epsilon|^2},\, d_{A_\epsilon} u_\epsilon \right) \,=\, -\, h_\epsilon. \qquad\qquad \text{(V.6)}$$

∎

(Recall that $d_A\, u \,=\, du \,-\, iA\, u$, where A is considered as a form.)

Proof. – Since $|u| \geqslant 1/2$ on $\tilde{\Omega}$ and since (V.6) is a local property it suffices to verify it locally. We may write

$$u = |u| e^{i\phi} \qquad \text{locally on} \quad \tilde{\Omega},$$

and hence

$$(iu, d_A u) = |u|^2 (d\phi - A).$$

Thus

$$d\left(i\frac{u}{|u|^2}, d_A u \right) = -dA = - \star h.$$

PROPOSITION V.2. – *Set $\rho_\epsilon = |u_\epsilon|$. The magnetic field h_ϵ satisfies*

$$- div\left(\frac{1}{\rho_\epsilon^2} \nabla h_\epsilon \right) + h_\epsilon = 0 \qquad in \quad \tilde{\Omega}. \tag{V.7}$$

∎

Proof. – Recall the Ginzburg-Landau equation

$$- \star dh = (iu, d_A u).$$

Hence in $\tilde{\Omega}$ we may write

$$\star d\left(\star \frac{1}{\rho^2} dh \right) = - \star d\left(i\frac{u}{|u|^2}, d_A u \right) = h.$$

∎

Equation (V.7) is supplemented with the boundary condition

$$\frac{\partial h}{\partial \nu} = - g \qquad \text{on} \quad \partial\Omega. \tag{V.8}$$

This leads us to consider elliptic equations of divergence form with Neumann boundary condition. As in [2] we will use the method of Stampacchia [15] to derive L^p bounds for $|\nabla h|$ with $p < 2$. This is the purpose of the next section.

5.2. Auxiliary linear problems

In this section we consider a domain $\Omega \subset I\!\!R^2$, and a positive function $a : \Omega \to I\!\!R$, and a constant $1 > \gamma > 0$ such that

$$\gamma < a < \gamma^{-1}. \tag{V.9}$$

Let $f = (f_1, f_2)$ be a vector field in $L^q(\Omega)$, for some $q > 2$. We consider the solution $\xi \in H^1(\Omega)$ of the linear problem

$$\int_\Omega a \, \nabla\xi \, \nabla\phi + \xi \, \phi = \int_\Omega f \, \nabla\phi \quad , \quad \forall \phi \in H^1(\Omega). \qquad (V.10)$$

Clearly ξ is unique and can be found by minimization.

Testing with $\phi \equiv 1$, we obtain

$$\int_\Omega \xi = 0, \qquad (V.11)$$

and testing with $\phi = \xi$, we derive

$$\int_\Omega |\nabla\xi|^2 + \xi^2 \leqslant C \int_\Omega |f|^2, \qquad (V.12)$$

where C depends on α. Our main estimate is the following.

PROPOSITION V.3. – *For any $q > 2$ we have*

$$||\xi||_{L^\infty(\Omega)} \leqslant C_q \, ||f||_{L^q(\Omega)} \qquad (V.13)$$

where C_q is a constant depending only on α, Ω, and q. ∎

Proof. – The proof is due to G. Stampacchia [15] (in the case of Dirichlet boundary conditions). We closely follow his arguments (as in [2]).

To avoid problems with regularity we assume first that f and a are smooth and derive (V.13) in that case. The general case will then follow by density. Since f and a are smooth, ξ will be also.

For $\delta > 0$ consider the set

$$F(\delta) = \{ x \in \Omega \, , \, dist(x, \partial\Omega) < \delta \}$$

and

$$\Lambda(\delta) = \partial F(\delta) \setminus \partial\Omega.$$

Let $\pi : F(\delta) \to \partial\Omega$ be the nearest point projection onto $\partial\Omega$. There is some δ_0 such that for $\delta < \delta_0$, π is a smooth fibration. Consider the reflexion map

$$T : \; F(\delta_0) \longrightarrow \mathbb{R}^2 \setminus \Omega$$
$$x \longrightarrow T(x),$$

where $T(x)$ is such that $dist(T(x), \partial\Omega) = dist(x, \partial\Omega)$ and $x - T(x)$ is parallel to $x - \pi(x)$. Clearly T is smooth and

$$||\nabla T||_{L^\infty(F(\delta))} \leqslant C, \quad \forall \delta < \delta_0.$$

Next we consider the level sets of ξ. For $k > 0$ set

$$\Omega(k) = \{\, x \in \Omega, \quad \xi > k \,\},$$

$$\mu(k) = |\Omega(k)|,$$

and

$$V(k) = \partial\Omega(k).$$

We now divide the proof in several steps.

Step 1. – There is some constant D_1 depending only on Ω, γ, and $k_0 \in (\, D_1 \, |f|_{L^2}, \, 2\, D_1 \, |f|_{L^2} \,)$ such that $V(k_0)$ is smooth and

$$\mathcal{H}^1(V(k_0)) \leqslant \frac{1}{4}\, \delta_0. \tag{V.14}$$

Where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure on $I\!\!R^2$.

Proof. – By the coarea formula, for any $D_1 > 0$

$$\int_{D_1 \, |f|_{L^2}}^{2\, D_1 \, |f|_{L^2}} \mathcal{H}^1(V(k))\, dk \leqslant \int_\Omega |\nabla \xi|$$

$$\leqslant C \left(\int_\Omega |\nabla \xi|^2 \right)^{\frac{1}{2}} \leqslant C \, |f|_{L^2}.$$

We deduce that there is some $k_0 \in (\, D_1 \, |f|_{L^2}, \, 2\, D_1 \, |f|_{L^2} \,)$ such that $V(k_0)$ is smooth (by Sard's theorem, since ξ is smooth) and

$$\mathcal{H}^1(V(k_0)) \leqslant C/D_1.$$

Step 2. – There is a constant δ depending only on Ω such that for $k > k_0$, the following inequality holds

$$\gamma\, \delta\, \|(\xi - k)^+\|_{L^2(\Omega)} \leqslant \|\nabla(\xi - k)^+\|_{L^1(\Omega)} \tag{V.15}$$

Proof. – We consider the connected components of $V(k_0)$. And the maximal curves $V^1(k_0), ..., V^p(k_0)$. We divide these components in two sets

$$K_1 = \{\, i \in \{1, ..., p\} \; V^i(k_0) \cap \partial\Omega = \emptyset \,\}$$

$$K_2 = \{1, ..., p\} \setminus K_1.$$

Let W_i be the domains bounded by the curves V^i. Clearly, for $k > k_0$, we have

$$\Omega(k) \subset \Omega(k_0) \subset \cup_{i=1}^p W_i.$$

For $i \in K_1$ we easily verify that

$$(\xi - k)^+ = 0 \quad \text{on} \quad \partial W_i \quad \text{for } i \in K_1 \quad k > k_0.$$

Hence by Sobolev embedding we deduce that

$$
\begin{gathered}
\text{for } i \in K_1 \quad \text{and} \quad k > k_0, \\
\alpha\,\delta_0\,\|(\xi - k)^+\|_{L^2(W_i)} \leqslant \|\nabla(\xi - k)^+\|_{L^1(W_i)},
\end{gathered}
\tag{V.16}
$$

where δ_0 is an absolute constant.

