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SEMI-CLASSICAL SCHRÖDINGER EQUATIONS WITH HARMONIC POTENTIAL AND NONLINEAR PERTURBATION

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ABSTRACT. – Solutions of semi-classical Schrödinger equation with isotropic harmonic potential focus periodically in time. We study the perturbation of this equation by a nonlinear term. If the scaling of this perturbation is critical, each focus crossing is described by a nonlinear scattering operator, which is therefore iterated as many times as the solution passes through a focus. The study of this nonlinear problem is made possible by the introduction of two operators well adapted to Schrödinger equations with harmonic potential, and by suitable Strichartz inequalities.

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RÉSUMÉ. – Les solutions de l'équation de Schrödinger semi-classique avec potentiel harmonique isotrope focalisent périodiquement en temps. Nous étudions la perturbation de cette équation par un terme non linéaire. Pour une échelle critique de cette perturbation, chaque traversée de foyer est décrite par un opérateur de diffusion non linéaire, qui est par conséquent itéré autant de fois que la solution traverse une caustique. Cette étude est permise par l'usage de deux opérateurs qui s'avèrent bien adaptés à l'équation de Schrödinger avec potentiel harmonique, et par des estimations de Strichartz adéquates.

1. Introduction

Consider the initial value problem,

$$\begin{cases} i\varepsilon\partial_t v^\varepsilon + \frac{1}{2}\varepsilon^2\Delta v^\varepsilon = \frac{x^2}{2}v^\varepsilon, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ v^\varepsilon_{|t=0} = f(x), \end{cases} \tag{1.1}$$

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where $\varepsilon \in]0,1]$ is a parameter going to zero and f is a smooth function, say $f \in \mathcal{S}(\mathbb{R})$. The potential is the isotropic harmonic potential,

$$V(x) \equiv \frac{x^2}{2} = \frac{1}{2}\left(x_1^2 + \cdots + x_n^2\right). \tag{1.2}$$

The case of anisotropic harmonic potentials is discussed in Section 5. Even though no (rapid) oscillation is present in the initial data, the solution v^ε is rapidly oscillating (at frequency $1/\varepsilon$) for any positive time, and focuses at time $t = \frac{\pi}{2}$ (Section 2). This can be seen by a stationary phase argument applied to the Mehler's formula (see [12]),

$$v^\varepsilon(t,x) = \frac{1}{(2i\pi\varepsilon \sin t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i}{\varepsilon \sin t}\left(\frac{x^2+y^2}{2}\cos t - x\cdot y\right)} f(y)\,dy =: U^\varepsilon(t) f(x). \tag{1.3}$$

Perturbations of the harmonic potential by other potentials (sub-quadratic perturbation, see [30,14,21], or super-quadratic perturbation, see [29]) have been studied, and in particular the role of these perturbations on the singularities of the fundamental solution of the Schrödinger equation.

In physics, nonlinear perturbations are considered, for Bose–Einstein condensation (see [10]), where the harmonic potential is used for its confining properties,

$$i\hbar\partial_t\psi^\hbar + \frac{1}{2}\hbar^2\Delta\psi^\hbar = \frac{x^2}{2}\psi^\hbar + Ng|\psi^\hbar|^2\psi^\hbar,$$

where N stands for the number of particles and g is a coupling constant (in $\hbar^2$).

We study precisely the perturbation of (1.1) with a nonlinear term,

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = \frac{x^2}{2}u^\varepsilon + \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ u^\varepsilon_{|t=0} = f(x) + r^\varepsilon(x), \end{cases} \tag{1.4}$$

with $\sigma > 1/n$ if $n = 1,2$, and $\frac{2}{n+2} < \sigma < \frac{2}{n-2}$ if $n \geqslant 3$. We assume that the perturbation r^ε of the initial data is small in

$$\Sigma := H^1\left(\mathbb{R}^n\right) \cap \mathcal{F}\left(H^1\left(\mathbb{R}^n\right)\right), \tag{1.5}$$

where the Fourier transform is defined by

$$\mathcal{F}v(\xi) = \hat{v}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} v(x)\,dx,$$

and that $f \in \Sigma$. The space Σ is equipped with the norm

$$\|f\|_\Sigma = \|f\|_{L^2} + \|\nabla_x f\|_{L^2} + \|xf\|_{L^2},$$

and we assume $\|r^\varepsilon\|_\Sigma \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Remark. – Initial data with plane oscillations. Let $\xi_0 \in \mathbb{R}^n$, and introduce

$$\mathsf{u}^\varepsilon(t,x) = u^\varepsilon(t, x - \xi_0 \sin t)\, e^{i(x - \frac{\xi_0}{2}\sin t)\cdot \xi_0 \cos t/\varepsilon}.$$

Then u^ε solves the Schrödinger equation (1.4), with initial data

$$\mathsf{u}^\varepsilon{}_{|t=0} = \big(f(x) + r^\varepsilon(x)\big)\, e^{i\frac{x\cdot\xi_0}{\varepsilon}}.$$

Therefore, describing the solution of (1.4) is enough to describe the solution when the initial data have plane oscillations.

We can also prove some results with a *focusing* critical nonlinearity ($2\sigma = 4/n$),

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{1}{2}\varepsilon^2 \Delta u^\varepsilon = \dfrac{x^2}{2} u^\varepsilon - \varepsilon^2 |u^\varepsilon|^{4/n} u^\varepsilon, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u^\varepsilon_{|t=0} = f(x). \end{cases} \tag{1.6}$$

We will consider the focusing case only in the one-dimensional situation, and state the corresponding results at the end of this introduction. Similar results for the multi-dimensional case would be easy to prove.

The idea of this paper is the following. Initially, the nonlinear term is negligible, essentially because the term $|u^\varepsilon|^{2\sigma}$ is uniformly bounded in suitable Lebesgue spaces, therefore it vanishes in the limit $\varepsilon \to 0$ because of the factor $\varepsilon^{n\sigma}$. Meanwhile, the harmonic potential makes the solution focus near the origin at time $t = \pi/2$, as in the linear case (1.1). When the focusing effects become relevant, that is when u^ε becomes of order $\varepsilon^{-n/2}$, the nonlinear term is no longer negligible. On the other hand, if u^ε is localized near $x = 0$, the term $x^2 u^\varepsilon$ becomes negligible; only the nonlinear term is relevant near the focus. When the nonlinearity is *defocusing* (Eq. (1.4)), the solution u^ε passes through the focus, and the crossing is given by the (nonlinear) scattering operator associated to the unscaled Schrödinger equation,

$$i\partial_t \psi + \frac{1}{2}\Delta\psi = |\psi|^{2\sigma}\psi. \tag{1.7}$$

Since the nonlinearity is defocusing, dispersive effects in (1.7) are the same as for the free equation. Therefore, the solution u^ε leaves the focus along dispersive rays. When rays are dispersed, the energy is no longer localized, the nonlinear term becomes negligible again and the harmonic potential makes the rule, as before the focus (Theorem 1.2). This process can be iterated indefinitely, and each focus crossing is described by the scattering operator (Corollary 1.4).

When the nonlinearity is *focusing* (Eq. (1.6)), and when the mass of f is critical (see [24]), the solution blows up near $t = \pi/2$ (before or after, see Proposition 1.5). The focusing effects of the harmonic potential first, then of the nonlinear term, cumulate and ruin the existence of the solution (Proposition 1.5).

In both situations (focusing or defocusing nonlinearity), two distinct régimes occur. First, the harmonic potential leads the evolution of the solution, next the nonlinear term does so. The two dynamics superpose: they balance each other in the case of a defocusing nonlinearity, and cumulate in the case of a focusing nonlinearity. The matching of these two régimes occurs in a boundary layer of size ε around the focus, as in [3,15].

Formal WKB expansions suggest that with our choice $n\sigma > 1$, the nonlinear term is negligible so long as no focusing occurs. We prove that this holds true. It would not be so with the choice $n\sigma = 1$; the nonlinear term would be nowhere negligible, and we leave out this case.

On the other hand, we show that the nonlinear term alters the asymptotics of the exact solution near and past the (first) focus. More precisely, we prove that the caustic crossing is measured by the scattering operator associated to (1.7). This phenomenon is to be compared with the results of [5], where focusing is caused by initial oscillations, and with the results of [3] (see also [1,2]), where such a behavior was first noticed, for the wave equation. In the present case, focusing is caused by the oscillations created by the harmonic potential, but the description of the phenomena near the focal point is similar.

The asymptotic state for (1.7) we will consider is defined by

$$\psi_-(x) := \frac{1}{(2i\pi)^{n/2}} \hat{f}(x). \tag{1.8}$$

We assume that $f \in \Sigma$ and that the scattering operator S acts on ψ_-, with $\psi_+ = S\psi_- \in \Sigma$ (see Proposition 3.10), which is verified in either of the following cases,

- $\sigma > \frac{2-n+\sqrt{n^2+12n+4}}{4n}$, or
- $\|f\|_\Sigma$ is sufficiently small.

ASSUMPTION 1.1. – *Our hypotheses are the following*:

- $1 \leqslant n \leqslant 5$ *and* $\sigma > 1/2$, *so that the nonlinearity* $|z|^{2\sigma}z$ *is twice differentiable.*
- *If* $n = 1$, *we assume moreover that* $\sigma > 1$.
- *If* $3 \leqslant n \leqslant 5$, *we take* $\sigma < \frac{2}{n-2}$.
- *If* $n \leqslant 2$, *we assume*
  - *either* $\sigma > \frac{2-n+\sqrt{n^2+12n+4}}{4n}$,
  - *or* $\|f\|_\Sigma \leqslant \delta$ *sufficiently small.*

Remark. – We could treat the case $n \geqslant 6$ if we replaced the nonlinear term $\varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon$ by $F(\varepsilon^n|u^\varepsilon|^2)u^\varepsilon$, with F smooth and

$$F\left(|z|^2\right) \lesssim 1 + |z|^{2\sigma}.$$

THEOREM 1.2. – *Let* $2 < r < \frac{2n}{n-2}$. *If* $n = 1$, *take* $r = \infty$. *Then under Assumptions* 1.1, *the following asymptotics holds in* $L^2 \cap L^r$,

- *If* $0 \leqslant t < \pi/2$, *then*

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \frac{e^{in\frac{\pi}{4}}}{(2\pi|\cos t|)^{n/2}} \widehat{\psi_-}\left(\frac{-x}{\cos t}\right) e^{-i\frac{x^2}{2\varepsilon}\tan t}.$$

- *If $\pi/2 < t < 3\pi/2$, then*

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \frac{e^{in\frac{\pi}{4}-in\frac{\pi}{2}}}{(2\pi|\cos t|)^{n/2}} \widehat{\psi_+}\left(\frac{-x}{\cos t}\right) e^{-i\frac{x^2}{2\varepsilon}\tan t},$$

where ψ_- is defined by (1.8) *and $\psi_+ = S\psi_-$.*

Remark. – We will prove actually that these asymptotics hold in a stronger sense (see Corollary 2.5, Propositions 3.9 and 3.22).

We can restate this result when time $t = \pi/2$ is considered as the initial time, in place of $t = 0$.

COROLLARY 1.3. – *Let $\varphi \in \Sigma$. Assume that u^ε solves*

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{1}{2}\varepsilon^2\Delta u^\varepsilon = \dfrac{x^2}{2}u^\varepsilon + \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon, \quad (t,x)\in\mathbb{R}\times\mathbb{R}, \\ u^\varepsilon_{|t=0} = \dfrac{1}{\varepsilon^{n/2}}\varphi\left(\dfrac{x}{\varepsilon}\right) + \dfrac{1}{\varepsilon^{n/2}} r^\varepsilon\left(\dfrac{x}{\varepsilon}\right), \end{cases} \tag{1.9}$$

with $\|r^\varepsilon\|_\Sigma \underset{\varepsilon\to 0}{\longrightarrow} 0$, and φ satisfies the same assumptions as f. Denote $\psi_\pm = W_\pm^{-1}\varphi$, where $W_\pm$ are the wave operators (*see Proposition* 3.10)*. Then with r as in Theorem* 1.2 *and under Assumptions* 1.1*, the following asymptotics holds in $L^2 \cap L^r$,*

- *If $0 < t < \pi$, then*

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \left(\frac{-i}{2\pi\sin t}\right)^{n/2} \widehat{\psi_+}\left(\frac{x}{\sin t}\right) e^{i\frac{x^2}{2\varepsilon\tan t}}.$$

- *If $-\pi < t < 0$, then*

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \left(\frac{-i}{2\pi\sin t}\right)^{n/2} \widehat{\psi_-}\left(\frac{x}{\sin t}\right) e^{i\frac{x^2}{2\varepsilon\tan t}}.$$

Remark. – In [25], the author considers equations which can be compared to (1.1), that is

$$\begin{cases} i\varepsilon\partial_t v^\varepsilon + \dfrac{1}{2}\varepsilon^2\Delta v^\varepsilon = V(x)v^\varepsilon + U\left(\dfrac{x}{\varepsilon}\right)v^\varepsilon, \\ v^\varepsilon_{|t=0} = \dfrac{1}{\varepsilon^{n/2}}\varphi\left(\dfrac{x}{\varepsilon}\right), \end{cases} \tag{1.10}$$

where U is a short range potential. The potential V in that case cannot be the harmonic potential, for it has to be bounded as well as all its derivatives. In that paper, the author proved that under suitable assumptions, the influence of U occurs near $t = 0$ and is localized near the origin, while only the value $V(0)$ of V at the origin is relevant in this régime. For times $\varepsilon \ll |t| < T_*$, the situation is different: the potential U becomes negligible, while V dictates the propagation. As in our paper, the transition between these two régimes is measured by the scattering operator associated to U.

Our assumption $n\sigma > 1$ makes the nonlinear term short range. With our scaling for the nonlinearity, this perturbation is relevant only near the focus, where the harmonic potential is negligible, while the opposite occurs for $\varepsilon \ll |t| < \pi$. In this perspective, a new point in our paper (besides the fact that the problem is nonlinear) is that we can tell what happens for *any* time, as stated in the following corollary.

COROLLARY 1.4. – *Suppose Assumption* 1.1 *are satisfied. Let $k \in \mathbb{N}^*$. Then, with r as in Theorem* 1.2, *the asymptotics of u^ε for $\pi/2 + (k-1)\pi < t < \pi/2 + k\pi$ is given, in $L^2 \cap L^r$, by*

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \frac{e^{in\frac{\pi}{4} - ink\frac{\pi}{2}}}{(2\pi|\cos t|)^{n/2}} \widehat{S^k\psi_-}\left(\frac{-x}{\cos t}\right) e^{-i\frac{x^2}{2\varepsilon}\tan t},$$

*where S^k denotes the k*th *iterate of S* (*which is well defined under our assumptions on f*).

Remark. – The phase shift $e^{in\frac{\pi}{4} - ink\frac{\pi}{2}}$ is present in the linear case, for Eq. (1.1), and is explained in [11]. On the contrary, the presence of the scattering operator S is typically a nonlinear phenomenon, as in [5]. The new point here is that this operator is iterated, at each focus crossing.

Remark. – If the nonlinear perturbation was of the form $\varepsilon^{n\sigma_1}|u^\varepsilon|^{2\sigma_2}u^\varepsilon$, with $\sigma_1 > \sigma_2 > 0$ (no additional assumption on σ_2) and $n\sigma_1 > 1$, the nonlinear term would be everywhere negligible, that is, S^k should be replaced by the identity in Corollary 1.4. This can be seen by an easy adaptation of the proof of Theorem 1.2. This shows that the scaling (1.4) is critical for the nonlinearity to have a leading order influence near the singularities ($t = \pi/2 + k\pi$).

We conclude this introduction by stating our result when the nonlinearity is focusing (Eq. (1.6)).

PROPOSITION 1.5. – *Let $n = 1$ and let R be the unique solution* (*up to translation and sign change*) *of $-\frac{1}{2}R'' + R = R^5$, given by,*

$$R(x) = \frac{3^{1/4}}{\sqrt{\cosh(2x\sqrt{2})}}. \tag{1.11}$$

For $t_* \in \mathbb{R}$, define f by $f(x) = R(x)e^{i\frac{t_*}{2}x^2}$, and the approximate solution by

$$\tilde{v}^\varepsilon(t,x) = \frac{1}{\sqrt{\frac{\pi}{2} + \varepsilon t_* - t}} R\left(\frac{x}{\frac{\pi}{2} + \varepsilon t_* - t}\right) e^{i\frac{\varepsilon}{\pi/2 + \varepsilon t_* - t}} e^{i\frac{x^2}{2\varepsilon(t - \pi/2 - \varepsilon t_*)}}. \tag{1.12}$$

Let u^ε be the solution of (1.6). *Then for any $\lambda > 0$,*

$$\limsup_{\varepsilon\to 0} \sup_{\frac{\pi}{2} - \Lambda\varepsilon \leqslant t \leqslant \frac{\pi}{2} + \varepsilon t_* - \lambda\varepsilon} \left\| B^\varepsilon(t)\left(u^\varepsilon(t) - \tilde{v}^\varepsilon(t)\right)\right\|_{L^2} \underset{\Lambda\to+\infty}{\longrightarrow} 0,$$

where $B^\varepsilon(t)$ *is either of the operators* Id, $\varepsilon\partial_x$ *or* $x/\varepsilon + i(t-\pi/2)\partial_x$. *In particular,*

$$\liminf_{\varepsilon\to 0} \sup_{0\leqslant t\leqslant \pi/2+t_*\varepsilon-\lambda\varepsilon} \left\|\varepsilon\partial_x u^\varepsilon(t)\right\|_{L^2} \underset{\lambda\to 0^+}{\longrightarrow} +\infty,$$

$$\liminf_{\varepsilon\to 0} \sup_{0\leqslant t\leqslant \pi/2+t_*\varepsilon-\lambda\varepsilon} \left\|\sqrt{\varepsilon}u^\varepsilon(t)\right\|_{L^\infty} \underset{\lambda\to 0^+}{\longrightarrow} +\infty.$$

Remark. – The blow up occurs at $t = \frac{\pi}{2} + \varepsilon t_*$, no matter the sign of t_*. This means that u^ε can blow up before or after the focus.

