Ann. I. H. Poincaré – AN 19, 1 (2002) 1–11

S0294-1449(01)00079-8/FLA

EXISTENCE RESULTS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH SMALL MEASURE DATA

Nathalie GRENON

Faculté des Sciences de Bourges, rue Gaston Berger B.P. 4043, 18028 Bourges Cedex, France
Received 13 June 2000, revised 5 March 2001

ABSTRACT. – We give a smallness condition on $|m|$, and $\|f\|_q$ for the existence of a solution for the model problem: $-\Delta_p u = f(x)|u|^\gamma + m\mu$ with $u = 0$ on $\partial\Omega$, where Ω is a bounded open set of $\mathbb{R}^N$, $f(x) \in L^q(\Omega)$, $q \geqslant 1$, $m \in \mathbb{R}$ and μ is a Radon measure with bounded variation on Ω such that $|\mu|(\Omega) = 1$.

RÉSUMÉ. – Nous donnons une condition suffisante sur $|m|$, et $\|f\|_q$ pour l'existence de solution au problème modèle : $-\Delta_p u = f(x)|u|^\gamma + m\mu$ avec $u = 0$ sur $\partial\Omega$, où Ω est un ouvert borné de $\mathbb{R}^N$, $f(x) \in L^q(\Omega)$, $q \geqslant 1$, $m \in \mathbb{R}$ et μ est une mesure de Radon à variation bornée sur Ω telle que $|\mu(\Omega)| = 1$.

1. Introduction and main results

The main goal of this paper is to prove, if the data are small enough, the existence of a solution for the model problem

$$\begin{cases} -\Delta_p u = f(x)|u|^\gamma + m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $N \geqslant 1$, Ω is a bounded open subset of $\mathbb{R}^N$, $-\Delta_p$ is the so called p-Laplace operator, $f(x) \in L^q(\Omega)$, $q \geqslant 1$, $\mu \in M_B(\Omega)$ (that is to say μ is a Radon measure with bounded variation in Ω) such that $|\mu|(\Omega) = 1$ and $m \in \mathbb{R}$.
In fact we study the more general problem

$$\begin{cases} -\text{div}(a(x, Du)) = h(x, u) + m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $u \mapsto -\text{div}(a(x, Du))$ is a monotone operator defined on $W_0^{1,p}(\Omega)$ with values in $W^{-1,p'}(\Omega)$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. We suppose more precisely that,

$$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Caratheodory function,} \tag{1.3}$$

E-mail address: Nathalie.Grenon@univ-Orleans.fr (N. Grenon).

that is to say $a(.,\xi)$ is measurable on Ω for every ξ in $\mathbb{R}^N$, and $a(x,.)$ is continuous on $\mathbb{R}^N$ for almost every x in Ω, that,

$$a(x,\xi)\xi \geqslant \alpha|\xi|^p, \tag{1.4}$$

for almost every x in Ω and for every ξ in $\mathbb{R}^N$, where $\alpha > 0$ is a constant, that,

$$|a(x,\xi)| \leqslant d\big(b(x)+|\xi|\big)^{p-1}, \tag{1.5}$$

for almost every x in Ω and every ξ in $\mathbb{R}^N$, where $d > 0$ is a constant and b is a nonnegative function in $L^p(\Omega)$, and that,

$$\big(a(x,\xi)-a(x,\xi')\big)(\xi-\xi') > 0, \tag{1.6}$$

for almost x in Ω, and for every ξ,ξ' in $\mathbb{R}^N$, $\xi \neq \xi'$. We also assume that,

$$h:\Omega\times\mathbb{R}\to\mathbb{R} \quad \text{is a Caratheodory function,} \tag{1.7}$$

that is to say $h(.,t)$ is measurable on Ω for every t in $\mathbb{R}$, and $h(x,.)$ is continuous on $\mathbb{R}$ for almost every x in Ω, and that,

$$\begin{cases} |h(x,t)| \leqslant f(x)|t|^\gamma, \\ \text{for some } 1 \leqslant \gamma < +\infty \text{ and some } f \in L^q(\Omega), \\ \text{where } 1 \leqslant q \leqslant +\infty, \end{cases} \tag{1.8}$$

for almost every x in Ω for every t in $\mathbb{R}$.
Observe that there is no sign assumption on $h(x,t)$, only the growth on t is considered.