Next we consider K_2. We deduce from step 1 that there is some δ_1 in $(\delta_0/2, \delta_0)$ such that

$$V(k_0) \cap \Lambda(\delta_1) = \emptyset. \tag{V.17}$$

It follows that, for $i \in K_2$,

$$W_i \subset F(\delta_1) \subset F(\delta_0). \tag{V.18}$$

Therefore for $i \in K_2$, we may consider the set

$$\tilde{W}_i = W_i \cup T(W_i),$$

and the function $\tilde{\xi}$ defined on $\tilde{W}_i$ by

$$\tilde{\xi}(x) = \xi(x) \quad \text{if} \quad x \in W_i$$

$$\tilde{\xi}(x) = \xi(T^{-1}(x)) \quad \text{if} \quad x \in \tilde{W}_i.$$

We verify that $\tilde{\xi}$ is lipschitz and that $(\tilde{\xi} - k)^+ = 0$ on $\partial \tilde{W}_i$. Therefore

$$
\begin{gathered}
\text{for } i \in K_2 \quad k > k_0, \\
\alpha\,\delta_0\,\|(\tilde{\xi} - k)^+\|_{L^2(\tilde{W}_i)} \leqslant \|\nabla(\tilde{\xi} - k)^+\|_{L^1(\tilde{W}_i)}.
\end{gathered}
\tag{V.19}
$$

On the other hand, we have

$$\|(\tilde{\xi} - k)^+\|_{L^2(\tilde{W}_i)} \geqslant \|(\xi - k)^+\|_{L^2(W_i)} \tag{V.20}$$

and

$$\|\nabla(\tilde{\xi} - k)^+\|_{L^1(\tilde{W}_i)} \leqslant C\,\|\nabla(\xi - k)^+\|_{L^1(W_i)}. \tag{V.21}$$

Combining (V.19), (V.20), (V.21) we obtain

$$
\begin{gathered}
\text{for } i \in K_2 \quad k > k_0, \\
C\,\alpha\,\delta_0\,\|(\xi - k)^+\|_{L^2(W_i)} \leqslant \|\nabla(\xi - k)^+\|_{L^1(W_i)}.
\end{gathered}
\tag{V.22}
$$

Finally combining (V.16) and (V.22) we obtain (V.15).

Step 3. – We complete the proof of proposition V.3 as in Stampacchia [15]. Applying (V.10) with the test function $\phi = (\xi - k)^+$ we obtain

$$\gamma \int_\Omega |\nabla(\xi - k)^+|^2 + \xi(\xi - k)^+ \leqslant \int_\Omega |f.\nabla(\xi - k)^+|$$

which yields, by Hölder inequality, (note that $\xi.(\xi - k)^+ \geqslant 0$)

$$\gamma \|\nabla(\xi - k)^+\|_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega_k)} \leqslant \mu(k)^{\frac{1}{2} - \frac{1}{q}} \|f\|_{L^q(\Omega)}.$$

Hence

$$\|\nabla(\xi - k)^+\|_{L^1(\Omega)} \leqslant \frac{1}{\gamma} \mu(k)^{1 - \frac{1}{q}} \|f\|_{L^q(\Omega)}$$

By the result of step 2, we obtain for $k > k_0$

$$\|(\xi - k)^+\|_{L^1(\Omega)} \leqslant \frac{1}{\alpha \delta} \mu(k)^\nu \|f\|_{L^q(\Omega)} \qquad (\text{V.23})$$

where $\nu = \dfrac{3}{2} - \dfrac{1}{q}$. We have

$$\|(\xi - k)^+\|_{L^1(\Omega)} = -\int_k^{+\infty} (t - k) \, d\mu(t) \equiv H(k),$$

and we are led to the differential inequality

$$H'(k) = -\mu(k) \leqslant -\left(\frac{H(k)}{\beta}\right)^{\frac{1}{\nu}}$$

where

$$\beta = \frac{1}{\gamma \delta} \|f\|_{L^q}.$$

integrating, we obtain

$$H(k) = 0 \qquad \text{for} \quad k \geqslant \beta^{\frac{1}{\nu}} \frac{\nu}{\nu - 1} H(k_0)^{1 - \frac{1}{\nu}} + k_0.$$

On the other hand, by (V.23), we have

$$H(k_0) = \|(\xi - k_0)^+\|_{L^1(\Omega)} \leqslant \beta |\Omega|^\nu$$

Hence

$$H(k) = 0 \qquad \text{for} \quad k \geqslant \beta \frac{\nu}{\nu - 1} |\Omega|^{\nu - 1} + k_0,$$

Which shows that

$$\|\xi^+\|_{L^\infty} \leqslant \frac{1}{\gamma\,\delta}\,\frac{\nu}{\nu-1}\,|\Omega|^{\frac{1}{2}-\frac{1}{q}}\,\|f\|_{L^q} + 2\,D_1\,\|f\|_{L^2} \leqslant C\,|\Omega|^{\frac{1}{2}-\frac{1}{q}}\,\|f\|_{L^q}$$

This proves proposition V.3, since a similar estimate holds for ξ^-.   ■

5.3. Proof of theorem V.I

We divide the proof in several steps.

Step 1. – Proof of (V.1). Let η be a smooth function defined on Ω such that

$$\eta \equiv 1 \quad \text{on} \quad \hat{\Omega} = \Omega \setminus \bigcup_{i\in J} B(x_i, 2\,\lambda\,\epsilon), \qquad (\text{V.24})$$

$$\eta \equiv 0 \quad \text{on} \quad \bigcup_{i\in J} B(x_i, \lambda\,\epsilon) \cap \Omega, \qquad (\text{V.25})$$

$$|\nabla\eta| \leqslant \frac{C}{\epsilon}, \qquad (\text{V.26})$$

where C is a constant depending on Ω. Let a be the function defined on Ω by

$$\begin{cases} a = \dfrac{1}{\rho^2} & \text{on} \quad \tilde{\Omega} = \Omega \setminus \bigcup_{i\in J} B(x_i, \lambda\,\epsilon) \\ a = 1 & \text{on} \quad B(x_i, \lambda\,\epsilon) \cap \Omega. \end{cases} \qquad (\text{V.27})$$

Thus we have $1 \leqslant a \leqslant 4$. Let q be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and f be any function in $L^q(\Omega)$ such that

$$f \equiv 0 \quad \text{on} \quad \Omega \cap B(x_i, 2\,\lambda\,\epsilon). \qquad (\text{V.28})$$

Consider the solution ξ to (V.10). In (V.10) we use the test function $\phi = \eta\,h$. This yields

$$\int_\Omega a\,\nabla\xi\,\nabla(\eta\,h) + \xi\,\eta\,h = \int_\Omega f\,\nabla(\eta\,h) = \int_\Omega f\,\nabla h, \qquad (\text{V.29})$$

where we have used (V.24) and (V.28). We have

$$\int_\Omega a\,\nabla\xi\,\nabla(\eta\,h) = \int_\Omega a\,\eta\,\nabla\xi\,\nabla h + \int_\Omega a\,h\,\nabla\xi\,\nabla\eta, \qquad (\text{V.30})$$

and

$$\begin{aligned} \int_\Omega a\,\eta\,\nabla\xi\,\nabla h &= \int_\Omega \frac{1}{\rho^2}\,\eta\nabla\xi\,\nabla h \\ &= \int_\Omega \frac{1}{\rho^2}\,\nabla(\xi\,\eta)\,\nabla h - \int_\Omega \frac{1}{\rho^2}\,\xi\,\nabla\eta\,\nabla h. \end{aligned} \qquad (\text{V.31})$$

Recall that

$$|\nabla h| \leqslant \frac{C}{\epsilon}. \qquad (V.32)$$

Hence

$$\left| \int_\Omega \frac{1}{\rho^2} \xi \, \nabla \eta \, \nabla h \right| \leqslant C \sum_{i \in J} \left(\int_{B(x_i, 2\lambda\epsilon)} \frac{1}{\epsilon^2} \right) ||\xi||_{L^\infty(\Omega)}$$
$$\leqslant C \, ||\xi||_{L^\infty(\Omega)}. \qquad (V.33)$$