This paper is organized as follows. In Section 2, we study the linear equation (1.1) using WKB methods, and introduce two operators (J^ε and H^ε) whose role is crucial in the nonlinear setting. In Section 3, we analyze the nonlinear equation (1.4), and we prove Theorem 1.2. In Section 4, we prove Proposition 1.5. Finally, Section 5 addresses the case of anisotropic harmonic potentials.

Some of the results presented in this paper were announced in [6].

2. WKB expansion for the linear equation

We seek an approximate solution of the linear equation (1.1) of the form,

$$v^\varepsilon_{\mathrm{app}}(t,x) = v_0(t,x)\, e^{i\varphi(t,x)/\varepsilon}. \tag{2.1}$$

To cancel the term ε^0 when plugging this approximate solution in (1.1), the phase φ must satisfy the eikonal equation,

$$\partial_t\varphi + \frac{1}{2}(\nabla_x\varphi)^2 = -\frac{x^2}{2}. \tag{2.2}$$

To cancel the term ε^1, the amplitude v_0 must satisfy the transport equation,

$$\partial_t v_0 + \nabla_x\varphi\cdot\nabla_x v_0 + \frac{1}{2}v_0\Delta\varphi = 0. \tag{2.3}$$

To solve the eikonal equation, one computes the bicharacteristic curves associated to the classical Hamiltonian

$$p(t,x,\tau,\xi) = \tau + \frac{1}{2}\xi^2 + \frac{x^2}{2},$$

given by

$$\begin{cases} \dot{t} = 1, \\ \dot{x} = \xi, \\ \dot{\tau} = 0, \\ \dot{\xi} = -x. \end{cases}$$

Therefore,

$$x(t) = x_0\cos t + \xi_0\sin t, \qquad \xi(t) = \xi_0\cos t - x_0\sin t.$$

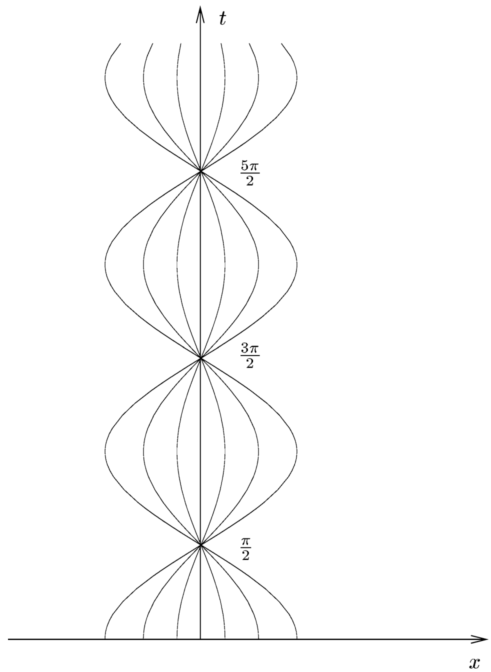

Fig. 1. Rays of geometric optics.

Since no oscillation is present in the initial data, $\xi_0 = 0$, and the rays of geometric optics are sinusoids,

$$x(t) = x_0 \cos t. \tag{2.4}$$

They all meet at the origin at time $t = \pi/2$, and periodically at time $t = \pi/2 + k\pi$ for any $k \in \mathbb{N}^*$ (Fig. 1).

Given $\xi(t) = \nabla_x \varphi(t)$, one can solve (2.2) for $0 \leqslant t < \pi/2$, by

$$\varphi(t,x) = -\frac{x^2}{2}\tan t,$$

and (2.3) is solved by

$$v_0(t,x) = \frac{1}{(\cos t)^{n/2}} f\left(\frac{x}{\cos t}\right),$$

therefore

$$v^\varepsilon_{\mathrm{app}}(t,x) = \frac{1}{(\cos t)^{n/2}} f\left(\frac{x}{\cos t}\right) e^{-i\frac{x^2}{2\varepsilon}\tan t}. \tag{2.5}$$

Recall that $V(x) = \frac{x^2}{2}$. The approximate solution solves

$$\begin{cases} i\varepsilon\partial_t v^\varepsilon_{\mathrm{app}} + \dfrac{1}{2}\varepsilon^2 \Delta v^\varepsilon_{\mathrm{app}} = V(x) v^\varepsilon_{\mathrm{app}} + \dfrac{1}{2}\varepsilon^2 e^{i\varphi(t,x)/\varepsilon}\Delta v_0, \\ v^\varepsilon_{\mathrm{app}|t=0} = f(x). \end{cases} \tag{2.6}$$

Denote the remainder $w^\varepsilon := v^\varepsilon - v^\varepsilon_{\text{app}}$. It solves,

$$\begin{cases} i\varepsilon\partial_t w^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w^\varepsilon = V(x)w^\varepsilon - \frac{1}{2}\varepsilon^2 e^{i\varphi(t,x)/\varepsilon}\Delta v_0, \\ w^\varepsilon_{|t=0} = 0. \end{cases} \tag{2.7}$$

Recall that

$$\Delta v_0(t,x) = \frac{1}{\cos^2 t}\frac{1}{(\cos t)^{n/2}}\Delta f\left(\frac{x}{\cos t}\right).$$

Recall the classical result,

LEMMA 2.1. – *Assume a function w^ε satisfies*

$$i\varepsilon\partial_t w^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w^\varepsilon = U(t,x)w^\varepsilon + S^\varepsilon(t,x), \quad (t,x) \in I \times \mathbb{R}^n, \tag{2.8}$$

where U is a real-valued potential, I is an interval, and $S^\varepsilon \in C_t(I, L^2)$. Then the following estimate holds for $t \in I$,

$$\varepsilon\partial_t \|w^\varepsilon(t)\|_{L^2} \leqslant 2\|S^\varepsilon(t)\|_{L^2}.$$

Applying this lemma, it follows,

$$\varepsilon\partial_t \|w^\varepsilon(t)\|_{L^2} \leqslant \varepsilon^2\frac{1}{\cos^2 t}\|\Delta f\|_{L^2}. \tag{2.9}$$

With the idea of a nonlinear perturbation in mind, it is natural to seek estimates in other spaces than L^2, in particular Sobolev like spaces. In geometrical optics, it is classical to assess ε-derivatives to get nonlinear estimates (see for instance [26]). This is because ε-oscillating solutions are studied. This approach is sharp for multi-phase problems, but it contains no geometric information (given by the phase(s)). In our case, only one phase is present, and in the nonlinear setting (1.4), it remains so. In the linear case, this means that controlling $v^\varepsilon_{\text{app}}$ in Lebesgue's spaces L^p is equivalent to controlling v_0 in L^p. With Gagliardo–Nirenberg inequalities in mind, it is therefore natural to introduce the operator

$$J^\varepsilon(t) = -i(\cos t)e^{i\varphi/\varepsilon}\nabla_x(e^{-i\varphi/\varepsilon}.) = \frac{x}{\varepsilon}\sin t - i\cos t\nabla_x. \tag{2.10}$$

Given the dynamics of the harmonic potential, it is also natural to introduce the "orthogonal" operator,

$$H^\varepsilon(t) = x\cos t + i\varepsilon\sin t\nabla_x. \tag{2.11}$$

When $n \geqslant 2$, we write, for $1 \leqslant j \leqslant n$,

$$\begin{aligned} H^\varepsilon_j(t) &= x_j\cos t + i\varepsilon\sin t\,\partial_{x_j}, \\ J^\varepsilon_j(t) &= \frac{x_j}{\varepsilon}\sin t - i\cos t\,\partial_{x_j}. \end{aligned} \tag{2.12}$$

We now state all the properties we will need, including the action on nonlinear terms.

LEMMA 2.2. – *The operators H^ε and J^ε satisfy the following properties.*

- *The commutation relation,*

$$\left[H_j^\varepsilon(t), i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta - \frac{x^2}{2}\right] = \left[J_j^\varepsilon(t), i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta - \frac{x^2}{2}\right] = 0. \tag{2.13}$$

- *Denote $M^\varepsilon(t) = e^{-i\frac{x^2}{2\varepsilon}\tan t}$, and $Q^\varepsilon(t) = e^{i\frac{x^2}{2\varepsilon\tan t}}$, then $J^\varepsilon(t)$ and $H^\varepsilon(t)$ read, for $t \notin \frac{\pi}{2}\mathbb{Z}$,*

$$J^\varepsilon(t) = -i\cos t M^\varepsilon(t)\nabla_x M^\varepsilon(-t), \qquad H^\varepsilon(t) = i\varepsilon\sin t Q^\varepsilon(t)\nabla_x Q^\varepsilon(-t). \tag{2.14}$$

- *The modified Sobolev inequalities. For $n = 1$ and $t \notin \frac{\pi}{2}\mathbb{Z}$,*

$$\begin{aligned}\|w(t)\|_{L^\infty} &\leqslant \frac{C}{|\cos t|^{1/2}}\|w(t)\|_{L^2}^{1/2}\|J^\varepsilon(t)w(t)\|_{L^2}^{1/2},\\ \|w(t)\|_{L^\infty} &\leqslant \frac{C}{|\varepsilon\sin t|^{1/2}}\|w(t)\|_{L^2}^{1/2}\|H^\varepsilon(t)w(t)\|_{L^2}^{1/2}.\end{aligned} \tag{2.15}$$

For $n \geqslant 2$, and $2 \leqslant r < \frac{2n}{n-2}$, define $\delta(r)$ by

$$\delta(r) \equiv n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Then for any $2 \leqslant r < \frac{2n}{n-2}$, there exists C_r such that, for $t \notin \frac{\pi}{2}\mathbb{Z}$,

$$\begin{aligned}\|w(t)\|_{L^r} &\leqslant \frac{C_r}{|\cos t|^{\delta(r)}}\|w(t)\|_{L^2}^{1-\delta(r)}\|J^\varepsilon(t)w(t)\|_{L^2}^{\delta(r)},\\ \|w(t)\|_{L^r} &\leqslant \frac{C_r}{|\varepsilon\sin t|^{\delta(r)}}\|w(t)\|_{L^2}^{1-\delta(r)}\|H^\varepsilon(t)w(t)\|_{L^2}^{\delta(r)}.\end{aligned} \tag{2.16}$$

- *For any function $F \in C^1(\mathbb{C},\mathbb{C})$ satisfying the gauge invariance condition*

$$\exists G \in C(\mathbb{R}_+,\mathbb{R}),\ F(z) = zG(|z|^2),$$

one has, for $t \notin \frac{\pi}{2}\mathbb{Z}$,

$$\begin{aligned}H^\varepsilon(t)F(w) &= \partial_z F(w)H^\varepsilon(t)w - \partial_{\bar z}F(w)\overline{H^\varepsilon(t)w},\\ J^\varepsilon(t)F(w) &= \partial_z F(w)J^\varepsilon(t)w - \partial_{\bar z}F(w)\overline{J^\varepsilon(t)w}.\end{aligned} \tag{2.17}$$

Remarks. –

- Estimates (2.15) are easy consequences of the conjugation properties (2.14).
- With the WKB approximation (2.5) in mind, the $|\cos t|^{-1/2}$ term in (2.15) gives optimal time dependence of the L_x^∞ estimates of the solution of (1.1) away from

the focus. This is the main advantage of this operator over all the others one could think of (such as ∇_x in particular).
- The $|\varepsilon \sin t|^{-1/2}$ term in (2.15) gives optimal L^∞_x estimates of the solution of (1.1) near the focus (where $|\sin t| \sim 1$).
- The operator J^ε can be considered as the modification of the Galilean operator $x + it\nabla_x$, which is very useful in scattering theory (see [8,16,17]). For semi-classical problems where focusing at the origin occurs, it was used in [5] and [7], with the rescaling $\frac{x}{\varepsilon} + i(t - t_*)\nabla_x$, where t_* is the focusing time. The operator J^ε is that operator, transported to the case of a harmonic potential.
- Property (2.17) states that H^ε and J^ε act on nonlinearities satisfying the gauge invariance condition like derivatives (Eq. (2.17) holds for the operator $i\nabla_x$).
- The fact that all these identities, except the first one, hold only for almost all $t \in \mathbb{R}$ is not a problem, since in any case integrations with respect to time will be performed.
- The operators J^ε and H^ε are known in quantum mechanics, as Heisenberg observables (metaplectic transforms, see [19,13]),

$$J^\varepsilon(t) = U^\varepsilon(t)(-i\nabla_x)U^\varepsilon(-t) \tag{2.18}$$

$$= U^\varepsilon\left(t - \frac{\pi}{2}\right)\frac{x}{\varepsilon}U^\varepsilon\left(\frac{\pi}{2} - t\right), \tag{2.19}$$

$$H^\varepsilon(t) = U^\varepsilon(t)xU^\varepsilon(-t) \tag{2.20}$$

$$= U^\varepsilon\left(t - \frac{\pi}{2}\right)(i\varepsilon\nabla_x)U^\varepsilon\left(\frac{\pi}{2} - t\right), \tag{2.21}$$

where $U^\varepsilon(t)$ is the propagator defined by Mehler's formula (1.3), that is

$$U^\varepsilon(t) = e^{i\frac{t}{\varepsilon}(-\varepsilon^2/2\cdot\Delta + x^2/2)}.$$

The commutation properties (2.13) are straightforward consequences of the conjugation relations (2.18) and (2.20). Identities between (2.18) and (2.19) on the one hand, (2.20) and (2.21) on the other hand, are due to the geometric properties of the harmonic oscillator, that rotates the phase space. It is easy to check that Sobolev inequalities (2.15) follow from (2.19), (2.20) and the estimate

$$\left\|U^\varepsilon(t)f\right\|_{L^\infty_x} \lesssim \frac{1}{(\varepsilon|\sin t|)^{1/2}}\|f\|_{L^1}.$$

The most remarkable fact is certainly that in the case of the harmonic potential, one can estimate the action of these observables of Heisenberg on a large class of nonlinearities, through (2.17).

Lemma 2.2 makes it possible to get more precise estimates of the approximation given by (the first term of) WKB methods. Denote

$$\mathcal{H} := \left\{f \in H^3(\mathbb{R}^n), \text{ such that } xf \in H^2(\mathbb{R}^n)\right\}. \tag{2.22}$$

PROPOSITION 2.3. – *Assume $f \in \mathcal{H}$. Then there exists $C = C(\|f\|_{H^3}, \|xf\|_{H^2})$ such that the remainder $v^\varepsilon - v^\varepsilon_{\text{app}}$ satisfies, for $0 \leqslant t < \pi/2$,*

$$\begin{aligned}&\big\|\big(v^\varepsilon - v^\varepsilon_{\text{app}}\big)(t)\big\|_{L^2} + \big\|J^\varepsilon\big(v^\varepsilon - v^\varepsilon_{\text{app}}\big)(t)\big\|_{L^2} + \big\|H^\varepsilon\big(v^\varepsilon - v^\varepsilon_{\text{app}}\big)(t)\big\|_{L^2}\\ &\quad\leqslant C\left(\varepsilon\int_0^t \frac{ds}{\cos^2 s} + \frac{\varepsilon^2}{\cos^2 t}\right).\end{aligned}$$

Remark. – As mentioned in the introduction (Eq. (1.3)), the expression of v^ε is given explicitly by an oscillatory integral, and the above result could be proved by stationary phase methods. Nevertheless, we do not use this approach, and rather present the approach whose spirit is the same as in the nonlinear setting.

Proof. – The first estimate is given by (2.9), with $C = \|\Delta f\|_{L^2}$. For the second estimate, apply $J^\varepsilon(t)$ to (2.7). The commutation property (2.13) yields,

$$\begin{cases} i\varepsilon\partial_t J^\varepsilon w^\varepsilon + \dfrac{1}{2}\varepsilon^2\Delta J^\varepsilon w^\varepsilon = V(x)J^\varepsilon w^\varepsilon - \dfrac{1}{2}\varepsilon^2 J^\varepsilon(t)\big(e^{i\varphi(t,x)/\varepsilon}\Delta v_0\big),\\ J^\varepsilon w^\varepsilon_{|t=0} = 0.\end{cases} \tag{2.23}$$

One has explicitly,

$$\begin{aligned}J^\varepsilon(t)\big(e^{i\varphi(t,x)/\varepsilon}\Delta v_0\big) &= \left(\frac{x}{\varepsilon}\sin t - i\cos t\nabla_x\right)\left(e^{-i\frac{x^2}{2\varepsilon}\tan t}\frac{1}{(\cos t)^{n/2+2}}\Delta f\left(\frac{x}{\cos t}\right)\right)\\ &= -ie^{-i\frac{x^2}{2\varepsilon}\tan t}\cos t\nabla_x\left(\frac{1}{(\cos t)^{n/2+2}}\Delta f\left(\frac{x}{\cos t}\right)\right)\\ &= -ie^{-i\frac{x^2}{2\varepsilon}\tan t}\frac{1}{(\cos t)^{n/2+2}}\nabla_x\Delta f\left(\frac{x}{\cos t}\right),\end{aligned}$$

and the same estimate as for the L^2 case follows, with $C = \|f\|_{H^3}$. For the last estimate of the proposition, apply $H^\varepsilon(t)$ to (2.7). Because of the commutation property (2.13), the remainder $H^\varepsilon(t)w^\varepsilon$ is estimated by the L^2 norm of

$$\begin{aligned}H^\varepsilon(t)\big(e^{i\varphi(t,x)/\varepsilon}\Delta v_0\big) &= \big(x\cos t + i\varepsilon\sin t\nabla_x\big)\left(e^{-i\frac{x^2}{2\varepsilon}\tan t}\frac{1}{(\cos t)^{n/2+2}}\Delta f\left(\frac{x}{\cos t}\right)\right)\\ &= \frac{x}{\cos t}e^{-i\frac{x^2}{2\varepsilon}\tan t}\frac{1}{(\cos t)^{n/2+2}}\Delta f\left(\frac{x}{\cos t}\right)\\ &\qquad + i\varepsilon\tan t\frac{1}{(\cos t)^{n/2+2}}\nabla_x\Delta f\left(\frac{x}{\cos t}\right).\end{aligned}$$

The L^2 norm of the first term is $\frac{1}{\cos^2 t}\|x\Delta f\|_{L^2}$, and the L^2 norm of the second term is $\varepsilon\frac{\sin t}{\cos^3 t}\|f\|_{\dot H^3}$. This completes the proof of Proposition 2.3. □

From Proposition 2.3, WKB methods provide a good approximation of the exact solution before focusing. More precisely, the remainder will be small up to a boundary layer of size ε around $t = \pi/2$.