We now recall some well known results about measures.
For every measure $\mu \in M_B(\Omega)$ there exists a unique pair of measures (μ_0,μ_s) such that $\mu = \mu_0 + \mu_s$ (see [5] and [10]) with μ_0 in $M_0(\Omega)$ (that is to say the set of all measures in $M_B(\Omega)$ which are absolutely continuous with respect to the p-capacity) and μ_s in $M_S(\Omega)$ (that is to say the set of all measures in $M_B(\Omega)$ which are singular with the p-capacity). In other words, μ_s is concentrated on a subset E of Ω with zero p-capacity, and μ_0 does not charge the set of zero p-capacity. Moreover it is equivalent for a measure to be in $M_0(\Omega)$ and to belong to $L^1(\Omega)+W^{-1,p'}(\Omega)$, that is to say every μ_0 can be written as $\mu_0 = f - \operatorname{div} g$ with $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$. In short, every $\mu \in M_B(\Omega)$ can be decomposed as follows,

$$\mu = f - \operatorname{div} g + \mu_s^+ - \mu_s^-$$

where $f \in L^1(\Omega)$, $g \in (L^{p'}(\Omega))^N$, μ_s^+, μ_s^- (the positive part and negative part of μ_s) are two nonnegative measures in $M_s(\Omega)$ which are concentrated on two disjoint subsets E^+ and E^- of zero p-capacity. Recall also (see [3,7,8]) that if u is a measurable function defined on Ω, which is finite almost everywhere, and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$ (where $T_k(u)$ is the truncate at level k), then there exists a measurable function $v:\Omega\to\mathbb{R}^N$ such that $DT_k(u) = v\chi_{\{|u|\leqslant k\}}$ almost everywhere in Ω, for every $k > 0$, which is unique up to almost everywhere equivalence. We define the gradient Du of u as this function v.

Let us recall the definition of a renormalized solution (see [7,8]).

DEFINITION 1.1. – *We suppose* (1.3)–(1.6), $p > 1$, $\mu \in M_B(\Omega)$. *We say that u is a renormalized solution of*

$$\begin{cases} -\mathrm{div}(a(x, Du)) = \mu & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{1.9}$$

if,

- *the function u is measurable and finite everywhere and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$,*
- *the gradient Du in the previous sense satisfies,*

$$|Du|^{p-1} \in L^q(\Omega), \quad \forall q,\ 1 \leqslant q < \frac{N}{N-1},$$

- *if w belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exists $k > 0$ and $w^{+\infty}, w^{-\infty} \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ with $r > N$ such that,*

$$w = w^{+\infty} \quad \textit{a.e. on the set } \{u > k\},$$

$$w = w^{-\infty} \quad \textit{a.e. on the set } \{u < -k\},$$

then,

$$\int_\Omega a(x, Du) Dw \, \mathrm{d}x = \int_\Omega w \, \mathrm{d}\mu_0 + \int_\Omega w^{+\infty} \, \mathrm{d}\mu_s^+ - \int_\Omega w^{-\infty} \, \mathrm{d}\mu_s^-. \tag{1.10}$$

In [8] the authors give equivalent definitions of renormalized solutions. When $\mu \in M_0(\Omega)$, this definition is equivalent to the definition of an entropy solution (see [3] and [5]).