By proposition V.2, we may write

$$\int_\Omega \frac{1}{\rho^2} \nabla(\xi\eta) \, \nabla h \; + \; \xi\eta h = \int_{\partial\Omega} \frac{1}{\rho^2} \frac{\partial h}{\partial \nu} (\xi\eta)$$
$$= - \int_{\partial\Omega} \frac{1}{\rho^2} \, g \, (\xi\eta).$$

Hence

$$\left| \int_\Omega \frac{1}{\rho^2} \nabla(\xi\eta) \, \nabla h \; + \; \xi\eta h \right| \leqslant C \, ||\xi||_{L^\infty(\Omega)}. \qquad (V.34)$$

Finally, it remains to estimate

$$\int_\Omega a \, h \, \nabla\xi \, \nabla\eta \; = \; \sum_{i \in J} \int_{B(x_i, 2\lambda\epsilon)} a \, h \, \nabla\xi \, \nabla\eta.$$

Set $h_i = h(x_i)$. We have by (V.32) (proposition II.6),

$$|h - h_i| \leqslant C \qquad \text{on} \quad B(x_i, 2\lambda\epsilon).$$

Hence, using (V.12)

$$\left| \int_{B(x_i, 2\lambda\epsilon)} a \, (h - h_i) \, \nabla\xi \, \nabla\eta \right| \leqslant C \int_{B(x_i, 2\lambda\epsilon)} |\nabla\xi| \, |\nabla\eta|$$
$$\leqslant C \left(\int_{B(x_i, 2\lambda\epsilon)} |\nabla\xi|^2 \right)^{\frac{1}{2}}$$
$$\leqslant C \, ||f||_{L^2(\Omega)}. \qquad (V.35)$$

On the other hand

$$\int_{B(x_i, 2\lambda\epsilon)} a \, h_i \, \nabla\xi \, \nabla\eta \; = \; \int_\Omega a \, \nabla\xi \, \nabla(\eta_i - 1) \, h_i,$$

where η_i is the map

$$\eta_i \equiv \eta \qquad \text{on} \quad B(x_i, 2\lambda\epsilon)$$
$$\eta_i \equiv 1 \qquad \text{on} \quad \Omega \setminus B(x_i, 2\lambda\epsilon).$$

By the equation (V.10), satisfied by ξ we have

$$\int_\Omega a \, \nabla\xi \, \nabla((\eta_i - 1) \, h_i) = -\int_\Omega (\eta_i - 1) \, h_i \, \xi$$
$$= -\int_{B(x_i, 2\lambda\epsilon)} (\eta_i - 1) \, h_i \, \xi.$$

Therefore

$$\left| \int_\Omega a \, \nabla\xi \, \nabla((\eta_i - 1) \, h_i) \right| \leqslant \|\xi\|_{L^\infty} \left(\int_{B(x_i, 2\lambda\epsilon)} |h| + C \right)$$
$$\leqslant C \, \|\xi\|_{L^\infty}, \qquad \qquad \text{(V.36)}$$

where we have used the proposition III.1 (the local estimates). Finally, combining (V.29), (V.30), (V.31), (V.33), (V.34), (V.35), and (V.36), we obtain

$$\left| \int_\Omega f.\nabla h \right| \leqslant C \, (\|\xi\|_{L^\infty} + \|f\|_{L^2}),$$

and hence by proposition V.3

$$\left| \int_\Omega f.\nabla h \right| \leqslant C \, \|f\|_{L^q(\hat\Omega)}.$$

Therefore, since f was arbitrary, satisfying (V.28), we deduce by duality that

$$\|\nabla h\|_{L^p(\hat\Omega)} \leqslant C_p \qquad \forall p < 2. \qquad \text{(V.37)}$$

(Recall that $\hat\Omega = \Omega \setminus \bigcup_{i \in J} B(x_i, 2\lambda\epsilon)$). On the other hand, since $|\nabla h| \leqslant C/\epsilon$ we have

$$\sum_{i \in J} \int_{B(x_i, 2\lambda\epsilon)} |\nabla h|^2 \leqslant C. \qquad \text{(V.38)}$$

Combining (V.37), and (V.38) we prove (V.1).

Step 2. – Proof of (V.2). We have

$$\left| \int_\Omega h \right| = \left| \int_{\partial\Omega} A.\tau \right| = \left| 2\pi \, d - \int_{\partial\Omega} g \right| < C. \qquad \text{(V.39)}$$

Hence (V.2) follows from (V.1), (V.39) and Sobolev imbedding.

Step 3. – Proof of (V.3).
We have

$$- \star dh = (iu , d_A u)$$

and

$$(u , d_A u) = \frac{1}{2} d|u|^2 = \frac{1}{2} d\rho^2.$$

Therefore, we have, for $\rho \neq 0$,

$$|\nabla_A u|^2 \leqslant \frac{1}{\rho^2} \left(|\nabla h|^2 + |\nabla \rho|^2 \right).$$

It follows that

$$|\nabla_A u| \leqslant C \left(|\nabla h| + |\nabla \rho| \right) \quad \text{in} \quad \hat{\Omega} \tag{V.40}$$

and, in order to prove (V.3), it suffices to establish that

$$\int_\Omega |\nabla \rho|^p \leqslant C_p \qquad \forall p < 2, \tag{V.41}$$

where C_p depends only on p and Ω.

Proof of (V.41)
Set

$$\bar{\rho} = Max\left(\rho, 1 - \frac{1}{|log\epsilon|^2} \right),$$

and consider the set

$$K = \left\{ x \in \Omega , \rho \geqslant 1 - \frac{1}{1 - 1/log^2\epsilon} \right\}.$$

Clearly $\nabla \bar{\rho} = \nabla \rho$ on K and $\nabla \bar{\rho} = 0$ on $\Omega \setminus K$. Moreover since

$$\frac{1}{\epsilon^2} \int_K (1 - \rho)^2 \leqslant C |log\epsilon|,$$

we deduce

$$meas(\Omega \setminus K) \leqslant C \epsilon^2 |log\epsilon|^5.$$

Recall equation (II.6)

$$-\Delta \rho^2 = \frac{2}{\epsilon^2} \rho^2 (1 - \rho^2) - 2 |\nabla_A u|^2 \quad \text{in} \Omega, \tag{V.42}$$

and multiply it by $(1 - \bar{\rho})$, this yields

$$\int_K 2\,\rho\,|\nabla\rho|^2 \leqslant \int_\Omega 2\,(1 - \bar{\rho})|\nabla_A u|^2 \leqslant 2\,||1 - \bar{\rho}||_{L^\infty} \int_\Omega |\nabla_A u|^2$$
$$\leqslant 2\,||1 - \bar{\rho}||_{L^\infty}\ |log\epsilon|$$

Since $0 \leqslant 1 - \bar{\rho} \leqslant 1/|log\epsilon|^2$, we obtain

$$\int_K |\nabla\rho|^2 \leqslant \frac{C}{|log\epsilon|^2} \longrightarrow 0.$$

On the other hand, by Hölder's inequality we have

$$\int_{\Omega\backslash K} |\nabla\rho|^p \leqslant \left(\int_{\Omega\backslash K} |\nabla\rho|^2 \right)^{p/2} meas(\Omega\backslash K)^{1-p/2}$$
$$\leqslant C\ (|log\epsilon|)^{p/2}\ (\epsilon^2\,|log\epsilon|^5)1 - p/2.$$

Combining the two previous inequalities we get (V.41).   ■

6. CONVERGENCE OF $(u_{\epsilon_n}, A_{\epsilon_n})$

We may extract a subsequence $\epsilon_n \to 0$ ($n \to +\infty$) such that

$$\#J_{\epsilon_n} = const = N_1, \tag{VI.1}$$

$$x^{\epsilon_n}{}_i \to l_i \in \overline{\Omega} \qquad \forall\, i \in J. \tag{VI.2}$$

We cannot exclude, at this point, the possibility that some of the points $x^{\epsilon_n}{}_i$ converge to the same limit. We denote by $a_1, ..., a_{N_0}$ the collection of distinct limits l_i with $N_0 \leqslant N_1$.