The assumption $f \in \mathcal{H}$ is necessary to estimate precisely the validity of WKB approximation, but is not really essential. Since the set of such f is dense in Σ, the following lemma shows that this extra regularity can be introduced without modifying the asymptotics.

LEMMA 2.4. – *Assume $f \in \Sigma$, and let v^ε be the solution of* (1.1). *Then for any $t > 0$,*

$$\left\|v^\varepsilon(t)\right\|_{L^2} = \|f\|_{L^2}; \qquad \left\|J^\varepsilon(t)v^\varepsilon\right\|_{L^2} = \|\nabla f\|_{L^2}; \qquad \left\|H^\varepsilon(t)v^\varepsilon\right\|_{L^2} = \|xf\|_{L^2}.$$

Proof. – This lemma is a straightforward consequence of Lemma 2.1 and of the commutation property (2.13). □

Notice that the L^2-norm of $v^\varepsilon_{\mathrm{app}}(t)$ does not depend on time, nor that of $J^\varepsilon(t)v^\varepsilon_{\mathrm{app}}$ or $H^\varepsilon(t)v^\varepsilon_{\mathrm{app}}$. We can therefore remove the smoothness assumption of Proposition 2.3.

COROLLARY 2.5. – *Assume $f \in \Sigma$. Then,*

$$\limsup_{\varepsilon\to 0} \sup_{0\leqslant t\leqslant \frac{\pi}{2}-\Lambda\varepsilon} \left\|A^\varepsilon(t)\left(v^\varepsilon - v^\varepsilon_{\mathrm{app}}(t)\right)\right\|_{L^2} \underset{\Lambda\to+\infty}{\longrightarrow} 0,$$

where $A^\varepsilon(t)$ is either of the operators Id, $J^\varepsilon(t)$ *or* $H^\varepsilon(t)$.

3. The nonlinear case

The proof for asymptotics in the nonlinear setting relies on Strichartz estimates (even though we could do without when $n = 1$). We first recall how we get them in the present case, then prove a general estimate. Then the proof of Theorem 1.2 is essentially split into three parts: the asymptotics before the focus ($0 \ll \pi/2 - t$), the matching between the two régimes (linear and nonlinear), and the asymptotics around the focus ($|t - \pi/2| \lesssim \varepsilon$).

3.1. Strichartz inequalities

First, recall the classical definition (see, e.g., [8]),

DEFINITION 3.1. – *A pair (q, r) is admissible if $2 \leqslant r < \frac{2n}{n-2}$ (resp. $2 \leqslant r \leqslant \infty$ if $n = 1$, $2 \leqslant r < \infty$ if $n = 2$) and*

$$\frac{2}{q} = \delta(r) \equiv n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Strichartz estimates provide mixed type estimates (that is, in spaces of the form $L^q_t(L^r_x)$ with (q, r) admissible) of quantities involving the unitary group $e^{i\frac{t}{2}\Delta}$ (see [27, 18,22,28,8,16,17]). With the scaling of Eq. (1.4), the natural unitary group to consider is

$$U^\varepsilon_0(t) := e^{i\varepsilon\frac{t}{2}\Delta}. \tag{3.1}$$

Now we can state the Strichartz estimates obtained by a scaling argument from the usual ones (with $\varepsilon = 1$). The notation r' stands for the Hölder conjugate exponent of r.

PROPOSITION 3.2 (Scaled Strichartz inequalities). –

(1) *For any admissible pair* (q,r), *there exists* C_r *such that*

$$\varepsilon^{\frac{1}{q}}\left\|U_0^\varepsilon(t)u\right\|_{L^q(\mathbb{R};L^r)} \leqslant C_r\|u\|_{L^2}. \tag{3.2}$$

(2) *For any admissible pairs* (q_1,r_1) *and* (q_2,r_2), *and any interval* I, *there exists* C_{r_1,r_2} *such that*

$$\varepsilon^{\frac{1}{q_1}+\frac{1}{q_2}}\left\|\int\limits_{I\cap\{s\leqslant t\}} U_0^\varepsilon(t-s)F(s)\,ds\right\|_{L^{q_1}(I;L^{r_1})} \leqslant C_{r_1,r_2}\|F\|_{L^{q_2'}(I;L^{r_2'})}. \tag{3.3}$$

The above constants are independent of ε *and* I.

The proof of this result relies on two properties (see [8], or [23] for a more general statement):

- The group U_0^ε is unitary on L^2, $\|U_0^\varepsilon(t)\|_{L^2\to L^2}=1$.
- For $t\neq 0$, it maps $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$,

$$\left\|U_0^\varepsilon(t)\right\|_{L^1\to L^\infty} \lesssim \frac{1}{(\varepsilon|t|)^{n/2}}.$$

As a matter of fact, these two estimates also hold for the propagator associated to the Schrödinger equation with a harmonic potential (1.1). Therefore we can obtain similar Strichartz estimates (see [8]).

If v^ε solves (1.1), then Mehler's formula yields, for $t\notin\pi\mathbb{Z}$ (see [12]),

$$v^\varepsilon(t,x)=\frac{1}{(2i\pi\varepsilon\sin t)^{n/2}}\int\limits_{\mathbb{R}^n} e^{\frac{i}{\varepsilon\sin t}\left(\frac{x^2+y^2}{2}\cos t - x.y\right)} f(y)\,dy =: U^\varepsilon(t)f(x).$$

Therefore:

- The group U^ε is unitary on L^2, $\|U^\varepsilon\|_{L^2\to L^2}=1$.
- For $t\in\,]-\pi,0[\,\cup\,]0,\pi[$, it maps $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$,

$$\left\|U^\varepsilon\right\|_{L^1\to L^\infty} \lesssim \frac{1}{(\varepsilon|\sin t|)^{n/2}}.$$

Since for $|t|\leqslant\pi/2$, $|\sin t|\geqslant\frac{2}{\pi}|t|$, the proof of Proposition 3.2 still works when U_0^ε is replaced by U^ε, provided that only *finite* time intervals are considered.

PROPOSITION 3.3. –

(1) *For any admissible pair* (q,r), *for any finite interval* I, *there exists* $C_r(I)$ *such that*

$$\varepsilon^{\frac{1}{q}}\left\|U^\varepsilon(t)u\right\|_{L^q(I;L^r)} \leqslant C_r(I)\|u\|_{L^2}. \tag{3.4}$$

(2) *For any admissible pairs* (q_1, r_1) *and* (q_2, r_2)*, and any finite interval* I*, there exists* $C_{r_1,r_2}(I)$ *such that*

$$\varepsilon^{\frac{1}{q_1}+\frac{1}{q_2}} \left\| \int_{I\cap\{s\leqslant t\}} U_0^\varepsilon(t-s)F(s)\,ds \right\|_{L^{q_1}(I;L^{r_1})} \leqslant C_{r_1,r_2}(I)\|F\|_{L^{q_2'}(I;L^{r_2'})}. \tag{3.5}$$

The above constants are independent of ε.

3.2. A general estimate

We start with an algebraic lemma.

LEMMA 3.4. – *Let* $n \geqslant 2$*, and assume* $\frac{2}{n+2} < \sigma < \frac{2}{n-2}$*. There exists* $\underline{q}$*,* $\underline{r}$*,* $\underline{s}$ *and* $\underline{k}$ *satisfying*

$$\begin{cases} \dfrac{1}{\underline{r}'} = \dfrac{1}{\underline{r}} + \dfrac{2\sigma}{\underline{s}}, \\ \dfrac{1}{\underline{q}'} = \dfrac{1}{\underline{q}} + \dfrac{2\sigma}{\underline{k}}, \end{cases} \tag{3.6}$$

and the additional conditions:
- *The pair* $(\underline{q}, \underline{r})$ *is admissible,*
- $0 < \frac{1}{\underline{k}} < \delta(\underline{s}) < 1$.

If $n = 1$*, we take* $(\underline{q}, \underline{r}) = (\infty, 2)$ *and* $(\underline{k}, \underline{s}) = (2\sigma, \infty)$.

Proof. – With $\delta(\underline{s}) = 1$, the first part of (3.6) becomes

$$\delta(\underline{r}) = \sigma\left(\frac{n}{2} - 1\right),$$

and this expression is less than 1 for $\sigma < \frac{2}{n-2}$. Still with $\delta(\underline{s}) = 1$, the second part of (3.6) yields

$$\frac{2}{\underline{k}} = 1 - \frac{n}{2} + \frac{1}{\sigma},$$

which lies in $]0, 2[$ for $\frac{2}{n+2} < \sigma < \frac{2}{n-2}$. By continuity, these conditions are still satisfied for $\delta(\underline{s})$ close to 1 and $\delta(\underline{s}) < 1$. □

From now on, we assume $n \geqslant 2$ and $\frac{2}{n+2} < \sigma < \frac{2}{n-2}$. We state a general estimate that can be applied to nonlinear Schrödinger equations with or without harmonic potential. Let $\mathcal{U}^\varepsilon(t)$ be a group for which Proposition 3.3 holds (typically, U_0^ε or U^ε in our situation). We seek a general estimate for the integral equation,

$$u^\varepsilon(t) = \mathcal{U}^\varepsilon(t-t_0)u_0^\varepsilon - i\varepsilon^{n\sigma-1}\int_{t_0}^{t}\mathcal{U}^\varepsilon(t-s)F^\varepsilon(u^\varepsilon)(s)\,ds$$
$$- i\varepsilon^{-1}\int_{t_0}^{t}\mathcal{U}^\varepsilon(t-s)h^\varepsilon(s)\,ds. \tag{3.7}$$

This equation generalizes the Duhamel formula for Eq. (1.4),

- to the case of the same equation without potential (take U_0^ε in place of U^ε),
- to the case of any initial time and any initial data (u_0^ε and t_0 are general),
- to the possibility of having a nonlinear term which is not a power, $F^\varepsilon(u^\varepsilon)$,
- to the possibility of having a source term, h^ε.

PROPOSITION 3.5. – *Let $t_1 > t_0$, with $|t_1 - t_0| \leqslant \pi$. Assume that there exists a constant C independent of t and ε such that for $t_0 \leqslant t \leqslant t_1$,*

$$\left\| F^\varepsilon(u^\varepsilon)(t)\right\|_{L_x^{\underline{r}'}} \leqslant \frac{C}{(|\cos t| + \varepsilon)^{2\sigma\delta(\underline{s})}}\left\| u^\varepsilon(t)\right\|_{L_x^{\underline{r}}}, \tag{3.8}$$

and define

$$A^\varepsilon(t_0,t_1) := \left(\int_{t_0}^{t_1}\frac{dt}{(|\cos t| + \varepsilon)^{\underline{k}\delta(\underline{s})}}\right)^{2\sigma/\underline{k}}.$$

Then there exist C^* independent of ε, t_0 and t_1 such that for any admissible pair (q,r),

$$\|u^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})} \leqslant C^*\varepsilon^{-1/\underline{q}}\left\|u_0^\varepsilon\right\|_{L^2} + C_{\underline{q},q}\varepsilon^{-1-\frac{1}{\underline{q}}-\frac{1}{q}}\|h^\varepsilon\|_{L^{q'}(t_0,t_1;L^{r'})}$$
$$+ C^*\varepsilon^{2\sigma(\delta(\underline{s})-\frac{1}{\underline{k}})}A^\varepsilon(t_0,t_1)\|u^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})}. \tag{3.9}$$

We will rather use the following corollary,

COROLLARY 3.6. – *Suppose the assumptions of Proposition 3.5 are satisfied. Assume moreover that $C^*\varepsilon^{2\sigma(\delta(\underline{s})-\frac{1}{\underline{k}})}A^\varepsilon(t_0,t_1) \leqslant 1/2$, which holds in either of the two cases,*

- $0 \leqslant t_0 \leqslant t_1 \leqslant \frac{\pi}{2} - \Lambda\varepsilon$, *with $\Lambda \geqslant \Lambda_0$ sufficiently large,*
- $t_0, t_1 \in [\frac{\pi}{2} - \Lambda\varepsilon, \frac{\pi}{2} + \Lambda\varepsilon]$, *with $\frac{t_1-t_0}{\varepsilon} \leqslant \eta$ sufficiently small.*

Then

$$\|u^\varepsilon\|_{L^\infty(t_0,t_1;L^2)} \leqslant C\left\|u_0^\varepsilon\right\|_{L^2} + C_{\underline{q},q}\varepsilon^{-1-\frac{1}{q}}\|h^\varepsilon\|_{L^{q'}(t_0,t_1;L^{r'})}. \tag{3.10}$$

Proof of Proposition 3.5. – Apply Strichartz inequalities (3.4) and (3.5) to (3.7) with $q_1 = \underline{q}$, $r_1 = \underline{r}$, and $q_2 = \underline{q}$, $r_2 = \underline{r}$ for the term with $F^\varepsilon(u^\varepsilon)$, $q_2 = q$, $r_2 = r$ for the term with h^ε, it yields

$$\|u^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})} \leqslant C\varepsilon^{-1/\underline{q}}\|u_0^\varepsilon\|_{L^2} + C_{\underline{q},q}\varepsilon^{-1-\frac{1}{\underline{q}}-\frac{1}{q}}\|h^\varepsilon\|_{L^{q'}(t_0,t_1;L^{r'})} + C\varepsilon^{n\sigma-1-\frac{2}{\underline{q}}}\|F^\varepsilon(u^\varepsilon)\|_{L^{\underline{q}'}(t_0,t_1;L^{\underline{r}'})}.$$

Then estimate the space norm of the last term by (3.8) and apply Hölder inequality in time, thanks to (3.6), it yields (3.9). □

Proof of Corollary 3.6. – The additional assumption implies that the last term in (3.9) can be "absorbed" by the left-hand side, up to doubling the constants,

$$\|u^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})} \leqslant C\varepsilon^{-1/\underline{q}}\|u_0^\varepsilon\|_{L^2} + C\varepsilon^{-1-\frac{1}{\underline{q}}-\frac{1}{q}}\|h^\varepsilon\|_{L^{q'}(t_0,t_1;L^{r'})}. \tag{3.11}$$

Now apply Strichartz inequalities (3.4) and (3.5) to (3.7) again, but with $q_1=\infty$, $r_1=2$, and $q_2=\underline{q}$, $r_2=\underline{r}$ for the term with $F^\varepsilon(u^\varepsilon)$, $q_2=q$, $r_2=r$ for the term with h^ε. It yields

$$\|u^\varepsilon\|_{L^\infty(t_0,t_1;L^2)} \leqslant C\|u_0^\varepsilon\|_{L^2} + C\varepsilon^{-1-\frac{1}{q}}\|h^\varepsilon\|_{L^{q'}(t_0,t_1;L^{r'})} + C\varepsilon^{n\sigma-1-\frac{1}{\underline{q}}}\|F^\varepsilon(u^\varepsilon)\|_{L^{\underline{q}'}(t_0,t_1;L^{\underline{r}'})}.$$

Like before,

$$\begin{aligned}\varepsilon^{n\sigma-1-\frac{1}{\underline{q}}}\|F^\varepsilon(u^\varepsilon)\|_{L^{\underline{q}'}(t_0,t_1;L^{\underline{r}'})} &\leqslant C\varepsilon^{\frac{1}{\underline{q}}}\varepsilon^{2\sigma(\delta(\underline{s})-\frac{1}{\underline{k}})}A^\varepsilon(t_0,t_1)\|u^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})}\\ &\leqslant C\varepsilon^{\frac{1}{\underline{q}}}\|u^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})},\end{aligned}$$

and the corollary follows from (3.11). □

3.3. Existence results

Local existence in Σ stems from the well-known case of the nonlinear Schrödinger equation (1.7), once we noticed that the operators H^ε and J^ε are the exact substitutes for the usual operators $\varepsilon\nabla$ and $\frac{x}{\varepsilon}+i(t-\frac{\pi}{2})\nabla$, by Lemma 2.2. Duhamel's formula for (1.4) writes

$$u^\varepsilon(t) = U^\varepsilon(t)\big(f+r^\varepsilon\big) - i\varepsilon^{n\sigma-1}\int_0^t U^\varepsilon(t-s)\big(|u^\varepsilon|^{2\sigma}u^\varepsilon\big)(s)\,ds. \tag{3.12}$$

Replacing U^ε with U_0^ε would yield the Duhamel's formula for the same equation with no harmonic potential. From the above remark (the essential point is that H^ε and J^ε commute with U^ε) and the fact that the same Strichartz inequalities hold for U^ε and U_0^ε when time is bounded, local existence is actually a byproduct of the existence theory for (1.4) (which relies essentially on the results of Section 3.2, see [22,8,16,17]). For (q_0,r_0) admissible, introduce the spaces

$$Y_{r_0}^\varepsilon(I) = \big\{u^\varepsilon \in C(I,\Sigma),\, u^\varepsilon,\, H^\varepsilon u^\varepsilon,\, J^\varepsilon u^\varepsilon \in L^{q_0}_{\mathrm{loc}}\big(I, L^{r_0}_x\big)\big\},$$
$$Y^\varepsilon(I) = \big\{u^\varepsilon \in C(I,\Sigma),\, \forall(q,r) \text{ admissible},\ u^\varepsilon,\, H^\varepsilon u^\varepsilon,\, J^\varepsilon u^\varepsilon \in L^{q}_{\mathrm{loc}}\big(I, L^{r}_x\big)\big\}.$$

PROPOSITION 3.7. – *Fix* $\varepsilon \in]0,1]$*, and let* $f, r^\varepsilon \in \Sigma$*. There exists* $t^\varepsilon > 0$ *such that* (1.4) *has a unique solution* $u^\varepsilon \in Y^\varepsilon_{2\sigma+2}(0,t^\varepsilon)$*. Moreover, this solution belongs to* $Y^\varepsilon(0,t^\varepsilon)$*. The same result holds for Eq.* (1.6) *and for any initial time.*

We can take $t^\varepsilon = +\infty$ when the nonlinearity is defocusing (Eq. (1.4)), thanks to the conservations of mass and energy,

$$\|u^\varepsilon(t)\|_{L^2} = \|u^\varepsilon(0)\|_{L^2} = \mathrm{O}(1), \tag{3.13}$$

$$\begin{aligned} E^\varepsilon(t) &:= \frac{1}{2}\|\varepsilon\nabla_x u^\varepsilon(t)\|^2_{L^2} + \int_{\mathbb{R}^n} V(x)|u^\varepsilon(t,x)|^2\,dx + \frac{\varepsilon^{n\sigma}}{\sigma+1}\|u^\varepsilon(t)\|^{2\sigma+2}_{L^{2\sigma+2}} \\ &= E^\varepsilon(0) = \mathrm{O}(1). \end{aligned} \tag{3.14}$$

The conservation of energy provides an a priori estimate for $H^\varepsilon u^\varepsilon$ and $J^\varepsilon u^\varepsilon$ thanks to the identity,

$$\forall t,x,\ |H^\varepsilon(t)u^\varepsilon(t,x)|^2 + \varepsilon^2|J^\varepsilon(t)u^\varepsilon(t,x)|^2 = x^2|u^\varepsilon(t,x)|^2 + |\varepsilon\nabla_x u^\varepsilon(t,x)|^2. \tag{3.15}$$

PROPOSITION 3.8. – *Fix* $\varepsilon \in]0,1]$ *and let* $f, r^\varepsilon \in \Sigma$*. Then* (1.4) *has a unique solution* $u^\varepsilon \in Y^\varepsilon(\mathbb{R})$ *and there exists* C *such that for any* $t \geqslant 0$ *and any* $\varepsilon \in]0,1]$*,*

$$\|u^\varepsilon(t)\|_{L^2} + \|\varepsilon\nabla_x u^\varepsilon(t)\|_{L^2_x} + \|x u^\varepsilon(t,x)\|_{L^2_x} \leqslant C. \tag{3.16}$$

3.4. Propagation before the focus

Before the focus, we take as an approximate solution the solution of the linear problem, that is, v^ε defined by (1.1).

