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LIMIT BEHAVIOUR OF THIN INSULATING LAYERS AROUND MULTICONNECTED DOMAINS

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ABSTRACT. – Let Ω be a bounded domain of $\mathbb{R}^N$, with boundary $\partial\Omega$. Let Γ_0 and Γ be connected components of $\partial\Omega$. We assume that Ω is surrounded along Γ_0 and Γ by thin insulating layers Σ_0^ε and Σ^ε of varying respective thicknesses $h_0^\varepsilon(s)$ and $h^\varepsilon(s)$, s being the generic point of Γ_0 and Γ. We denote by Γ_0^ε and Γ^ε the parts of $\partial\Sigma_0^\varepsilon$ and $\partial\Sigma^\varepsilon$ which do not meet Ω. We consider a class of quasilinear elliptic problems with different exponents (p in Ω, q_0 in Σ_0^ε, q in Σ^ε) and with the following boundary conditions:

- on Γ_0^ε, $u^\varepsilon = 0$,
- on Γ^ε, the total flux is prescribed and u^ε is constant, but unprescribed,
- on Γ_0 and Γ, the natural transmission conditions.

The restricted equations in Σ_0^ε and Σ^ε have nonconstant coefficients, μ_0^ε and μ^ε, in the form $\mu_0^\varepsilon(x) = \mu_0^\varepsilon(\sigma_0(x))$ (respectively $\mu^\varepsilon(x) = \mu^\varepsilon(\sigma(x))$), σ_0 and σ being the respective projections on Γ_0 and Γ. We predict the asymptotic behaviour of this problem as h_0^ε and μ_0^ε (respectively h^ε and μ^ε) tend to zero in a suitable sense, provided they are related in a convenient way.

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RÉSUMÉ. – Soit Ω un domaine borné de $\mathbb{R}^N$, de frontière $\partial\Omega$. Soient Γ_0 et Γ des composantes connexes de $\partial\Omega$. Nous supposons que Ω est entouré le long de Γ_0 et Γ par de fines couches isolantes Σ_0^ε et Σ^ε d'épaisseurs variables respectives $h_0^\varepsilon(s)$ et $h^\varepsilon(s)$, s étant le point générique de Γ_0 ou Γ. Nous désignons par Γ_0^ε et Γ^ε les parties de $\partial\Sigma_0^\varepsilon$ et $\partial\Sigma^\varepsilon$ qui ne bordent pas Ω. Nous considérons une classe de problèmes elliptiques quasilinéaires, avec des exposants de Sobolev différents (p dans Ω, q_0 dans Σ_0^ε, q dans Σ^ε) et avec les conditions au bord suivantes :

- sur Γ_0^ε, $u^\varepsilon = 0$,

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• sur Γ^ε, le flux total est prescrit et u^ε est constant, mais indéterminé,
• sur Γ_0 et Γ, les conditions de transmission naturelles.

Les équations restreintes à Σ_0^ε et Σ^ε ont des coefficients non-constants, μ_0^ε et μ^ε, de la forme $\mu_0^\varepsilon(x) = \mu_0^\varepsilon(\sigma_0(x))$ (resp. $\mu^\varepsilon(x) = \mu^\varepsilon(\sigma(x))$), σ_0 et σ étant les projections respectives sur Γ_0 et Γ. Nous prédisons le comportement asymptotique de ce problème, lorsque h_0^ε et μ_0^ε (resp. h^ε et μ^ε) tendent vers zéro simultanément, tout en vérifiant une relation de corrélation convenable.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ and let Γ_0 and Γ be connected components of the boundary $\partial\Omega$ of Ω. We assume that Ω is surrounded along Γ_0 and Γ by thin reinforcements

$$\Sigma_0^\varepsilon = \{s + tn(s), s \in \Gamma_0, 0 < t < h_0^\varepsilon(s)(\leqslant \varepsilon)\}, \tag{1.1}$$

$$\Sigma^\varepsilon = \{s + tn(s), s \in \Gamma, 0 < t < h^\varepsilon(s)(\leqslant \varepsilon)\}, \tag{1.2}$$

where $n(s)$ denotes the outer normal to Ω at the point s of Γ_0 or Γ. Then we denote by $\Omega^\varepsilon = \Omega \cup \Sigma_0^\varepsilon \cup \Sigma^\varepsilon \cup \Gamma_0 \cup \Gamma$ the reinforced domain and we define Γ_0^ε and Γ^ε as the parts of $\partial\Sigma_0^\varepsilon$ and $\partial\Sigma^\varepsilon$ which do not meet Ω. We study the limit behaviour (when ε tends to zero) of some quasilinear problems with three (possibly different) exponents $p, q_0, q \in (1, \infty)$ of the type:

$$\begin{cases} -\operatorname{div}\big(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon\big) = f^\varepsilon & \text{in } \Omega,\\ -\operatorname{div}\big(\mu_0^\varepsilon|\nabla u^\varepsilon|^{q_0-2}\nabla u^\varepsilon\big) = g_0^\varepsilon & \text{in } \Sigma_0^\varepsilon,\\ -\operatorname{div}\big(\mu^\varepsilon|\nabla u^\varepsilon|^{q-2}\nabla u^\varepsilon\big) = g^\varepsilon & \text{in } \Sigma^\varepsilon,\\ u^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon = \big(\partial\Sigma_0^\varepsilon\big) \setminus \Gamma_0,\\ u^\varepsilon \text{ is constant (undetermined)} & \text{on } \Gamma^\varepsilon = (\partial\Sigma^\varepsilon) \setminus \Gamma,\\ \displaystyle\int_{\Gamma^\varepsilon} \mu^\varepsilon|\nabla u^\varepsilon|^{q-2}\frac{\partial u^\varepsilon}{\partial n}\,\mathrm{d}s = I^\varepsilon & \text{(given)},\\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \setminus (\Gamma_0^\varepsilon \cup \Gamma^\varepsilon),\\ + \text{transmission conditions} & \text{on } \Gamma_0 \text{ and } \Gamma, \end{cases} \tag{$\mathcal{E}$}$$

where we write “transmission conditions” for:

$$u^\varepsilon_{|\Omega} = \begin{cases} u^\varepsilon_{|\Sigma_0^\varepsilon} & \text{on } \Gamma_0,\\ u^\varepsilon_{|\Sigma^\varepsilon} & \text{on } \Gamma, \end{cases}$$

$$\big|\nabla u^\varepsilon_{|\Omega}\big|^{p-2}\frac{\partial(u^\varepsilon_{|\Omega})}{\partial n} = \begin{cases} \mu^\varepsilon\big|\nabla u^\varepsilon_{|\Sigma_0^\varepsilon}\big|^{q_0-2}\dfrac{\partial(u^\varepsilon_{|\Sigma_0^\varepsilon})}{\partial n} & \text{on } \Gamma_0,\\ \mu^\varepsilon\big|\nabla u^\varepsilon_{|\Sigma^\varepsilon}\big|^{q-2}\dfrac{\partial(u^\varepsilon_{|\Sigma^\varepsilon})}{\partial n} & \text{on } \Gamma, \end{cases}$$

and where n denotes the outer normal to $\partial\Omega$ or $\partial\Omega^\varepsilon$. In an electric (or heat propagation) setting, if Ω^ε is doubly connected, with Γ^ε as inner part of the boundary $\partial\Omega^\varepsilon$ and Γ_0^ε as outer part, the boundary constraints mean that the potential (or temperature) has a given value on Γ_0^ε and that Γ^ε surrounds a perfect conductor, in which consequently the potential (or temperature) is constant, but unprescribed. The total flux, i.e. I^ε, is then given in terms of the integral of the source term over the perfectly conducting region. The assumption that μ_0^ε and μ^ε are small means that Σ_0^ε and Σ^ε modelize (unperfect) insulating layers. The torsional rigidity problem for a cable of multiconnected cross section also is of the same form.

To be more explicit, consider $\mathcal{D}^\varepsilon$ such that $\partial\mathcal{D}^\varepsilon = \Gamma_0^\varepsilon$ and $\mathcal{C}^\varepsilon$ such that $\partial\mathcal{C}^\varepsilon = \Gamma^\varepsilon$ ($\mathcal{D}^\varepsilon = \Omega^\varepsilon \cup \overline{\mathcal{C}^\varepsilon}$); the model problem consists in minimizing the energy

$$\mathcal{E}^\varepsilon(v) = \int\limits_{\mathcal{D}^\varepsilon} c^\varepsilon |\nabla v|^2\,\mathrm{d}x - \int\limits_{\mathcal{D}^\varepsilon} f^\varepsilon v\,\mathrm{d}x,$$

over the subset of functions v in $H_0^1(\mathcal{D}^\varepsilon)$ such that $\nabla v = 0$ in $\mathcal{C}^\varepsilon$, with a conductivity coefficient c^ε having value 1 in Ω, μ^ε in Σ^ε, μ_0^ε in Σ_0^ε. This problem is equivalent to minimizing

$$\mathcal{E}^\varepsilon(v) = \int\limits_{\Omega^\varepsilon} c^\varepsilon |\nabla v|^2\,\mathrm{d}x - \int\limits_{\Omega^\varepsilon} f^\varepsilon v\,\mathrm{d}x - v_{|\Gamma^\varepsilon} \int\limits_{\mathcal{C}^\varepsilon} f^\varepsilon\,\mathrm{d}x,$$

over the subset of functions v in $H^1(\Omega^\varepsilon)$ such that $v = 0$ on Γ_0^ε and v is constant (but unprescribed) on Γ^ε. Now the Euler equation of the above minimization problem is

$$\begin{cases} -\Delta u^\varepsilon = f^\varepsilon & \text{in } \Omega,\\ -\operatorname{div}\big(\mu_0^\varepsilon \nabla u^\varepsilon\big) = f^\varepsilon & \text{in } \Sigma_0^\varepsilon,\\ -\operatorname{div}\big(\mu^\varepsilon \nabla u^\varepsilon\big) = f^\varepsilon & \text{in } \Sigma^\varepsilon,\\ u^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon,\\ u^\varepsilon \text{ is constant (undetermined)} & \text{on } \Gamma^\varepsilon,\\ \displaystyle\int\limits_{\Gamma^\varepsilon} \mu^\varepsilon \frac{\partial u^\varepsilon}{\partial n}\,\mathrm{d}s = \int\limits_{\mathcal{C}^\varepsilon} f^\varepsilon\,\mathrm{d}x, & \\ u^\varepsilon_{|\Omega} = u^\varepsilon_{|\Sigma_0^\varepsilon} \quad \text{and} \quad \dfrac{\partial u^\varepsilon_{|\Omega}}{\partial n} = \mu_0^\varepsilon \dfrac{\partial (u^\varepsilon_{|\Sigma^\varepsilon})}{\partial n} & \text{on } \Gamma_0,\\ u^\varepsilon_{|\Omega} = u^\varepsilon_{|\Sigma^\varepsilon} \quad \text{and} \quad \dfrac{\partial u^\varepsilon_{|\Omega}}{\partial n} = \mu^\varepsilon \dfrac{\partial (u^\varepsilon_{|\Sigma^\varepsilon})}{\partial n} & \text{on } \Gamma, \end{cases}$$

which is the model equation for $(\mathcal{E})$.

Similar problems (not involving Σ^ε) were considered by Boutkrida, Mossino and Moussa in [4] and [5]. Previous works on reinforcement problems were done by Sanchez-Palencia [13,14], Acerbi and Buttazzo [1], Brezis, Caffarelli and Friedman [6], Buttazzo and Kohn [7] and by Buttazzo, Dal Maso and Mosco [8]. To our knowledge, the condition on the given flux and the undetermined constant boundary value appeared only very recently in this context of boundary layers [11].

As in [4] and [5] the shape of the reinforcement may depend on ε, that is we consider "general" functions h_0^ε and h^ε, and the insulating (or reinforcing) material may be inhomogeneous along Γ_0 and Γ, but μ_0^ε and μ^ε are constant along each normal to Γ_0 and Γ: in other words $\mu_0^\varepsilon(x) = \mu_0^\varepsilon(\sigma_0(x))$ (respectively $\mu^\varepsilon(x) = \mu^\varepsilon(\sigma(x))$), where $\sigma_0(x)$ (respectively $\sigma(x)$) denotes the projection of $x \in \Sigma_0^\varepsilon$ (respectively Σ^ε) on Γ_0 (respectively Γ). We define $a_0^\varepsilon : \Gamma_0 \to \mathbb{R}$ (respectively $a^\varepsilon : \Gamma \to \mathbb{R}$) by $a_0^\varepsilon = \mu_0^\varepsilon (h_0^\varepsilon)^{1-q_0}$ (respectively $a^\varepsilon = \mu^\varepsilon (h^\varepsilon)^{1-q}$). We assume essentially that a_0^ε are positive functions in $L^\infty(\Gamma_0)$ (and the same for $a^\varepsilon \in L^\infty(\Gamma)$), with uniformly bounded inverses, and that Γ_0 (respectively Γ) is divided into two parts:

- $\underline{\Gamma}_0$ (respectively $\underline{\Gamma}$) such that either $\underline{\Gamma}_0$ (respectively $\underline{\Gamma}$) is empty or $\frac{1}{a_0^\varepsilon}$ tends to zero in $L^{q_0'-1}(\underline{\Gamma}_0)$ (respectively $\frac{1}{a^\varepsilon} \to 0$ in $L^{q'-1}(\underline{\Gamma})$), q_0' and q' denoting the conjugates of q_0 and q (e.g. $\frac{1}{q} + \frac{1}{q'} = 1$),
- $\overline{\Gamma}_0$ (respectively $\overline{\Gamma}$) such that either $\overline{\Gamma}_0$ (respectively $\overline{\Gamma}$) is empty or a_0^ε tends to a_0 in weak $\star - L^\infty(\overline{\Gamma}_0)$ (respectively $a^\varepsilon \to a$ in weak $\star - L^\infty(\overline{\Gamma})$) and such that h_0^ε (respectively h^ε) does not oscillate too much on $\overline{\Gamma}_0$ (respectively $\overline{\Gamma}$).