For sake of simplicity we will use sometimes the notation $h_n = h_{\epsilon_n}$, $A_n = A_{\epsilon_n}$, $u_n = u_{\epsilon_n}$. Recall that the gauge (u_n, A_n) is choosen so that

$$\begin{cases} d^* A_n = 0 & \text{in} \quad \Omega \\ A_n.\nu = 0 & \text{on} \quad \partial\Omega,_c r \end{cases} \tag{VI.3}$$

and that $A_n = \left(-\xi_{n x_2}, \xi_{n x_1} \right)$ where ξ_n is a solution of

$$\begin{cases} \Delta\xi_n = h_n & \text{in} \quad \Omega \\ \xi_n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{VI.4}$$

We deduce from theorem V.1 and the L^p bound on ∇h , extracting a further subsequence if necessary, that there is some $h_* \in W^{1,p}(\Omega, I\!R)$ such that

$$h_n \rightharpoonup h_* \quad \text{weakly in } W^{1,p}(\Omega, I\!R) \quad \forall p < 2. \qquad \text{(VI.5)}$$

It follows that there is some A_* in $W^{2,p}(\Omega, I\!R^2)$ and ξ_* in $W^{3,p}(\Omega, I\!R)$ such that

$$A_n \rightharpoonup A_* \quad \text{weakly in } W^{2,p}(\Omega, I\!R^2) \quad \forall p < 2, \qquad \text{(VI.6)}$$

$$\xi_n \rightharpoonup \xi_* \quad \text{weakly in } W^{3,p}(\Omega, I\!R) \quad \forall p < 2. \qquad \text{(VI.7)}$$

Finally, we have by (V.3)

$$\int_\Omega |\nabla_{A_n} u_n|^p \leqslant C_p \qquad \forall p < 2, \qquad \text{(VI.8)}$$

which yields, in view of the following estimate, (which follows from (V.2)),

$$\int_\Omega |A_n|^q + \int_\Omega |\nabla A_n|^q \leqslant C_q \qquad \forall q < +\infty, \qquad \text{(VI.9)}$$

that

$$\int_\Omega |\nabla u_n|^p \leqslant C_p \qquad \forall p < 2. \qquad \text{(VI.10)}$$

Hence passing to a further subsequence if necessary, we may assume that there is some map $u_* \in W^{1,p}(\Omega, I\!R^2)$ such that

$$u_n \rightharpoonup u_* \quad \text{weakly in } W^{1,p}(\Omega, I\!R^2) \quad \forall p < 2. \qquad \text{(VI.11)}$$

Clearly since

$$\int_\Omega (1 - |u_n|^2)^2 \leqslant C \epsilon_n^2 |log\epsilon_n| \longrightarrow 0, \qquad \text{(VI.12)}$$

we obtain

$$|u_*| = 1 \quad \text{almost everywhere.} \qquad \text{(VI.13)}$$

PROPOSITION VI.1. – *Let K be a compact subset of $\overline{\Omega} \setminus \bigcup_{i=1}^{N_0} \{a_i\}$ we have*

$$\int_K |\nabla h_n|^2 + |\nabla u_n|^2 < C_K$$

where C is some constant, and moreover

$$h_n \to h_* \qquad \text{strongly in } H^1(K), \qquad (\text{VI}.14)$$

$$u_n \to u_* \qquad \text{strongly in } H^1(K), \qquad (\text{VI}.15)$$

$$\text{and} \quad \frac{1}{\epsilon_n^2} \int_K (1 - |u_n|^2)^2 \longrightarrow 0. \qquad (\text{VI}.16)$$

$\blacksquare$

Proof. – The proof is similar to the proof of theorem X.2 of [2]. Let η be a smooth function compactly supported in $\overline{\Omega} \setminus \bigcup_{i=1}^{N_0} \{a_i\}$ such that $\eta \equiv 1$ on K. Recall that on $\tilde{\Omega}_{\epsilon_n}$, h_n verifies (see proposition V.2)

$$- div\left(\frac{1}{\rho_n^2} \nabla h_n \right) + h_n = 0 \quad \text{in} \quad \tilde{\Omega}_{\epsilon_n}. \qquad (\text{VI}.17)$$

For n large enough, the support of η is contained in $\tilde{\Omega}_{\epsilon_n}$. Hence we may multiply (VI.17) by $\eta\, h_n$. Integrating we obtain

$$\int_K |\nabla h_n|^2 + h_n^2 \leqslant C \left| \int_K |\nabla \eta| \, |\nabla h_n| \, h_n \right| + \int_{\partial\Omega} \left| \frac{\partial h_n}{\partial \nu} \right| . |h_n|$$

$$\leqslant C \int_\Omega |\nabla h_n| \, |h_n| + \int_{\partial\Omega} |h_n|.$$

Since h_n is bounded in $W^{1,p}(\Omega)$ $(\forall p < 2)$ we have

$$\int_\Omega |\nabla h_n| \, |h_n| < C.$$

Likewise, by the trace theorem,

$$\int_{\partial\Omega} |h_n|^q \leqslant C_q \qquad \forall q < +\infty.$$

Hence

$$\int_K |\nabla h_n|^2 + |h_n|^2 \leqslant C.$$

Moreover, since $h_n \rightharpoonup h_*$ we deduce that

$$\int_K |\nabla h_*|^2 + h_*^2 \leqslant C. \qquad (\text{VI}.18)$$

Next we multiply (VI.17) by $\eta \, (h_n - h_*)$. We obtain

$$\int_\Omega \frac{\eta}{\rho_n^2} \, |\nabla h_n|^2 + h_n \, \eta \, (h_n - h_*) = \int_\Omega (h_* - h_n) \, \frac{1}{\rho_n^2} \, \nabla h_n \, \nabla \eta$$
$$+ \int_\Omega \frac{1}{\rho_n^2} \eta \, \nabla h_n \, \nabla h_*. \quad \text{(VI.19)}$$

Since $h_n \rightharpoonup h_*$ weakly in $H^1(K)$ we deduce by Sobolev imbedding that

$$\| \, h_n - h_* \, \|_{L^q(K)} \longrightarrow 0 \qquad \forall \, q < +\infty.$$

Hence

$$\int_\Omega \frac{1}{\rho_n^2} \, \eta \, |\nabla h_n|^2 \longrightarrow \int_\Omega \eta \, |\nabla h_*|^2.$$

Since $\rho \leqslant 1$, this implies, by lower semi-continuity that

$$\nabla h_n \longrightarrow \nabla h_* \quad \text{strongly in} \quad L^2(K)$$

and establishes (V.14).

We turn now to (VI.15) and (VI.16). We consider the equation for ρ_n (see (II.6)),

$$-\Delta \rho_n + \rho_n^{-3} \, |\nabla h_n|^2 = \frac{1}{\epsilon_n^2} \, \rho_n \, (1 - \rho_n^2) \quad \text{in} \quad \Omega_{\epsilon_n}. \quad \text{(VI.20)}$$

Multiplying (VI.20) by $\eta \, (1 - \rho_n)$ and integrating we obtain

$$\int_\Omega \eta \, (|\nabla \rho_n|^2 + \frac{1}{\epsilon_n^2} \rho_n (1 - \rho_n)^2 \, (1 + \rho_n))$$
$$\leqslant \int_{\Omega_{\epsilon_n}} \eta \, \frac{(1 - \rho_n)}{\rho_n^3} |\nabla h_n|^2$$
$$+ \int_\Omega |\nabla \eta \, \nabla \rho_n \, (1 - \rho_n)|. \quad \text{(VI.21)}$$

Since $\rho_n \to 1$ and is bounded in $W^{1,p}$ we deduce from (V.14) and dominated convergence that the right hand side on (VI.21) goes to zero. Hence

$$\int_K |\nabla \rho_n|^2 + \frac{1}{\epsilon_n} \, (1 - \rho_n^2)^2 \longrightarrow 0; \quad \text{(VI.22)}$$

this proves (VI.16). Moreover we have

$$|\nabla_{A_n} u_n|^2 = \frac{1}{\rho_n^2} \, (iu_n, \nabla_{A_n} u_n)^2 + |\nabla \rho_n|^2$$
$$= \frac{1}{\rho_n^2} |\nabla h_n|^2 + |\nabla \rho_n|^2.$$

Hence we deduce that

$$\nabla_{A_n} u_n \longrightarrow \nabla_{A_*} u_* \qquad \text{strongly in } L^2(K), \qquad (\text{VI}.23)$$

and (VI.15) follows from the strong convergence of A_n to A_* in L^∞ and (VI.23).