Notice that from Proposition 2.3, we know the asymptotic behavior of v^ε before the focus. We prove that in the very same region, v^ε is a good approximation of the nonlinear problem.

PROPOSITION 3.9. – *Assume* $f, r^\varepsilon \in \Sigma$*. Then*

$$\limsup_{\varepsilon\to 0} \sup_{0\leqslant t\leqslant \frac{\pi}{2}-\Lambda\varepsilon} \|A^\varepsilon(t)\big(u^\varepsilon(t) - v^\varepsilon(t)\big)\|_{L^2} \underset{\Lambda\to+\infty}{\longrightarrow} 0,$$

where $A^\varepsilon(t)$ *is either of the operators* Id, $J^\varepsilon(t)$ *or* $H^\varepsilon(t)$*.*

Proof. – Define the remainder $w^\varepsilon = u^\varepsilon - v^\varepsilon$. It solves

$$\begin{cases} i\varepsilon\partial_t w^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w^\varepsilon = V(x)w^\varepsilon + \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon, \\ w^\varepsilon_{|t=0} = r^\varepsilon. \end{cases}$$

From Duhamel's principle, this writes,

$$w^\varepsilon(t) = U^\varepsilon(t)r^\varepsilon - i\varepsilon^{n\sigma-1}\int_0^t U^\varepsilon(t-s)\big(|u^\varepsilon|^{2\sigma}u^\varepsilon\big)(s)\,ds. \tag{3.17}$$

Since v^ε solves the linear equation (1.1), so does $J^\varepsilon(t)v^\varepsilon$, and

$$\left\|v^\varepsilon(t)\right\|_{L^2} = \|f\|_{L^2}, \qquad \left\|J^\varepsilon(t)v^\varepsilon\right\|_{L^2} = \|\nabla f\|_{L^2}.$$

From Sobolev inequality (2.16),

$$\left\|v^\varepsilon(t)\right\|_{L^{\underline{s}}} \leqslant \frac{C}{|\cos t|^{\delta(\underline{s})}}\|f\|_{L^2}^{1-\delta(\underline{s})}\|\nabla f\|_{L^2}^{\delta(\underline{s})}.$$

Therefore there exists C_0 such that

$$\left\|v^\varepsilon(t)\right\|_{L^{\underline{s}}} \leqslant \frac{C_0}{|\cos t|^{\delta(\underline{s})}}. \tag{3.18}$$

From Sobolev inequality, for ε sufficiently small, $\|w^\varepsilon(0)\|_{L^{\underline{s}}} < C_0$. From Proposition 3.8, for fixed $\varepsilon > 0$, $u^\varepsilon \in C(\mathbb{R}, \Sigma)$, and the same obviously holds for v^ε. Therefore, there exists $t^\varepsilon > 0$ such that

$$\left\|w^\varepsilon(t)\right\|_{L^{\underline{s}}} \leqslant \frac{C_0}{|\cos t|^{\delta(\underline{s})}}, \tag{3.19}$$

for any $t \in [0, t^\varepsilon]$. So long as (3.19) holds, we have

$$\left\|u^\varepsilon(t)\right\|_{L^{\underline{s}}} \leqslant \frac{2C_0}{|\cos t|^{\delta(\underline{s})}},$$

and we can apply Proposition 3.5. Indeed, take $\mathcal{U}^\varepsilon = U^\varepsilon$, $h^\varepsilon = \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}v^\varepsilon$ and $F^\varepsilon(w^\varepsilon) = |u^\varepsilon|^{2\sigma}w^\varepsilon$. From Hölder inequality and the above estimate,

$$\left\|F^\varepsilon(w^\varepsilon)(t)\right\|_{L^{\underline{r}'}} \leqslant \left\|u^\varepsilon(t)\right\|_{L^{\underline{s}}}^{2\sigma}\left\|w^\varepsilon(t)\right\|_{L^{\underline{r}}} \leqslant \frac{(2C_0)^{2\sigma}}{(|\cos t|)^{2\sigma\delta(\underline{s})}}\left\|w^\varepsilon(t)\right\|_{L^{\underline{r}}}.$$

Assume (3.19) holds for $0 \leqslant t \leqslant T$. If $0 \leqslant t \leqslant T \leqslant \frac{\pi}{2} - \Lambda\varepsilon$, then $\varepsilon \lesssim \cos t$, and the above estimate shows that F^ε satisfies assumption (3.8).

From Corollary 3.6, if Λ is sufficiently large, then for $0 \leqslant t \leqslant T \leqslant \frac{\pi}{2} - \Lambda\varepsilon$, and for any (q, r) admissible,

$$\|w^\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant C\|r^\varepsilon\|_{L^2} + C\varepsilon^{n\sigma-1-\frac{1}{q}}\left\||u^\varepsilon|^{2\sigma}v^\varepsilon\right\|_{L^{q'}(0,T;L^{r'})}.$$

Taking $(q, r) = (\underline{q}, \underline{r})$ yields, from Hölder inequality,

$$\left\||u^\varepsilon|^{2\sigma}v^\varepsilon\right\|_{L^{\underline{q}'}(0,T;L^{\underline{r}'})} \leqslant \|u^\varepsilon\|_{L^{\underline{k}}(0,T;L^{\underline{s}})}^{2\sigma}\|v^\varepsilon\|_{L^{\underline{q}}(0,T;L^{\underline{r}})}.$$

The first term of the right-hand side is estimated through (3.18) and (3.19). The last term is estimated the same way, for (3.18) still holds when replacing $\underline{s}$ with $\underline{r}$. Therefore,

$$\left\||u^\varepsilon|^{2\sigma}v^\varepsilon\right\|_{L^{\underline{q}'}(0,T;L^{\underline{r}'})} \leqslant \frac{C}{(\frac{\pi}{2} - T)^{n\sigma-1-\frac{1}{\underline{q}}}},$$

and

$$\|w^\varepsilon\|_{L^\infty(0,T;L^2)} \leqslant C\|r^\varepsilon\|_{L^2} + C\left(\frac{\varepsilon}{\frac{\pi}{2}-T}\right)^{n\sigma-1-\frac{1}{\underline{q}}}. \tag{3.20}$$

Now apply the operator J^ε to (3.17). Since J^ε and U^ε commute, it yields,

$$J^\varepsilon(t)w^\varepsilon = U^\varepsilon(t)J^\varepsilon(0)r^\varepsilon - i\varepsilon^{n\sigma-1}\int_0^t U^\varepsilon(t-s)J^\varepsilon(s)\big(|u^\varepsilon|^{2\sigma}u^\varepsilon\big)(s)\,ds.$$

Because J^ε acts on this nonlinear like a derivative, we have an equation which is very similar to (3.17), with w^ε replaced by $J^\varepsilon w^\varepsilon$ and r^ε replaced by $-i\nabla r^\varepsilon$. Therefore the same computation as above yields

$$\big\|J^\varepsilon w^\varepsilon\big\|_{L^\infty(0,T;L^2)} \leqslant C\|\nabla r^\varepsilon\|_{L^2} + C\left(\frac{\varepsilon}{\frac{\pi}{2}-T}\right)^{n\sigma-1-\frac{1}{\underline{q}}}. \tag{3.21}$$

Combining (3.20) and (3.21) yields, along with (2.16),

$$\forall t\in[0,T],\quad \big\|w^\varepsilon(t)\big\|_{L^{\underline{s}}} \leqslant \frac{C}{|\cos t|^{\delta(\underline{s})}}\left(\|r^\varepsilon\|_{H^1} + \left(\frac{\varepsilon}{\frac{\pi}{2}-t}\right)^{n\sigma-1-\frac{1}{\underline{q}}}\right).$$

Therefore, choosing ε sufficiently small and Λ sufficiently large, we deduce that we can take $T=\frac{\pi}{2}-\Lambda\varepsilon$. This yields Proposition 3.9 for $A^\varepsilon = \mathrm{Id}$ and J^ε. The case $A^\varepsilon = H^\varepsilon$ is now straightforward. $\square$

3.5. Matching linear and nonlinear regimes

When time approaches $\pi/2$, the nonlinear term cannot be neglected. On the other hand, since the solution tends to concentrate at the origin, the potential becomes negligible. It is then natural to seek an approximate solution $\tilde{v}^\varepsilon$ that solves

$$i\varepsilon\partial_t\tilde{v}^\varepsilon + \frac{1}{2}\varepsilon^2\Delta\tilde{v}^\varepsilon = \varepsilon^{n\sigma}|\tilde{v}^\varepsilon|^{2\sigma}\tilde{v}^\varepsilon.$$

The question that arises naturally is, how can we match $\tilde{v}^\varepsilon$ and v^ε? With the results of [5] in mind, we can expect that $\tilde{v}^\varepsilon$ is exactly a concentrating profile,

$$\tilde{v}^\varepsilon(t,x) = \frac{1}{\varepsilon^{n/2}}\psi\left(\frac{t-\frac{\pi}{2}}{\varepsilon},\frac{x}{\varepsilon}\right). \tag{3.22}$$

The function ψ must be defined to match the solution u^ε, or one of its approximations v^ε or $v^\varepsilon_{\mathrm{app}}$, when $t=\pi/2-\Lambda\varepsilon$, for Λ sufficiently large. Notice that this problem was already encountered by Bahouri and Gérard in [3] (see also [1], Gallagher and Gérard [15]). We prove that for $\Lambda>0$ sufficiently large, the propagation for $\pi/2-\Lambda\varepsilon\leqslant t\leqslant \pi/2+\Lambda\varepsilon$ is described by $\tilde{v}^\varepsilon$.

Write $t_*^\varepsilon = \pi/2 - \Lambda\varepsilon$, and assume from now on that $\Lambda > 1$. For large Λ, Propositions 2.3 and 3.9 imply

$$u^\varepsilon\left(t_*^\varepsilon, x\right) \sim v^\varepsilon\left(t_*^\varepsilon, x\right) \sim v_{\mathrm{app}}^\varepsilon\left(t_*^\varepsilon, x\right) = \frac{1}{(\sin(\Lambda\varepsilon))^{n/2}} f\left(\frac{x}{\sin(\Lambda\varepsilon)}\right) e^{-i\frac{x^2}{2\varepsilon\tan(\Lambda\varepsilon)}}.$$

For $\Lambda\varepsilon$ close to zero, the following approximation is expected,

$$\frac{1}{(\sin(\Lambda\varepsilon))^{n/2}} f\left(\frac{x}{\sin(\Lambda\varepsilon)}\right) e^{-i\frac{x^2}{2\varepsilon\tan(\Lambda\varepsilon)}} \sim \frac{1}{(\Lambda\varepsilon)^{n/2}} f\left(\frac{x}{\Lambda\varepsilon}\right) e^{-i\frac{x^2}{2\varepsilon(\Lambda\varepsilon)}}.$$

We prove that this approximation is correct in Lemma 3.13 below. From (3.22), this should also be close to

$$\frac{1}{\varepsilon^{n/2}}\psi\left(-\Lambda, \frac{x}{\varepsilon}\right).$$

Recall the classical result,

PROPOSITION 3.10 ([20], Theorem 1.1; [9], Theorem 4.2). – *Assume $\psi_- \in \Sigma$ and $\frac{2}{n+2} < \sigma < \frac{2}{n-2}$ if $n \geqslant 2$, $\sigma > 1$ if $n = 1$. Denote*

$$\sigma_0(n) := \frac{2 - n + \sqrt{n^2 + 12n + 4}}{4n}.$$

If $\sigma > \sigma_0(n)$ or if $\|\psi_-\|_\Sigma$ is sufficiently small, then

- *There exists a unique $\psi \in C(\mathbb{R}_t, \Sigma)$ solution of* (1.7)*, such that*

$$\lim_{t\to-\infty}\left\|\psi_- - U_0(-t)\psi(t)\right\|_\Sigma = 0, \quad \text{where } U_0(t) = e^{i\frac{t}{2}\Delta}.$$

- *There exists a unique $\psi_+ \in \Sigma$ such that*

$$\lim_{t\to+\infty}\left\|\psi_+ - U_0(-t)\psi(t)\right\|_\Sigma = 0.$$

Recall that the asymptotic state ψ_- was defined in introduction by,

$$\psi_- := \frac{1}{(2i\pi)^{n/2}}\hat{f},$$

and the approximate solution (near $t = \pi/2$) is given by

$$\tilde{v}^\varepsilon(t, x) = \frac{1}{\varepsilon^{n/2}}\psi\left(\frac{t - \frac{\pi}{2}}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

We prove,

PROPOSITION 3.11. – *Assume $f, r^\varepsilon \in \Sigma$. Take ψ_- defined by* (1.8)*. Then*

$$\limsup_{\varepsilon\to 0}\left\|u^\varepsilon\left(\frac{\pi}{2} - \Lambda\varepsilon, \cdot\right) - \frac{1}{\varepsilon^{n/2}}\left(U_0(-\Lambda)\psi_-\right)\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2} \underset{\Lambda\to+\infty}{\longrightarrow} 0,$$

and the same holds when applying either of the operators $\varepsilon\nabla_x$ *or* $\frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x$ *to the considered functions.*

Proof. – From Corollary 2.5 (from which $v^\varepsilon \sim v^\varepsilon_{\mathrm{app}}$) and Proposition 3.9 (from which $v^\varepsilon \sim u^\varepsilon$),

$$\limsup_{\varepsilon\to 0}\left\| u^\varepsilon\left(t^\varepsilon_*, x\right) - \frac{1}{(\sin(\Lambda\varepsilon))^{n/2}} f\left(\frac{x}{\sin(\Lambda\varepsilon)}\right) e^{-i\frac{x^2}{2\varepsilon\tan(\Lambda\varepsilon)}}\right\|_{L^2} \underset{\Lambda\to+\infty}{\longrightarrow} 0, \tag{3.23}$$

and the same result holds when applying either of the operators $J^\varepsilon(t^\varepsilon_*)$ or $H^\varepsilon(t^\varepsilon_*)$. Notice that applying $J^\varepsilon(t^\varepsilon_*)$ or $H^\varepsilon(t^\varepsilon_*)$ is not so different from applying $\varepsilon\nabla_x$ or $\frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x$, for when $\Lambda\varepsilon$ goes to zero,

$$J^\varepsilon\left(t^\varepsilon_*\right) \sim \frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x, \qquad H^\varepsilon\left(t^\varepsilon_*\right) \sim i\varepsilon\nabla_x.$$

Recall that $t^\varepsilon_* = \pi/2 - \Lambda\varepsilon$.

LEMMA 3.12. – *Let* $a^\varepsilon(t^\varepsilon_*, \cdot) \in \Sigma$ *be a family of functions such that there exists* C_* *independent of* $\varepsilon \in]0,1]$ *such that,*

$$\left\|xa^\varepsilon\left(t^\varepsilon_*, x\right)\right\|_{L^2} + \left\|\varepsilon\nabla_x a^\varepsilon\left(t^\varepsilon_*, x\right)\right\|_{L^2} \leqslant C_*. \tag{3.24}$$

Then for any $\Lambda > 1$,

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left\|\left(J^\varepsilon\left(t^\varepsilon_*\right) - \frac{x}{\varepsilon} + i\Lambda\varepsilon\nabla_x\right)a^\varepsilon\left(t^\varepsilon_*\right)\right\|_{L^2}\\ &\quad= \limsup_{\varepsilon\to 0}\left\|\left(H^\varepsilon\left(t^\varepsilon_*\right) - i\varepsilon\nabla_x\right)a^\varepsilon\left(t^\varepsilon_*\right)\right\|_{L^2} = 0.\end{aligned}$$

In particular, we can take $a^\varepsilon = u^\varepsilon$ *or* $a^\varepsilon = v^\varepsilon_{\mathrm{app}}$.