We prove that the limit problem has the form

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f & \text{in } \Omega,\\ u = 0 & \text{on } \underline{\Gamma}_0,\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} + a_0|u|^{q_0-2}u = 0 & \text{on } \overline{\Gamma}_0,\\ u = k (\text{undetermined constant}) & \text{on } \underline{\Gamma},\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} + a|u-k|^{q-2}(u-k) = 0 & \text{on } \overline{\Gamma},\\ \displaystyle\int_\Gamma |\nabla u|^{p-2}\frac{\partial u}{\partial n}\,\mathrm{d}s = I, & \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma), \end{cases} \tag{$\mathcal{L}$}$$

where f and I are respective limits of f^ε and I^ε. As Acerbi and Buttazzo did in [1], we use the Γ-convergence theory introduced by De Giorgi [10] (see also Attouch [2] and Dal Maso [9]) and we actually are able to predict the explicit limit of more general minimization problems than those associated with (1.1), (1.2) and $(\mathcal{E})$:

- One can introduce additional constraints on $u^\varepsilon_{|\Omega}$;
- The reinforcements (1.1) and (1.2) can be generalized to

$$\Sigma_0^\varepsilon = \big\{s + t n_{H_0}(s), s \in \Gamma_0, 0 < t < h_0^\varepsilon(s) (\leqslant \varepsilon)\big\},$$
$$\Sigma^\varepsilon = \big\{s + t n_H(s), s \in \Gamma, 0 < t < h^\varepsilon(s) (\leqslant \varepsilon)\big\},$$

where $n_{H_0}(s)$ (respectively $n_H(s)$) is supported by the line of points having H_0 (respectively H)-projection s on Γ_0 (respectively Γ) and where H_0 and H are general norms ($n_H = n$, the unit normal vector, if H is the euclidian norm);

- In the energy functional, the term

$$\frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x + \frac{1}{q_0}\int_{\Sigma_0^\varepsilon} \mu_0^\varepsilon \circ \sigma_0 |\nabla v|^{q_0}\,\mathrm{d}x + \frac{1}{q}\int_{\Sigma^\varepsilon} \mu^\varepsilon \circ \sigma |\nabla v|^q\,\mathrm{d}x$$

can be generalized to the anisotropic one

$$F(v_{|\Omega}) + \int_{\Sigma_0^\varepsilon} \frac{a_0^\varepsilon \circ \sigma_0}{h_0^\varepsilon \circ \sigma_0} G_0\big(h_0^\varepsilon \circ \sigma_0 H_0^o(\nabla v)\big)\, \mathrm{d}x + \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\, \mathrm{d}x,$$

where (e.g.) $(\mu^\varepsilon \circ \sigma)(x) = \mu^\varepsilon(\sigma(x))$, $\sigma(x)$ (respectively $\sigma_0(x)$) is the projection of x for the H-norm (respectively H_0-norm) and where (e.g.) H^o denotes the dual norm of H, defined as

$$H^o(\xi^o) = \sup_{\xi \neq 0} \frac{\xi^o.\xi}{H(\xi)}.$$

We give the precise assumptions on F, G, G_0, H, H_0 in the following.

We would like to emphasize the connection between the geometry of the reinforcements Σ^ε, Σ_0^ε (defined by H and H_0) and the energy functional, whose integrands on Σ^ε and Σ_0^ε are defined in terms of the dual norms H^o and H_0^o.

Finally, let us mention that the result can be easily generalized to reinforcements along a finite number of components of $\partial\Omega$.

2. Statement of the problem and of the result

Let $H : \mathbb{R}^N \to \mathbb{R}^+$ be a norm. In particular

$$\forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \quad H(t\xi) = |t| H(\xi), \tag{2.1}$$

$$\exists \delta_1, \delta_2 > 0, \forall \xi \in \mathbb{R}^N, \quad \delta_1|\xi| \leqslant H(\xi) \leqslant \delta_2|\xi|, \tag{2.2}$$

$$\forall \xi_1, \xi_2 \in \mathbb{R}^N, \quad H(\xi_1 + \xi_2) \leqslant H(\xi_1) + H(\xi_2) \tag{2.3}$$

and from (2.1), (2.3), H is convex. The dual function of H, defined as

$$H^o(\xi^o) = \sup\left\{\frac{\xi^o.\xi}{H(\xi)}, \xi \in \mathbb{R}^N, \xi \neq 0\right\} = \sup\{\xi^o.\xi, \xi \in \mathbb{R}^N, 0 < H(\xi) \leqslant 1\} \tag{2.4}$$

is also a norm, with

$$\forall \xi^o \in \mathbb{R}^N, \quad \frac{1}{\delta_2}|\xi^o| \leqslant H^o(\xi^o) \leqslant \frac{1}{\delta_1}|\xi^o|; \tag{2.5}$$

H and H^o are dual to each other and satisfy

$$\forall \xi \in \mathbb{R}^N, \forall \xi^o \in \mathbb{R}^N, \quad |\xi^o.\xi| \leqslant H(\xi) H^o(\xi^o); \tag{P.1}$$

In this paper we consider two such norms H and H_0; we assume that they are differentiable at any point but zero and that H^o and H_0^o are strictly convex.

Now consider a bounded regular domain Ω in $\mathbb{R}^N$ and let Γ and Γ_0 be connected components of $\partial\Omega$. We introduce the distances (related to H and H_0) from x to Γ and Γ_0, defined by

$$t(x) = \min_{s \in \Gamma} H(x - s), \qquad t_0(x) = \min_{s \in \Gamma_0} H_0(x - s).$$

Let us remark that the above minima are achieved. Moreover if, for example, σ minimizes $H(x-s)$ for $s\in\Gamma$ and if y is on the segment $[\sigma,x]$, then σ also minimizes $H(y-s)$ for $s\in\Gamma$. (Actually assume σ' is on Γ with $H(y-\sigma')<H(y-\sigma)$. Then $H(x-\sigma')\leqslant H(x-y)+H(y-\sigma')<H(x-y)+H(y-\sigma)=H(x-\sigma)$ and we get a contradiction.) It follows that the set of points having H-projection σ on Γ is a line emanating from σ.

Let $\Omega'\supset\Omega$ be the domain having boundary $(\partial\Omega\setminus(\Gamma\cup\Gamma_0))\cup\Gamma'\cup\Gamma_0'$, with Γ' and Γ_0' at given small distance t' (relative to H and H_0respectively) from Γ and Γ_0. We have $\Omega'\setminus\overline{\Omega}=\Sigma'\cup\Sigma_0'$ and we assume that Σ' and Σ_0' are $\mathcal{C}^1$-diffeomorphic to $\Gamma\times(0,t')$ and $\Gamma_0\times(0,t')$ by the mappings $\mathcal{D}$ and $\mathcal{D}_0$:

$$\mathcal{D}: x\in\Sigma'\to\big(\sigma(x),t(x)\big)\in\Gamma\times(0,t'),$$

with $t(x)=\min_{s\in\Gamma}H(x-s)$ as above and $\sigma(x)=\arg\min_{s\in\Gamma}H(x-s)$. ($\mathcal{D}_0$ is defined similarly). We notice that (e.g.) $\mathcal{D}^{-1}(\sigma,t)=\sigma+tn_H(\sigma)$ with $H(n_H(\sigma))=1$ (since $t=H(tn_H(\sigma))$).

Let $\varepsilon<t'$ be a small parameter (hereafter ε will describe a sequence of positive numbers tending to zero) and let $h^\varepsilon:\Gamma\to\mathbb{R}^+\setminus\{0\}$ be a positive $\mathcal{C}^1$-function such that

$$\forall\sigma\in\Gamma,\quad h^\varepsilon(\sigma)\leqslant\varepsilon; \tag{2.6}$$

h^ε defines the reinforcement Σ^ε of Ω along Γ:

$$\Sigma^\varepsilon=\big\{x\in\Sigma',0<t(x)<h^\varepsilon\big(\sigma(x)\big)\big\}=\big\{\sigma+tn_H(\sigma),\sigma\in\Gamma,0<t<h^\varepsilon(\sigma)\big\}.$$

We set

$$\Gamma^\varepsilon=\big\{\sigma+h^\varepsilon(\sigma)n_H(\sigma),\sigma\in\Gamma\big\}.$$

We consider also a similar function h_0^ε defined on Γ_0 and we associate with it the reinforcement Σ_0^ε (of Ω along Γ_0) and the part Γ_0^ε of its boundary which does not meet Ω. We denote by $\Omega^\varepsilon=\Omega\cup\Sigma^\varepsilon\cup\Sigma_0^\varepsilon\cup\Gamma\cup\Gamma_0$ the reinforced domain. Note that $\Omega\subset\Omega^\varepsilon\subset\Omega'=\Omega\cup\Sigma'\cup\Sigma_0'\cup\Gamma\cup\Gamma_0$.

With the above geometrical data and given p,q,q_0 in $(1,\infty)$, we consider the functional space

$$V^\varepsilon=\big\{v:\Omega^\varepsilon\to\mathbb{R}, v_{|\Omega}\in W^{1,p}(\Omega), v_{|\Sigma^\varepsilon}\in W^{1,q}(\Sigma^\varepsilon), v_{|\Sigma_0^\varepsilon}\in W^{1,q_0}(\Sigma_0^\varepsilon),$$
$$v_{|\Omega}=v_{|\Sigma^\varepsilon}\text{ on }\Gamma, v_{|\Omega}=v_{|\Sigma_0^\varepsilon}\text{ on }\Gamma_0, v_{|\Gamma^\varepsilon}=\text{(undetermined) constant}, v_{|\Gamma_0^\varepsilon}=0\big\}.$$

We are given data I^ε, f^ε, g^ε, g_0^ε, F, K, G, G_0, a^ε, a_0^ε such that

- $I^\varepsilon\in\mathbb{R}$, $f^\varepsilon\in L^{p'}(\Omega)$, $g^\varepsilon\in L^{q'}(\Sigma^\varepsilon)$, $g_0^\varepsilon\in L^{q_0'}(\Sigma_0^\varepsilon)$, with p', q' and q'_0 the conjugates of p, q and q_0,
- $F:W^{1,p}(\Omega)\to\mathbb{R}$ is a continuous strictly convex functional such that

$$\exists\lambda>0,\exists\lambda'>0,\forall v\in W^{1,p}(\Omega),\quad F(v)\geqslant\lambda\|\nabla v\|^p_{L^p(\Omega)^N}-\lambda'\|v\|_{W^{1,p}(\Omega)}, \tag{2.7}$$

- K is a nonempty closed convex subset of $W^{1,p}(\Omega)$, corresponding to conditions that concern neither Γ nor Γ_0,

- G and $G_0 : \mathbb{R}^+ \to \mathbb{R}^+$ are increasing, strictly convex functions and

$$\exists \mu_1, \mu_2 > 0, \forall \eta \in \mathbb{R}^+, \quad \mu_1 \eta^q \leqslant G(\eta) \leqslant \mu_2 \eta^q \tag{2.8}$$

(respectively $\mu_1 \eta^{q_0} \leqslant G_0(\eta) \leqslant \mu_2 \eta^{q_0}$),
- $a^\varepsilon \in L^\infty(\Gamma)$, $a^\varepsilon > 0$ a.e. and $\frac{1}{a^\varepsilon} \in L^\infty(\Gamma)$, $a_0^\varepsilon \in L^\infty(\Gamma_0)$, $a_0^\varepsilon > 0$ a.e. and $\frac{1}{a_0^\varepsilon} \in L^\infty(\Gamma_0)$.

Now we are able to define $J^\varepsilon : V^\varepsilon \to \mathbb{R}$ by

$$J^\varepsilon(v) = F(v_{|\Omega}) + \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\,\mathrm{d}x + \int_{\Sigma_0^\varepsilon} \frac{a_0^\varepsilon \circ \sigma_0}{h_0^\varepsilon \circ \sigma_0} G_0\big(h_0^\varepsilon \circ \sigma_0 H_0^o(\nabla v)\big)\,\mathrm{d}x$$
$$- \int_\Omega f^\varepsilon v\,\mathrm{d}x - \int_{\Sigma^\varepsilon} g^\varepsilon v\,\mathrm{d}x - \int_{\Sigma_0^\varepsilon} g_0^\varepsilon v\,\mathrm{d}x - I^\varepsilon v_{|\Gamma^\varepsilon},$$

where (e.g.) $(a^\varepsilon \circ \sigma)(x) = a^\varepsilon(\sigma(x))$ and the first integral over Σ^ε is meaningful since by (2.5), (2.6), (2.8) the nonnegative integrand is bounded by $\mu_2 \varepsilon^{q-1} \delta_1^{-q} ||a^\varepsilon||_{L^\infty(\Gamma)} |\nabla v|^q$.

Our aim is to study the limit, as ε tends to zero, of the sequence of minimization problems

$$\operatorname{Inf}\big\{J^\varepsilon(v), v \in V^\varepsilon, v_{|\Omega} \in K\big\}. \tag{$\mathcal{P}^\varepsilon$}$$

PROPOSITION 1. – *($\mathcal{P}^\varepsilon$) has a unique solution u^ε.*

Proof. – In this proof, as well as in the whole paper, C denotes various constants and we write C^ε for constants depending on ε. It follows from (2.5), (2.8) and the Lebesgue dominated convergence theorem that $v \to G(h^\varepsilon \circ \sigma H^o(\nabla v))$ is continuous from $W^{1,q}(\Sigma^\varepsilon)$ to $L^1(\Sigma^\varepsilon)$, so that the first integral over Σ^ε is a continuous function on $W^{1,q}(\Sigma^\varepsilon)$. The same is true for Σ_0^ε.

Let us prove that

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^p(\Omega)^N} + \|\nabla v\|_{L^q(\Sigma^\varepsilon)^N} + \|\nabla v\|_{L^{q_0}(\Sigma_0^\varepsilon)^N}$$

is a norm on V^ε which is equivalent to the usual one induced by $W^{1,p}(\Omega) \times W^{1,q}(\Sigma^\varepsilon) \times W^{1,q_0}(\Sigma_0^\varepsilon)$. Actually by Poincaré inequality

$$\|v\|_{W^{1,q_0}(\Sigma_0^\varepsilon)} \leqslant C^\varepsilon \|\nabla v\|_{L^{q_0}(\Sigma_0^\varepsilon)^N}$$

and

$$\|v\|_{L^p(\Omega)} \leqslant C\big[\|\nabla v\|_{L^p(\Omega)^N} + \|v_{|\Gamma_0}\|_{L^1(\Gamma_0)}\big],$$

$$\|v\|_{L^q(\Sigma^\varepsilon)} \leqslant C^\varepsilon\big[\|\nabla v\|_{L^q(\Sigma^\varepsilon)^N} + \|v_{|\Gamma}\|_{L^1(\Gamma))}\big].$$

Moreover

$$\|v_{|\Gamma_0}\|_{L^1(\Gamma_0)} \leqslant C\|v_{|\Gamma_0}\|_{L^{q_0}(\Gamma_0)} \leqslant C^\varepsilon \|v\|_{W^{1,q_0}(\Sigma_0^\varepsilon)},$$

$$\|v_{|\Gamma}\|_{L^1(\Gamma)} \leqslant C\|v_{|\Gamma}\|_{L^p(\Gamma)} \leqslant C\|v\|_{W^{1,p}(\Omega)}.$$

It follows

$$\begin{aligned}\|v\|_{L^p(\Omega)} &\leqslant C\big[\|\nabla v\|_{L^p(\Omega)^N} + C^\varepsilon\|v\|_{W^{1,q_0}(\Sigma_0^\varepsilon)}\big]\\ &\leqslant C\big[\|\nabla v\|_{L^p(\Omega)^N} + C^\varepsilon\|\nabla v\|_{L^{q_0}(\Sigma_0^\varepsilon)^N}\big],\\ \|v\|_{L^q(\Sigma^\varepsilon)} &\leqslant C^\varepsilon\big[\|\nabla v\|_{L^q(\Sigma^\varepsilon)^N} + \|v\|_{W^{1,p}(\Omega)}\big]\\ &\leqslant C^\varepsilon\big[\|\nabla v\|_{L^q(\Sigma^\varepsilon)^N} + \|\nabla v\|_{L^p(\Omega)^N} + \|\nabla v\|_{L^{q_0}(\Sigma_0^\varepsilon)^N}\big].\end{aligned}$$

This gives the equivalence of the two norms under consideration.