PROPOSITION VI.2. – *We have*

$$\frac{1}{\epsilon_n^2} \int_\Omega (1 - |u_n|^2)^2 \leqslant C \qquad (\text{VI}.24)$$

where C depends only on g, d and Ω. ∎

Proof. – By proposition VII.1 below (which of course does not rely on (VI.24)), for all i, a_i is not in $\partial\Omega$. Let $\mu > 0$, be small, such that

$$B(a_i, \mu) \cap B(a_j, \mu) = \emptyset \qquad i \neq j \quad \text{and} \quad B(a_i, \mu) \cap \partial\Omega = \emptyset.$$

In view of proposition VI.1 we may find, by Fubini's theorem, some $\mu_n \in (\mu, 2\mu)$ such that, for n large enough

$$\int_{\partial B(a_i, \mu_n)} \frac{1}{\epsilon_n^2} (1 - |u_n|^2)^2 + |\nabla_{A_n} u_n|^2 + |h_n|^2 \leqslant C(\mu) \qquad (\text{VI}.25)$$

where $C(\mu)$ depends only on μ, g, d, and Ω. Applying proposition II.4 to (u_n, A_n) on the set $B(a_i, \mu) \cap \Omega$ we deduce

$$\int_{B(a_i, \mu_n) \cap \Omega} \frac{1}{\epsilon_n^2} (1 - |u_n|^2)^2 \leqslant C + \int_{B(a_i, \mu_n) \cap \Omega} |h_n|^2 \leqslant C. \qquad (\text{VI}.26)$$

Hence (VI.24) follows from (VI.16) and (VI.26). ∎

7. PROPERTIES OF (u_*, A_*)

PROPOSITION VII.1. – *We have*

$$N_0 = d \qquad (\text{VII}.1)$$

$$a_i \notin \partial\Omega \qquad \forall i = 1, ..., d \qquad (\text{VII}.2)$$

$$d_i = deg(u_*, a_i) = 1 \qquad \forall i = 1, ..., d. \qquad (\text{VII}.3)$$

∎

Proof. – The argument is readily the same as in [2]. We briefly sketch it. As in [2], consider a domain Ω' such that $\Omega \subset\subset \Omega'$. We are going to extend first u_n to Ω'. Note that

$$u_n \times \frac{\partial u_n}{\partial \tau} = g + A_n . \tau$$

and hence, by (VI.6)

$$\int_{\partial \Omega} \left| \frac{\partial u_n}{\partial \tau} \right|^2 < C.$$

It follows that there is some map which extends u_n to Ω' (which will still be denoted by u_n), such that $u_n \in H^1(\Omega' \setminus \Omega, S^1)$ and

$$\int_{\Omega' \setminus \Omega} |\nabla u_n|^2 \leqslant C. \tag{VII.4}$$

Similarly, we extend u_* to Ω'. We first establish (VII.3).

Let $\delta > 0$, and consider the balls $B(a_i, \delta)$ and the domain $\Omega_\delta = \Omega' \setminus \bigcup_{i=1}^{N_0} B(a_i, \delta)$. by theorem I.8 of [2] we have

$$\int_{\Omega_\delta} |\nabla u_*|^2 \geqslant 2\pi \sum_{i=1}^{N_0} |d_i|^2 \log(1/\delta) - C$$

where C depends only on the points a_i, and g. Hence, from the strong convergence (VI.15), we deduce

$$\int_{\Omega_\delta} |\nabla u_n|^2 \geqslant 2\pi \sum_{i=1}^{N_0} |d_i|^2 \log(1/\delta) - C, \tag{VII.5}$$

for n sufficiently large depending on δ. On the other hand, it follows from a result of [5], that

$$\int_{B(a_i, \delta)} |\nabla u_n|^2 \geqslant 2\pi |d_i| \log(\delta/\epsilon_n) - C. \tag{VII.6}$$

Combining (VII.4), (VII.5) and (VII.6), we are led to

$$\int_\Omega |\nabla u_n|^2 \geqslant 2\pi \sum_{i=1}^{N_0} \left(|d_i|^2 \log(1/\delta) + |d_i| \log(\delta/\epsilon_n) \right) - C \tag{VII.7}$$

for any $\delta > 0$, and n sufficiently large (depending on ρ). We also have, by (VI.7)

$$\int_\Omega |\nabla u_n|^2 \ = \ \int_\Omega |\nabla u_n - iA_n u_n|^2 \ + \ R_n, \qquad (VII.8)$$

where $R_n < C$. Combining (VII.7), (VII.8) and proposition II.5 we obtain (VII.3). Property (VII.2) can be derived similarly adapting the arguments of [2], theorem VI.2.

PROPOSITION VII.2. – *The magnetic field h_* verifies the equation*

$$\begin{cases} -\Delta h_* + h_* \ = \ 2\pi \sum_{i=1}^d \delta_{a_i} & in \quad \Omega \\ \dfrac{\partial h_*}{\partial \nu} = -g & on \quad \partial\Omega. \end{cases} \qquad (VII.9)$$

∎

Proof. – Let K be some compact subset of $\Omega \setminus \bigcup_{i=1}^d \{a_i\}$. We easily verify that

$$-\Delta h_* + h_* \ = \ 0 \qquad on \quad K.$$

Indeed we may pass to the limit in equation V.6, thanks to the L^p bound for ∇h_n. Let $\delta > 0$, we have

$$\int_{\partial B(a_i,\delta)} \frac{1}{\rho_n{}^2} \frac{\partial h_n}{\partial \nu}$$

$$= - \int_{\partial B(a_i,\delta)} \frac{1}{\rho_n{}^2} \left(i u_n, \tau . \nabla_{A_n} u_n \right)$$

$$- \int_{\partial B(a_i,\delta)} \frac{1}{\rho_n{}^2} \left(i u_n, \tau . \nabla u_n \right) + \int_{\partial B(a_i,\delta)} A_n . \tau$$

$$= - 2\pi\, d_i \ + \int_{B(a_i,\delta)} h \ = \ - 2\pi\, d_i \ - \ O(\delta) \qquad (VII.10)$$

where ν is the exterior normal of $\partial B(a_i, \delta)$. By standard arguments, passing to the limit in (V.6) on the whole of Ω, this implies (VII.9). ∎

PROPOSITION VII.3. – *The map u_* is a harmonic map on $\Omega \setminus \bigcup_{k=1}^d \{a_k\}$.* i.e. *verifies*

$$-\Delta u_* \ = \ u_* |\nabla u_*|^2 \qquad on \quad \Omega \setminus \bigcup_{k=1...d} \{a_k\}. \qquad (VII.11)$$