Remark. – Lemma 3.12 has a simple geometric interpretation. Near the focus, rays of geometric optics, given by (2.4), are straightened (Fig. 2). Thus in the neighborhood of $t = \pi/2$, rays are almost straight lines, that is, the geometry is nearly the same as in [5]. In that case, with the natural scaling (3.22), the "good" operators are $\varepsilon\nabla_x$ and $\frac{x}{\varepsilon} + i(t - \pi/2)\nabla_x$.

Proof of Lemma 3.12. – Fix $\Lambda > 1$.

$$\left(J^\varepsilon\left(t^\varepsilon_*\right) - \frac{x}{\varepsilon} + i\Lambda\varepsilon\nabla_x\right)a^\varepsilon\left(t^\varepsilon_*, x\right) = \left(\left(\cos(\Lambda\varepsilon) - 1\right)\frac{x}{\varepsilon} - i\left(\sin(\Lambda\varepsilon) - \Lambda\varepsilon\right)\nabla_x\right)a^\varepsilon\left(t^\varepsilon_*, x\right).$$

Taking the L^2 norm yields,

$$\left\|\left(J^\varepsilon\left(t^\varepsilon_*\right) - \frac{x}{\varepsilon} + i\Lambda\varepsilon\nabla_x\right)a^\varepsilon\left(t^\varepsilon_*\right)\right\|_{L^2} \leqslant C(\Lambda\varepsilon)^2\left\|\frac{x}{\varepsilon}a^\varepsilon\left(t^\varepsilon_*, x\right)\right\|_{L^2_x} + C(\Lambda\varepsilon)^3\left\|\nabla_x a^\varepsilon\left(t^\varepsilon_*\right)\right\|_{L^2_x}.$$

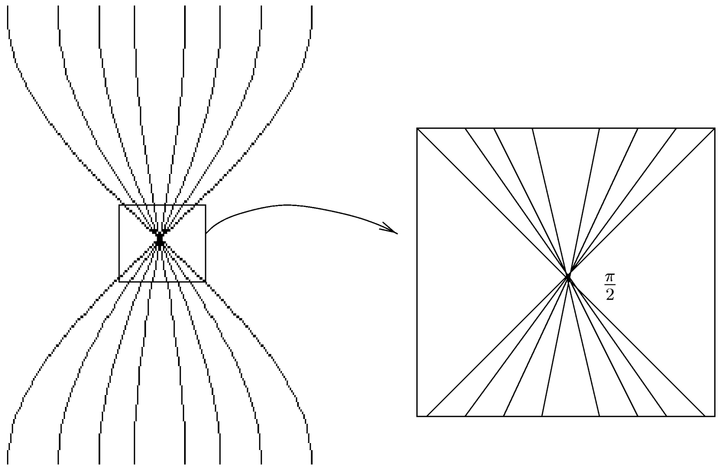

Fig. 2. Rays of geometric optics are straightened near $t = \pi/2$.

The assumption (3.24) (which is a consequence of (3.16) for u^ε, and straightforward for $v^\varepsilon_{\rm app}$) implies

$$\left\| \left(J^\varepsilon(t^\varepsilon_*) - \frac{x}{\varepsilon} + i\Lambda\varepsilon\nabla_x \right) a^\varepsilon(t^\varepsilon_*) \right\|_{L^2} \leqslant C\Lambda^2\varepsilon + C\Lambda^3\varepsilon^2,$$

which proves the first part of the lemma. Similarly,

$$\begin{aligned}\left\| \left(H^\varepsilon(t^\varepsilon_*) - i\varepsilon\nabla_x\right) a^\varepsilon(t^\varepsilon_*) \right\|_{L^2} &\leqslant C(\Lambda\varepsilon) \left\| x a^\varepsilon(t^\varepsilon_*, x) \right\|_{L^2_x} + C(\Lambda\varepsilon)^2 \left\| \varepsilon\nabla_x a^\varepsilon(t^\varepsilon_*) \right\|_{L^2_x} \\ &\leqslant C(\Lambda\varepsilon) + C(\Lambda\varepsilon)^2.\end{aligned}$$

This completes the proof of the lemma. □

Now we prove that in (3.23), we can replace $\sin(\Lambda\varepsilon)$ and $\tan(\Lambda\varepsilon)$ with $\Lambda\varepsilon$ up to a small error term. Denote

$$\tilde{v}^\varepsilon_{\rm app}(t,x) = \frac{1}{(\frac{\pi}{2} - t)^{n/2}} f\left(\frac{x}{\frac{\pi}{2} - t} \right) e^{-i\frac{x^2}{2\varepsilon(\pi/2 - t)}}.$$

LEMMA 3.13. – *Assume $f \in \Sigma$. For any $\Lambda > 1$,*

$$\begin{aligned}\limsup_{\varepsilon\to 0} \left\| \left(v^\varepsilon_{\rm app} - \tilde{v}^\varepsilon_{\rm app}\right)(t^\varepsilon_*) \right\|_{L^2} &= \limsup_{\varepsilon\to 0} \left\| \varepsilon\nabla_x \left(v^\varepsilon_{\rm app} - \tilde{v}^\varepsilon_{\rm app}\right)(t^\varepsilon_*) \right\|_{L^2} \\ &= \limsup_{\varepsilon\to 0} \left\| \left(\frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x \right) \left(v^\varepsilon_{\rm app} - \tilde{v}^\varepsilon_{\rm app}\right)(t^\varepsilon_*) \right\|_{L^2} = 0.\end{aligned}$$

Proof. – Write $\lambda = \Lambda\varepsilon$. For fixed Λ, λ is a small parameter when ε goes to zero, and

$$\begin{aligned}\left(v^\varepsilon_{\rm app} - \tilde{v}^\varepsilon_{\rm app}\right)(t^\varepsilon_*, x) &= \frac{1}{(\sin\lambda)^{n/2}} f\left(\frac{x}{\sin\lambda}\right) e^{-i\frac{x^2}{2\varepsilon\tan\lambda}} - \frac{1}{\lambda^{n/2}} f\left(\frac{x}{\lambda}\right) e^{-i\frac{x^2}{2\varepsilon\lambda}} \\ &= \left(\frac{1}{(\sin\lambda)^{n/2}} f\left(\frac{x}{\sin\lambda}\right) - \frac{1}{\lambda^{n/2}} f\left(\frac{x}{\lambda}\right) \right) e^{-i\frac{x^2}{2\varepsilon\tan\lambda}}\end{aligned}$$

$$+\frac{1}{\lambda^{n/2}} f\left(\frac{x}{\lambda}\right)\left(e^{-i\frac{x^2}{2\varepsilon\tan\lambda}} - e^{-i\frac{x^2}{2\varepsilon\lambda}}\right).$$

Taking the L^2 norm yields,

$$\begin{aligned}
\left\|\left(v^\varepsilon_{\mathrm{app}} - \tilde{v}^\varepsilon_{\mathrm{app}}\right)\left(t^\varepsilon_*\right)\right\|_{L^2} &\leqslant \left\|\frac{1}{(\sin\lambda)^{n/2}} f\left(\frac{\cdot}{\sin\lambda}\right) - \frac{1}{\lambda^{n/2}} f\left(\frac{\cdot}{\lambda}\right)\right\|_{L^2} \\
&\quad + \left\| f(x)\left(e^{-i\frac{\lambda^2 x^2}{2\varepsilon\tan\lambda}} - e^{-i\frac{\lambda x^2}{2\varepsilon}}\right)\right\|_{L^2} \\
&\leqslant \left\|\left(\frac{1}{(\sin\lambda)^{n/2}} - \frac{1}{\lambda^{n/2}}\right) f\left(\frac{\cdot}{\sin\lambda}\right)\right\|_{L^2} \\
&\quad + \left\|\frac{1}{\lambda^{n/2}}\left(f\left(\frac{\cdot}{\sin\lambda}\right) - f\left(\frac{\cdot}{\lambda}\right)\right)\right\|_{L^2} \\
&\quad + \left\| f(x)\left(e^{-i\frac{x^2}{2\varepsilon}\left(\frac{\lambda^2}{\tan\lambda}-\lambda\right)} - 1\right)\right\|_{L^2} \\
&\leqslant \left|\left(\frac{\sin\lambda}{\lambda}\right)^{n/2} - 1\right| \|f\|_{L^2} + \left\| f\left(\frac{\lambda\,\cdot}{\sin\lambda}\right) - f(\cdot)\right\|_{L^2} \\
&\quad + \left\| f(x)\left(e^{-i\frac{x^2}{2\varepsilon}\left(\frac{\lambda^2}{\tan\lambda}-\lambda\right)} - 1\right)\right\|_{L^2}.
\end{aligned}$$

The first term of the right-hand side clearly goes to zero with λ. So does the second one: if $f \in C_0^\infty(\mathbb{R})$, it is $\mathrm{O}(\lambda^2)$, and by density, it is $o(1)$ when λ goes to zero for any $f \in L^2$. Recalling that $\lambda = \Lambda\varepsilon$, we have

$$\frac{1}{\varepsilon}\left(\frac{\lambda^2}{\tan\lambda} - \lambda\right) = \Lambda\left(\frac{\lambda}{\tan\lambda} - 1\right).$$

Thus, for any *fixed* $\Lambda > 1$, this term goes to zero when ε goes to zero. Therefore, from dominated convergence, for any fixed $\Lambda > 1$,

$$\limsup_{\varepsilon\to 0} \left\| f(x)\left(e^{-i\frac{x^2}{2\varepsilon}\left(\frac{\lambda^2}{\tan\lambda}-\lambda\right)} - 1\right)\right\|_{L^2} = 0.$$

Computations for $\|\varepsilon\nabla_x(v^\varepsilon_{\mathrm{app}} - \tilde{v}^\varepsilon_{\mathrm{app}})(t^\varepsilon_*)\|_{L^2}$ and $\|(\frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x)(v^\varepsilon_{\mathrm{app}} - \tilde{v}^\varepsilon_{\mathrm{app}})(t^\varepsilon_*)\|_{L^2}$ are similar and essentially involve one more derivative or one more momentum. Indeed, $v^\varepsilon_{\mathrm{app}}$ and $\tilde{v}^\varepsilon_{\mathrm{app}}$ behave well with respect to the operators $\varepsilon\nabla_x$ and $\frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x$, thus we can use the same density argument as above. □

The next step to prove Proposition 3.11 consists in comparing $\tilde{v}^\varepsilon_{\mathrm{app}}$ and the rescaled free evolution of the asymptotic state ψ_-.

LEMMA 3.14. – *Assume $f \in \Sigma$. The following limits hold, uniformly with respect to $\varepsilon \in]0,1]$,*

$$\begin{aligned}
&\lim_{\Lambda\to+\infty} \left\|\tilde{v}^\varepsilon_{\mathrm{app}}\left(t^\varepsilon_*\right) - \frac{1}{\varepsilon^{n/2}}\left(U_0(-\Lambda)\psi_-\right)\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2} \\
&\quad = \lim_{\Lambda\to+\infty} \left\|\varepsilon\nabla_x\left(\tilde{v}^\varepsilon_{\mathrm{app}}\left(t^\varepsilon_*\right) - \frac{1}{\varepsilon^{n/2}}\left(U_0(-\Lambda)\psi_-\right)\left(\frac{\cdot}{\varepsilon}\right)\right)\right\|_{L^2}
\end{aligned}$$

$$= \lim_{\Lambda\to+\infty}\left\|\left(\frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x\right)\left(\tilde{v}^\varepsilon_{\mathrm{app}}(t^\varepsilon_*) - \frac{1}{\varepsilon^{n/2}}\big(U_0(-\Lambda)\psi_-\big)\left(\frac{\cdot}{\varepsilon}\right)\right)\right\|_{L^2} = 0.$$

Proof. – From the Fourier Inversion Formula, we have

$$\begin{aligned}\tilde{v}^\varepsilon_{\mathrm{app}}(t^\varepsilon_*, x) &= \frac{1}{(\Lambda\varepsilon)^{n/2}} f\left(\frac{x}{\Lambda\varepsilon}\right) e^{-i\frac{x^2}{2\varepsilon^2\Lambda}}\\ &= \frac{1}{(2\pi)^n}\frac{1}{(\Lambda\varepsilon)^{n/2}} e^{-i\frac{x^2}{2\varepsilon^2\Lambda}} \int e^{i\frac{x.y}{\varepsilon\Lambda}} \hat{f}(y)\,dy.\end{aligned}$$

On the other hand, the expression of the free Schrödinger group U_0 implies, along with definition (1.8),

$$\begin{aligned}\frac{1}{\varepsilon^{n/2}}\big(U_0(-\Lambda)\psi_-\big)\left(\frac{x}{\varepsilon}\right) &= \left(\frac{i}{2\pi\Lambda\varepsilon}\right)^{n/2} e^{-i\frac{x^2}{2\varepsilon^2\Lambda}} \int e^{i\frac{xy}{\varepsilon\Lambda} - i\frac{y^2}{2\Lambda}} \psi_-(y)\,dy\\ &= \frac{1}{(2\pi)^n}\frac{1}{(\Lambda\varepsilon)^{n/2}} e^{-i\frac{x^2}{2\varepsilon^2\Lambda}} \int e^{i\frac{xy}{\varepsilon\Lambda} - i\frac{y^2}{2\Lambda}} \hat{f}(y)\,dy.\end{aligned}$$

Thus the remainder we have to assess writes

$$\frac{1}{(2\pi)^n}\frac{1}{(\Lambda\varepsilon)^{n/2}} e^{-i\frac{x^2}{2\varepsilon^2\Lambda}} \int e^{i\frac{xy}{\varepsilon\Lambda}}\big(1 - e^{-i\frac{y^2}{2\Lambda}}\big)\hat{f}(y)\,dy,$$

which is also,

$$\frac{1}{(\Lambda\varepsilon)^{n/2}} e^{-i\frac{x^2}{2\varepsilon^2\Lambda}} \big(\big(1 - e^{i\frac{\Delta}{2\Lambda}}\big) f\big)\left(\frac{x}{\Lambda\varepsilon}\right).$$

The lemma then follows from the strong convergence in L^2, $e^{i\delta\Delta} \xrightarrow[\delta\to 0]{} 1$. □

Lemmas 3.12, 3.13 and 3.14 imply Proposition 3.11.

3.6. Description of the solution near the focus

Propositions 3.10 and 3.11 imply that

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left\|u^\varepsilon\left(\frac{\pi}{2} - \Lambda\varepsilon, \cdot\right) - \tilde{v}^\varepsilon\left(\frac{\pi}{2} - \Lambda\varepsilon, \cdot\right)\right\|_{L^2} +\\ &\quad + \limsup_{\varepsilon\to 0}\left\|\varepsilon\nabla_x\left(u^\varepsilon\left(\frac{\pi}{2} - \Lambda\varepsilon, \cdot\right) - \tilde{v}^\varepsilon\left(\frac{\pi}{2} - \Lambda\varepsilon, \cdot\right)\right)\right\|_{L^2}\\ &\quad + \limsup_{\varepsilon\to 0}\left\|\left(\frac{x}{\varepsilon} - i\Lambda\varepsilon\nabla_x\right)\left(u^\varepsilon\left(\frac{\pi}{2} - \Lambda\varepsilon, \cdot\right) - \tilde{v}^\varepsilon\left(\frac{\pi}{2} - \Lambda\varepsilon, \cdot\right)\right)\right\|_{L^2} \xrightarrow[\Lambda\to+\infty]{} 0.\end{aligned}$$

This means that taking Λ large enough, and ε small enough, the difference $u^\varepsilon - \tilde{v}^\varepsilon$ is small at time $t^\varepsilon_* = \pi/2 - \Lambda\varepsilon$, which is the "initial" time in the boundary layer where nonlinear effects take place (and where the potential is negligible). Since the role of r^ε is negligible, we first assume $r^\varepsilon \equiv 0$.

PROPOSITION 3.15. – *Assume $f \in \mathcal{H}$, and that the nonlinearity is C^2, that is, $\sigma > 1/2$, which is possible only if $n \leqslant 5$. Then the difference $u^\varepsilon - \tilde{v}^\varepsilon$ is small around* *the focus.*

$$\limsup_{\varepsilon\to 0}\sup_{\frac{\pi}{2}-\Lambda\varepsilon\leqslant t\leqslant\frac{\pi}{2}+\Lambda\varepsilon}\left\|A^\varepsilon(t)\big(u^\varepsilon(t)-\tilde{v}^\varepsilon(t)\big)\right\|_{L^2}\underset{\Lambda\to+\infty}{\longrightarrow}0,$$

where $A^\varepsilon(t)$ *is either of the operators* Id, $J^\varepsilon(t)$ *or* $H^\varepsilon(t)$.

Remark. – The assumption $\sigma>\frac{1}{2}$ is needed to prove Lemma 3.17 below. It seems purely technical, and one expects Lemma 3.17 to hold without this assumption. If $n=2$, the nonlinearity is automatically C^2 thanks to the assumption $\sigma>\frac{2}{n+2}$. If $n=3$, then we have to restrict our study to the case $\frac{1}{2}<\sigma<2$. In particular, the value $\sigma=1$, which corresponds to a cubic nonlinearity, is accepted.