Clearly J^ε is a continuous strictly convex functional on V^ε. Moreover J^ε is coercive since by (2.5), (2.7), (2.8) and the above equivalence, one has with α^ε such that $0 < \alpha^\varepsilon \leqslant \min(\inf_{\sigma\in\Gamma} a^\varepsilon(\sigma)h^\varepsilon(\sigma)^{q-1}, \inf_{\sigma\in\Gamma_0} a_0^\varepsilon(\sigma)h_0^\varepsilon(\sigma)^{q_0-1})$,

$$\begin{aligned}J^\varepsilon(v) &\geqslant \lambda\|\nabla v\|^p_{L^p(\Omega)^N} - \lambda'\|v\|_{W^{1,p}(\Omega)}\\ &\quad + \mu_1\alpha^\varepsilon\delta_2^{-q}\|\nabla v\|^q_{L^q(\Sigma^\varepsilon)^N} + \mu_1\alpha^\varepsilon\delta_2^{-q_0}\|\nabla v\|^{q_0}_{L^{q_0}(\Sigma_0^\varepsilon)^N}\\ &\quad - \|f^\varepsilon\|_{L^{p'}(\Omega)}\|v\|_{W^{1,p}(\Omega)} - \|g^\varepsilon\|_{L^{q'}(\Sigma^\varepsilon)}\|v\|_{W^{1,q}(\Sigma^\varepsilon)}\\ &\quad - \|g^\varepsilon\|_{L^{q_0'}(\Sigma_0^\varepsilon)}\|v\|_{W^{1,q_0}(\Sigma_0^\varepsilon)} - C^\varepsilon\|v\|_{W^{1,q}(\Sigma^\varepsilon)}\\ &\geqslant \lambda\|\nabla v\|^p_{L^p(\Omega)^N} + \mu_1\alpha^\varepsilon\big[\delta_2^{-q}\|\nabla v\|^q_{L^q(\Sigma^\varepsilon)^N} + \delta_2^{-q_0}\|\nabla v\|^{q_0}_{L^{q_0}(\Sigma_0^\varepsilon)^N}\big]\\ &\quad - C^\varepsilon\big[\|v\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,q}(\Sigma^\varepsilon)} + \|v\|_{W^{1,q_0}(\Sigma_0^\varepsilon)}\big]\\ &\geqslant \big[\lambda\|\nabla v\|^p_{L^p(\Omega)^N} - C^\varepsilon\|\nabla v\|_{L^p(\Omega)}\big]\\ &\quad + \big[\mu_1\alpha^\varepsilon\delta_2^{-q}\|\nabla v\|^q_{L^q(\Sigma^\varepsilon)^N} - C^\varepsilon\|\nabla v\|_{L^q(\Sigma^\varepsilon)^N}\big]\\ &\quad + \big[\mu_1\alpha^\varepsilon\delta_2^{-q_0}\|\nabla v\|^{q_0}_{L^{q_0}(\Sigma_0^\varepsilon)^N} - C^\varepsilon\|\nabla v\|_{L^{q_0}(\Sigma_0^\varepsilon)^N}\big]\end{aligned}$$

and since, when $\|v\|_{V^\varepsilon} \to +\infty$, at least one of $\|\nabla v\|_{L^p(\Omega)^N}$, $\|\nabla v\|_{L^q(\Sigma^\varepsilon)^N}$ or $\|\nabla v\|_{L^{q_0}(\Sigma_0^\varepsilon)^N}$ tends to infinity. □

We study the limit behaviour of $(\mathcal{P}^\varepsilon)$ under the following additional assumptions on a^ε, a_0^ε, h^ε, h_0^ε, I^ε, f^ε, g^ε, g_0^ε, valid when ε tends to zero. First we assume that $\{\frac{1}{a^\varepsilon}\}_\varepsilon$ and $\{\frac{1}{a_0^\varepsilon}\}_\varepsilon$ are respectively bounded in $L^\infty(\Gamma)$ and $L^\infty(\Gamma_0)$:

$$\exists\alpha > 0,\ \text{a.e. } s\in\Gamma,\ \text{a.e. } s_0\in\Gamma_0,\quad \forall\varepsilon, a^\varepsilon(s)\geqslant\alpha, a_0^\varepsilon(s_0)\geqslant\alpha. \tag{2.9}$$

Moreover we assume that, up to a set of $(N-1)$-dimensional measure zero, there exists a partition of Γ (respectively Γ_0) into two open regular subsets $\underline{\Gamma}$ and $\overline{\Gamma}$ (respectively $\underline{\Gamma}_0$ and $\overline{\Gamma}_0$) independent of ε (one of them being possibly empty, none of them being necessarily connected), such that

$$\begin{gathered}\text{either } \underline{\Gamma} = \emptyset \quad\text{or}\quad \frac{1}{a^\varepsilon_{|\underline{\Gamma}}} \to 0 \quad \text{in } L^{q'-1}(\underline{\Gamma})\\ \left(\text{respectively either } \underline{\Gamma}_0 = \emptyset \quad\text{or}\quad \frac{1}{a^\varepsilon_{0|\underline{\Gamma}_0}} \to 0 \quad \text{in } L^{q_0'-1}(\underline{\Gamma}_0)\right)\end{gathered} \tag{2.10}$$

and

$$\text{either } \overline{\Gamma} = \emptyset \quad \text{or} \quad \exists a \in L^\infty(\overline{\Gamma}),\, a^\varepsilon_{|\overline{\Gamma}} \rightharpoonup a \quad \text{in weak} \star\text{-}L^\infty(\overline{\Gamma}) \tag{2.11}$$

$$\big(\text{respectively either } \overline{\Gamma}_0 = \emptyset \quad \text{or} \quad \exists a_0 \in L^\infty(\overline{\Gamma}_0),\, a^\varepsilon_{0|\overline{\Gamma}_0} \rightharpoonup a_0 \quad \text{in weak} \star\text{-}L^\infty(\overline{\Gamma}_0)\big),$$

$$\nabla h^\varepsilon \to 0 \quad \text{in } L^q(\overline{\Gamma}) \quad \big(\text{respectively } \nabla h^\varepsilon_0 \to 0 \quad \text{in } L^{q_0}(\overline{\Gamma}_0)\big). \tag{2.12}$$

We also assume that

$$\begin{aligned}
&\text{for any } v \in K \text{ with } v_{|\underline{\Gamma}} = l \text{ (constant)},\, v_{|\underline{\Gamma}_0} = 0,\, v_{|\overline{\Gamma}} \in L^q(\overline{\Gamma}),\, v_{|\overline{\Gamma}_0} \in L^{q_0}(\overline{\Gamma}_0),\\
&\text{there exists a sequence of elements } v_n \in \mathcal{C}^1(\overline{\Omega}) \cap K \text{ such that}\\
&v_{n|\underline{\Gamma}} = l,\, v_{|\underline{\Gamma}_0} = 0,\, v_n \to v \text{ in } W^{1,p}(\Omega),\, v_{n|\overline{\Gamma}} \to v_{|\overline{\Gamma}} \text{ in } L^q(\overline{\Gamma}),\, v_{n|\overline{\Gamma}_0} \to v_{|\overline{\Gamma}_0} \text{ in } L^{q_0}(\overline{\Gamma}_0).
\end{aligned} \tag{2.13}$$

Finally we assume that

$$\begin{aligned}
&I^\varepsilon \to I,\, \big\{\|g^\varepsilon\|_{L^{q'}(\Sigma^\varepsilon)}\big\}_\varepsilon \text{ and } \big\{\|g^\varepsilon_0\|_{L^{q'_0}(\Sigma^\varepsilon_0)}\big\}_\varepsilon \text{ are bounded and}\\
&\exists f \in L^{p'}(\Omega),\, f^\varepsilon \rightharpoonup f \text{ weakly in } L^{p'}(\Omega).
\end{aligned} \tag{2.14}$$

Let us comment (2.12). It means that the oscillations of h^ε and h^ε_0 on $\overline{\Gamma}$ and $\overline{\Gamma}_0$ are small. Of course this holds true if (e.g.) $h^\varepsilon(s) \equiv \varepsilon h(s)$ with h of class $\mathcal{C}^1(\partial\Omega)$, $0 < h \leqslant 1$, but also if (e.g.) $N = 2$, Γ is a closed curve, $\varepsilon = n^{-r}$, $h^\varepsilon(s) = n^{-r}\mathcal{H}(ny(s))$ for $s \in \Gamma$, $y(s) =$ arc length (s), $\mathcal{H}$ periodic of period Y where Y is the length of Γ, $0 < \mathcal{H} \leqslant 1$, $\mathcal{H}$ of class $\mathcal{C}^1(\mathbb{R}^+)$ and $r > 1$. (Actually in this case $|\nabla h^\varepsilon| \to 0$ in $L^\infty(\overline{\Gamma})$.) On the contrary it does not hold true in the periodic case of [7] or [11], where $r = 1$, unless if, in the notations of [7], $R^\varepsilon/S^\varepsilon$ tends to zero.

Under the above assumptions we have

THEOREM 1. – *Let u^ε be the solution of*

$$\operatorname{Inf}\big\{J^\varepsilon(v),\, v \in V^\varepsilon,\, v_{|\Omega} \in K\big\}, \tag{$\mathcal{P}^\varepsilon$}$$

where K is a nonempty closed convex subset of $W^{1,p}(\Omega)$, corresponding to conditions that concern neither Γ nor Γ_0,

$$\begin{aligned}
V^\varepsilon = \{&v : \Omega^\varepsilon \to \mathbb{R},\, v_{|\Omega} \in W^{1,p}(\Omega),\, v_{|\Sigma^\varepsilon} \in W^{1,q}(\Sigma^\varepsilon),\, v_{|\Sigma^\varepsilon_0} \in W^{1,q_0}(\Sigma^\varepsilon_0),\\
&v_{|\Omega} = v_{|\Sigma^\varepsilon} \textit{ on } \Gamma,\, v_{|\Omega} = v_{|\Sigma^\varepsilon_0} \textit{ on } \Gamma_0,\, v_{|\Gamma^\varepsilon} \textit{ is constant},\, v_{|\Gamma^\varepsilon_0} = 0\big\},
\end{aligned}$$

$$\begin{aligned}
J^\varepsilon(v) = F(v_{|\Omega}) &+ \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\,\mathrm{d}x + \int_{\Sigma^\varepsilon_0} \frac{a^\varepsilon_0 \circ \sigma_0}{h^\varepsilon_0 \circ \sigma_0} G_0\big(h^\varepsilon_0 \circ \sigma_0 H^o_0(\nabla v)\big)\,\mathrm{d}x\\
&- \int_\Omega f^\varepsilon v\,\mathrm{d}x - \int_{\Sigma^\varepsilon} g^\varepsilon v\,\mathrm{d}x - \int_{\Sigma^\varepsilon_0} g^\varepsilon_0 v\,\mathrm{d}x - I^\varepsilon v_{|\Gamma^\varepsilon}.
\end{aligned}$$

Let us define $(\mathcal{P})$ by

$$\operatorname{Inf}\big\{J(v,l);\, (v,l) \in W,\, v \in K\big\}, \tag{$\mathcal{P}$}$$

$$W = \{(v,l) \in W^{1,p}(\Omega) \times \mathbb{R}, v_{|\overline{\Gamma}} \in L^q(\overline{\Gamma}), v_{|\overline{\Gamma}_0} \in L^{q_0}(\overline{\Gamma}_0),$$
$$v_{|\underline{\Gamma}} = l \text{ (if } \underline{\Gamma} \neq \emptyset), v_{|\underline{\Gamma}_0} = 0 (\text{if } \underline{\Gamma}_0 \neq \emptyset)\}$$

(*the restriction* $v_{|\overline{\Gamma}} \in L^q(\overline{\Gamma})$ *being effective only if* $\overline{\Gamma} \neq \emptyset$ *and* $q > p$, *similarly for* $\overline{\Gamma}_0$),

$$J(v,l) = F(v) + \int_{\overline{\Gamma}} aG(|v-l|)H^o(n)\,\mathrm{d}s + \int_{\overline{\Gamma}_0} a_0 G_0(|v|)H_0^o(n)\,\mathrm{d}s - \int_{\Omega} fv\,\mathrm{d}x - Il,$$

where $n = n(s)$ *is the unit normal to* $\overline{\Gamma}$ *or* $\overline{\Gamma}_0$ *at the point* s.

Then $(\mathcal{P})$ *has a unique solution* $(u,k) \in W$ *with* $u \in K$. *Moreover when* $\varepsilon \to 0$,

(1) ${u^\varepsilon}_{|\Omega}$ *tends to* u *in weak-*$W^{1,p}(\Omega)$ *and in* $L^p(\Omega)$;

(2) *the function* $\tilde{v}^\varepsilon : \Sigma' \to \mathbb{R}$ *given by* $\tilde{v}^\varepsilon = u^\varepsilon - u^\varepsilon_{|\Gamma^\varepsilon}$ *in* Σ^ε, 0 *in* $\Sigma' \setminus \Sigma^\varepsilon$ *tends to zero in* $L^q(\Sigma')$; *the function* $\tilde{v}_0^\varepsilon : \Sigma_0' \to \mathbb{R}$ *given by* $\tilde{v}_0^\varepsilon = u^\varepsilon$ *in* Σ_0^ε, 0 *in* $\Sigma_0' \setminus \Sigma_0^\varepsilon$ *tends to zero in* $L^{q_0}(\Sigma_0')$;

(3) ${u^\varepsilon}_{|\Gamma} \to u_{|\Gamma}$ *in* $L^p(\Gamma)$, ${u^\varepsilon}_{|\Gamma} \rightharpoonup u_{|\Gamma}$ *in weak-*$L^q(\Gamma)$, ${u^\varepsilon}_{|\Gamma_0} \to u_{|\Gamma_0}$ *in* $L^p(\Gamma_0)$, ${u^\varepsilon}_{|\Gamma_0} \rightharpoonup u_{|\Gamma_0}$ *in weak-*$L^{q_0}(\Gamma_0)$, *the weak convergences being of interest only if* $q > p$ (*respectively* $q_0 > p$)), $u^\varepsilon_{|\Gamma^\varepsilon} \to k$ *in* $\mathbb{R}$;

(4) $J^\varepsilon(u^\varepsilon) \to J(u,k)$.

Except for an example given in the last section, the rest of the paper is devoted to the proof of this theorem.

