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A note on weak approximation of minors

by

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ABSTRACT. – Let $A_{p,q}(\Omega)$, where $\Omega \subseteq R^n$ is a bounded open domain, be a set of all mappings $u \in W^{1,p}(\Omega, R^n)$ such that $\operatorname{adj} Du \in L^q$. Among other results we prove that if $n-1 \leq p < n$, $1 < q < n/(n-1)$, then the subclass of $A_{p,q}$ mappings, which consists of mappings with bounded $(n-1)$-dimensional image, is dense in the sequential weak topology of $A_{p,q}$. We also extend this result to other $A_{p,q}$ type spaces.

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1. INTRODUCTION AND STATEMENT

Let $\Omega \subseteq R^n$ be a bounded domain. We define

$$A_{p,q}(\Omega) = \{u : \Omega \to R^n \,|\, u \in W^{1,p}(\Omega, R^n), \operatorname{adj} Du \in L^q\},$$

where the matrix $\operatorname{adj} Du$ satisfies the identity $(\det Du)\mathrm{Id} = Du \operatorname{adj} Du$.

Spaces of this type arise in a natural way as classes on which variational functionals related to nonlinear elasticity are defined (*see* e.g. [1], [5], [6],

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[7] [8], [10], [16], [19], [21], [22], [24], [29]). It is very important to note that if $q > p/(n-1)$, then the space $A_{p,q}$ is not linear. The space $A_{p,q}$ is endowed both with strong and weak topology. We say that $u_k \to u$ strongly in $A_{p,q}$, if $u_k \to u$ strongly in $W^{1,p}$ and $\operatorname{adj} Du_k \to \operatorname{adj} Du$ strongly in L^q; $u_k \to u$ weakly in $A_{p,q}$ if $u_k \to u$ weakly in $W^{1,p}$ and $\operatorname{adj} Du_k \to \operatorname{adj} Du$ weakly in L^q. All kinds of weak convergence, as usual, will be denoted by "harpoon" $\rightharpoonup$ in place of "arrow". Our main results are Theorem 1 and its generalisation, Theorem 3.

THEOREM 1. – *If $u \in A_{p,q}(\Omega)$, $\Omega \subseteq R^n$, $n-1 \le p < n$, $1 < q < n/(n-1)$, then there exists a sequence $u^{(\nu)} \in A_{p,q}(\Omega)$, such that values of $u^{(\nu)}$ belong to a certain bounded $(n-1)$-dimensional simplicial set (which depends on ν) and $u^{(\nu)} \rightharpoonup u$ weakly in $A_{p,q}(\Omega)$. In the other words the subset of $A_{p,q}$, which consists of mappings with bounded $(n-1)$-dimensional range is dense in $A_{p,q}$ in the sequential weak topology.*

Remark. – Note that it is not possible to substitute in the above theorem weak convergence in $A_{p,q}$ by strong convergence. Othervise we would have (after passing to a subsequence), $Du^{(\nu)} \to Du$ a.e. and hence $0 \equiv \det Du^{(\nu)} \to \det Du$ a.e.

COROLLARY 1. – *If u is as above, then there exists $u^{(\nu)} \in A_{p,q}(\Omega)$, such that $u^{(\nu)} \rightharpoonup u$ weakly in $A_{p,q}$ and* $\det Du^{(\nu)} \equiv 0$.

COROLLARY 2. – *If $u \in A_{n-1,\frac{n}{n-1}}(\Omega)$, then for each $q < n/(n-1)$ there exists a sequence $u^{(\nu)}$, such that $u^{(\nu)} \in A_{n-1,q}(\Omega)$, $u^{(\nu)} \rightharpoonup u$ weakly in $A_{n-1,q}$ and* $\det Du^{(\nu)} \equiv 0$.

The above corollaries are in contrast with the following theorem of Müller-Qi-Yan [24].