Let us observe that when $p > N$, the renormalized solution is just a usual weak solution and belongs to some $C^{0,\alpha}(\Omega)$; therefore the notion of renormalized solution is not really needed. This is also the case for example in the linear case where $a(x,\xi) = A(x)\xi$ when the matrix A has smooth coefficients. However, when the coefficients are not smooth, a new notion is necessary even in the linear case in order to obtain both existence and uniqueness results (see [16]). Observe in particular that the test function w which is used in (1.10) actually depends on the solution u itself, and that in some sense $u = +\infty$ on the set where μ_s^+ is concentrated, while $u = -\infty$ on the set where μ_s^- is concentrated since the action of μ_s on the set where $|u| \leqslant k$ does not appear in (1.10). For more comments on the notion of renormalized solutions, see [8]. These equations have been widely studied. Especially in [1,2,11], the authors give a sufficient and necessary condition for the existence of a solution of equations closed to (1.2) in the case $p = 2$, but their method doesn't extend to $p \neq 2$. See also [15] for the case of an

eigenvalue problem. Let us also quote [4] in which the authors give counter examples to the existence for the equation of the type (1.2). Quasilinear equations have been studied with more regular data in [9,12,14] for instance. In these papers existence results are obtained assuming that the data are small enough relatively to a convenient norm.
The main result of this paper is the following,

THEOREM 1.1. – *Assume* (1.3)–(1.8), *let* $m \in \mathbb{R}$ *and* $\mu \in M_B(\Omega)$, *such that* $|\mu|(\Omega) = 1$, $1 \leqslant \gamma < +\infty$, $1 \leqslant q \leqslant +\infty$ *with* $q \neq 1$ *if* $N = p$ *and* $\gamma q' < \frac{(p-1)N}{N-p}$ *if* $N > p$. *Then there exists a renormalized solution of* (1.2)
(1) *if* $1 \leqslant \gamma < p-1$ (*thus* $p > 2$)

$$\textit{with no additionnal condition on} \|f\|_q, m;$$

(1) *if* $\gamma \geqslant p-1$ *then the condition is*

$$\|f\|_q |m|^{\frac{\gamma-p+1}{p-1}} \leqslant \frac{C}{|\Omega|^{\frac{1}{q'}+\frac{\gamma}{p-1}(-1+\frac{p}{N})}} \tag{1.11}$$

for some constant $C = C(N, p, \gamma)$.

Remarks. –

- First observe that when $p < N$, there exists some q with $1 \leqslant q \leqslant +\infty$ and some $\gamma \geqslant 1$ such that $\gamma q' < \frac{(p-1)N}{N-p}$ if and only if $p > \frac{2N}{N+1}$.
This is a restriction on the values of γ and q, which is natural. Indeed, in order to define a renormalized solution of (1.2), we need $h(x,u)$ to belong to $L^1(\Omega)$. But even if $h(x,u) \equiv 0$, the renormalized solution u of (1.2) belongs to $L^r(\Omega)$ for any r, $1 \leqslant r < \frac{(p-1)N}{N-p}$ and is not in general in $L^{\frac{(p-1)N}{N-p}}(\Omega)$. Consequently if $\gamma q' \geqslant \frac{(p-1)N}{N-p}$ we shall not have $h(x,u) \in L^1(\Omega)$.
- If $\gamma = p-1$ condition (1.11) reads

$$\|f\|_q \leqslant C|\Omega|^{\frac{1}{q}-\frac{p}{N}}$$

with no condition on m. Actually, if u solves

$$-\Delta_p u = f(x)|u|^{p-1} + m\mu,$$

then for any $c > 0$, $v = cu$ solves

$$-\Delta_p v = f(x)|v|^{p-1} + c^{p-1} m\mu.$$

That is to say, if there is a solution for m and μ given, then there is a solution for every $|m|$.
- If $\mu \geqslant 0$ and $h \geqslant 0$, then a solution of (1.2) is nonnegative. Indeed, we can use $w = -T_k(u^-)$ as test function in the equation satisfied by u and then (observe that $\mu_s^- = 0$ and $w^{+\infty} = 0$)

$$-\int_\Omega a(x, Du) DT_k(u^-)\,\mathrm{d}x = \int_\Omega h(x,u)\bigl(-T_k(u^-)\bigr)\,\mathrm{d}x + \int_\Omega -T_k(u^-)\,\mathrm{d}\mu_0 \leqslant 0,$$

from (1.4), we deduce that,

$$\alpha \| DT_k(u^-)\|_p \leqslant 0$$

for any $k > 0$, and then $u^- = 0$. It means that Theorem 1.1 gives conditions for the existence of a positive renormalized solution of

$$\begin{cases} -\Delta_p u = h(x,u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2. Estimates and preliminary lemmas