3. Existence and uniqueness of the solution (u,k) of $(\mathcal{P})$

- If $\overline{\Gamma} = \emptyset$, that is if $\underline{\Gamma} = \Gamma$, $(\mathcal{P})$ reduces to

$$\operatorname{Inf}\{J'(v), v \in V' \cap K\}, \tag{$\mathcal{P}'$}$$

where

$$V' = \{v \in W^{1,p}(\Omega), v_{|\Gamma} \text{ is constant }, v_{|\overline{\Gamma}_0} \in L^{q_0}(\overline{\Gamma}_0), v_{|\underline{\Gamma}_0} = 0\},$$

$$J'(v) = F(v) + \int_{\overline{\Gamma}_0} a_0 G_0(|v|)H_0^o(n)\,\mathrm{d}s - \int_{\Omega} fv\,\mathrm{d}x - Iv_{|\Gamma}.$$

If also $\overline{\Gamma}_0$ (together with $\overline{\Gamma}$) is empty, then V' reduces to

$$\{v \in W^{1,p}(\Omega), v_{|\Gamma_0} = 0, v_{|\Gamma} \text{ is constant}\},$$

$J'(v)$ reduces to

$$F(v) - \int_{\Omega} fv\,\mathrm{d}x - Iv_{|\Gamma}$$

and it is very classical that $(\mathcal{P}')$ admits a unique solution $u \in V' \cap K$. If $\overline{\Gamma} = \emptyset$ and $\overline{\Gamma}_0 \neq \emptyset$, then V' is a Banach space for

$$\|v\|_{V'} = \|\nabla v\|_{L^p(\Omega)^N} + \|v\|_{L^{q_0}(\overline{\Gamma}_0)}$$

and for any $v \in V'$, one has by Poincaré inequality

$$\|v\|_{L^p(\Omega)} \leqslant C\|\nabla v\|_{L^p(\Omega)^N} + C\|v\|_{L^1(\overline{\Gamma}_0)} \leqslant C\|\nabla v\|_{L^p(\Omega)^N} + C\|v\|_{L^{q_0}(\overline{\Gamma}_0)},$$

so that J' is coercive on V'.

• Otherwise, that is if $\overline{\Gamma} \neq \emptyset$, W is a Banach space for

$$\|(v,l)\|_W = \|\nabla v\|_{L^p(\Omega)^N} + \|v\|_{L^{q_0}(\overline{\Gamma}_0)} + \|v-l\|_{L^q(\overline{\Gamma})}.$$

Note that this norm is equivalent to the natural one:

$$\||(v,l)|\| = \|v\|_{W^{1,p}(\Omega)} + \|v\|_{L^{q_0}(\overline{\Gamma}_0)} + \|v\|_{L^q(\overline{\Gamma})} + |l|.$$

Actually since

$$\|v-l\|_{L^q(\overline{\Gamma})} \leqslant \|v\|_{L^q(\overline{\Gamma})} + \|l\|_{L^q(\overline{\Gamma})} \leqslant \|v\|_{L^q(\overline{\Gamma})} + C|l|,$$

we get clearly $\|(v,l)\|_W \leqslant \||(v,l)|\|$; on the contrary by Poincaré inequality,

$$\begin{aligned}
\|v\|_{W^{1,p}(\Omega)} &\leqslant C\|\nabla v\|_{L^p(\Omega)^N} + C\|v\|_{L^1(\overline{\Gamma}_0)} \\
&\leqslant C\|\nabla v\|_{L^p(\Omega)^N} + C\|v\|_{L^{q_0}(\overline{\Gamma}_0)} \leqslant C\|(v,l)\|_W, \\
|l| &= \frac{1}{|\overline{\Gamma}|}\left|\int_{\overline{\Gamma}} v\,\mathrm{d}s - \int_{\overline{\Gamma}} (v-l)\,\mathrm{d}s\right| \leqslant C\int_{\overline{\Gamma}} |v|\,\mathrm{d}s + C\int_{\overline{\Gamma}} |v-l|\,\mathrm{d}s \\
&\leqslant C\|v\|_{W^{1,p}(\Omega)} + C\|v-l\|_{L^q(\overline{\Gamma})} \leqslant C\|(v,l)\|_W, \\
\|v\|_{L^q(\overline{\Gamma})} &\leqslant \|v-l\|_{L^q(\overline{\Gamma})} + C|l| \leqslant \|v-l\|_{L^q(\overline{\Gamma})} + C\|(v,l)\|_W \leqslant C\|(v,l)\|_W
\end{aligned}$$

and we get $\||(v,l)|\| \leqslant C\|(v,l)\|_W$. The functional J is strictly convex and continuous on W and it is coercive since by (2.5), (2.7), (2.8), (2.9), (2.11), (2.14) and by the above inequalities

$$\begin{aligned}
J(v,l) &\geqslant \lambda\|\nabla v\|^p_{L^p(\Omega)^N} - \lambda'\|v\|_{W^{1,p}(\Omega)} + \alpha\mu_1\delta_2^{-1}\left[\|v-l\|^q_{L^q(\overline{\Gamma})} + \|v\|^{q_0}_{L^{q_0}(\overline{\Gamma}_0)}\right] \\
&\quad - \|f\|_{L^{p'}(\Omega)}\|v\|_{W^{1,p}(\Omega)} - |I|\left[C\|v\|_{W^{1,p}(\Omega)} + C\|v-l\|_{L^q(\overline{\Gamma})}\right] \\
&\geqslant \lambda\|\nabla v\|^p_{L^p(\Omega)^N} - C\|\nabla v\|_{L^p(\Omega)^N} + \left[\alpha\mu_1\delta_2^{-1}\|v-l\|^q_{L^q(\overline{\Gamma})} - C\|v-l\|_{L^q(\overline{\Gamma})}\right] \\
&\quad + \left[\alpha\mu_1\delta_2^{-1}\|v\|^{q_0}_{L^{q_0}(\overline{\Gamma}_0)} - C\|v\|_{L^{q_0}(\overline{\Gamma}_0)}\right]
\end{aligned}$$

and since, when $\|(v,l)\|_W \to +\infty$, either $\|\nabla v\|_{L^p(\Omega)^N}$ or $\|v\|_{L^{q_0}(\overline{\Gamma}_0)}$, or $\|v-l\|_{L^q(\overline{\Gamma})}$ tends to infinity.

Remark 1. – Note that if $\underline{\Gamma} = \emptyset$, there is no link between v and l for (v,l) in W. Otherwise, that is if $\underline{\Gamma} \neq \emptyset$, $(\mathcal{P})$ can be formulated as the nonlocal problem:

$$\operatorname{Inf}\{J''(v),\, v \in V'' \cap K\}, \qquad (\mathcal{P}'')$$

with

$$V'' = \left\{v \in W^{1,p}(\Omega),\, v_{|\underline{\Gamma}_0} = 0,\, v_{|\overline{\Gamma}_0} \in L^{q_0}(\overline{\Gamma}_0),\, v_{|\overline{\Gamma}} \in L^q(\overline{\Gamma}),\, v_{|\underline{\Gamma}} \text{ is constant}\right\},$$

$$J''(v) = F(v) + \int_{\overline{\Gamma}} aG(|v - v_{|\underline{\Gamma}}|)H^o(n)\,\mathrm{d}s + \int_{\overline{\Gamma}_0} a_0 G_0(|v|)H_0^o(n)\,\mathrm{d}s - \int_{\Omega} f v\,\mathrm{d}x - I v_{|\underline{\Gamma}}.$$

4. A priori estimates and consequences for u^ε

The a priori estimates are given in

LEMMA 1. – (1) $\exists C > 0, \forall\varepsilon, \forall v \in W^{1,q}(\Sigma^\varepsilon), v_{|\Gamma^\varepsilon} = k^\varepsilon$ (*constant*) $\Rightarrow$

$$\int_{\Sigma^\varepsilon} |v - k^\varepsilon|^q\,\mathrm{d}x \leqslant C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^q H^o(\nabla v)^q\,\mathrm{d}x,$$

$$\int_{\Gamma} |v - k^\varepsilon|^q\,\mathrm{d}s \leqslant C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla v)^q\,\mathrm{d}x,$$

which applies to u^ε*, giving also*

$$\int_{\Sigma_0^\varepsilon} |u^\varepsilon|^{q_0}\,\mathrm{d}x \leqslant C \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x,$$

$$\int_{\Gamma_0} |u^\varepsilon|^{q_0}\,\mathrm{d}s \leqslant C \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x.$$

(2) ${u^\varepsilon}_{|\Omega}$ *is bounded in* $W^{1,p}(\Omega)$*,* ${u^\varepsilon}_{|\Gamma^\varepsilon}$ *is bounded in* $\mathbb{R}$.

(3) $F({u^\varepsilon}_{|\Omega})$*,* $\int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(h^\varepsilon \circ \sigma H^o(\nabla u^\varepsilon))\,\mathrm{d}x$ *and* $\int_{\Sigma_0^\varepsilon} \frac{a_0^\varepsilon \circ \sigma_0}{h_0^\varepsilon \circ \sigma_0} G(h_0^\varepsilon \circ \sigma H_0^o(\nabla u^\varepsilon))\,\mathrm{d}x$ *are bounded.*

(4) ${u^\varepsilon}_{|\Gamma}$ *is bounded in* $L^q(\Gamma)$ *and* ${u^\varepsilon}_{|\Gamma_0}$ *is bounded in* $L^{q_0}(\Gamma_0)$.

(5) $\int_{\Sigma^\varepsilon} |u^\varepsilon - u^\varepsilon_{|\Gamma^\varepsilon}|^q\,\mathrm{d}x$ *and* $\int_{\Sigma_0^\varepsilon} |u^\varepsilon|^{q_0}\,\mathrm{d}x$ *tend to zero.*

Proof of (1). – By using the $\mathcal{C}^1$-diffeomorphism of Σ_0' onto $\Gamma_0 \times (0, t')$, it is proved in [5] that $\exists C > 0, \forall\varepsilon, \forall w \in W^{1,q_0}(\Sigma_0^\varepsilon), w_{|\Gamma_0^\varepsilon} = 0 \Rightarrow$

$$\int_{\Sigma_0^\varepsilon} |w|^{q_0}\,\mathrm{d}x \leqslant C \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0} H_0^o(\nabla w)^{q_0}\,\mathrm{d}x,$$

$$\int_{\Gamma_0} |w|^{q_0}\,\mathrm{d}s \leqslant C \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla w)^{q_0}\,\mathrm{d}x.$$

Applying this to $w = v - k^\varepsilon$ and deleting the subscript zero gives the first two estimates.

Proof of (2), (3) and (4). – Let $k^\varepsilon = u^\varepsilon_{|\Gamma^\varepsilon}$. It follows from (1) that

$$\|u^\varepsilon - k^\varepsilon\|_{L^1(\Gamma)} \leqslant C\|u^\varepsilon - k^\varepsilon\|_{L^q(\Gamma)} \leqslant C\left[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q\,\mathrm{d}x\right]^{1/q}$$

and hence

$$C|k^\varepsilon| = \|k^\varepsilon\|_{L^1(\Gamma)} \leqslant \|u^\varepsilon\|_{L^1(\Gamma)} + C\Bigg[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x\Bigg]^{1/q}$$
$$\leqslant C\|u^\varepsilon\|_{W^{1,p}(\Omega)} + C\Bigg[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x\Bigg]^{1/q}$$
$$\leqslant C\|\nabla u^\varepsilon\|_{L^p(\Omega)^N} + C\|u^\varepsilon\|_{L^{q_0}(\Gamma_0)} + C\Bigg[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x\Bigg]^{1/q}$$

and it follows that

$$|k^\varepsilon| \leqslant C\|\nabla u^\varepsilon\|_{L^p(\Omega)^N} + C\Bigg[\int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0} \,\mathrm{d}x\Bigg]^{1/q_0} + C\Bigg[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x\Bigg]^{1/q}. \tag{4.1}$$

Moreover for small ε ($\varepsilon \leqslant 1$), we have using (1) again

$$\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)} \leqslant C\|k^\varepsilon\|_{L^q(\Sigma^\varepsilon)} + C\Bigg[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x\Bigg]^{1/q}$$
$$\leqslant C|k^\varepsilon| + C\Bigg[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x\Bigg]^{1/q}$$

and by means of (4.1)

$$\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)} \leqslant C\|\nabla u^\varepsilon\|_{L^p(\Omega)^N} + C\Bigg[\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x\Bigg]^{1/q} + C\Bigg[\int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u_0^\varepsilon)^{q_0} \,\mathrm{d}x\Bigg]^{1/q_0}. \tag{4.2}$$

Similarly one has by making $k^\varepsilon = 0$ in the penultimate inequality

$$\|u^\varepsilon\|_{L^{q_0}(\Sigma_0^\varepsilon)} \leqslant C\Bigg[\int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0} H_0^o(\nabla u^\varepsilon)^{q_0} \,\mathrm{d}x\Bigg]^{1/q_0}. \tag{4.3}$$

Now let α be as in (2.9). One has by (2.8)

$$\alpha \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x \leqslant \int_{\Sigma^\varepsilon} (a^\varepsilon \circ \sigma)(h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \,\mathrm{d}x$$
$$\leqslant \frac{1}{\mu_1} \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla u^\varepsilon)\big) \,\mathrm{d}x \tag{4.4}$$

and similarly

$$\alpha \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x \leqslant \frac{1}{\mu_1} \int_{\Sigma_0^\varepsilon} \frac{a_0^\varepsilon \circ \sigma_0}{h_0^\varepsilon \circ \sigma_0} G_0\big(h_0^\varepsilon \circ \sigma_0 H_0^o(\nabla u^\varepsilon)\big)\,\mathrm{d}x, \tag{4.5}$$

so that by (2.7), (2.14), (4.1) to (4.3), one has with $v \in V^\varepsilon$, $v_{|\Omega} \in K$, $v = 0$ in Σ^ε and Σ_0^ε,

$$\begin{aligned}
&\lambda \|\nabla u^\varepsilon\|^p_{L^p(\Omega)^N} - \lambda' \|u^\varepsilon\|_{W^{1,p}(\Omega)} \\
&\quad + \mu_1 \alpha \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q\,\mathrm{d}x + \mu_1 \alpha \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x \\
&\leqslant F(u^\varepsilon{}_{|\Omega}) + \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla u^\varepsilon)\big)\,\mathrm{d}x \\
&\quad + \int_{\Sigma_0^\varepsilon} \frac{a_0^\varepsilon \circ \sigma_0}{h_0^\varepsilon \circ \sigma_0} G_0\big(h_0^\varepsilon \circ \sigma_0 H_0^o(\nabla u^\varepsilon)\big)\,\mathrm{d}x \\
&\leqslant F(v_{|\Omega}) + \int_\Omega f^\varepsilon u^\varepsilon\,\mathrm{d}x - \int_\Omega f^\varepsilon v dx + \int_{\Sigma^\varepsilon} g^\varepsilon u^\varepsilon\,\mathrm{d}x + \int_{\Sigma_0^\varepsilon} g_0^\varepsilon u^\varepsilon\,\mathrm{d}x + I^\varepsilon k^\varepsilon \\
&\quad \text{(as } u^\varepsilon \text{ solves } (\mathcal{P}^\varepsilon) \text{ and as } H^o(0) = H_0^o(0) = 0 \text{ and } G(0) = G_0(0) = 0) \\
&\leqslant C + C\|u^\varepsilon\|_{W^{1,p}(\Omega)} + C\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)} + C\|u^\varepsilon\|_{L^{q_0}(\Sigma_0^\varepsilon)} + C|k^\varepsilon| \\
&\leqslant C + C\|u^\varepsilon\|_{W^{1,p}(\Omega)} + C\Big(\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q\,\mathrm{d}x\Big)^{1/q} \\
&\quad + C\Big(\int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x\Big)^{1/q_0}.
\end{aligned} \tag{4.6}$$