THEOREM 2 (Müller-Qi-Yan [24, Lemma 4.1]). – *Let $p \ge n-1$, $q \ge \dfrac{n}{n-1}$ and $u^{(\nu)} \rightharpoonup u$ weakly in $A_{p,q}(\Omega)$.*

1. *If $q > \dfrac{n}{n-1}$, then* $\det Du^{(\nu)} \rightharpoonup \det Du$ *in* $L^{q(n-1)/n}$.

2. *If $q = \dfrac{n}{n-1}$ and* $\det Du^{(\nu)} \ge 0$, *then* $\det Du^{(\nu)} \rightharpoonup \det Du$ *in* $L^1(K)$ *for all compact sets $K \subseteq \Omega$.*

The following theorem generalizes Theorem 1.

THEOREM 3. – *If $u \in W^{1,p}(\Omega, R^n)$, $p < n$, then there exists a sequence $u^{(\nu)} \in W^{1,p}(\Omega, R^n)$ such that values of $u^{(\nu)}$ belong to a certain bounded $[p]$-dimensional simplicial set (this set depends on ν) and $u^{(\nu)} \rightharpoonup u$ weakly in $W^{1,p}$. Moreover if* $\operatorname{adj}_s Du \in L^{q_s}$, *for some s, q_s*

such that $1 < q_s < ([p]+1)/s$, *then we can additionally obtain that* $\operatorname{adj}_s Du^{(\nu)} \rightharpoonup \operatorname{adj}_s Du$ *weakly in* L^{q_s}.

Remarks. – 1) Up to the sign and order, $\operatorname{adj}_s Du$ is a multidimensional matrix consisting of all $s \times s$ minors of Du (*cf.* [5]). By $[p]$ we denote the greatest integer, less then or equal to p.

2) By the same reason as in the case of Theorem 1, we cannot substitute weak convergence by strong convergence.

Theorems 1 and 3 are in some sense related to the results about the approximation of Sobolev mappings between manifolds $W^{1,p}(M,N)$ ([28], [3] [2], [15], [4], [13], [12]). At first sight the problem seems to be different because of a different nature of Sobolev mappings $W^{1,p}(M,N)$ and mappings $A_{p,q}(\Omega)$. However the careful study shows some deep connections between these problems. In particular the crucial (for us) method of retractions (Lemma 1) is a modification of the analogous method previously used in the context of approximation of Sobolev mappings between manifolds [4], [13].

To see further connections between the theory of $W^{1,p}(M,N)$ and $A_{p,q}(\Omega)$ mappings we refer to [14].

By Q^n we will denote a "general" n-dimensional cube. By C we will denote a general constant. It can change its value even in the same proof. Writting for example $C(n,p)$, we will show that this constant depends on n and p only.

2. PROOFS OF THEOREMS 1 AND 3

Before we proceed to the proofs of these theorems, we shall state and prove main technical lemma (Lemma 1 below). The following mapping is defined in $R^n \setminus \{x\}$:

$$\pi_{x,\varepsilon}(z) = \begin{cases} z & \text{when } |x-z| \geq \varepsilon \\ \dfrac{z-x}{|z-x|}\varepsilon + x & \text{when } |x-z| \leq \varepsilon. \end{cases}$$

In other words $\pi_{x,\varepsilon}$ is a mapping which is an identity on the complement of the ball $B^n(x,\varepsilon)$, and which is a projection along radii onto the boundary $S^{n-1}(x,\varepsilon)$ inside the ball. Evidently $\pi_{x,\varepsilon}$ is discontinuous at $z = x$. Moreover

$$|D_z(\pi_{x,\varepsilon})(z)| \leq C(n)\frac{\varepsilon}{|z-x|}, \tag{1}$$

for $z \in B^n(x,\varepsilon)$. In the sequel we will use the following notation. If $\overline{x} = (x_1,\dots,x_k)$ is a sequence of points of R^n, then we set

$$\pi_{\overline{x},\varepsilon} = \pi_{x_1,\varepsilon} \circ \cdots \circ \pi_{x_k,\varepsilon}.$$

In the proof of Theorems 1 and 3 the following lemma will play a fundamental role.