Recall the following estimates,

LEMMA 2.1. – *We suppose* (1.3)–(1.6), $\mu \in M_B(\Omega)$, *such that* $|\mu|(\Omega) = 1$, $m \in \mathbb{R}$ *and* $p > 1$. *Let u be a renormalized solution of*

$$\begin{cases} -\operatorname{div}(a(x, Du)) = m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then the following estimate holds

$$\|u\|_r \leqslant C|\Omega|^{\frac{1}{r}+\frac{1}{p-1}(-1+\frac{p}{N})}|m|^{\frac{1}{p-1}}, \tag{2.1}$$

for some positive constant $C = C(N, p, r)$ *and for any* $r \in [1, +\infty]$ *if* $p > N$, $r \in [1, +\infty)$ *if* $p = N$, *and* $r \in [1, \frac{N(p-1)}{N-p})$ *if* $p < N$.

This estimate is proven in [13] for instance, where explicit value for C is explicitely given in a more general context. It can also be proven by symmetrization techniques (see [17]). We have to specify that in [13], the right-hand side is in $L^1(\Omega)$, but the proof extends to $\mu \in M_B(\Omega)$ without difficulty.

COROLLARY 2.1. – *Assume* (1.3)–(1.8), $1 \leqslant \gamma < +\infty$, $1 \leqslant q \leqslant +\infty$. *If* $v \in L^{\gamma q'}(\Omega)$, $m \in \mathbb{R}$ *and* $\mu \in M_B(\Omega)$ *such that* $|\mu|(\Omega) = 1$, *if* $q \neq 1$ *when* $N = p$ *and if* $\gamma q' < \frac{(p-1)N}{N-p}$ (*thus* $p > \frac{2N}{N+1}$) *when* $N > p$, *and if u is a renormalized solution of*

$$\begin{cases} -\operatorname{div}(a(x, Du)) = h(x, v) + m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

then,

$$\|u\|_{\gamma q'} \leqslant A + B\|v\|_{\gamma q'}^{\frac{\gamma}{p-1}}$$

where

$$A = C|\Omega|^{\frac{1}{\gamma q'}+\frac{1}{p-1}(-1+\frac{p}{N})}|m|^{\frac{1}{p-1}}, \qquad B = C|\Omega|^{\frac{1}{\gamma q'}+\frac{1}{p-1}(-1+\frac{p}{N})}\|f\|_q^{\frac{1}{p-1}},$$

for some positive constant $C = C(N, p, \gamma)$.

Proof. – We have

$$\left(|h(x,v)+m\mu|(\Omega)\right)^{\frac{1}{p-1}} \leqslant \left(\|h(x,v)\|_1+|m|\right)^{\frac{1}{p-1}},$$

then from (1.8), and Hölder inequality,

$$\left(|h(x,v)+m\mu|(\Omega)\right)^{\frac{1}{p-1}} \leqslant \left(\|v\|_{\gamma q'}^{\gamma}\|f\|_q+|m|\right)^{\frac{1}{p-1}}$$

and then,

- if $\frac{1}{p-1}<1$,

$$\left(|h(x,v)+m\mu|(\Omega)\right)^{\frac{1}{p-1}} \leqslant \|f(x)\|_q^{\frac{1}{p-1}}\|v\|_{\gamma q'}^{\frac{\gamma}{p-1}}+|m|^{\frac{1}{p-1}},$$

- if $\frac{1}{p-1}\geqslant 1$,

$$\left(|h(x,v)+m\mu|(\Omega)\right)^{\frac{1}{p-1}} \leqslant 2^{\frac{2-p}{p-1}}\|f(x)\|_q^{\frac{1}{p-1}}\|v\|_{\gamma q'}^{\frac{\gamma}{p-1}}+2^{\frac{2-p}{p-1}}|m|^{\frac{1}{p-1}}$$

and we get the corollary from (2.1) with $r=\gamma q'$.