As already noticed

$$\|u^\varepsilon\|_{W^{1,p}(\Omega)} \leqslant C\|\nabla u^\varepsilon\|_{L^p(\Omega)^N} + C\Big(\int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x\Big)^{1/q_0} \tag{4.7}$$

and it follows from (4.6) and (4.7) that

$$\begin{aligned}
&\mu_1 \alpha \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q\,\mathrm{d}x - C\Big(\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q\,\mathrm{d}x\Big)^{1/q} \\
&\quad + \mu_1 \alpha \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x - C\Big(\int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0}\,\mathrm{d}x\Big)^{1/q_0} \\
&\quad + \lambda \|\nabla u^\varepsilon\|^p_{L^p(\Omega)^N} - C\|\nabla u^\varepsilon\|_{L^p(\Omega)^N} \leqslant C.
\end{aligned}$$

We remark that each of the above lines has the form $Ct^p - C't$, hence it is bounded below and tends to infinity with t. We conclude that

$$\|\nabla u^\varepsilon\|_{L^p(\Omega)^N}, \quad \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \, \mathrm{d}x \quad \text{and} \quad \int_{\Sigma_0^\varepsilon} (h_0^\varepsilon \circ \sigma_0)^{q_0-1} H_0^o(\nabla u^\varepsilon)^{q_0} \, \mathrm{d}x$$

are bounded. By (4.7), u^ε is bounded in $W^{1,p}(\Omega)$ and by (4.1), $k^\varepsilon = u^\varepsilon_{|\Gamma^\varepsilon}$ is bounded in $\mathbb{R}$. From the arguments following (4.5),

$$F(u^\varepsilon{}_{|\Omega}), \quad \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla u^\varepsilon)\big) \, \mathrm{d}x \quad \text{and} \quad \int_{\Sigma_0^\varepsilon} \frac{a_0^\varepsilon \circ \sigma_0}{h_0^\varepsilon \circ \sigma_0} G_0\big(h_0^\varepsilon \circ \sigma_0 H_0^o(\nabla u^\varepsilon)\big) \, \mathrm{d}x$$

are bounded. By (1-) it is clear that $u^\varepsilon{}_{|\Gamma_0}$ is bounded in $L^{q_0}(\Gamma_0)$. Moreover we get that $u^\varepsilon_{|\Gamma} - k^\varepsilon$ is bounded in $L^q(\Gamma)$ and as k^ε is bounded, it follows that $u^\varepsilon_{|\Gamma}$ is bounded in $L^q(\Gamma)$.

Proof of (5). – From (1) and (2.6)

$$\int_{\Sigma^\varepsilon} |u^\varepsilon - u^\varepsilon_{|\Gamma^\varepsilon}|^q \, \mathrm{d}x \leqslant C \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^q H^o(\nabla u^\varepsilon)^q \, \mathrm{d}x \leqslant C\varepsilon \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \, \mathrm{d}x \leqslant C\varepsilon,$$

as we have just seen that $\int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} H^o(\nabla u^\varepsilon)^q \, \mathrm{d}x$ is bounded. Similarly

$$\int_{\Sigma_0^\varepsilon} |u^\varepsilon|^{q_0} \, \mathrm{d}x \leqslant C\varepsilon.$$

From the a-priori estimates we are going to deduce

LEMMA 2. – *There exists a subsequence ε' of ε and an element (u,k) of W such that $u \in K$ and*

$$\begin{aligned}
&u^{\varepsilon'}{}_{|\Omega} \rightharpoonup u \quad \textit{in weak-}W^{1,p}(\Omega), \qquad u^{\varepsilon'}{}_{|\Omega} \to u \quad \textit{in } L^p(\Omega),\\
&u^{\varepsilon'}{}_{|\Gamma^{\varepsilon'}} \to k \quad (\textit{in } \mathbb{R}),\\
&u^{\varepsilon'}{}_{|\Gamma} \to u_{|\Gamma} \quad \textit{in } L^p(\Gamma) \quad (\textit{hence in } L^q(\Gamma) \textit{ if } q \leqslant p),\\
&u^{\varepsilon'}{}_{|\Gamma_0} \to u_{|\Gamma_0} \quad \textit{in } L^p(\Gamma_0) \quad (\textit{hence in } L^{q_0}(\Gamma_0) \textit{ if } q_0 \leqslant p),\\
&u^{\varepsilon'}{}_{|\Gamma} \rightharpoonup u_{|\Gamma} \quad \textit{in weak-}L^q(\Gamma),\\
&u^{\varepsilon'}{}_{|\Gamma_0} \rightharpoonup u_{|\Gamma_0} \quad \textit{in weak-}L^{q_0}(\Gamma_0),
\end{aligned}$$

(and $\tilde v^\varepsilon = (u^\varepsilon - u^\varepsilon_{|\Gamma^\varepsilon}$ in $\Sigma^\varepsilon, 0$ in $\Sigma' \setminus \Sigma^\varepsilon) \to 0$ in $L^q(\Sigma')$, $\tilde v_0^\varepsilon = (u^\varepsilon$ in $\Sigma_0^\varepsilon, 0$ in $\Sigma_0' \setminus \Sigma_0^\varepsilon) \to 0$ in $L^{q_0}(\Sigma_0')$). Moreover

$$\liminf \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G\big(h^{\varepsilon'} \circ \sigma H^o(\nabla u^{\varepsilon'})\big) \, \mathrm{d}x \geqslant \int_{\overline{\Gamma}} a H^o(n) G(|u-k|) \, \mathrm{d}s, \tag{4.8}$$

$$\liminf \int_{\Sigma_0^{\varepsilon'}} \frac{a_0^{\varepsilon'} \circ \sigma_0}{h_0^{\varepsilon'} \circ \sigma_0} G_0\big(h_0^{\varepsilon'} \circ \sigma_0 H_0^o\big(\nabla u^{\varepsilon'}\big)\big)\,\mathrm{d}x \geqslant \int_{\overline{\Gamma}_0} a_0 H_0^o(n) G_0(|u|)\,\mathrm{d}s. \tag{4.9}$$

Proof. – We recall that the injection mapping: $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact and that the trace mappings $v \in W^{1,p}(\Omega) \to v_{|\partial\Gamma} \in L^p(\partial\Gamma)$ or $v_{|\partial\Gamma_0} \in L^p(\partial\Gamma_0)$ also are compact. Hence all the convergences follow from Lemma 1 and from compactness and we know that $u \in K$, $u_{|\Gamma} \in L^q(\Gamma)$ and $u_{|\Gamma_0} \in L^{q_0}(\Gamma_0)$. It just remains to prove that $u_{|\underline{\Gamma}} = k$ if $\underline{\Gamma} \neq \emptyset$, $u_{|\underline{\Gamma}_0} = 0$ if $\underline{\Gamma}_0 \neq \emptyset$, and to prove (4.8), (4.9).

⋆ We begin by recalling the following properties of the distance function t. (These properties are already mentionned in [5], where a proof is given.)

$$\nabla t(x).n_H\big(\sigma(x)\big) = 1, \tag{4.10}$$

$$H^o\big(\nabla t(x)\big) = 1, \tag{4.11}$$

$$\frac{1}{|\nabla t(x)|} = H^o\big(n(x)\big), \tag{4.12}$$

where $n(x)$ is the normal at x to the hypersurface $t = t(x)$. Of course one has similar versions of (4.10) to (4.12) with subscripts zero.

⋆ For any $v \in \mathcal{C}^1(\overline{\Sigma^\varepsilon})$, $v_{|\Gamma^\varepsilon} = l$ (constant), we have, using the diffeomorphism of Σ' on $\partial\Omega \times (0, t')$, refining an argument already used above and denoting $s_t = s + t n_H(s)$,

$$\int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\,\mathrm{d}x = \int_{\Gamma} \int_0^{h^\varepsilon(s)} \frac{a^\varepsilon(s)}{h^\varepsilon(s)} G(h^\varepsilon(s) H^o\big(\nabla v(s_t)\big)\psi(s,t)\,\mathrm{d}t\,\mathrm{d}s$$

and

$$\psi(s,t) = \psi(s,0) + \frac{\partial\psi}{\partial t}\big(s, \theta(s,t)\big)t = \frac{1}{|\nabla t|} + \Phi(s,t)t = H^o(n) + \Phi(s,t)t$$

by (4.12), where n is the normal to Γ and where Φ is bounded ($|\Phi| \leqslant M$). Hence

$$\begin{aligned} &\int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\,\mathrm{d}x \\ &\quad = \int_{\Gamma} \int_0^{h^\varepsilon(s)} \frac{a^\varepsilon(s)}{h^\varepsilon(s)} G\big(h^\varepsilon(s) H^o(\nabla v(s_t))\big) H^o(n)\,\mathrm{d}t\,\mathrm{d}s + A^\varepsilon \end{aligned} \tag{4.13}$$

with

$$\begin{aligned} |A^\varepsilon| &\leqslant \int_{\Gamma} \int_0^{h^\varepsilon(s)} \frac{a^\varepsilon(s)}{h^\varepsilon(s)} G\big(h^\varepsilon(s) H^o(\nabla v(s_t))\big) |\Phi(s,t)| t\,\mathrm{d}t\,\mathrm{d}s \\ &\leqslant \varepsilon M \int_{\Gamma} \int_0^{h^\varepsilon(s)} \frac{a^\varepsilon(s)}{h^\varepsilon(s)} G\big(h^\varepsilon(s) H^o(\nabla v(s_t))\big)\,\mathrm{d}t\,\mathrm{d}s \end{aligned}$$

(as $|\Phi| \leqslant M$ and $0 \leqslant t \leqslant h^\varepsilon(s) \leqslant \varepsilon$)

$$\leqslant C\varepsilon M \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\,\mathrm{d}x.$$

We get from (4.13)

$$\int_\Gamma \int_0^{h^\varepsilon(s)} \frac{a^\varepsilon(s)}{h^\varepsilon(s)} G\big(h^\varepsilon(s) H^o(\nabla v(s_t))\big) H^o(n)\,\mathrm{d}t\,\mathrm{d}s$$
$$\leqslant (1 + C\varepsilon M) \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\,\mathrm{d}x. \tag{4.14}$$

As in [5]

$$|v(s) - l| \leqslant \int_0^{h^\varepsilon(s)} H^o\big(\nabla v(s_t)\big)\,\mathrm{d}t$$

and as G is monotone nondecreasing

$$G(|v(s) - l|) \leqslant G\left(\int_0^{h^\varepsilon(s)} H^o\big(\nabla v(s_t)\big)\,\mathrm{d}t \right) = G\left(\frac{1}{h^\varepsilon(s)} \int_0^{h^\varepsilon(s)} h^\varepsilon(s) H^o\big(\nabla v(s_t)\big)\,\mathrm{d}t \right)$$
$$\leqslant \frac{1}{h^\varepsilon(s)} \int_0^{h^\varepsilon(s)} G\big(h^\varepsilon(s) H^o\big(\nabla v(s_t)\big)\big)\,\mathrm{d}t$$

(by Jensen inequality). It follows, by means of (4.14),

$$\int_\Gamma a^\varepsilon(s) G(|v(s) - l|) H^o(n)\,\mathrm{d}s$$
$$= \int_\Gamma \frac{a^\varepsilon(s)}{h^\varepsilon(s)} h^\varepsilon(s) G(|v(s) - l|) H^o(n)\,\mathrm{d}s$$
$$\leqslant \int_\Gamma \frac{a^\varepsilon(s)}{h^\varepsilon(s)} \left(\int_0^{h^\varepsilon(s)} G\big(h^\varepsilon(s) H^o(\nabla v(s_t))\big)\,\mathrm{d}t \right) H^o(n)\,\mathrm{d}s$$
$$\leqslant (1 + C\varepsilon M) \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o(\nabla v)\big)\,\mathrm{d}x.$$

This is also true by density for any $v \in W^{1,q}(\Sigma^\varepsilon)$ such that $v_{|\Gamma^\varepsilon} = l$ and in particular for u^ε, with $u^\varepsilon_{|\Gamma^\varepsilon}$ denoted by k^ε:

$$\int_\Gamma a^\varepsilon(s) G\big(|u^\varepsilon - k^\varepsilon|\big) H^o(n)\,\mathrm{d}s$$

$$\leqslant (1+C\varepsilon M)\int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}G\big(h^\varepsilon\circ\sigma H^o\big(\nabla u^\varepsilon\big)\big)\,\mathrm{d}x$$

$$\leqslant C(1+C\varepsilon M)\ \text{(by Lemma 1)}\ \leqslant C, \tag{4.15}$$

which gives from (2.5), (2.8), (2.10)

$$\int_{\underline{\Gamma}}\big|u^\varepsilon(s)-k^\varepsilon\big|\,\mathrm{d}s$$

$$=\int_{\underline{\Gamma}}\big(a^\varepsilon(s)\big)^{-1/q}\big(a^\varepsilon(s)\big)^{1/q}|u^\varepsilon(s)-k^\varepsilon|\,\mathrm{d}s$$

$$\leqslant\bigg(\int_{\underline{\Gamma}}\big(a^\varepsilon(s)\big)^{-q'/q}\,\mathrm{d}s\bigg)^{1/q'}\bigg(\int_{\underline{\Gamma}}a^\varepsilon(s)\big|u^\varepsilon(s)-k^\varepsilon\big|^q\,\mathrm{d}s\bigg)^{1/q}$$

$$\leqslant\bigg(\int_{\underline{\Gamma}}\big[\big(a^\varepsilon(s)\big)^{-1}\big]^{q'-1}\,\mathrm{d}s\bigg)^{1/q'}(\mu_1)^{-1/q}\delta_2^{1/q}\bigg(\int_{\underline{\Gamma}}a^\varepsilon(s)G\big(\big|u^\varepsilon(s)-k^\varepsilon\big|\big)H^o(n)\,\mathrm{d}s\bigg)^{1/q}$$

$$\leqslant C\bigg(\int_{\underline{\Gamma}}\big[\big(a^\varepsilon(s)\big)^{-1}\big]^{q'-1}\,\mathrm{d}s\bigg)^{1/q'}\to 0.$$

As ${u^{\varepsilon'}}_{|\underline{\Gamma}}\to u_{|\underline{\Gamma}}$ in (strong) $L^p(\underline{\Gamma})$ and as $k^{\varepsilon'}\to k$, we get $u_{|\underline{\Gamma}}=k$. By the same way one can prove that $u_{|\underline{\Gamma}_0}=0$.