Proof. – Propositions 3.10 and 3.11 imply that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left\|u^\varepsilon\left(\frac{\pi}{2}-\Lambda\varepsilon,\cdot\right)-\tilde{v}^\varepsilon\left(\frac{\pi}{2}-\Lambda\varepsilon,\cdot\right)\right\|_{L^2}\\
&\quad+\limsup_{\varepsilon\to 0}\left\|\varepsilon\nabla\left(u^\varepsilon\left(\frac{\pi}{2}-\Lambda\varepsilon,\cdot\right)-\tilde{v}^\varepsilon\left(\frac{\pi}{2}-\Lambda\varepsilon,\cdot\right)\right)\right\|_{L^2}\\
&\quad+\limsup_{\varepsilon\to 0}\left\|\left(\frac{x}{\varepsilon}-i\Lambda\varepsilon\nabla\right)\left(u^\varepsilon\left(\frac{\pi}{2}-\Lambda\varepsilon,\cdot\right)-\tilde{v}^\varepsilon\left(\frac{\pi}{2}-\Lambda\varepsilon,\cdot\right)\right)\right\|_{L^2}\underset{\Lambda\to+\infty}{\longrightarrow}0.
\end{aligned}$$

Define the remainder $\widetilde{w}^\varepsilon=u^\varepsilon-\tilde{v}^\varepsilon$, and keep the notation $t_*^\varepsilon=\pi/2-\Lambda\varepsilon$. From Proposition 3.11,

$$\limsup_{\varepsilon\to 0}\left\|B^\varepsilon(t_*^\varepsilon)\widetilde{w}^\varepsilon(t_*^\varepsilon)\right\|_{L^2}\underset{\Lambda\to+\infty}{\longrightarrow}0,$$

where $B^\varepsilon(t)$ is either of the operators Id, $\frac{x}{\varepsilon}+i(t-\pi/2)\nabla$ or $\varepsilon\nabla$. From Lemma 3.12, this implies

$$\limsup_{\varepsilon\to 0}\left\|A^\varepsilon(t_*^\varepsilon)\widetilde{w}^\varepsilon(t_*^\varepsilon)\right\|_{L^2}\underset{\Lambda\to+\infty}{\longrightarrow}0,$$

where $A^\varepsilon(t)$ is either of the operators Id, $J^\varepsilon(t)$ or $H^\varepsilon(t)$.

From the conservation of energy (3.14), we have

$$\left\|\varepsilon\nabla u^\varepsilon(t)\right\|_{L^2}\leqslant C.$$

From the conservation of energy for (1.7),

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\nabla\psi(t)\right\|_{L^2}^2+\frac{1}{\sigma+1}\left\|\psi(t)\right\|_{L^{2\sigma+2}}^{2\sigma+2}\right)=0,$$

we have

$$\left\|\varepsilon\nabla\tilde{v}^\varepsilon(t)\right\|_{L^2}\leqslant C.$$

Therefore, since $\widetilde{w}^\varepsilon=u^\varepsilon-\tilde{v}^\varepsilon$,

$$\left\|\varepsilon\nabla\widetilde{w}^\varepsilon(t)\right\|_{L^2}\leqslant C.$$

From Sobolev inequality,

$$\left\|\widetilde{w}^{\varepsilon}(t)\right\|_{L^{\underline{s}}} \leqslant C\left\|\widetilde{w}^{\varepsilon}(t)\right\|_{L^2}^{1-\delta(\underline{s})}\left\|\nabla\widetilde{w}^{\varepsilon}(t)\right\|_{L^2}^{\delta(\underline{s})},$$

and there exists C_1 such that for any $t\in\mathbb{R}$,

$$\left\|\widetilde{w}^{\varepsilon}(t)\right\|_{L^{\underline{s}}} \leqslant \frac{C_1}{\varepsilon^{\delta(\underline{s})}}. \tag{3.25}$$

This estimate will be useful for $|t-\pi/2|\leqslant\Lambda_0\varepsilon$, where Λ_0 is given by Corollary 3.6. For $|t-\pi/2|\geqslant\Lambda_0\varepsilon$, sharper estimates are provided by J^{ε}, along with Sobolev inequality (2.16).

The first step of the proof consists in showing that the harmonic potential can be truncated near the origin without altering the asymptotics. Let $\chi\in C_0^{\infty}(\mathbb{R}^n)$ be a cut-off function, with

$$\operatorname{supp}\chi\subset B(0,2),\quad 0\leqslant\chi\leqslant 1\quad\text{and}\quad\forall x\in B(0,1),\ \chi(x)=1.$$

For $R>0$, define

$$u_R^{\varepsilon}(t,x)=\chi\left(\frac{x}{R}\right)u^{\varepsilon}(t,x).$$

LEMMA 3.16. – *Assume $f\in\mathcal{H}$, $\sigma>\frac{1}{2}$ and take $R=\varepsilon^{\alpha}$. Then for any $0<\alpha<1$,*

$$\limsup_{\varepsilon\to 0}\ \sup_{\frac{\pi}{2}-\Lambda\varepsilon\leqslant t\leqslant\frac{\pi}{2}+\Lambda\varepsilon}\left\|A^{\varepsilon}(t)\big(u^{\varepsilon}(t)-u_R^{\varepsilon}(t)\big)\right\|_{L^2}\underset{\Lambda\to+\infty}{\longrightarrow}0,$$

where $A^{\varepsilon}(t)$ is either of the operators Id, $J^{\varepsilon}(t)$ *or* $H^{\varepsilon}(t)$.

Proof of Lemma 3.16. – The function u_R^{ε} satisfies,

$$\left(i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta-\frac{x^2}{2}\right)u_R^{\varepsilon}=\varepsilon^{n\sigma}|u^{\varepsilon}|^{2\sigma}u_R^{\varepsilon}+\frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right)\cdot\nabla u^{\varepsilon}+\left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)u^{\varepsilon},$$

therefore the difference $w_R^{\varepsilon}:=u^{\varepsilon}-u_R^{\varepsilon}$ solves,

$$\left(i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta-\frac{x^2}{2}\right)w_R^{\varepsilon}=\varepsilon^{n\sigma}|u^{\varepsilon}|^{2\sigma}w_R^{\varepsilon}-\frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right)\nabla u^{\varepsilon}-\left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)u^{\varepsilon}.$$

From Lemma 2.1, and because the term $\varepsilon^{n\sigma}|u^{\varepsilon}|^{2\sigma}$ can be considered as a real potential,

$$\varepsilon\partial_t\left\|w_R^{\varepsilon}(t)\right\|_{L^2}\leqslant C\frac{\varepsilon}{R}\left\|\varepsilon\nabla u^{\varepsilon}(t)\right\|_{L^2}+C\left(\frac{\varepsilon}{R}\right)^2\left\|u^{\varepsilon}(t)\right\|_{L^2},$$

which implies, from (3.13) and (3.16),

$$\varepsilon\partial_t\left\|w_R^{\varepsilon}(t)\right\|_{L^2}\leqslant C\frac{\varepsilon}{R}+C\left(\frac{\varepsilon}{R}\right)^2.$$

Integrating this inequality on $[\frac{\pi}{2}-\Lambda\varepsilon, \frac{\pi}{2}+\Lambda\varepsilon]$ gives

$$\sup_{\frac{\pi}{2}-\Lambda\varepsilon\leqslant t\leqslant\frac{\pi}{2}+\Lambda\varepsilon}\left\|w_R^\varepsilon(t)\right\|_{L^2}\leqslant\left\|w_R^\varepsilon(\pi/2-\Lambda\varepsilon)\right\|_{L^2}+C\Lambda\frac{\varepsilon}{R}+C\Lambda\left(\frac{\varepsilon}{R}\right)^2.$$

Taking $R=\varepsilon^\alpha$ with $0<\alpha<1$ yields,

$$\limsup_{\varepsilon\to0}\sup_{\frac{\pi}{2}-\Lambda\varepsilon\leqslant t\leqslant\frac{\pi}{2}+\Lambda\varepsilon}\left\|w_R^\varepsilon(t)\right\|_{L^2}\leqslant\limsup_{\varepsilon\to0}\left\|w_R^\varepsilon(\pi/2-\Lambda\varepsilon)\right\|_{L^2}.$$

Now since $\psi_-\in L^2$, $0<\alpha<1$ implies, along with the dominated convergence theorem,

$$\left\|\left(1-\chi\left(\frac{\cdot}{\varepsilon^\alpha}\right)\right)\frac{1}{\varepsilon^{n/2}}\left(U_0(-\Lambda)\psi_-\right)\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2}\underset{\varepsilon\to0}{\longrightarrow}0.$$

From Proposition 3.11, the first part of Lemma 3.16 (with $A^\varepsilon=\mathrm{Id}$) follows.

To estimate $J^\varepsilon w_R^\varepsilon$, notice that

$$J^\varepsilon(t)w_R^\varepsilon(t,x)=\left(1-\chi\left(\frac{x}{R}\right)\right)J^\varepsilon(t)u^\varepsilon(t,x)+i\frac{\cos t}{R}\nabla\chi\left(\frac{x}{R}\right)u^\varepsilon(t,x),$$

and for $\frac{\pi}{2}-\Lambda\varepsilon\leqslant t\leqslant\frac{\pi}{2}+\Lambda\varepsilon$,

$$\left\|\frac{\cos t}{R}\nabla\chi\left(\frac{\cdot}{R}\right)u^\varepsilon(t,\cdot)\right\|_{L^2}\leqslant C\frac{|\cos t|}{R}\leqslant C\frac{\Lambda\varepsilon}{R}.$$

Therefore to prove Lemma 3.16 when $A^\varepsilon=J^\varepsilon$, it is enough to prove,

$$\limsup_{\varepsilon\to0}\sup_{\frac{\pi}{2}-\Lambda\varepsilon\leqslant t\leqslant\frac{\pi}{2}+\Lambda\varepsilon}\left\|\left(1-\chi\left(\frac{\cdot}{R}\right)\right)J^\varepsilon(t)u^\varepsilon(t,\cdot)\right\|_{L^2}=0.$$

The function $J^\varepsilon(t)u^\varepsilon$ satisfies, from the commutation property (2.13),

$$\left(i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta-\frac{x^2}{2}\right)J^\varepsilon(t)u^\varepsilon=\varepsilon^\sigma J^\varepsilon(t)\left(|u^\varepsilon|^{2\sigma}u^\varepsilon\right).\tag{3.26}$$

Notice that from Proposition 3.9 and (3.16), Sobolev inequality implies that there exists $C=C(\Lambda)$ such that for any $t\in[0,\pi/2+\Lambda\varepsilon]$,

$$\left\|u^\varepsilon(t)\right\|_{L^{\underline{s}}}\leqslant\frac{C}{(|\cos t|+\varepsilon)^{\delta(\underline{s})}}.\tag{3.27}$$

At this stage, C might depend on Λ (even though we will know it does not, afterward). Therefore, Corollary 3.6, applied to (3.26) a finite number of times to cover the interval $[\frac{\pi}{2}-\Lambda_0\varepsilon, \frac{\pi}{2}+\Lambda\varepsilon]$, implies that for any $\Lambda\geqslant\Lambda_0$, $J^\varepsilon(t)u^\varepsilon$ is bounded in L^2 for $t\in[0,\pi/2+\Lambda\varepsilon]$. Next, commuting the cut-off function χ with (3.26) yields,

$$\left(i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta-\frac{x^2}{2}\right)\left(1-\chi\left(\frac{x}{R}\right)\right)J^\varepsilon(t)u^\varepsilon$$
$$=\varepsilon^{n\sigma}\left(1-\chi\left(\frac{x}{R}\right)\right)J^\varepsilon(t)\big(|u^\varepsilon|^{2\sigma}u^\varepsilon\big)-\frac{\varepsilon^2}{2R^2}\Delta\chi\left(\frac{x}{R}\right)J^\varepsilon(t)u^\varepsilon$$
$$-\frac{\varepsilon^2}{R}\nabla\chi\left(\frac{x}{R}\right)\nabla J^\varepsilon(t)u^\varepsilon.$$

From Corollary 3.6 and (3.27), if we denote $t_-^\varepsilon=\pi/2-\Lambda\varepsilon$, $t_+^\varepsilon=\pi/2+\Lambda\varepsilon$, we have

$$\left\|\left(1-\chi\left(\frac{\cdot}{R}\right)\right)J^\varepsilon(t)u^\varepsilon\right\|_{L^\infty(t_-^\varepsilon,t_+^\varepsilon;L^2)}\leqslant\left\|\left(1-\chi\left(\frac{\cdot}{R}\right)\right)J^\varepsilon\big(t_-^\varepsilon\big)u^\varepsilon\right\|_{L^2}$$
$$+C\frac{\varepsilon}{2R^2}\big\|J^\varepsilon(t)u^\varepsilon\big\|_{L^1(t_-^\varepsilon,t_+^\varepsilon;L^2)}+C\frac{\varepsilon}{R}\big\|\nabla J^\varepsilon(t)u^\varepsilon\big\|_{L^1(t_-^\varepsilon,t_+^\varepsilon;L^2)}$$
$$\leqslant\left\|\left(1-\chi\left(\frac{\cdot}{R}\right)\right)J^\varepsilon\big(t_-^\varepsilon\big)u^\varepsilon\right\|_{L^2}$$
$$+C\Lambda\frac{\varepsilon^2}{2R^2}\big\|J^\varepsilon(t)u^\varepsilon\big\|_{L^\infty(t_-^\varepsilon,t_+^\varepsilon;L^2)}+C\Lambda\frac{\varepsilon^2}{R}\big\|\nabla J^\varepsilon(t)u^\varepsilon\big\|_{L^\infty(t_-^\varepsilon,t_+^\varepsilon;L^2)}.$$

We can conclude with the following lemma, whose proof is postponed to Section 3.7.

LEMMA 3.17. – *Assume $f\in\mathcal{H}$ and $\sigma>1/2$. Let $\Lambda>1$. There exists $C=C(\Lambda)$ such that for any $t\in[\pi/2-\Lambda\varepsilon,\pi/2+\Lambda\varepsilon]$,*

$$\big\|\varepsilon\nabla J^\varepsilon(t)u^\varepsilon\big\|_{L^2}+\big\|\varepsilon\nabla H^\varepsilon(t)u^\varepsilon\big\|_{L^2}\leqslant C.$$

This completes the proof of Lemma 3.16, the computations with H^ε being similar.

To prove Proposition 3.15, we now have to compare $\tilde{v}^\varepsilon$ and the truncated exact solution u_R^ε.

LEMMA 3.18. – *Assume $f\in\mathcal{H}$ and take $R=\varepsilon^\alpha$. Then for any $0<\alpha<1$,*

$$\limsup_{\varepsilon\to0}\sup_{\frac{\pi}{2}-\Lambda\varepsilon\leqslant t\leqslant\frac{\pi}{2}+\Lambda\varepsilon}\big\|A^\varepsilon(t)\big(u_R^\varepsilon(t)-\tilde{v}^\varepsilon(t)\big)\big\|_{L^2}\underset{\Lambda\to+\infty}{\longrightarrow}0,$$

where $A^\varepsilon(t)$ is either of the operators Id, $J^\varepsilon(t)$ *or* $H^\varepsilon(t)$.

Proof of Lemma 3.18. – Denote $\widetilde{w}_R^\varepsilon=u_R^\varepsilon-\tilde{v}^\varepsilon$. Recall that u_R^ε solves

$$\left(i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta-\frac{x^2}{2}\right)u_R^\varepsilon=\varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u_R^\varepsilon+\frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right).\nabla u^\varepsilon+\left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)u^\varepsilon,$$

and notice that with our choice for the cut-off function χ,

$$\chi\left(\frac{x}{R}\right)=\chi\left(\frac{x}{2R}\right)\chi\left(\frac{x}{R}\right),$$

therefore

$$\left(i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta\right)u_R^\varepsilon=V_R(x)u_R^\varepsilon+\varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u_R^\varepsilon + \frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right)\nabla u^\varepsilon+\left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)u^\varepsilon,$$

where

$$V_R(x)=\chi\left(\frac{x}{2R}\right)\frac{x^2}{2}.$$

The remainder $\widetilde{w}_R^\varepsilon$ solves

$$\begin{aligned}\left(i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta\right)\widetilde{w}_R^\varepsilon = {} & V_R(x)u_R^\varepsilon+\varepsilon^{n\sigma}\left(|u^\varepsilon|^{2\sigma}u_R^\varepsilon-|\tilde{v}^\varepsilon|^{2\sigma}\tilde{v}^\varepsilon\right)\\ & +\frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right)\nabla u^\varepsilon+\left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)u^\varepsilon.\end{aligned}\tag{3.28}$$

Apply Proposition 3.5, with now $\mathcal{U}^\varepsilon=U_0^\varepsilon$, $F^\varepsilon=0$ and $h^\varepsilon=h_1^\varepsilon+h_2^\varepsilon$, where

$$h_1^\varepsilon=V_R(x)u_R^\varepsilon+\frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right)\nabla u^\varepsilon+\left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)u^\varepsilon,$$

and

$$h_2^\varepsilon=\varepsilon^{n\sigma}\left(|u^\varepsilon|^{2\sigma}u_R^\varepsilon-|\tilde{v}^\varepsilon|^{2\sigma}\tilde{v}^\varepsilon\right).$$

This yields, for $\pi/2-\Lambda\varepsilon\leqslant t_0\leqslant t_1\leqslant\pi/2+\Lambda\varepsilon$,

$$\begin{aligned}\|\widetilde{w}_R^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})} &\leqslant C\varepsilon^{-\frac{1}{\underline{q}}}\|\widetilde{w}_R^\varepsilon(t_0)\|_{L^2}+C\varepsilon^{-1-\frac{1}{\underline{q}}}\|h_1^\varepsilon\|_{L^1(t_0,t_1;L^2)}\\ &\quad +C\varepsilon^{-1-\frac{2}{\underline{q}}}\|h_2^\varepsilon\|_{L^{\underline{q}'}(t_0,t_1;L^{\underline{r}'})}\\ &\leqslant C\varepsilon^{-\frac{1}{\underline{q}}}\|\widetilde{w}_R^\varepsilon(t_0)\|_{L^2}+C\varepsilon^{-1-\frac{1}{\underline{q}}}\int_{t_0}^{t_1}\left(R^2+\frac{\varepsilon}{R}+\left(\frac{\varepsilon}{R}\right)^2\right)dt\\ &\quad +\underline{C}\left(\frac{t_1-t_0}{\varepsilon}\right)^{2\sigma/\underline{k}}\left(\|\widetilde{w}_R^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})}+\|w_R^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})}\right),\end{aligned}$$

from Hölder inequality. Taking

$$\underline{C}\left(\frac{t_1-t_0}{\varepsilon}\right)^{2\sigma/\underline{k}}\leqslant\frac{1}{2},$$

we have

$$\begin{aligned}\|\widetilde{w}_R^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})} &\leqslant C\varepsilon^{-\frac{1}{\underline{q}}}\|\widetilde{w}_R^\varepsilon(t_0)\|_{L^2}+C\varepsilon^{-\frac{1}{\underline{q}}}\left(R^2+\frac{\varepsilon}{R}+\left(\frac{\varepsilon}{R}\right)^2\right)\\ &\quad +C\|w_R^\varepsilon\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})}.\end{aligned}$$