Moreover we have

$$u_* = \prod_{k=1\ldots d} \frac{z - a_k}{|z - a_k|} e^{i\phi_*} \qquad\qquad (\text{VII}.12)$$

where ϕ_* verifies (17) and (18) (in the introduction). ∎

Proof. – Applying the operator d^* to the equation (II.2), we obtain

$$d^* \left[u_n \times d_{A_n} u_n \right] = 0 \qquad \text{in} \quad \mathcal{D}'(\Omega). \qquad\qquad (\text{VII}.13)$$

Passing to the limit in (VII.13), we are led to

$$d^* \left[u_* \times d_{A_*} u_* \right] = 0 \qquad \text{in} \quad \mathcal{D}'(\Omega). \qquad\qquad (\text{VII}.14)$$

Since $d^* A_* = 0$, and $|u_*| = 1$, we deduce that

$$d^* \left[u_* \times d u_* \right] = 0 \qquad \text{in} \quad \mathcal{D}'(\Omega). \qquad\qquad (\text{VII}.15)$$

Hence, we obtain (VII.11). Moreover, (VII.12) follows from (VII.11), the L^p bounds of u_* and the analysis of [2], section I.3. Finally combining the propositons of sections VI and VII we have proved theorem 3. ∎

8. RENORMALIZED ENERGIES

The aim of this section is to prove Theorem 4, *i.e.* to determine the configuration (a_i). For that purpose, we introduce an auxiliary problem and follow the strategy of [2], section VIII.

Consider d points $b_1, ..., b_d$ in Ω, and $\delta > 0$. Clearly for δ sufficiently small (say $\delta < \delta_0$, depending on the configuration (b_i))

$$B(b_i, \delta) \subset \Omega \qquad \forall i \in \{1, ..., d\}$$

$$B(b_i, \delta) \cap B(b_j, \delta) = \emptyset \qquad \text{for} \quad i \neq j.$$

Set $\Omega_\delta = \Omega \setminus \bigcup_{i=1}^{d} B(a_i, \delta)$ and consider now the space W_δ defined by

$$W_\delta = \left\{ \begin{array}{c} (u, A) \in H^1(\Omega_\delta, S^1) \times H^1(\Omega, \mathbb{R}^2), \text{ such that} \\ (iu, \tau.(\nabla_A u)) = g \quad \text{on} \quad \partial\Omega \\ deg(u, \partial B(b_i, \delta)) = 1 \end{array} \right\} \qquad (\text{VIII}.1)$$

and the functional E_δ defined on W_δ by

$$E_\delta(u, A) = \frac{1}{2} \int_{\Omega_\delta} |\nabla_A u|^2 + \frac{1}{2} \int_\Omega |dA|^2.$$

Set

$$\mu_\delta = \underset{(u,A)\in W_\delta}{Inf} E_\delta(u, A). \qquad \text{(VIII.2)}$$

PROPOSITION VIII.1. – *We have:* μ_δ *is achieved and for* $\delta < \delta_0$

$$\mu_\delta \leqslant \pi d \log\frac{1}{\delta} + C \qquad \text{(VIII.3)}$$

where C *depends only on the configuration* (b_i) *and* δ_0.   ∎

Proof. – The proof is similar to the proof of proposition II.5, therefore we omit it.   ∎

PROPOSITION VIII.2. – *We have, for* $\delta < \delta_0$, *and for a minimizer* (v_δ, B_δ) *of (VIII.2)*

$$\int_\Omega |dB_\delta|^2 \leqslant C, \qquad \text{(VIII.4)}$$

and

$$\int_{\Omega_\delta} |\nabla_{B_\delta} v_\delta|^2 \geqslant \pi d \log\frac{1}{\delta} - C, \qquad \text{(VIII.5)}$$

where C *depends only on* (b_i) *on* δ_0.   ∎

Proof. – Let (v_δ, B_δ) be a minimizer of (VIII.2) such that

$$\begin{cases} d^*B = 0 & \text{in} \quad \Omega \\ B.\nu = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(In this proof we omit the subscript δ for sake of simplicity). Hence we have

$$B = \star d\xi$$

where ξ solves

$$\begin{cases} \Delta\xi = h = \star dB & \text{in} \quad \Omega \\ \xi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We have

$$\int_{\Omega_\delta} |\nabla_B v|^2 = \int_{\Omega_\delta} |\nabla v|^2 + |\nabla\xi|^2 + 2[\xi, v] \qquad \text{(VIII.6)}$$

where we have used the notation $[\xi, v] = d\xi \wedge (iv, dv)$. We notice that

$$
\int_{\Omega_\delta} d\xi \wedge (iv, dv)
$$

$$
= -\int_{\Omega_\delta} \xi \, d(iv, dv) + \sum_{i=1}^{d} \int_{\partial B(b_i, \delta)} \xi \, (iv, v_\tau)
$$

$$
= \sum_{i=1}^{d} \int_{\partial B(b_i, \delta)} \xi \, (iv, v_\tau), \tag{VIII.7}
$$

where τ is the unit tangential vector to $\partial B(b_i, \delta)$ such that (ν, τ) is direct, ν is the exterior normal to $\partial B(b_i, \delta)$. Here we have used the fact that $d(iv, dv) = (idv \wedge dv) = 0$. Since $deg(v, \partial B(b_i, \delta)) = 1$, we may write

$$
v = e^{i(\theta + \psi)} \quad \text{on} \quad U_i \setminus B(b_i, \delta),
$$

where U_i is some fixed neighborhood of b_i, and where ψ is some (singled) valued function on $U_i \setminus B(b_i, \delta)$. We verify that, for $x \in U_i \setminus B(b_i, \delta)$

$$
|\nabla \psi|^2 \leqslant 2 \left(|\nabla v|^2 + \frac{1}{|x - b_i|^2} \right). \tag{VIII.8}
$$

Integrating by parts, we obtain

$$
\int_{\partial B(b_i, \delta)} \xi \, (iv, v_\tau) = 2\pi \overline{\xi}_i + \int_{\partial B(b_i, \delta)} \xi_\tau \, (\psi - \overline{\psi}_i) \tag{VIII.9}
$$

where $\overline{\xi}_i = \dfrac{1}{|\partial B(b_i, \delta)|} \displaystyle\int_{\partial B(b_i, \delta)} \xi$ and $\overline{\psi}_i = \dfrac{1}{|\partial B(b_i, \delta)|} \displaystyle\int_{\partial B(b_i, \delta)} \psi$. By Sobolev imbedding and elliptic estimates, we have for any $\alpha < 1$,

$$
|\xi|_{C^{0,\alpha}} \leqslant C_\alpha \, |\Delta \xi|_{L^2}. \tag{VIII.10}
$$

On the other hand

$$
\int_{B(b_i, \delta)} \frac{\partial \xi}{\partial x_j} = \int_{\partial B(b_i, \delta)} \xi \, e_j.\nu = \int_{\partial B(b_i, \delta)} (\xi - \xi(b_i)) \, e_j.\nu
$$

and by (VIII.10)

$$
\frac{1}{|B(b_i, \delta)|} \left| \int_{B(b_i, \delta)} \frac{\partial \xi}{\partial x_j} \right| \leqslant 2\pi \, C_\alpha \delta^{\alpha - 1} \, |\Delta \xi|_{L^2}. \tag{VIII.11}
$$

By the trace Theorem, we obtain

$$\int_{\partial B(b_i,\delta)} \left| \frac{\partial \xi}{\partial x_j} - \frac{1}{|B(b_i,\delta)|} \int_{B(b_i,\delta)} \frac{\partial \xi}{\partial x_j} \right|^2 \leqslant C\delta \int_{\Omega} |\nabla^2 \xi|^2$$

$$\leqslant C\delta \int_{\Omega} |\Delta \xi|^2. \quad \text{(VIII.12)}$$

Combining (VIII.11) and (VIII.12), we are led to

$$\int_{\partial B(b_i,\delta)} |\nabla \xi|^2 \leqslant C\left(\delta + \delta^{2\alpha-1}\right) \int_{\Omega} |\Delta \xi|^2. \quad \text{(VIII.13)}$$