⋆ Now we prove (4.8). This is trivial if $\overline{\Gamma}$ is empty and is easy to prove from (4.15) if $q\leqslant p$ since then ${u^{\varepsilon'}}_{|\Gamma}\to u_{|\Gamma}$ in (strong-) $L^q(\Gamma)$ and $k^{\varepsilon'}\to k$ imply that $G(|{u^{\varepsilon'}}_{|\Gamma}-k^{\varepsilon'}|)\to G(|u_{|\Gamma}-k|)$ in (strong-) $L^1(\Gamma)$, so that as $a^\varepsilon\rightharpoonup a$ in weak$\star-L^\infty(\overline{\Gamma})$

$$\int_{\overline{\Gamma}}aG(|u-k|)H^o(n)\,\mathrm{d}s=\lim\int_{\overline{\Gamma}}a^{\varepsilon'}G\big(\big|u^{\varepsilon'}-k^{\varepsilon'}\big|\big)H^o(n)\,\mathrm{d}s$$

$$\leqslant\liminf\int_{\Sigma^{\varepsilon'}}\frac{a^{\varepsilon'}\circ\sigma}{h^{\varepsilon'}\circ\sigma}G\big(h^{\varepsilon'}\circ\sigma H^o\big(\nabla u^{\varepsilon'}\big)\big)\,\mathrm{d}x\quad\text{by (4.15)}.$$

If $\overline{\Gamma}$ is not empty and $q>p$, the proof (also valid otherwise) goes as follows. By convexity

$$G\big(\big|u^\varepsilon-k^\varepsilon\big|\big)\geqslant G(|u-k|)+D\big(\big|u^\varepsilon-k^\varepsilon\big|-|u-k|\big)$$
$$\geqslant G(|u-k|)-D\big|u^\varepsilon-k^\varepsilon-u+k\big|,$$

for any $D\in\partial G(|u-k|)$, the subdifferential of G at the point $|u-k|$. We notice that (2.8) and the monotonicity of G imply

$$\exists\mu>0,\quad\forall\xi\in\mathbb{R}^+,\quad\forall d\in\partial G(\xi),\quad 0\leqslant d\leqslant\mu\xi^{q-1}$$

and hence $D\in L^{q'}(\Gamma)$,

$$\int_{\Gamma} a^{\varepsilon'} G\big(\big|u^{\varepsilon'}-k^{\varepsilon'}\big|\big) H^o(n)\,\mathrm{d}s$$
$$\geqslant \int_{\overline{\Gamma}} a^{\varepsilon'} G\big(\big|u^{\varepsilon'}-k^{\varepsilon'}\big|\big) H^o(n)\,\mathrm{d}s$$
$$\geqslant \int_{\overline{\Gamma}} a^{\varepsilon'} G(|u-k|) H^o(n)\,\mathrm{d}s - \int_{\overline{\Gamma}} D a^{\varepsilon'} \big|u^{\varepsilon'}-k^{\varepsilon'}-u+k\big| H^o(n)\,\mathrm{d}s.$$

Now

$$\int_{\overline{\Gamma}} a^{\varepsilon'} G(|u-k|) H^o(n)\,\mathrm{d}s \to \int_{\overline{\Gamma}} a G(|u-k|) H^o(n)\,\mathrm{d}s.$$

Moreover

$$\int_{\overline{\Gamma}} D a^{\varepsilon'} \big|u^{\varepsilon'}-k^{\varepsilon'}-u+k\big| H^o(n)\,\mathrm{d}s \to 0$$

since $DH^o(n)$ belongs to $L^{q'}(\overline{\Gamma})$ and $a^{\varepsilon'}|u^{\varepsilon'}-k^{\varepsilon'}-u+k| \rightharpoonup 0$ in weak-$L^q(\overline{\Gamma})$, because $a^{\varepsilon'}|u^{\varepsilon'}-k^{\varepsilon'}-u+k|$ is bounded in $L^q(\overline{\Gamma})$ and $a^{\varepsilon'}|u^{\varepsilon'}-k^{\varepsilon'}-u+k| \to 0$ in (strong) $L^p(\overline{\Gamma})$. Therefore by (4.15)

$$\liminf \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'}\circ\sigma}{h^{\varepsilon'}\circ\sigma} G\big(h^{\varepsilon'}\circ\sigma\, H^o\big(\nabla u^{\varepsilon'}\big)\big)\,\mathrm{d}x \geqslant \liminf \int_{\Gamma} a^{\varepsilon'} G\big(\big|u^{\varepsilon'}-k^{\varepsilon'}\big|\big) H^o(n)\,\mathrm{d}s$$
$$\geqslant \int_{\overline{\Gamma}} a G(|u-k|) H^o(n)\,\mathrm{d}\sigma.$$

Inequality (4.9) is proved similarly (see also [5]). □

5. Proof of the convergence of $(\mathcal{P}^\varepsilon)$ to $(\mathcal{P})$

We first prove it from Lemma 3, whose proof is postponed.

LEMMA 3. – *Let $v \in \mathcal{C}^1(\overline{\Omega}) \cap K$, $v_{|\underline{\Gamma}_0} = 0$ and let $l \in \mathbb{R}$. If $\underline{\Gamma} \neq \emptyset$, we assume that $v_{|\underline{\Gamma}} = l$, otherwise there is no additional restriction on v and l. For any such v and l, there exists a sequence of elements $v^\varepsilon \in V^\varepsilon$ such that $v^\varepsilon_{|\Omega} \in K$ and*

$$v^\varepsilon{}_{|\Omega} = v, \quad v^\varepsilon_{|\Gamma^\varepsilon} = l, \qquad \|v^\varepsilon - l\|_{L^q(\Sigma^\varepsilon)} \to 0, \quad v^\varepsilon_{|\Gamma_0^\varepsilon} = 0, \quad \big\|v^\varepsilon\big\|_{L^{q_0}(\Sigma_0^\varepsilon)} \to 0, \tag{5.1}$$

$$\limsup \int_{\Sigma^\varepsilon} \frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma} G\big(h^\varepsilon\circ\sigma\, H^o\big(\nabla v^\varepsilon\big)\big)\,\mathrm{d}x \leqslant \int_{\overline{\Gamma}} a G(|v-l|) H^o(n)\,\mathrm{d}s, \tag{5.2}$$

$$\limsup \int_{\Sigma_0^\varepsilon} \frac{a_0^\varepsilon\circ\sigma_0}{h_0^\varepsilon\circ\sigma_0} G_0\big(h_0^\varepsilon\circ\sigma_0 H_0^o\big(\nabla v^\varepsilon\big)\big)\,\mathrm{d}x \leqslant \int_{\overline{\Gamma}_0} a_0 G_0(|v|) H^o(n)\,\mathrm{d}s. \tag{5.3}$$

Proof of Theorem 1 assuming Lemma 3. – Let u^ε be the solution of $(\mathcal{P}^\varepsilon)$. From Lemma 1,

$$\int_{\Sigma^\varepsilon} \left|u^\varepsilon - u^\varepsilon{}_{|\Gamma^\varepsilon}\right|^q \mathrm{d}x \quad \text{and} \quad \int_{\Sigma_0^\varepsilon} \left|u^\varepsilon\right|^{q_0} \mathrm{d}x \to 0,$$

so that (2) is proved. Let $(u,k) \in W$ and ε' be as in Lemma 2. As the solution of $(\mathcal{P})$ is unique it just remains to prove that (u,k) solves $(\mathcal{P})$ and that $J^{\varepsilon'}(u^{\varepsilon'}) \to J(u,k)$. (Then a classical argument gives $\varepsilon' \equiv \varepsilon$: one has the convergences for the whole sequence ε.) Let $v \in \mathcal{C}^1(\overline{\Omega}) \cap K$, $v_{|\underline{\Gamma}_0} = 0$ and let $l \in \mathbb{R}$. If $\underline{\Gamma} \neq \emptyset$, assume that $v_{|\underline{\Gamma}} = l$. Finally let v^ε be as in Lemma 3.

$$\begin{aligned}
\text{(A)} \quad \liminf J^{\varepsilon'}(u^{\varepsilon'}) &\leqslant \limsup J^{\varepsilon'}(u^{\varepsilon'}) \leqslant \limsup J^{\varepsilon'}(v^{\varepsilon'}) \\
&= \limsup \Bigg\{ F(v) + \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G\big(h^{\varepsilon'} \circ \sigma H^o(\nabla v^{\varepsilon'})\big)\,\mathrm{d}x \\
&\quad + \int_{\Sigma_0^{\varepsilon'}} \frac{a_0^{\varepsilon'} \circ \sigma_0}{h_0^{\varepsilon'} \circ \sigma_0} G_0\big(h_0^{\varepsilon'} \circ \sigma_0 H_0^o(\nabla v^{\varepsilon'})\big)\,\mathrm{d}x \\
&\quad - \int_\Omega f^{\varepsilon'} v^{\varepsilon'}\,\mathrm{d}x - \int_{\Sigma^{\varepsilon'}} g^{\varepsilon'} v^{\varepsilon'}\,\mathrm{d}x - \int_{\Sigma_0^{\varepsilon'}} g_0^{\varepsilon'} v^{\varepsilon'}\,\mathrm{d}x - I^{\varepsilon'} l \Bigg\}.
\end{aligned}$$

But $\int_\Omega f^{\varepsilon'} v\,\mathrm{d}x \to \int_\Omega f v\,\mathrm{d}x$, since $f^{\varepsilon'} \rightharpoonup f$ in weak-$L^{p'}(\Omega)$, $v \in L^p(\Omega)$, and $I^{\varepsilon'} \to I$ in $\mathbb{R}$. Moreover

$$\begin{aligned}
\left|\int_{\Sigma^{\varepsilon'}} g^{\varepsilon'} v^{\varepsilon'}\,\mathrm{d}x\right| &= \left|\int_{\Sigma^{\varepsilon'}} g^{\varepsilon'}(v^{\varepsilon'} - l + l)\,\mathrm{d}x\right| \\
&\leqslant \|g^{\varepsilon'}\|_{L^{q'}(\Sigma^{\varepsilon'})}\big(\|v^{\varepsilon'} - l\|_{L^q(\Sigma^{\varepsilon'})} + |l|\,|\Sigma^{\varepsilon'}|^{1/q}\big) \\
&\leqslant C\big(\|v^{\varepsilon'} - l\|_{L^q(\Sigma^{\varepsilon'})} + |l|\,|\Sigma^{\varepsilon'}|^{1/q}\big)
\end{aligned}$$

(by (2.14)), which tends to zero thanks to (5.1). Similarly,

$$\int_{\Sigma_0^{\varepsilon'}} g_0^{\varepsilon'} v^{\varepsilon'}\,\mathrm{d}x \to 0,$$

so that by Lemma 3,

$$\begin{aligned}
\liminf J^{\varepsilon'}(u^{\varepsilon'}) &\leqslant \limsup J^{\varepsilon'}(u^{\varepsilon'}) \\
&\leqslant F(v) - \int_\Omega f v\,\mathrm{d}x - Il + \limsup \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G\big(h^{\varepsilon'} \circ \sigma H^o(\nabla v^{\varepsilon'})\big)\,\mathrm{d}x \\
&\quad + \limsup \int_{\Sigma_0^{\varepsilon'}} \frac{a_0^{\varepsilon'} \circ \sigma_0}{h_0^{\varepsilon'} \circ \sigma_0} G_0\big(h_0^{\varepsilon'} \circ \sigma_0 H_0^o(\nabla v^{\varepsilon'})\big)\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leqslant F(v) - \int_\Omega f v \,\mathrm{d}x - Il \\
&\quad + \int_{\overline{\Gamma}} aG(|v-l|)H^o(n)\,\mathrm{d}s + \int_{\overline{\Gamma}_0} a_0 G_0(|v|)H_0^o(n)\,\mathrm{d}s \\
&= J(v,l).
\end{aligned}
$$

$$
\begin{aligned}
\text{(B)}\quad J^{\varepsilon'}\big(u^{\varepsilon'}\big) &= F\big(u^{\varepsilon'}{}_{|\Omega}\big) + \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'}\circ\sigma}{h^{\varepsilon'}\circ\sigma} G\big(h^{\varepsilon'}\circ\sigma H^o\big(\nabla u^{\varepsilon'}\big)\big)\,\mathrm{d}x \\
&\quad + \int_{\Sigma_0^{\varepsilon'}} \frac{a_0^{\varepsilon'}\circ\sigma_0}{h_0^{\varepsilon'}\circ\sigma_0} G_0\big(h_0^{\varepsilon'}\circ\sigma_0 H_0^o\big(\nabla u^{\varepsilon'}\big)\big)\,\mathrm{d}x \\
&\quad - \int_\Omega f^{\varepsilon'} u^{\varepsilon'}\,\mathrm{d}x - \int_{\Sigma^{\varepsilon'}} g^{\varepsilon'} u^{\varepsilon'}\,\mathrm{d}x - \int_{\Sigma_0^{\varepsilon'}} g_0^{\varepsilon'} u^{\varepsilon'}\,\mathrm{d}x - I^{\varepsilon'} u^{\varepsilon'}_{|\Gamma^{\varepsilon'}}.
\end{aligned}
$$

As above

$$
\left|\int_{\Sigma^{\varepsilon'}} g^{\varepsilon'} u^{\varepsilon'}\,\mathrm{d}x\right| \leqslant C\big(\big\|u^{\varepsilon'} - u^{\varepsilon'}_{|\Gamma^{\varepsilon'}}\big\|_{L^q(\Sigma^{\varepsilon'})} + \big|u^{\varepsilon'}_{|\Gamma^{\varepsilon'}}\big|\big|\Sigma^{\varepsilon'}\big|^{1/q}\big)
$$

tends to zero. Of course the same holds for the corresponding integral over $\Sigma_0^{\varepsilon'}$. Then from Lemma 2 and (2.14) we get

$$
\begin{aligned}
\liminf J^{\varepsilon'}\big(u^{\varepsilon'}\big) &\geqslant \liminf F\big(u^{\varepsilon'}{}_{|\Omega}\big) + \liminf \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'}\circ\sigma}{h^{\varepsilon'}\circ\sigma} G\big(h^{\varepsilon'}\circ\sigma H^o\big(\nabla u^{\varepsilon'}\big)\big)\,\mathrm{d}x \\
&\quad + \liminf \int_{\Sigma_0^{\varepsilon'}} \frac{a_0^{\varepsilon'}\circ\sigma_0}{h_0^{\varepsilon'}\circ\sigma_0} G_0\big(h_0^{\varepsilon'}\circ\sigma_0 H_0^o\big(\nabla u^{\varepsilon'}\big)\big)\,\mathrm{d}x - \int_\Omega f u\,\mathrm{d}x - Ik \\
&\geqslant F(u) + \int_{\overline{\Gamma}} aG(|u-k|)H^o(n)\,\mathrm{d}s + \int_{\overline{\Gamma}_0} a_0 G_0(|u|)H_0^o(n)\,\mathrm{d}s \\
&\quad - \int_\Omega f u\,\mathrm{d}x - Ik \\
&= J(u,k)
\end{aligned}
$$

(by using the lower semi-continuity of F, the weak-$W^{1,p}(\Omega)$ convergence of $u^{\varepsilon'}{}_{|\Omega}$ and (4.8)).