Repeating this manipulation a finite number of times covers the whole interval $t\in[\pi/2-\Lambda_0\varepsilon,\pi/2+\Lambda_0\varepsilon]$. Doing this, we get a possibly large, but finite, constant, which can be seen as the analogue of the exponential term in Gronwall lemma. For $\Lambda_0\varepsilon \leqslant |t-\pi/2| \leqslant \Lambda\varepsilon$, we use time decay estimates provided by J^ε; bearing the comparison with Gronwall lemma in mind, this means that the operator J^ε provides some integrability for $\Lambda_0\varepsilon \leqslant |t-\pi/2|$, which is stated in (3.8), and implies the first condition in Corollary 3.6. This integrability is needed to get a bound independent of $\Lambda \geqslant \Lambda_0$. When $A^\varepsilon = H^\varepsilon$, from (2.13),

$$\left[H^\varepsilon(t), i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta\right] = \left[H^\varepsilon(t), \frac{x^2}{2}\right] = i\varepsilon x\sin t,$$

and $H^\varepsilon u_R^\varepsilon$ satisfies,

$$\begin{aligned}\left(i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta\right)H^\varepsilon u_R^\varepsilon = {} & i\varepsilon x(\sin t)u_R^\varepsilon + \varepsilon^{n\sigma}H^\varepsilon\left(|u^\varepsilon|^{2\sigma}u_R^\varepsilon\right)\\ & + V_R(x)H^\varepsilon u_R^\varepsilon + i\varepsilon(\sin t)\nabla V_R(x)u_R^\varepsilon\\ & + \frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right)H^\varepsilon\partial_x u^\varepsilon + i\varepsilon\sin t\frac{\varepsilon^2}{2R^2}\Delta\chi\left(\frac{x}{R}\right)\nabla u^\varepsilon\\ & + \left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)H^\varepsilon u^\varepsilon + i\varepsilon\sin t\frac{\varepsilon^2}{R^3}\nabla\Delta\chi\left(\frac{x}{R}\right)u^\varepsilon.\end{aligned}$$

It follows that the remainder $H^\varepsilon\widetilde{w}_R^\varepsilon$ satisfies,

$$\begin{aligned}\left(i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta\right)H^\varepsilon\widetilde{w}_R^\varepsilon = {} & \varepsilon^{n\sigma}H^\varepsilon\left(|u^\varepsilon|^{2\sigma}u_R^\varepsilon - |\tilde{v}^\varepsilon|^{2\sigma}\tilde{v}^\varepsilon\right)\\ & + ix\varepsilon\chi\left(\frac{x}{R}\right)(\sin t)u^\varepsilon + i\varepsilon(\sin t)\nabla V_R(x)u_R^\varepsilon\\ & + V_R(x)\chi\left(\frac{x}{R}\right)H^\varepsilon u^\varepsilon + i\frac{\varepsilon}{R}V_R(x)(\sin t)\nabla\chi\left(\frac{x}{R}\right)u^\varepsilon\\ & + \frac{\varepsilon^2}{2R}\nabla\chi\left(\frac{x}{R}\right)H^\varepsilon\nabla u^\varepsilon + i\varepsilon\sin t\frac{\varepsilon^2}{2R^2}\Delta\chi\left(\frac{x}{R}\right)\nabla u^\varepsilon\\ & + \left(\frac{\varepsilon}{R}\right)^2\Delta\chi\left(\frac{x}{R}\right)H^\varepsilon u^\varepsilon + i\varepsilon\sin t\frac{\varepsilon^2}{R^3}\nabla\Delta\chi\left(\frac{x}{R}\right)u^\varepsilon.\end{aligned}$$

We can estimate the term in $H^\varepsilon\partial_x u^\varepsilon$ because we can estimate $\partial_x H^\varepsilon u^\varepsilon$ (Lemma 3.17) and the following holds,

$$\left[H^\varepsilon(t), \nabla\right] = -\cos t = \mathrm{O}(\varepsilon) \quad \text{for } \pi/2 - \Lambda\varepsilon \leqslant t \leqslant \pi/2 + \Lambda\varepsilon.$$

The proof then proceeds as above. □

Lemmas 3.16 and 3.18 clearly imply Proposition 3.15.

The assumption $f \in \mathcal{H}$ turns out to be unnecessary. Indeed, we can use a density argument for $\tilde{v}^\varepsilon$, and approach $f \in \Sigma$ by functions in $\mathcal{H}$ up to a small error in the norms that are considered in Proposition 3.15; this stems from global well-posedness of (1.7) (see, e.g., [8,17]). We can mimic the proof of this result for u^ε, thanks to J^ε and H^ε.

PROPOSITION 3.19. – *Proposition* 3.15 *still holds if we assume* $f \in \Sigma$ *and* $r^\varepsilon \not\equiv 0$.

3.7. Proof of Lemma 3.17

We first use the following remark.

LEMMA 3.20. – *Assume a function $u^\varepsilon(x)$ satisfies*

$$\left\|\left(-\varepsilon^2\Delta + x^2\right)u^\varepsilon\right\|_{L^2} + \|\varepsilon\nabla u^\varepsilon\|_{L^2} + \|xu^\varepsilon\|_{L^2} \leqslant \underline{C},$$

where $\underline{C}$ does not depend on ε. Then

$$\|\varepsilon^2\Delta u^\varepsilon\|_{L^2} + \|x^2u^\varepsilon\|_{L^2} \leqslant \underline{C}.$$

Now the idea is to differentiate (1.4) with respect to time. This is classical for the case of the nonlinear Schrödinger equation (1.7), see, e.g., [8], Section 5.2. Thanks to the above lemma, we can adapt the mentioned results to prove the following proposition.

PROPOSITION 3.21. – *Assume $f \in \mathcal{H}$. Let $\Lambda > 1$. Then*

$$u^\varepsilon \in C\left(0, \frac{\pi}{2} + \Lambda\varepsilon; H^2 \cap \mathcal{F}\left(H^2\right)\right) \cap C^1\left(0, \frac{\pi}{2} + \Lambda\varepsilon; L^2\right),$$

and there exists $C = C(\Lambda)$ independent of ε such that

$$\sup_{0\leqslant t\leqslant \frac{\pi}{2}+\Lambda\varepsilon} \left\|\varepsilon\partial_t u^\varepsilon(t)\right\|_{L^2} + \sup_{0\leqslant t\leqslant \frac{\pi}{2}+\Lambda\varepsilon} \left\|\varepsilon^2\Delta u^\varepsilon(t)\right\|_{L^2} + \sup_{0\leqslant t\leqslant \frac{\pi}{2}+\Lambda\varepsilon} \left\|x^2u^\varepsilon(t)\right\|_{L^2} \leqslant C.$$

Idea of the proof. – As in [8], the idea of the proof consists in differentiating the equation satisfied by u^ε with respect to time, and estimate $\varepsilon\partial_t u^\varepsilon$. Since the harmonic potential commutes with the time derivative, one can mimic the proof given in [8], Section 5.2. When there is no potential, like in (1.7), the control of the nonlinear term and the time derivative give some control on $\varepsilon^2\Delta u^\varepsilon$. In our case, this controls $\varepsilon^2\Delta u^\varepsilon - x^2u^\varepsilon$. From the above lemma, this means that we can estimate each of these two terms. □

Notice that the following algebraic identity holds point-wise, for any j, k,

$$\begin{aligned}&\left|x_k\varepsilon J_j^\varepsilon(t)u^\varepsilon\right|^2 + \left|\varepsilon^2\partial_k J_j^\varepsilon(t)u^\varepsilon\right|^2 + \left|x_k H_j^\varepsilon(t)u^\varepsilon\right|^2 + \left|\varepsilon\partial_k H_j^\varepsilon(t)u^\varepsilon\right|^2\\ &\quad= \left|x_jx_ku^\varepsilon(t)\right|^2 + \left|\varepsilon x_k\partial_j u^\varepsilon(t)\right|^2 + \left|\varepsilon\delta_{jk}u^\varepsilon(t) + \varepsilon x_j\partial_k u^\varepsilon(t)\right|^2 + \left|\varepsilon^2\partial_{jk}^2u^\varepsilon(t)\right|^2,\end{aligned} \tag{3.29}$$

where δ_{jk} stands for the Kronecker symbol. From Proposition 3.21, the right-hand side is bounded in L_x^1, uniformly for $0 \leqslant t \leqslant \frac{\pi}{2} + \Lambda\varepsilon$. This implies the boundedness of $\varepsilon\nabla H^\varepsilon(t)u^\varepsilon$ stated in Lemma 3.17, and even a little more, that is,

$$\forall t \in [0, \pi/2 + \Lambda\varepsilon], \quad \left\|\varepsilon\nabla H^\varepsilon(t)u^\varepsilon\right\|_{L^2} \leqslant C. \tag{3.30}$$

At this stage, we have not assumed that the nonlinearity was twice differentiable. On the other hand, we just have

$$\left\|\varepsilon\nabla J^\varepsilon(t)u^\varepsilon\right\|_{L^2} \leqslant \frac{C}{\varepsilon}.$$

The idea is that for this term, (3.29) is far from giving a sharp estimate. Indeed, for $|t-\pi/2| = \mathrm{O}(\varepsilon)$, we guess that the main contribution of u^ε lies in $|x| = \mathrm{O}(\varepsilon)$ (semi-classical Schrödinger equations are morally hyperbolic). This is precisely what we have to prove. With the additional remark that near $t=\pi/2$, one can replace H^ε with $\varepsilon\nabla$ up to a small error term, this suggests that the leading order term of the left-hand side is $|\varepsilon\partial_{x_k} H_j^\varepsilon(t)u^\varepsilon|^2$, and the leading order term of the right-hand side is $|\varepsilon^2\partial^2_{x_j x_k} u^\varepsilon(t)|^2$. Thus there is nothing more to hope from this identity.

This in fact must not be surprising. The only additional estimates we obtained are those stated in Proposition 3.21,

$$\sup_{0\leqslant t\leqslant \frac{\pi}{2}+\Lambda\varepsilon} \left\|\varepsilon\partial_t u^\varepsilon(t)\right\|_{L^2} + \sup_{0\leqslant t\leqslant \frac{\pi}{2}+\Lambda\varepsilon} \left\|\varepsilon^2\Delta u^\varepsilon(t)\right\|_{L^2} + \sup_{0\leqslant t\leqslant \frac{\pi}{2}+\Lambda\varepsilon} \left\|x^2 u^\varepsilon(t)\right\|_{L^2} \leqslant C.$$

The boundedness of the first two terms means that u^ε is ε-oscillating, and the boundedness of the last term means that the solution remains confined. This is due to the fact that we work with an unbounded potential, but not to the fact that we consider the harmonic potential in particular. Therefore, there is no precise geometric information in this estimate. As a matter of fact, away from the focus, this kind of information is given by the operator J^ε.

We assume that the nonlinearity is twice differentiable. Recall that for $1\leqslant j\leqslant n$, $J_j^\varepsilon(t)u^\varepsilon$ satisfies

$$\left(i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta - \frac{x^2}{2}\right)J_j^\varepsilon(t)u^\varepsilon = \varepsilon^{n\sigma} J_j^\varepsilon(t)\left(|u^\varepsilon|^{2\sigma}u^\varepsilon\right).$$

Differentiating this equation with respect to x_k yields

$$\left(i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta - \frac{x^2}{2}\right)\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon = \varepsilon^{n\sigma+1}\partial_k J_j^\varepsilon(t)\left(|u^\varepsilon|^{2\sigma}u^\varepsilon\right) + \varepsilon x_k J_j^\varepsilon(t)u^\varepsilon. \tag{3.31}$$

The last term comes from the commutation of the harmonic potential with ∂_{x_k}. From (2.17), the following point-wise estimate holds,

$$\left|\varepsilon\partial_k J_j^\varepsilon(t)\left(|u^\varepsilon|^{2\sigma}u^\varepsilon\right)\right| \lesssim \left|u^\varepsilon\right|^{2\sigma-1}\left|\varepsilon\partial_k u^\varepsilon\right|\cdot\left|J_j^\varepsilon(t)u^\varepsilon\right| + |u^\varepsilon|^{2\sigma}\left|\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon\right|. \tag{3.32}$$

The idea is that the last term is well prepared to apply Gronwall lemma. For the first term of the left-hand side, we have to work a little more. Apply Corollary 3.6 to (3.31), with

$$\begin{aligned}
&\left|F^\varepsilon\left(\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon\right)\right| \lesssim |u^\varepsilon|^{2\sigma}\left|\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon\right|,\\
&|h^\varepsilon| \lesssim \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma-1}\left|\varepsilon\partial_k u^\varepsilon\right|\cdot\left|J_j^\varepsilon(t)u^\varepsilon\right| + \left|\varepsilon x_k J_j^\varepsilon(t)u^\varepsilon\right|.
\end{aligned}$$

We already know that F^ε satisfies (3.8). Corollary 3.6 yields,

$$\begin{aligned}
\left\|\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)} &\leqslant \left\|\varepsilon\partial_k(x_j f)\right\|_{L^2} + C\left\|x_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^1(t_0,t_1;L^2)}\\
&\quad + C\varepsilon^{n\sigma-1-\frac{1}{\underline{q}}}\left\||u^\varepsilon|^{2\sigma-1}\varepsilon\partial_k u^\varepsilon J_j^\varepsilon(t)u^\varepsilon\right\|_{L^{\underline{q}'}(t_0,t_1;L^{\underline{r}'})}.
\end{aligned} \tag{3.33}$$

For fixed t, Hölder inequality yields,

$$\left\| |u^\varepsilon|^{2\sigma-1}\varepsilon\partial_k u^\varepsilon J_j^\varepsilon(t)u^\varepsilon \right\|_{L^{\underline{r}'}} \leqslant \|u^\varepsilon\|_{L^{a_1}}^{2\sigma-1}\left\|\varepsilon\partial_k u^\varepsilon\right\|_{L^{a_2}}\left\|J_j^\varepsilon(t)u^\varepsilon\right\|_{L^{\underline{r}}},$$

with

$$\frac{2\sigma-1}{a_1}+\frac{1}{a_2}=\frac{2\sigma}{\underline{s}}.$$

We can take for instance $a_1=a_2=\underline{s}$. This implies, along with (2.16), since $\|\varepsilon\partial_k H^\varepsilon(t)u^\varepsilon\|_{L^2}$ is uniformly bounded

$$\left\| |u^\varepsilon|^{2\sigma-1}\varepsilon\partial_k u^\varepsilon J_j^\varepsilon(t)u^\varepsilon \right\|_{L^{\underline{r}'}} \leqslant \frac{C}{(\cos t+\varepsilon)^{2\sigma\delta(\underline{s})}}\left(\left\|J^\varepsilon(t)\varepsilon\partial_k u^\varepsilon\right\|_{L^2}^{\delta(a_2)}+1\right)\left\|J_j^\varepsilon(t)u^\varepsilon\right\|_{L^{\underline{r}}}.$$

Now apply Hölder inequality in time, with

$$\frac{1}{\underline{q}'}=\frac{2\sigma}{\underline{k}}+\frac{1}{\infty}+\frac{1}{\underline{q}}.$$

This yields

$$\begin{aligned}\left\| |u^\varepsilon|^{2\sigma-1}\varepsilon\partial_k u^\varepsilon J_j^\varepsilon(t)u^\varepsilon \right\|_{L^{\underline{q}'}(t_0,t_1;L^{\underline{r}'})} &\lesssim A^\varepsilon(t_0,t_1)\left\|J_j^\varepsilon(t)u^\varepsilon\right\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})}\\ &\quad + A^\varepsilon(t_0,t_1)\left\|J^\varepsilon(t)\varepsilon\partial_k u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)}^{\delta(a_2)}\left\|J_j^\varepsilon(t)u^\varepsilon\right\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})},\end{aligned}\tag{3.34}$$

where A^ε is defined in Proposition 3.5. We also know that

$$\left\|J_j^\varepsilon(t)u^\varepsilon\right\|_{L^{\underline{q}}(t_0,t_1;L^{\underline{r}})} \leqslant C\varepsilon^{-1/\underline{q}},$$

therefore (3.33) yields,

$$\begin{aligned}\left\|\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)} &\leqslant \left\|\varepsilon\partial_{jk}^2 f\right\|_{L^2}+C\left\|x_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^1(t_0,t_1;L^2)}\\ &\quad + C\varepsilon^{n\sigma-1-\frac{2}{\underline{q}}}A^\varepsilon(t_0,t_1)\left(1+\left\|J^\varepsilon(t)\varepsilon\partial_k u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)}^{\delta(a_2)}\right).\end{aligned}$$

But from Lemma 3.4,

$$n\sigma-1-\frac{2}{\underline{q}}=2\sigma\left(\delta(\underline{s})-\frac{1}{\underline{k}}\right),$$

and we find the same quantity as in Proposition 3.5, that is $\varepsilon^{2\sigma(\delta(\underline{s})-\frac{1}{\underline{k}})}A^\varepsilon(t_0,t_1)$. With the remarks that

$$\left[\varepsilon\partial_k, J_j^\varepsilon(t)\right]=\delta_{jk}\sin t,$$

and $\delta(a_2)\leqslant 1$, we have also,

$$\begin{aligned}\left\|\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)} &\leqslant \left\|\varepsilon\partial_{jk}^2 f\right\|_{L^2}+C\left\|x_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^1(t_0,t_1;L^2)}\\ &\quad + C\varepsilon^{n\sigma-1-\frac{2}{\underline{q}}}A^\varepsilon(t_0,t_1)\left(1+\left\|\varepsilon\partial_k J^\varepsilon(t)u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)}\right).\end{aligned}\tag{3.35}$$

Now it is natural to study $x_k J_j^\varepsilon(t)u^\varepsilon$. It satisfies,

$$\left(i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta - \frac{x^2}{2}\right)x_k J_j^\varepsilon(t)u^\varepsilon = \varepsilon^{n\sigma} x_k J_j^\varepsilon(t)\left(|u^\varepsilon|^{2\sigma}u^\varepsilon\right) + \varepsilon^2\partial_k J_j^\varepsilon(t)u^\varepsilon.$$

The same computation as above, minus the three terms estimate which is not needed here, yields

$$\begin{aligned}\left\|x_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)} &\leqslant \|x_k\partial_j f\|_{L^2} + C\left\|\varepsilon\partial_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^1(t_0,t_1;L^2)} \\ &\quad + C\varepsilon^{n\sigma-1-\frac{2}{q}} A^\varepsilon(t_0,t_1)\left\|x_k J_j^\varepsilon(t)u^\varepsilon\right\|_{L^\infty(t_0,t_1;L^2)}. \qquad (3.36)\end{aligned}$$

Summing (3.35) and (3.36) over j and k, Lemma 3.17 follows from the Gronwall lemma. □

3.8. Past the first focus

After the first focus, we can proceed like before the focus, and iterate this process. Notice that if $n \geqslant 3$, then $\frac{1}{2} > \sigma_0(n)$, and we always have $\sigma > \sigma_0(n)$. Next, we can prove the analogous of Proposition 3.9, using Proposition 3.11 and Corollary 2.5.