We have also (by the trace Theorem and Sobolev imbedding)

$$\int_{\partial B(b_i,\delta)} |\psi - \overline{\psi}_i|^2 \leqslant C\delta \int_{U} |\nabla \psi|^2$$

$$\leqslant C\delta \left(\int_{\Omega_\delta} |\nabla v|^2 + 2\pi \, log(1/\delta) \right). \quad \text{(VIII.14)}$$

combining (VIII.9), (VIII.13) and (VIII.14), we conclude

$$\left| \int_{\partial B(b_i,\delta)} \xi_\tau \left(\psi - \overline{\psi}_i \right) \right|$$

$$\leqslant C\,\delta^\alpha \,|\Delta \xi|_{L^2(\Omega)} \left(|\nabla v|_{L^2(\Omega_\delta)} + (log(1/\delta))^{1/2} \right) \quad \text{(VIII.15)}$$

Turning to the first term on the r.h.s. of (VIII.6), we have by a result of [2], theorem I.8.

$$\int_{\Omega_\delta} |\nabla v|^2 \geqslant 2\pi d \, log 1/\delta \, - \, C. \quad \text{(VIII.16)}$$

Combining (VIII.6), (VIII.7), (VIII.9), (VIII.16), and proposition VIII.1, we obtain

$$\left| \int_{\Omega_\delta} |\Delta \xi|^2 + |\nabla \xi|^2 + 4\pi \sum_{i=1}^{d} \xi(b_i) + R(\delta) \right| \leqslant C \quad \text{(VIII.17)}$$

where

$$R(\delta) = 2 \sum_{i=1}^{d} \left[\left(\overline{\xi}_i - \xi(b_i) \right) + \int_{\partial B(b_i,\delta)} \xi_\tau \left(\psi - \overline{\psi}_i \right) \right]. \quad \text{(VIII.18)}$$

Combining (VIII.15) and (VIII.10) we verify that

$$|R(\delta)| \leqslant C \, \|\Delta\xi\|_{L^2(\Omega)} (\delta^\alpha \, |\nabla v|_{L^2(\Omega_\delta)} + \delta^{\alpha/2}) \qquad \text{(VIII.19)}$$

Since $\nabla v = \nabla_B v + iv\,B$ we obtain

$$\int_{\Omega_\delta} |\nabla v|^2 \leqslant 2\left[\int_{\Omega_\delta} |\nabla_B v|^2 + |B|^2\right]$$
$$\leqslant 2\left[\int_{\Omega_\delta} |\nabla_B v|^2 + |\nabla\xi|^2\right] \leqslant C\,log(1/\delta) \quad \text{(VIII.20)}$$

(by proposition VIII.1). Going back to (VIII.19) we deduce that

$$R(\delta) \longrightarrow 0 \quad \text{as} \quad \delta \to 0. \qquad \text{(VIII.21)}$$

Since, by standard estimates

$$\left|\sum_{i=1}^{d} \xi(b_i)\right| \leqslant C \, |\Delta\xi|_{L^2(\Omega)} \qquad \text{(VIII.22)}$$

(VIII.21), (VIII.22) and (VIII.117) yield

$$|\Delta\xi|_{L^2(\Omega)} \leqslant C.$$

(VIII.4) and (VIII.5) follow. ■

PROPOSITION VIII.3. – *Let ξ_* be the solution of (20). We have $\xi_\delta \to \xi_*$ in* $W^{2,2}(\Omega)$ *as $\delta \to 0$.* ■

Proof. – We have on Ω_δ, by minimality of (v_δ, B_δ)

$$\begin{cases} -\Delta^2\xi_\delta + \Delta\xi_\delta = 0 & \text{in} \quad \Omega \\ \dfrac{\partial}{\partial\nu}\Delta\xi_\delta = -g & \text{on} \quad \partial\Omega. \end{cases} \qquad \text{(VIII.23)}$$

On the other hand

$$\int_{\partial B(a_i,\delta)} \frac{\partial}{\partial\nu}\Delta\xi_\delta = -\int_{\partial B(a_i,\delta)} (iv_\delta, \tau.\nabla_B v_\delta) = -2\pi + o(1). \quad \text{(VIII.24)}$$

From proposition VIII.2 we know that

$$|\Delta\xi_\delta|_{L^2(\Omega)} + |\nabla\xi_\delta|_{L^2(\Omega)} \leqslant |\Delta\xi_\delta|_{L^2(\Omega)} \leqslant C$$

and hence (passing to a subsequence if necessary) we may assert that

$$\xi_\delta \rightharpoonup \overline{\xi} \quad \text{in} \quad W^{2,2}(\Omega) \quad \text{weakly.}$$

Passing to the limit in (VIII.23) and using (VIII.24), we see that $\overline{\xi}$ verifies (20) and hence $\overline{\xi} = \xi_*$ (and the full sequence converges by uniqueness of the limit). By minimality of (v_δ, B_δ) we have

$$E_\delta(v_\delta, B_\delta) \leqslant E_\delta(v_\delta, A_*) \tag{VIII.25}$$

that is

$$\int_{\Omega_\delta} |\nabla_{B_\delta} v_\delta|^2 + \int_\Omega |dB_\delta|^2 \leqslant \int_{\Omega_\delta} |\nabla_{A_*} v_\delta|^2 + \int_\Omega |dA_*|^2.$$

Arguing as for (VIII.6) we obtain

$$\int_{\Omega_\delta} |dB_\delta|^2 + |\nabla \xi_\delta|^2 + 2\,[\xi_\delta, v_\delta]$$

$$\leqslant \int_\Omega |dA_*|^2 + |\nabla \xi_*|^2 + 2\,[\xi_*, v_*].$$

Since $\xi_\delta \rightharpoonup \xi_*$ weakly in $W^{2,2}(\Omega)$, we obtain

$$\int_\Omega |dB_\delta|^2 \leqslant \int_\Omega |dA_*|^2 + o(1),$$

that is

$$\int_\Omega |\Delta \xi_\delta|^2 \leqslant \int_\Omega |d\xi_*|^2 + o(1).$$

That implies the strong convergence of $\Delta \xi_\delta$ to $\Delta \xi_*$ in L^2 and Proposition VIII.3 follows. ∎

PROPOSITION VIII.4. – *Let*

$$\overline{\mu}_\delta = Min \left\{ \begin{array}{c} \dfrac{1}{2} \displaystyle\int_{\Omega_\delta} |\nabla u|^2, \quad u \in H^1(\Omega_\delta, S^1), \; deg(u, \partial B(b_i, \delta)) = 1 \\[2mm] \dfrac{\partial u}{\partial \tau} = i\left(g + \dfrac{\partial \xi_*}{\partial \nu}\right) u \quad on \quad \partial \Omega \end{array} \right\}.$$

We have

$$\left| \frac{1}{2} \int_{\Omega_\delta} |\nabla v_\delta|^2 - \overline{\mu}_\delta \right| \longrightarrow 0 \; as \; \delta \to 0. \tag{VIII.26}$$

∎

Proof. – Since

$$\frac{\partial v_\delta}{\partial \tau} = i\left(g + \frac{\partial \xi_\delta}{\partial \nu}\right) v_\delta \quad \text{on} \quad \partial\Omega,$$

we see easily (by proposition VIII.3 and standard elliptic estimates) that

$$\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\delta|^2 \geqslant \overline{\mu}_\delta + o(1). \qquad (VIII.27)$$