(C) From (A) and (B): Fo r any $l \in \mathbb{R}$, $v \in \mathcal{C}^1(\overline{\Omega}) \cap K$ such that $v_{|\underline{\Gamma}_0} = 0$ and $v_{|\underline{\Gamma}} = l$ (only if $\underline{\Gamma} \neq \emptyset$),

$$
J(u,k) \leqslant \liminf J^{\varepsilon'}\big(u^{\varepsilon'}\big) \leqslant \limsup J^{\varepsilon'}\big(u^{\varepsilon'}\big) \leqslant J(v,l).
$$

By the density property (2.13) and by continuity, this is also true for any $(v,l)\in W$, $v\in K$, that is (u,k) solves $(\mathcal{P})$ and $J^{\varepsilon'}(u^{\varepsilon'})$ tends to $J(u,k)$. This completes the proof of Theorem 1, except that we have to prove Lemma 3. □

Proof of Lemma 3. – Let $v\in\mathcal{C}^1(\overline{\Omega})\cap K$, $v_{|\underline{\Gamma}_0}=0$ and let $l\in\mathbb{R}$. If $\underline{\Gamma}\neq\emptyset$, we assume that $v_{|\underline{\Gamma}}=l$. Let v^ε be the continuous function defined in $\overline{\Omega}^\varepsilon$ by

$$v^\varepsilon(x)=\begin{cases} v(x) & \text{if } x\in\Omega,\\ w(x)\varphi^\varepsilon(x)+l(1-\varphi^\varepsilon(x)) & \text{if } x\in\overline{\Sigma}^\varepsilon,\\ w_0(x)\varphi_0^\varepsilon(x) & \text{if } x\in\overline{\Sigma}_0^\varepsilon,\end{cases}$$

with $w(x)=(v\circ\sigma)(x)$, $w_0(x)=(v\circ\sigma_0)(x)$, $\varphi^\varepsilon(x)=1-\frac{t(x)}{(h^\varepsilon\circ\sigma)(x)}$, $\varphi_0^\varepsilon(x)=1-\frac{t_0(x)}{(h_0^\varepsilon\circ\sigma_0)(x)}$, so that $v^\varepsilon=v$ in $\overline{\Omega}$, $v^\varepsilon=l$ on Γ^ε, $v^\varepsilon=0$ on Γ_0^ε. Clearly $v^\varepsilon\in V^\varepsilon$. Moreover

$$\|v^\varepsilon-l\|^q_{L^q(\Sigma^\varepsilon)}=\|(w-l)\varphi^\varepsilon\|^q_{L^q(\Sigma^\varepsilon)}\leqslant\int_{\Sigma^\varepsilon}|w-l|^q\,\mathrm{d}x\to 0,$$

since $w-l$ is bounded and since $|\Sigma^\varepsilon|$ tends to zero. Similarly $\|v^\varepsilon\|_{L^{q_0}(\Sigma_0^\varepsilon)}$ tends to zero. It just remains to prove the inequalities (5.2), (5.3) in Lemma 3. In the following we consider only (5.2), since the proof of (5.3) is very similar and was published in [5]. One writes for simplicity

$$K^\varepsilon(x,\xi)=\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}(x)G\big((h^\varepsilon\circ\sigma)(x)H^o(\xi)\big),$$

so that, denoting $\Sigma_r^\varepsilon=\{s+tn_H(s),\, s\in\overline{\Gamma},\, 0<t<h^\varepsilon(s)\}$ the reduced part of Σ^ε with H-projection in $\overline{\Gamma}$,

$$\int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}G\big(h^\varepsilon\circ\sigma H^o(\nabla v^\varepsilon)\big)\,\mathrm{d}x=\int_{\Sigma_r^\varepsilon}K^\varepsilon(x,\nabla v^\varepsilon)\,\mathrm{d}x,\tag{5.4}$$

where K^ε is convex in ξ. Using a classical convexity argument valid for any $\theta\in(0,1)$,

$$\begin{aligned}\int_{\Sigma_r^\varepsilon}K^\varepsilon(x,\nabla v^\varepsilon)\,\mathrm{d}x&=\int_{\Sigma_r^\varepsilon}K^\varepsilon\left(x,\theta\varphi^\varepsilon\frac{\nabla w}{\theta}+(1-\theta)(w-l)\frac{\nabla\varphi^\varepsilon}{1-\theta}\right)\mathrm{d}x\\ &\leqslant\theta\int_{\Sigma_r^\varepsilon}K^\varepsilon\left(x,\varphi^\varepsilon\frac{\nabla w}{\theta}\right)\mathrm{d}x+(1-\theta)\int_{\Sigma_2^\varepsilon}K^\varepsilon\left(x,(w-l)\frac{\nabla\varphi^\varepsilon}{1-\theta}\right)\mathrm{d}x.\end{aligned}$$

By definition of K^ε and by (2.5), (2.6), (2.8), (2.11),

$$\begin{aligned}\theta\int_{\Sigma_r^\varepsilon}K^\varepsilon\left(x,\varphi^\varepsilon\frac{\nabla w}{\theta}\right)\mathrm{d}x&=\theta\int_{\Sigma_r^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}G\left(h^\varepsilon\circ\sigma\frac{\varphi^\varepsilon H^o(\nabla w)}{\theta}\right)\mathrm{d}x\\ &\leqslant\frac{\mu_2}{\delta_1^q\theta^{q-1}}\int_{\Sigma_r^\varepsilon}(a^\varepsilon\circ\sigma)(h^\varepsilon\circ\sigma)^{q-1}(\varphi^\varepsilon)^q|\nabla w|^q\,\mathrm{d}x\leqslant\frac{C\mu_2}{\delta_1^q\theta^{q-1}}\varepsilon^{q-1}\int_{\Sigma_r^\varepsilon}|\nabla w|^q\,\mathrm{d}x\to 0,\end{aligned}$$

since ∇w is bounded, so that

$$\limsup \int_{\Sigma_r^\varepsilon} K^\varepsilon\big(x, \nabla v^\varepsilon\big)\,\mathrm{d}x \leqslant (1-\theta)\limsup \int_{\Sigma_r^\varepsilon} K^\varepsilon\left(x, \frac{(w-l)\nabla\varphi^\varepsilon}{1-\theta}\right)\mathrm{d}x$$

for any $\theta \in (0,1)$ and letting θ tend to zero

$$\limsup \int_{\Sigma_r^\varepsilon} K^\varepsilon\big(x, \nabla v^\varepsilon\big)\,\mathrm{d}x \leqslant \limsup \int_{\Sigma_r^\varepsilon} K^\varepsilon\big(x, (w-l)\nabla\varphi^\varepsilon\big)\,\mathrm{d}x. \tag{5.5}$$

Now on Σ_r^ε, we have by the classical properties of the norm H^o and (4.11)

$$\begin{aligned} H^o\big(\nabla\varphi^\varepsilon\big) = H^o\left(-\frac{\nabla t}{h^\varepsilon\circ\sigma} + \frac{t\nabla(h^\varepsilon\circ\sigma)}{(h^\varepsilon\circ\sigma)^2}\right) &\leqslant \frac{H^o(\nabla t)}{h^\varepsilon\circ\sigma} + \frac{t}{h^\varepsilon\circ\sigma}\frac{H^o(\nabla(h^\varepsilon\circ\sigma))}{h^\varepsilon\circ\sigma} \\ &\leqslant \frac{1}{h^\varepsilon\circ\sigma} + \frac{H^o(\nabla(h^\varepsilon\circ\sigma))}{h^\varepsilon\circ\sigma} \end{aligned}$$

which implies, by the same convexity argument and by monotonicity, that

$$\begin{aligned} \int_{\Sigma_r^\varepsilon} K^\varepsilon\big(x, (w-l)\nabla\varphi^\varepsilon\big)\,\mathrm{d}x &\leqslant \int_{\Sigma_r^\varepsilon} \frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma} G\big(|w-l| + |w-l| H^o\big(\nabla\big(h^\varepsilon\circ\sigma\big)\big)\big)\,\mathrm{d}x \\ &\leqslant \theta \int_{\Sigma_r^\varepsilon} \frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma} G\left(\frac{|w-l|}{\theta}\right)\mathrm{d}x + (1-\theta)\int_{\Sigma_r^\varepsilon} \frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma} G\left(\frac{|w-l| H^o(\nabla(h^\varepsilon\circ\sigma))}{1-\theta}\right)\mathrm{d}x. \end{aligned}$$

Using (2.5) and (2.8), the last integral is bounded by

$$\begin{aligned} \frac{\mu_2}{\delta_1^q(1-\theta)^q}\int_{\Sigma_r^\varepsilon} a^\varepsilon\circ\sigma \frac{|\nabla(h^\varepsilon\circ\sigma)|^q}{h^\varepsilon\circ\sigma}|w-l|^q\,\mathrm{d}x &\leqslant \frac{C}{(1-\theta)^q}\int_{\Sigma_2^\varepsilon} a^\varepsilon\circ\sigma\frac{|\nabla(h^\varepsilon\circ\sigma)|^q}{h^\varepsilon\circ\sigma}\,\mathrm{d}x \\ &\leqslant C\int_{\overline{\Gamma}} a^\varepsilon\big|\nabla h^\varepsilon\big|^q\,\mathrm{d}s \end{aligned}$$

and therefore by (2.11), (2.12) it tends to zero with ε: one obtains by letting θ tend to one

$$\limsup \int_{\Sigma_r^\varepsilon} K^\varepsilon\big(x, (w-l)\nabla\varphi^\varepsilon\big)\,\mathrm{d}x \leqslant \limsup \int_{\Sigma_r^\varepsilon} \frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma} G(|w-l|)\,\mathrm{d}x. \tag{5.6}$$

Finally thanks to the usual diffeomorphism argument

$$\begin{aligned} \int_{\Sigma_r^\varepsilon} \frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma} G(|w-l|)\,\mathrm{d}x &= \int_{\overline{\Gamma}}\int_0^{h^\varepsilon(s)} \frac{a^\varepsilon(s)}{h^\varepsilon(s)} G(|v(s)-l|)\big(H^o(n(s)) + \Phi(s,t)t\big)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{\overline{\Gamma}} a^\varepsilon(s) G(|v(s)-l|) b^\varepsilon(s)\,\mathrm{d}s, \end{aligned}$$

where

$$b^\varepsilon(s) = \frac{1}{h^\varepsilon(s)} \int_0^{h^\varepsilon(s)} \big(H^o(n(s)) + \Phi(s,t)t\big)\,\mathrm{d}t.$$

As b^ε is uniformly bounded and $b^\varepsilon(s) \to H^o(n(s))$ for a.e. s of $\overline{\Gamma}$, we have $b^\varepsilon \to H^o(n)$ in $L^1(\overline{\Gamma})$. Then, as $a^\varepsilon \rightharpoonup a$ in weak$\star - L^\infty(\overline{\Gamma})$ (see (2.11)), one obtains

$$\int_{\Sigma_r^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(|w - l|)\,\mathrm{d}x \to \int_{\overline{\Gamma}} aG(|v - l|)H^o(n)\,\mathrm{d}s$$

and by means of (5.5), (5.6) this implies

$$\limsup \int_{\Sigma_r^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\big(h^\varepsilon \circ \sigma H^o\big(\nabla v^\varepsilon\big)\big)\,\mathrm{d}x$$
$$= \limsup \int_{\Sigma_r^\varepsilon} K^\varepsilon\big(x, \nabla v^\varepsilon\big)\,\mathrm{d}x \leqslant \int_{\overline{\Gamma}} aG(|v - l|)H^o(n)\,\mathrm{d}s$$

and the proof is complete. □

6. Example

For given Σ^ε, Σ_0^ε and μ^ε (respectively μ_0^ε) in $L^\infty(\Gamma)$ (respectively $L^\infty(\Gamma_0)$), let us consider the problem of minimizing on $\{v : \Omega^\varepsilon \to \mathbb{R},\ v \in V^\varepsilon,\ v_{|\Omega} \in K\}$ the functional

$$\begin{aligned} J^\varepsilon(v) = {} & \frac{1}{p}\int_\Omega (A\nabla v, \nabla v)^{p/2}\,\mathrm{d}x + \int_\Omega \Psi(v)\,\mathrm{d}x + \frac{1}{q}\int_{\Sigma^\varepsilon} \mu^\varepsilon \circ \sigma (B\nabla v, \nabla v)^{q/2}\,\mathrm{d}x \\ & + \frac{1}{q_0}\int_{\Sigma_0^\varepsilon} \mu_0^\varepsilon \circ \sigma_0 (B_0\nabla v, \nabla v)^{q_0/2}\,\mathrm{d}x - \int_\Omega f^\varepsilon v\,\mathrm{d}x \\ & - \int_{\Sigma^\varepsilon} g^\varepsilon v\,\mathrm{d}x - \int_{\Sigma_0^\varepsilon} g_0^\varepsilon v\,\mathrm{d}x - I^\varepsilon v_{|\Gamma^\varepsilon}, \end{aligned}$$

where A, B and B_0 are given coercive symmetric matrices with constant coefficients, $\Psi : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^1$, convex, nonnegative and satisfies the growth condition:

$$\Psi(\theta) \leqslant C|\theta|^p + C',$$

for some positive constants C and C'. Denoting by ϕ the derivative of Ψ, it follows that ϕ satisfies a similar growth condition:

$$|\phi(\theta)| \leqslant C|\theta|^{p-1} + C'.$$

Then it is classical that $v \to \Psi(v)$ is continuous from $L^p(\Omega)$ into $L^1(\Omega)$ and that $v \to \phi(v)$ is continuous from $L^p(\Omega)$ into $L^{p'}(\Omega)$.