We now study the function, $\varphi:\mathbb{R}^+\to\mathbb{R}$ defined by,

$$\varphi(X)=A+BX^{\frac{\gamma}{p-1}}-X,$$

where $A,\ B\geqslant 0$.

- If $\gamma>p-1$, then, $\varphi(0)=A\geqslant 0$ and $\lim_{X\to+\infty}\varphi(X)=+\infty$, moreover, by calculation of the derivative, we get that φ has a minimum at the point,

$$X_0=\left(\frac{p-1}{B\gamma}\right)^{\frac{p-1}{\gamma-p+1}}$$

with

$$\varphi(X_0)=A+\frac{1}{\gamma^{\frac{\gamma}{\gamma-p+1}}}\frac{(p-1)^{\frac{p-1}{\gamma-p+1}}}{B^{\frac{p-1}{\gamma-p+1}}}(p-1-\gamma),$$

then φ has at least one root if and only if $\varphi(X_0)\leqslant 0$ that is to say if,

$$AB^{\frac{p-1}{\gamma-p+1}} \leqslant \frac{1}{\gamma^{\frac{\gamma}{\gamma-p+1}}}(p-1)^{\frac{p-1}{\gamma-p+1}}(\gamma+1-p), \tag{2.3}$$

and φ has two roots if,

$$AB^{\frac{p-1}{\gamma-p+1}} < \frac{1}{\gamma^{\frac{\gamma}{\gamma-p+1}}}(p-1)^{\frac{p-1}{\gamma-p+1}}(\gamma+1-p).$$

- If $\gamma = p - 1$, then,

$$\varphi(X) = (B-1)X + A,$$

then φ has a root if

$$B < 1, \quad \forall A \geqslant 0. \tag{2.4}$$

- If $\gamma < p - 1$, then,

$$\varphi(X) = A + BX^{\frac{\gamma}{p-1}} - X$$

and then,

$$\lim_{X \to +\infty} \varphi(X) = -\infty \quad \text{and} \quad \varphi(0) \geqslant 0,$$

then φ has a root for any $A, B \geqslant 0$.

We henceforth denote (when it exists),

$$Y\text{: the smallest root of } \varphi. \tag{2.5}$$

3. Proof of Theorem 1.1

First observe that,

- if $\gamma > p - 1$, condition (2.3) is equivalent to

$$|\Omega|^{\frac{1}{q'} + \frac{\gamma}{p-1}(-1+\frac{p}{N})} |m|^{\frac{\gamma-p+1}{p-1}} \|f(x)\|_q \leqslant C$$

for some constant $C = C(N, p, \gamma)$.

- if $\gamma = p - 1$, condition (2.4) is equivalent to

$$|\Omega|^{-\frac{1}{q}+\frac{p}{N}} \|f(x)\|_q \leqslant C$$

for some constant $C = C(N, p)$, and we recognize the condition which appear in the second case of Theorem 1.1.

We set

$$h_n(s) = T_n\big(h(s)\big),$$

where T_n is the truncate at level n.

LEMMA 3.1. – *We suppose* (1.3)–(1.8), *let* $\mu \in M_B(\Omega) \cap W^{-1,p'}(\Omega)$, *such that* $|\mu|(\Omega) = 1$ *and* $m \in \mathbb{R}$, *we suppose that* Y *defined by* (2.5) *exists, that is to say if the previous conditions are fulfilled. Then, for any* $\mu_n \in W^{-1,p'}(\Omega) \cap M_B(\Omega)$ *such that,* $|\mu_n|(\Omega) \leqslant m$ *there exists a solution* $u \in W_0^{1,p}(\Omega)$ *of the equation*:

$$\begin{cases} \displaystyle\int_\Omega a(x, Du)Dw \, dx = \int_\Omega h_n(x, u)w \, dx + \langle \mu_n, w \rangle \\ \forall w \in W_0^{1,p}(\Omega), \end{cases} \tag{3.1}$$

such that,

$$\|u\|_{\gamma q'} \leqslant Y,$$

where γ, q' satisfy the same conditions as in Corollary 2.1.