Recall

$$R(\delta) \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

We deduce

$$E_\delta(v_\delta, B_\delta) = \frac{1}{2}\int_{\Omega_\delta} |\nabla v_\delta|^2 + \frac{1}{2}\int_{\Omega}\left(|\Delta \xi_*|^2 + |\nabla \xi_*|^2\right)$$

$$+ 2\pi \sum_{i=1}^{d} \xi_*(b_i) + o(1). \qquad (VIII.28)$$

Hence, in view of (VIII.28), we have uncoupled v and B, and roughly speaking, in order to minimize $E_\delta(v, B)$, it suffices to minimize $\overline{\mu}_\delta$, and this yields

$$\frac{1}{2}\int_{\Omega_\delta} |\nabla v_\delta|^2 \leqslant \overline{\mu}_\delta + o(1).$$

Proposition VIII.4 follows. ∎

PROPOSITION VIII.5. – *Let* $1 < p < 2$ *the map* v_δ *remains bounded in* $W^{1,p}$ *and* v_δ *converges to* v_* *strongly in* $W^{1,p}$, *and in* $C_{loc}^k\left(\Omega \setminus \bigcup_{k=1}^{d}\{a_k\}\right)$, *where* v_* *is defined on* Ω *by*

$$v_* = \prod_{k=1\ldots d} \frac{z - a_k}{|z - a_k|} e^{i\phi_*}$$

and where ϕ_* *is harmonic and verifies*

$$\frac{\partial \phi_*}{\partial \tau} = -\left(i \prod_{k=1\ldots d} \frac{z - a_k}{|z - a_k|}, \frac{\partial}{\partial \tau}\left(\frac{z - a_k}{|z - a_k|}\right)\right) + \frac{\partial \xi_*}{\partial \nu} + g.$$

∎

Proof. – Since v_δ takes its values into S^1 we have

$$d^*(iv_\delta, \star dv_\delta) = 0 \quad \text{in} \quad \Omega_\delta.$$

Let Φ be the solution of

$$\begin{cases} \Delta\Phi = 2\pi \sum_{i=1}^{d} \delta_{b_i} \\ \dfrac{\partial\Phi}{\partial\nu} = g + \dfrac{\partial\xi_\delta}{\partial\nu}. \end{cases} \tag{VIII.29}$$

We have

$$d^*((iv_\delta, \star dv_\delta) + d\Phi) = 0,$$

and

$$\int_{\partial B(b_i,\delta)} ((iv_\delta, \star dv_\delta) + d\Phi).\nu = -\int_{\partial B(b_i,\delta)} \left(iv_\delta, \frac{\partial v_\delta}{\partial\tau}\right) + \int_{\partial B(b_i,\delta)} \frac{\partial\Phi}{\partial\nu} = 0.$$

Hence, there is some $H_\delta \in H^1(\Omega_\delta, I\!R)$, such that

$$(iv_\delta, \star dv_\delta) = -d\Phi + d^*H_\delta.$$

It follows (see [2])

$$\int_{\Omega_\delta} |\nabla v_\delta|^2 = \int_{\Omega_\delta} |\nabla\Phi|^2 + \int_{\Omega_\delta} |\nabla H_\delta|^2 + o(1).$$

From [2], section I, we have

$$\left| \int_{\Omega_\delta} |\nabla\Phi|^2 - \overline{\mu}_\delta \right| \longrightarrow 0 \qquad \text{as} \quad \delta \to 0, \tag{VIII.30}$$

and from proposition VIII.4 we deduce that

$$\int_{\Omega_\delta} |\nabla H_\delta|^2 \longrightarrow 0.$$

Hence,

$$\| (iv_\delta, \star dv_\delta) + d\Phi \|_{L^2(\Omega_\delta)} \longrightarrow 0 \qquad \text{as} \quad \delta \to 0,$$

and proposition VIII.5 follows from the analysis of [2], section I.

PROPOSITION VIII.6. – *We have for any configuration* (b_i)

$$\mu_\delta(b_i) = W(b_i) + \pi d \log(1/\delta) + o(1) \qquad as \quad \delta \to 0 \qquad (VIII.31)$$

where W *is defined by (19), (20), (21) and (22).* ∎

Proof. – We deduce from propositions VIII.3, VIII.4, and VIII.5 that, by (VIII.28)

$$\mu_\delta(b_i) = \frac{1}{2} \int_{\Omega_\delta} |\nabla v_\delta|^2 + \int_{\Omega_\delta} |\Delta \xi_*|^2 + |\nabla \xi_*|^2$$
$$+ 2\pi \sum_{i=1}^d \xi_*(b_i) + o(1). \qquad (VIII.32)$$

Moreover equation (20), established in proposition VIII.3, easily implies

$$\frac{1}{2} \int_\Omega |\Delta \xi_*|^2 + |\nabla \xi_*|^2 = -\pi \sum_{i=1}^d \xi_*(b_i) + \frac{1}{2} \int_{\partial\Omega} \Delta \xi_* \frac{\partial \xi_*}{\partial \nu}. \qquad (VIII.33)$$

(VIII.26) and the analysis of [2], section I (see Theorem I.7), imply

$$\frac{1}{2} \int_{\Omega_\delta} |\nabla v_\delta|^2 = \pi d \log(1/\delta) + \omega(b, d, g) + O(\delta), \qquad (VIII.34)$$

where

$$\omega(b, d, g) = -\pi \sum_{i \neq j} \log|b_i - b_j| - \pi \sum_{i=1}^d R(b_i)$$
$$+ \frac{1}{2} \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu} \Phi,$$

Φ is a function verifying (VIII.29) and R denotes

$$R(x) = \Phi(x) - \sum_{i=1}^d \log|x - b_i|.$$

Combining (VIII.32), (VIII.33) and (VIII.34) we obtain the desired result.

Proof of theorem 4. – Set

$$I(\epsilon, \delta) = Min\left\{\frac{1}{2} \int_{B(\delta)} |\nabla v|^2 + \frac{1}{4\epsilon^2}(1 - |v|^2)^2 , \quad v \in H^1(B(\delta), \mathbb{R}^2) \atop v = e^{i\theta} \quad \text{on} \quad \partial B(\delta)\right\}.$$

Let (b_i) be a configuration of d points. As in [2] we may construct a comparison function $v_{\epsilon,\delta}$ in $H^1(\Omega)$ such that

$$v_{\epsilon,\delta} = v_\delta \qquad \text{on} \quad \Omega_\delta$$

and

$$\frac{1}{2}\int_{B(b_i,\delta)} |\nabla v_{\epsilon,\delta}|^2 + \frac{1}{4\epsilon^2}(1 - |v_{\epsilon,\delta}|^2)^2 \leqslant I(\epsilon,\delta) + C(\delta),$$

where

$$|C(\delta)| \longrightarrow 0 \qquad \text{as} \quad \delta \to 0.$$

It follows that

$$
\begin{aligned}
G_\epsilon(v_\epsilon, A_\epsilon) &\leqslant G_\epsilon(v_{\epsilon,\delta}, A_\epsilon) \\
&\leqslant E_\delta(v_\delta, A_*) + d\,I(\epsilon,\delta) + C(\delta) \\
&\leqslant \mu_\delta(b_i) + d\,I(\epsilon,\delta) + C(\delta) \\
&\leqslant W(b_i) + \pi d\log(1/\delta) + d\,I(\epsilon,\delta) + C(\delta), \quad \text{(VIII.35)}
\end{aligned}
$$

where we have used proposition VIII.6, and where $C(\delta) \to 0$ as $\delta \to 0$. On the other hand, arguing as in the proof of lemma VIII.2 of [2], we have

$$G_\epsilon(u_\epsilon, A_\epsilon) \geqslant d\,I(\epsilon,\delta) + W(a_i) + \pi d\log(1/\delta) + C'(\delta), \quad \text{(VIII.36)}$$

where $C'(\delta) \to 0$ as $\delta \to 0$. Combining (VIII.35) and (VIII.36) we obtain, letting $\delta \to 0$,

$$W(a_i) \leqslant W(b_i) \qquad \forall~ (b_i),$$

and this proves theorem 4.   ∎

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