For Σ^ε (respectively Σ_0^ε) given, let us define $h^\varepsilon:\Gamma\to\mathbb{R}$ (respectively $h_0^\varepsilon:\Gamma_0\to\mathbb{R}$) by

$$\Sigma^\varepsilon=\{s+\theta Bn(s),0<\theta<h^\varepsilon(s)\big(Bn(s),n(s)\big)^{-1/2}\}$$

$$\big(\text{respectively } \Sigma_0^\varepsilon=\{s+\theta B_0n(s),0<\theta<h_0^\varepsilon(s)(B_0n(s),n(s))^{-1/2}\}\big),$$

where $n(s)$ is the normal to Γ (respectively Γ_0) at s, pointing outside Ω. Now let us assume that

- $\forall\varepsilon$, $h^\varepsilon\in\mathcal{C}^1(\Gamma)$ and $\forall s\in\Gamma$, $0<h^\varepsilon(s)\leqslant\varepsilon$ (respectively $h_0^\varepsilon\in\mathcal{C}^1(\Gamma_0), 0<h^\varepsilon(s)\leqslant\varepsilon$);
- $\exists\alpha>0$, a.e. $s\in\Gamma,\forall\varepsilon,\mu^\varepsilon(s)\geqslant\alpha(h^\varepsilon(s))^{q-1}$ (respectively a.e. $s\in\Gamma,\mu_0^\varepsilon(s)\geqslant\alpha(h_0^\varepsilon(s))^{q_0-1}$);
- there exists a partition $\underline{\Gamma}$, $\overline{\Gamma}$ of Γ and a partition $\underline{\Gamma}_0$, $\overline{\Gamma}_0$ of Γ_0 such that

$$\int_{\underline{\Gamma}} h^\varepsilon\big(\mu^\varepsilon\big)^{1-q'}\,\mathrm{d}s\to 0\quad\text{and}\quad\int_{\overline{\Gamma}}\big|\nabla h^\varepsilon\big|^q\,\mathrm{d}s\to 0$$

$$\left(\text{respectively }\int_{\underline{\Gamma}_0} h_0^\varepsilon\big(\mu_0^\varepsilon\big)^{1-q_0'}\,\mathrm{d}s\to 0\quad\text{and}\quad\int_{\overline{\Gamma}_0}\big|\nabla h_0^\varepsilon\big|^{q_0}\,\mathrm{d}s\to 0\right),$$

$$\mu^\varepsilon(h^\varepsilon)^{1-q}\rightharpoonup a \text{ in weak}\star\text{-}L^\infty(\overline{\Gamma})\ (\text{respectively } \mu_0^\varepsilon(h_0^\varepsilon)^{1-q_0}\rightharpoonup a_0$$

$$\text{in weak}\star\text{-}L^\infty(\overline{\Gamma}_0)),\text{ for some } a \text{ in } L^\infty(\overline{\Gamma}) \text{ and } a_0 \text{ in } L^\infty(\overline{\Gamma}_0).$$

(The data K, I^ε, f^ε, g^ε, g_0^ε satisfy the general assumptions.)

In this case one can apply the abstract result given in Theorem 1 with

$$F(v)=\frac{1}{p}\int_\Omega(A\nabla v,\nabla v)^{p/2}\,\mathrm{d}x+\int_\Omega\Psi(v)\,\mathrm{d}x,$$

$$G(\eta)=\frac{1}{q}\eta^q\quad\left(\text{respectively } G_0(\eta)=\frac{1}{q_0}\eta^{q_0}\right),$$

$$H(\xi)=\big(B^{-1}\xi,\xi\big)^{1/2}\quad\big(\text{respectively } H_0(\xi)=\big(B_0^{-1}\xi,\xi\big)^{1/2}\big),$$

$$H^o(\xi^o)=\big(B\xi^o,\xi^o\big)^{1/2}\quad\big(\text{respectively } H_0^o(\xi^o)=\big(B_0\xi^o,\xi^o\big)^{1/2}\big),$$

$$n_H(s)=Bn(s)\big(Bn(s),n(s)\big)^{-1/2}\quad\big(\text{respectively } n_{H_0}(s)=B_0n(s)(B_0n(s),n(s))^{-1/2}\big).$$

The application of Theorem 1 gives as limit problem

$$\operatorname{Inf}\{J(v,l),(v,l)\in Wv\in K\},\tag{$\mathcal{P}$}$$

where W is as before and where

$$J(v,l)=\frac{1}{p}\int_{\Omega}(A\nabla v,\nabla v)^{p/2}\,\mathrm{d}x+\int_{\Omega}\Psi(v)\,\mathrm{d}x+\frac{1}{q}\int_{\overline{\Gamma}}(Bn,n)^{1/2}a|v-l|^{q}\,\mathrm{d}s$$
$$+\frac{1}{q_0}\int_{\overline{\Gamma}_0}(B_0n,n)^{1/2}a_0|v|^{q_0}\,\mathrm{d}s-\int_{\Omega}fv\,\mathrm{d}x-Il.$$

Let us interpret this result. The solution u^ε of $(\mathcal{P}^\varepsilon)$ is characterized by the variational formulation:

$$u^\varepsilon\in V^\varepsilon,\ u^\varepsilon_{|\Omega}\in K,\ \text{ and for every } v\in V^\varepsilon,\ v_{|\Omega}\in K,$$
$$\int_{\Omega}\big(A\nabla u^\varepsilon,\nabla u^\varepsilon\big)^{\frac{p}{2}-1}\big(A\nabla u^\varepsilon,\nabla\big(v-u^\varepsilon\big)\big)\,\mathrm{d}x+\int_{\Omega}\phi(u^\varepsilon)(v-u^\varepsilon)\,\mathrm{d}x$$
$$+\int_{\Sigma^\varepsilon}\big(\mu^\varepsilon\circ\sigma\big)\big(B\nabla u^\varepsilon,\nabla u^\varepsilon\big)^{\frac{q}{2}-1}\big(B\nabla u^\varepsilon,\nabla\big(v-u^\varepsilon\big)\big)\,\mathrm{d}x$$
$$+\int_{\Sigma_0^\varepsilon}\big(\mu_0^\varepsilon\circ\sigma_0\big)\big(B_0\nabla u^\varepsilon,\nabla u^\varepsilon\big)^{\frac{q_0}{2}-1}\big(B_0\nabla u^\varepsilon,\nabla\big(v-u^\varepsilon\big)\big)\,\mathrm{d}x$$
$$-\int_{\Omega}f^\varepsilon\big(v-u^\varepsilon\big)\,\mathrm{d}x-\int_{\Sigma^\varepsilon}g^\varepsilon\big(v-u^\varepsilon\big)\,\mathrm{d}x-\int_{\Sigma_0^\varepsilon}g_0^\varepsilon\big(v-u^\varepsilon\big)\,\mathrm{d}x-I^\varepsilon\big(v-u^\varepsilon\big)_{|\Gamma^\varepsilon}\geqslant 0.$$

The solution u of the limit problem $(\mathcal{P})$ is characterized by the variational formulation:

$$u\in K,\,k\in\mathbb{R}u_{|\overline{\Gamma}}\in L^{q}(\overline{\Gamma}),\,u_{|\overline{\Gamma}_0}\in L^{q_0}(\overline{\Gamma}_0),\,u=k\text{ on }\underline{\Gamma},\,u=0\text{ on }\underline{\Gamma}_0$$

$$\text{and for any } v\in K,\,l\in\mathbb{R}v_{|\overline{\Gamma}}\in L^{q}(\overline{\Gamma}),\,v_{|\overline{\Gamma}_0}\in L^{q_0}(\overline{\Gamma}_0),\,v=l\text{ on }\underline{\Gamma},\,v=0\text{ on }\underline{\Gamma}_0,$$

$$\int_{\Omega}(A\nabla u,\nabla u)^{\frac{p}{2}-1}\big(A\nabla u,\nabla(v-u)\big)\,\mathrm{d}x+\int_{\Omega}\phi(u)(v-u)\,\mathrm{d}x$$
$$+\int_{\overline{\Gamma}}(Bn,n)^{1/2}a|u-k|^{q-2}(u-k)(v-l-u+k)\,\mathrm{d}s$$
$$+\int_{\overline{\Gamma}_0}(B_0n,n)^{1/2}a_0|u|^{q_0-2}u(v-u)\,\mathrm{d}s-\int_{\Omega}f(v-u)\,\mathrm{d}x-I(l-k)\geqslant 0.$$

For example, if $K=W^{1,p}(\Omega)$, u^ε is characterized by

$$u^\varepsilon\in V^\varepsilon\text{ and for every } v\in V^\varepsilon,$$
$$\int_{\Omega}\big(A\nabla u^\varepsilon,\nabla u^\varepsilon\big)^{\frac{p}{2}-1}\big(A\nabla u^\varepsilon,\nabla v\big)\,\mathrm{d}x+\int_{\Omega}\phi\big(u^\varepsilon\big)v\,\mathrm{d}x$$
$$+\int_{\Sigma^\varepsilon}\big(\mu^\varepsilon\circ\sigma\big)\big(B\nabla u^\varepsilon,\nabla u^\varepsilon\big)^{\frac{q}{2}-1}\big(B\nabla u^\varepsilon,\nabla v\big)\,\mathrm{d}x$$

$$+\int_{\Sigma_0^\varepsilon} (\mu_0^\varepsilon \circ \sigma_0)\big(B_0\nabla u^\varepsilon, \nabla u^\varepsilon\big)^{\frac{q_0}{2}-1}\big(B_0\nabla u^\varepsilon, \nabla v\big)\,\mathrm{d}x$$

$$-\int_\Omega f^\varepsilon v\,\mathrm{d}x - \int_{\Sigma^\varepsilon} g^\varepsilon v\,\mathrm{d}x - \int_{\Sigma_0^\varepsilon} g_0^\varepsilon v\,\mathrm{d}x - I^\varepsilon v_{|\Gamma^\varepsilon} = 0$$

and u is characterized by:

$$u \in W^{1,p}(\Omega), k \in \mathbb{R} u_{|\overline{\Gamma}} \in L^q(\overline{\Gamma}), u_{|\overline{\Gamma}_0} \in L^{q_0}(\overline{\Gamma}_0), u = k \text{ on } \underline{\Gamma}, u = 0 \text{ on } \underline{\Gamma}_0$$

and for any $v \in W^{1,p}(\Omega), l \in \mathbb{R} v_{|\overline{\Gamma}} \in L^q(\overline{\Gamma}), v_{|\overline{\Gamma}_0} \in L^{q_0}(\overline{\Gamma}_0), v = l$ on $\underline{\Gamma}$, $v = 0$ on $\underline{\Gamma}_0$,

$$\int_\Omega (A\nabla u, \nabla u)^{\frac{p}{2}-1}(A\nabla u, \nabla v)\,\mathrm{d}x + \int_\Omega \phi(u) v\,\mathrm{d}x$$

$$+\int_{\overline{\Gamma}} (Bn, n)^{1/2} a|u-k|^{q-2}(u-k)(v-l)\,\mathrm{d}s$$

$$+\int_{\overline{\Gamma}_0} (B_0 n, n)^{1/2} a_0 |u|^{q_0-2} u v\,\mathrm{d}s - \int_\Omega f v\,\mathrm{d}x - I l = 0.$$

In other terms, u^ε is a weak solution of

$$\begin{cases} -\operatorname{div}\big((A\nabla u^\varepsilon, \nabla u^\varepsilon)^{\frac{p}{2}-1} A\nabla u^\varepsilon\big) + \phi(u^\varepsilon) = f^\varepsilon & \text{in } \Omega, \\ -\operatorname{div}\big(\mu^\varepsilon \circ \sigma\,(B\nabla u^\varepsilon, \nabla u^\varepsilon)^{\frac{q}{2}-1} B\nabla u^\varepsilon\big) = g^\varepsilon & \text{in } \Sigma^\varepsilon, \\ -\operatorname{div}\big(\mu^\varepsilon \circ \sigma_0 (B_0\nabla u^\varepsilon, \nabla u^\varepsilon)^{\frac{q_0}{2}-1} B_0\nabla u^\varepsilon\big) = g_0^\varepsilon & \text{in } \Sigma_0^\varepsilon, \\ \big(A\nabla u^\varepsilon, \nabla u^\varepsilon\big)^{\frac{p}{2}-1}\big(A\nabla u^\varepsilon, n\big) = 0 & \text{on } \partial\Omega \setminus (\Gamma \cup \Gamma_0), \\ u^\varepsilon \text{ is contant on } \Gamma^\varepsilon, & \\ \int_{\Gamma^\varepsilon} \big(\mu^\varepsilon \circ \sigma\big)\big(B\nabla u^\varepsilon, \nabla u^\varepsilon\big)^{\frac{q}{2}-1}\big(B\nabla u^\varepsilon, n\big)\,\mathrm{d}s = I^\varepsilon, & \\ u^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ + \text{transmission conditions} & \text{on } \Gamma \cup \Gamma_0, \end{cases}$$

where the transmission conditions read

$$\begin{cases} u^\varepsilon_{|\Omega} = u^\varepsilon_{|\Sigma^\varepsilon} \text{ on } \Gamma, u^\varepsilon_{|\Omega} = u^\varepsilon_{|\Sigma_0^\varepsilon} & \text{on } \Gamma_0, \\ \big(A\nabla u^\varepsilon_{|\Omega}, \nabla u^\varepsilon_{|\Omega}\big)^{\frac{p}{2}-1}\big(A\nabla u^\varepsilon_{|\Omega}, n\big) & \\ \quad = \big(\mu^\varepsilon \circ \sigma\big)\big(B\nabla u^\varepsilon_{|\Sigma^\varepsilon}, \nabla u^\varepsilon_{|\Sigma^\varepsilon}\big)^{\frac{q}{2}-1}\big(B\nabla u^\varepsilon_{|\Sigma^\varepsilon}, n\big) & \text{on } \Gamma, \\ \big(A\nabla u^\varepsilon_{|\Omega}, \nabla u^\varepsilon_{|\Omega}\big)^{\frac{p}{2}-1}\big(A\nabla u^\varepsilon_{|\Omega}, n\big) & \\ \quad = \big(\mu_0^\varepsilon \circ \sigma_0\big)(B_0\nabla u^\varepsilon_{|\Sigma_0^\varepsilon}, \nabla u^\varepsilon_{|\Sigma_0^\varepsilon}\big)^{\frac{q_0}{2}-1}\big(B_0\nabla u^\varepsilon_{|\Sigma_0^\varepsilon}, n\big) & \text{on } \Gamma_0. \end{cases}$$

Moreover the limit u of u^ε is a weak solution of

$$\begin{cases} -\operatorname{div}\big((A\nabla u,\nabla u)^{\frac{p}{2}-1}A\nabla u\big)+\phi(u)=f & \text{in } \Omega,\\ (A\nabla u,\nabla u)^{\frac{p}{2}-1}(A\nabla u,n)=0 & \text{on } \partial\Omega\setminus(\Gamma\cup\Gamma_0),\\ u=k,\ \text{undetermined constant}, & \text{on } \underline{\Gamma},\\ (A\nabla u,\nabla u)^{\frac{p}{2}-1}(A\nabla u,n)+(Bn,n)^{1/2}a|u-k|^{q-2}(u-k)=0 & \text{on } \overline{\Gamma},\\ \displaystyle\int_\Gamma (A\nabla u,\nabla u)^{\frac{p}{2}-1}(A\nabla u,n)\,\mathrm{d}s=I, & \\ u=0 & \text{on } \underline{\Gamma}_0,\\ (A\nabla u,\nabla u)^{\frac{p}{2}-1}(A\nabla u,n)+(B_0n,n)^{1/2}a_0|u|^{q_0-2}u=0 & \text{on } \overline{\Gamma}_0. \end{cases}$$

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