Proof. – We shall use Schauder Fixed Point Theorem.

Let $v \in W_0^{1,p}(\Omega)$ then $h_n(x,v) + \mu_n \in W^{-1,p'}(\Omega)$ and there exists a unique $u \in W_0^{1,p}(\Omega)$, such that,

$$\begin{cases} \int_\Omega a(x,Du)Dw\,\mathrm{d}x = \int_\Omega h_n(x,v)w\,\mathrm{d}x + \langle \mu_n, w\rangle \\ \forall w \in W_0^{1,p}(\Omega). \end{cases} \tag{3.2}$$

Moreover since $|h_n(v)| \leqslant n$, using u as test function we easily get

$$\|Du\|_p \leqslant C_n, \tag{3.3}$$

where C_n is a constant which depends on n but not on v.
Let $v \in W_0^{1,p}(\Omega)$, we henceforth set $A_n(v) = u$ the solution of (3.2).
Let $E = \{v \in W_0^{1,p}(\Omega) \cap L^{\gamma q'}(\Omega), \|Dv\|_p \leqslant C_n, \|v\|_{\gamma q'} \leqslant Y\}$, then,

- E is a closed convex subset of $W_0^{1,p}(\Omega)$.
- Observe that from definition of Y, if $v \in E$ then

  $$\|u\|_\gamma \leqslant A + B\|v\|_\gamma^{\frac{\gamma}{p-1}} \leqslant A + BY^{\frac{\gamma}{p-1}} = Y.$$

  Moreover we have already seen that

  $$\|Du\|_p \leqslant C_n$$

  then,

  $$A_n : E \to E.$$

- Suppose that (v_ε) is a sequence in E such that $v_\varepsilon \to v$ in $W_0^{1,p}(\Omega)$ strong and let $u_\varepsilon = A(v_\varepsilon)$. Since (v_ε) is bounded in $W_0^{1,p}(\Omega)$ there exists a subsequence still denoted (u_ε) such that,

  $$u_\varepsilon \to u \ L^p(\Omega) \text{ strong, a.e. in } \Omega \text{ and } W_0^{1,p}(\Omega) \text{ weak.}$$

  Using $(u_\varepsilon - u)$ as test function in (3.2) we get,

  $$\int_\Omega a(x,Du_\varepsilon)D(u_\varepsilon - u)\,\mathrm{d}x = \int_\Omega h_n(v_\varepsilon)(u_\varepsilon - u)\,\mathrm{d}x + \langle \mu_n, u_\varepsilon - u\rangle.$$

  We can easily see that the right-hand side tends to zero as ε tends to zero, then, since,

$$\int_\Omega \big(a(x, Du_\varepsilon) - a(x, Du)\big) D(u_\varepsilon - u)\,\mathrm{d}x$$
$$= \int_\Omega a(x, Du_\varepsilon) D(u_\varepsilon - u)\,\mathrm{d}x - \int_\Omega a(x, Du) D(u_\varepsilon - u)\,\mathrm{d}x$$

we have,

$$\lim_{\varepsilon\to 0} \int_\Omega \big(a(x, Du_\varepsilon) - a(x, Du)\big) D(u_\varepsilon - u)\,\mathrm{d}x = 0$$

from a lemma of [6] it implies that,

$$\lim_{\varepsilon\to 0} \|D(u_\varepsilon - u)\|_p = 0.$$

This implies that we can pass to the limit in the equation satisfied by u_ε, and we get $u = A(v)$. Consequently the whole sequence (u_ε) converges to u and finally it proves that A is continuous.