PROPOSITION 3.22. – *The following asymptotics holds for $\pi/2 < t \leqslant \pi$,*

$$\limsup_{\varepsilon\to 0}\ \sup_{\pi/2+\Lambda\varepsilon\leqslant t\leqslant\pi}\left\|A^\varepsilon(t)\left(u^\varepsilon - v_1^\varepsilon\right)(t)\right\|_{L^2} \underset{\Lambda\to+\infty}{\longrightarrow} 0,$$

where $A^\varepsilon(t)$ is either of the operators Id, *$J^\varepsilon(t)$ or $H^\varepsilon(t)$.*

Finally, $v_{\mathrm{app},1}^\varepsilon$ approximates v_1^ε like in Corollary 2.5. Then Corollary 2.5, Propositions 3.9, 3.19 and 3.22 imply Theorem 1.2.

When $t = \pi$, the problem is almost the same as at time $t = 0$. The initial data f is replaced by f_1, and

$$\left\|u^\varepsilon(\pi,\cdot) - f_1\right\|_\Sigma = \mathrm{o}(1).$$

Therefore, Theorem 1.2 can be iterated, which yields Corollary 1.4, because of the property,

$$\forall\theta\in\mathbb{R},\ \forall\psi_-\in\Sigma,\ S\left(e^{i\theta}\psi_-\right) = e^{i\theta}S(\psi_-).$$

4. When the nonlinearity is focusing

In this section, we assume $n = 1$ for simplicity. The first remark to guess the result of Proposition 1.5 is that in the proof of Proposition 3.9, the sign of the nonlinearity in unimportant. One needs local existence results to start the “so long” argument, and general estimates on the nonlinear term that do not involve its sign. Therefore Proposition 3.9 still holds when u^ε is the solution of (1.6).

Next, assume for a moment that the matching argument can be used as in Proposition 3.11, and that afterward, the harmonic potential can be neglected because of concentration. The behavior of u^ε should then be the same as the solution of

$$i\varepsilon\partial_t v^\varepsilon + \frac{1}{2}\varepsilon^2\partial_x^2 v^\varepsilon = -\varepsilon^2|v^\varepsilon|^4 v^\varepsilon.$$

Resuming the scaling (3.22), we have to understand the behavior of the solution of the same equation with $\varepsilon = 1$. It is well known (see [8]) that for small initial data, the solution exists globally. The critical mass is the L^2-norm of the ground state R defined in Proposition 1.5. Recall what happens in this critical case.

THEOREM 4.1 ([24], case $n = 1$). – *Let $\varphi \in \Sigma$, with $\|\varphi\|_{L^2} = \|R\|_{L^2}$. Let ψ be the solution of the initial value problem,*

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\partial_x^2\psi = -|\psi|^4\psi, \\ \psi_{|t=0} = \varphi. \end{cases}$$

Assume that ψ blows up at time $t = t_*$. Then there exist $\theta, \omega, \xi_0, x_1 \in \mathbb{R}$ such that for $t < t_*$,

$$\psi(t,x) = \sqrt{\frac{\omega}{t_* - t}}R\left(\omega\left(\frac{x - x_1}{t_* - t} - \xi_0\right)\right)e^{i(\theta + \frac{\omega^2}{t_*-t} - \frac{(x-x_1)^2}{2(t_*-t)})}. \tag{4.1}$$

The second important remark is that such profiles as in (4.1) are dispersed when t goes to $-\infty$. If $\omega = 1$, $x_1 = \xi_0 = 0$, then

$$\left\|U_0(t_* - t)\psi(t) - \frac{1}{\sqrt{2i\pi}}\widehat{R}\right\|_\Sigma \underset{t\to-\infty}{\longrightarrow} 0. \tag{4.2}$$

From the uniqueness in the first part of Proposition 3.10, if ψ solves the critical nonlinear Schrödinger equation and behaves asymptotically when $t \to -\infty$ like the free evolution of

$$\psi_-^{t_*} := \frac{1}{\sqrt{2i\pi}}U_0(-t_*)\widehat{R},$$

then ψ is given by (4.1) with $\omega = 1$ and $x_1 = \xi_0 = 0$. Back to the scaling (3.22), this yields the definitions $f(x) = R(x)e^{i\frac{t_*}{2}x^2}$ (from (1.8) and the definition of $\psi_-^{t_*}$) and (1.12).

Now sketch the proof of Proposition 1.5. As we noticed, Proposition 3.9 describes the behavior of u^ε up to $t = \pi/2 - \Lambda\varepsilon$ for large Λ. What prevents us from mimicking the proof of Proposition 3.11? The limit (3.23) still holds, as well as Lemmas 3.13 and 3.14. However, one cannot apply Lemma 3.12 so easily to u^ε and $v^\varepsilon_{\mathrm{app}}$ for estimate (3.16) is not true when the nonlinearity is focusing. On the other hand, (3.16) is true up to time $t = \pi/2 - \Lambda\varepsilon$ for large Λ, from Proposition 3.9 and the algebraic identity (3.15). Therefore Proposition 3.11 still holds.

Finally, one can adapt Proposition 3.15 by replacing the time interval $[\pi/2 - \Lambda\varepsilon, \pi/2 + \Lambda\varepsilon]$ by $[\pi/2 - \Lambda\varepsilon, \pi/2 + t_*\varepsilon - \lambda\varepsilon]$, for any positive λ. The method of our proof does not allow to go further. Indeed, we have the following estimates,

$$\left\|\tilde{v}^\varepsilon\left(\frac{\pi}{2}+t_*\varepsilon-\lambda\varepsilon\right)\right\|_{L^\infty}=\frac{\|R\|_{L^\infty}}{\sqrt{\lambda\varepsilon}},\qquad \left\|\varepsilon\partial_x\tilde{v}^\varepsilon\left(\frac{\pi}{2}+t_*\varepsilon-\lambda\varepsilon\right)\right\|_{L^2}=\frac{C(R)}{\lambda}.$$

Therefore, one cannot hope that (3.25) holds beyond $t=\frac{\pi}{2}+t_*\varepsilon-\lambda\varepsilon$ (with C_1 proportional to $\lambda^{-1/2}$). On the other hand, if our final time is $t=\frac{\pi}{2}+t_*\varepsilon-\lambda\varepsilon$ with $\lambda>0$, we can prove the analogue of Proposition 3.15 by a "so long" argument (that is, (3.25) with C_1 proportional to $\lambda^{-1/2}$). As a result, we have the first part of Proposition 1.5. The last part follows from the remark we made above, that we know $\tilde{v}^\varepsilon$ explicitly, therefore in particular its value at time $t=\frac{\pi}{2}+t_*\varepsilon-\lambda\varepsilon$.

5. Anisotropic harmonic potential

Consider the general harmonic potential in $\mathbb{R}^n$,

$$V(x)=\frac{1}{2}\left(\omega_1^2x_1^2+\omega_2^2x_2^2+\cdots+\omega_n^2x_n^2\right),\tag{5.1}$$

with $\omega_j>0$ for all j. It is isotropic when all the ω_j's are equal, anisotropic otherwise. We suppose that the ω_j's take exactly d distinct values ($2\leqslant d\leqslant n$), and renaming the space variables if necessary, we can assume that

$$0<\omega_1<\omega_2<\cdots<\omega_d.$$

We denote i_j the multiplicity of ω_j, $1\leqslant j\leqslant d$ ($i_1+\cdots+i_d=n$). At least two possibilities occur, as for the result one can hope for, corresponding either to Theorem 1.2 or to Corrolary 1.3. In the former case, one would be interested in the Cauchy problem

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon+\dfrac{1}{2}\varepsilon^2\Delta u^\varepsilon=V(x)u^\varepsilon+\varepsilon^{k\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\\ u^\varepsilon_{|t=0}=f(x)+r^\varepsilon(x), \end{cases}\tag{5.2}$$

and in the latter, in

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon+\dfrac{1}{2}\varepsilon^2\Delta u^\varepsilon=V(x)u^\varepsilon+\varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\\ u^\varepsilon_{|t=0}=\dfrac{1}{\varepsilon^{n/2}}f\left(\dfrac{x}{\varepsilon}\right)+\dfrac{1}{\varepsilon^{n/2}}r^\varepsilon\left(\dfrac{x}{\varepsilon}\right), \end{cases}\tag{5.3}$$

where $f,r^\varepsilon\in\Sigma$ and $\|r^\varepsilon\|_\Sigma\underset{\varepsilon\to 0}{\longrightarrow}0$. We briefly discuss Eq. (5.2), and explain more precisely what happens for Eq. (5.3).

For (5.2), the same method as in Section 2 leads to the following phase, profile and operators,

$$\begin{aligned}
\varphi(t,x) &= -\frac{1}{2}\sum_{j=1}^{n}\omega_j x_j^2\tan(\omega_j t),\\
v_0(t,x) &= \left(\prod_{j=1}^{n}\frac{1}{\sqrt{\cos(\omega_j t)}}\right) f\left(\frac{x_1}{\cos(\omega_1 t)},\dots,\frac{x_n}{\cos(\omega_n t)}\right),\\
J_j^\varepsilon(t) &= \frac{\omega_j x_j}{\varepsilon}\sin(\omega_j t) - i\cos(\omega_j t)\partial_j,\\
H_j^\varepsilon(t) &= \omega_j x_j\cos(\omega_j t) + i\varepsilon\sin(\omega_j t)\partial_j.
\end{aligned}\tag{5.4}$$

The first focusing occurs for $t=\frac{\pi}{2\omega_d}$; the solution u^ε focuses on the i_d-dimensional vector space defined by

$$E_d = \{x_j = 0,\ \forall j \text{ such that } \omega_j = \omega_d\}.$$

Therefore, the critical index for the nonlinear term to be relevant in (5.2) would be $k = \dim E_d = i_d$. If $k > i_d$, then the nonlinear term remains negligible up to time $t = \frac{\pi}{2\omega_d}$ and before the next focusing, where the same discussion is valid. If $k > \max_{1\leqslant j\leqslant d} i_j$, then the nonlinear term is everywhere negligible, provided that no simultaneous focusings occur; indeed, the ω_j part of the harmonic potential will cause focusing at times

$$\frac{\pi}{2\omega_j} + \frac{\kappa\pi}{\omega_j},\quad \kappa\in\mathbb{Z}.$$

Two (or more) distinct ω_j's can cause cumulated focusing if they are rationally related. To simplify the discussion, we now assume $n=2$ and that ω_1 and ω_2 are irrationally related. In that case, u^ε focuses at time $t=\frac{\pi}{2\omega_2}$ on the line $\{x_2=0\}$. If $k=1$, then the nonlinear term becomes relevant near $\{(t,x_2)=(\frac{\pi}{2\omega_2},0)\}$. The case of a focusing on a line was treated in [4] without potential, with an initial oscillation that forces such a geometry for the caustic. With an anisotropic oscillator, the situation is technically much harder to handle. In [4], no oscillation was present in the other space variable, and this variable could be considered as a parameter. In the present case, oscillations are always present in both space variables, so it is harder to measure the dependence of u^ε with respect to x_1 when it focuses on $\{x_2=0\}$. We leave out the discussion at this stage.

On the other hand, it is possible to understand (and prove) what happens for Eq. (5.3). Because we altered the time origin, the operators we now use write,

$$\begin{aligned}
J_j^\varepsilon(t) &= \frac{\omega_j x_j}{\varepsilon}\cos(\omega_j t) + i\sin(\omega_j t)\partial_j,\\
H_j^\varepsilon(t) &= \omega_j x_j\sin(\omega_j t) - i\varepsilon\cos(\omega_j t)\partial_j.
\end{aligned}\tag{5.5}$$

We also have an explicit formula for the linear solution (the analogue of Eq. (1.3)), which yields in particular Strichartz estimates. The solution of

$$\begin{cases} i\varepsilon\partial_t v^\varepsilon + \frac{1}{2}\varepsilon^2\Delta v^\varepsilon = V(x)v^\varepsilon, \\ v^\varepsilon_{|t=0} = f(x), \end{cases}$$

is given by

$$v^\varepsilon(t,x) = \prod_{j=1}^n \left(\frac{\omega_j}{2i\pi\varepsilon\sin\omega_j t}\right)^{1/2} \int_{\mathbb{R}^n} e^{iS(t,x,y)/\varepsilon} f(y)\,dy,$$

where

$$S(t,x,y) = \sum_{j=1}^n \frac{\omega_j}{\sin\omega_j t}\left(\frac{x_j^2+y_j^2}{2}\cos\omega_j t - x_j y_j\right).$$

It is not hard to see that one can mimic the proof of Theorem 1.2 to get the following,

THEOREM 5.1. – *Assume* $2 \leqslant n \leqslant 5$, $\frac{1}{2} < \sigma < \frac{2}{n-2}$, *and let* $2 < r < \frac{2n}{n-2}$. *If* $n = 2$, *there exists* $\delta > 0$ *such that in either of the two cases,*

- $\sigma > \sigma_0(2)$, *or*
- $\|f\|_\Sigma \leqslant \delta$,

the following holds (*if* $3 \leqslant n \leqslant 5$, *no additional assumption is needed*). *Denote* $\psi_\pm = W_\pm^{-1} f$, *and*

$$\varphi(t,x) = \frac{1}{2}\sum_{j=1}^n \frac{\omega_j x_j^2}{\tan(\omega_j t)}.$$

Let u^ε *be the solution of* (5.3). *Then for* $|t| < \frac{\pi}{\omega_d}$ (*that is, before refocusing*), *and in* $L^2 \cap L^r$,

- *If* $0 < t < \frac{\pi}{\omega_d}$,

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \prod_{j=1}^n \left(\frac{\omega_j}{2i\pi\sin\omega_j t}\right)^{1/2} \widehat{\psi_+}\left(\frac{\omega_1 x_1}{\sin\omega_1 t}, \ldots, \frac{\omega_n x_n}{\sin\omega_n t}\right) e^{i\varphi(t,x)/\varepsilon}.$$

- *If* $-\frac{\pi}{\omega_d} < t < 0$,

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \prod_{j=1}^n \left(\frac{\omega_j}{2i\pi\sin\omega_j t}\right)^{1/2} \widehat{\psi_-}\left(\frac{\omega_1 x_1}{\sin\omega_1 t}, \ldots, \frac{\omega_n x_n}{\sin\omega_n t}\right) e^{i\varphi(t,x)/\varepsilon}.$$

At time $t = \frac{\pi}{\omega_d}$, the solution focuses on E_d, and the nonlinear term is negligible (use the operators H_j^ε for all indexes j such that $\omega_j = \omega_d$, and J_k^ε for the others). The nonlinear term will be relevant again only if there exists a time where the focusings caused by the different ω_j's ($1 \leqslant j \leqslant d$) occur simultaneously, that is if there are positive integers $\kappa_1, \dots, \kappa_d$ such that

$$t_1 = \frac{\kappa_1 \pi}{\omega_1} = \cdots = \frac{\kappa_d \pi}{\omega_d}.$$

This means that the ω_j's are pairwise rationally related. Therefore, at time $t = t_1$, the caustic crossing will be described again by the scattering operator. Notice that when t approaches t_1, the asymptotics given in Theorem 5.1 has been modified in terms of Maslov indexes (for instance, since the crossing of E_d is linear, only linear phenomenon occur at leading order, that is precisely a phase shift measured by the Maslov index). More precisely, for $(\kappa_d - 1)\pi/\omega_d < t < t_1$, every ω_j $(1 \leqslant j \leqslant d)$ part of the harmonic potential has caused $\kappa_j - 1$ (linear) caustic crossings, and

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \prod_{j=1}^{d} \left(\frac{\omega_j e^{-i(\kappa_j - 1)\pi}}{2i\pi\,|\sin \omega_j t|} \right)^{i_j/2} \widehat{\psi_+}\left(\frac{\omega_1 x_1}{\sin \omega_1 t}, \dots, \frac{\omega_n x_n}{\sin \omega_n t} \right) e^{i\varphi(t,x)/\varepsilon}.$$

For $t_1 < t < (\kappa_d + 1)\pi/\omega_d$, one has,

$$u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \prod_{j=1}^{d} \left(\frac{\omega_j e^{-i\kappa_j\pi}}{2i\pi\,|\sin \omega_j t|} \right)^{i_j/2} \widehat{S\psi_+}\left(\frac{\omega_1 x_1}{\sin \omega_1 t}, \dots, \frac{\omega_n x_n}{\sin \omega_n t} \right) e^{i\varphi(t,x)/\varepsilon},$$

and so on.

If the ω_j's are not pairwise rationally related, then only linear phenomena occur near caustics, and they are measured by Maslov indexes.

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