- With same arguments we can prove that $A(E)$ is precompact. Indeed if (u_ε) is a bounded sequence in $A(E)$ then $u_\varepsilon = A(v_\varepsilon)$ with (v_ε) or a subsequence is such that,

$$v_\varepsilon \to v \text{ a.e.} \quad \text{in } \Omega \text{ and } L^p(\Omega) \text{ strong}$$

and we deduce like previously that,

$$u_\varepsilon \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ strong.}$$

End of the proof of Theorem 1.1.
Let $\mu \in M_B(\Omega)$ such that $|\mu|(\Omega) = 1$ and $m \in \mathbb{R}$, then $m\mu$ can be decomposed as,

$$m\mu = f - \mathrm{div} g + \lambda^+ - \lambda^-.$$

Let (μ_n) a sequence of measures in $M_B(\Omega)$ such that,

$$\mu_n = f_n - \mathrm{div} g + \lambda_n^\oplus - \lambda_n^\ominus$$

with,

$$f_n \in L^{p'}(\Omega) \text{ and } (f_n) \text{ converges to } f \text{ weakly in } L^1(\Omega), \tag{3.4}$$

$$\begin{gathered}\lambda_n^\oplus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that} \\ \text{converges to } \mu_s^+ \text{ in the narrow topology of measures,}\end{gathered} \tag{3.5}$$

$$\begin{gathered}\lambda_n^\ominus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that} \\ \text{converges to } \mu_s^- \text{ in the narrow topology of measures,}\end{gathered} \tag{3.6}$$

$$|\mu_n|(\Omega) \leqslant m, \tag{3.7}$$

then there exists a solution u_n of the corresponding Eq. (3.1) which satisfies

$$\|u_n\|_{\gamma q'} \leqslant Y.$$

Observe that in (3.1) the right-hand side is bounded in $M_B(\Omega)$, then it is proven in [8] that we can extract a subsequence which converges in measure and a.e. in Ω to a measurable function u which is finite almost everywhere. Moreover since the right-hand side in (3.1) is bounded in $M_B(\Omega)$, from Lemma 2.1 we have, if $q \neq 1$, with a small δ

$$\|u_n\|_{\gamma q'+\delta} \leqslant C,$$

where C is a constant which does not depend on n. We deduce that $(u_n^{\gamma q'})$ converges to $(u^{\gamma q'})$ in $L^1(\Omega)$ strong (see [3]). Moreover, we have,

$$\big\| f(x)|u_n|^\gamma - f(x)|u|^\gamma \big\|_{L^1(\Omega)} \leqslant \|f\|_q \left(\int_\Omega \big(|u_n|^\gamma - |u|^\gamma\big)^{q'} \right)^{1/q'} \tag{3.8}$$

but, $(|u_n|^\gamma - |u|^\gamma)^{q'}$ tends to 0 a.e. in Ω and,

$$\big(|u_n|^\gamma - |u|^\gamma\big)^{q'} \leqslant 2^{q'-1}|u_n|^{\gamma q'} + 2^{q'-1}|u|^{\gamma q'}.$$

The right-hand side converges in $L^1(\Omega)$ strong. Then by Vitali Lemma and (3.8), we deduce that,

$$f(x)|u_n|^\gamma \text{ tends to } f(x)|u|^\gamma \text{ in } L^1(\Omega) \text{ strong.} \tag{3.9}$$

We assert again that $h_n(x,u_n)$ converges a.e. in Ω to $h(x,u)$ and by (1.8) and (3.9), we deduce that $h_n(x,u_n)$ converges to $h(x,u)$ in $L^1(\Omega)$ strong. The same conclusion holds when $q = 1$. So $f_n + h_n(x,u_n)$ converges in $L^1(\Omega)$ weak, and with the additionnal assumptions (3.5), (3.6) on $\lambda_n^\ominus$ and $\lambda_n^\oplus$ we can apply Theorem 3.2 of [8] and conclude that u is a renormalized solution of (3.1).

Acknowledgement

The author thanks Francois Murat for fruitful discussions in Bourges.

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