LEMMA 1. – *Let $u \in W^{1,p}(\Omega, R^n)$, $\Omega \subseteq R^n$, $p < n$. Let $A_1,\dots,A_k \subseteq R^n$ be a family of measurable sets such that $(\varepsilon/2)^n < |A_i| < \infty$ and* $\operatorname{dist}(A_i,A_j) \geq 2\varepsilon$ *for all $i \neq j$. Then for almost all $\overline{x} = (x_1,\dots,x_k) \in A_1 \times \cdots \times A_k$, $\pi_{\overline{x},\varepsilon} \circ u \in W^{1,p}(\Omega)$. Moreover there exists $\overline{x} \in A_1 \times \cdots \times A_k$ such that*

$$\int_\Omega |D(\pi_{\overline{x},\varepsilon} \circ u)(z)|^p\, dz \leq C(n,p) \int_\Omega |Du(z)|^p\, dz,$$

where the constant $C(n,p)$ depends on n and p only.

Remark. – Assumption $\operatorname{dist}(A_i,A_j) \geq 2\varepsilon$ guarantees that the sets $\mathcal{O}_\varepsilon(A_i) = \{x \in R^n \,|\, \operatorname{dist}(x,A_i) < \varepsilon\}$ are pairwise disjoint.

Before we proceed to the proof of Lemma 1 we shall be concerned with some other lemmas.

We say that $f \in \mathrm{ACL}(\Omega)$ if the function f is Borel measurable and absolutely continuous on almost all lines parallel to coordinate axes. Since absolutely continuous functions are almost everywhere differentiable, $f \in \mathrm{ACL}(\Omega)$ has partial derivatives a.e. and hence the gradient ∇f is defined a.e. Now we say that $f \in \mathrm{ACL}^p(\Omega)$ if $f \in L^p(\Omega) \cap \mathrm{ACL}(\Omega)$ and $|\nabla f| \in L^p$. The following characterization of Sobolev space is due to Nikodym ([25], [20, Section 1.1.3]).

THEOREM 4 (Nikodym). – $W^{1,p}(\Omega) = \mathrm{ACL}^p(\Omega)$.

Since maybe it is not evident how to understand this theorem, we shall comment it now. This theorem states that each $\mathrm{ACL}^p(\Omega)$ function belongs to $W^{1,p}(\Omega)$ and the gradient ∇f, which is defined a.e. for $f \in \mathrm{ACL}^p(\Omega)$ is just the distributional gradient. On the other hand, each element $f \in W^{1,p}(\Omega)$ (which is an equivalence class of functions equal exept the set of measure zero) admit a Borel representative, which belongs to the space $\mathrm{ACL}^p(\Omega)$.

LEMMA 2. – *Let $f : X \to Y$ be a mapping between separable metric spaces X and Y. If f is a Borel mapping, then the graph $G_f = \{(x,f(x)) \,|\, x \in X\} \subseteq X \times Y$ is a Borel set.*

Proof. – Let $\{A_i^{(n)}\}_{i=1}^{\infty}$ be a family of pairwise disjoint Borel subsets of Y such that $\bigcup_{i=1}^{\infty} A_i^{(n)} = Y$ and $\operatorname{diam} A_i^{(n)} < 1/n$. Then the set

$$B_n = \bigcup_{i=1}^{\infty} (f^{-1}(A_i^{(n)}) \times A_i^{(n)}) \subseteq X \times Y$$

is Borel and hence $G_f = \bigcap_{n=1}^{\infty} B_n$ is also a Borel set.

We also need the following famous theorem of Lusin and Sierpiński [17], [18], [9, Thm. 2.2.13].

THEOREM 5 (Lusin-Sierpiński). – *Let $P : R^{n+k} \to R^n$ be an orthogonal projection. If $B \subseteq R^{n+k}$ is a Borel set, then $P(B) \subseteq R^n$ is H^n-measurable.*

Remark. – Note that this theorem is no longer valid if we assume that B is H^{n+k} measurable, instead of being Borel. Note also that it is not true in general that $P(B)$ is Borel even if B is Borel.

We will use the above Lusin-Sierpiński's theorem in the proof of the following "Sard's type" lemma.

LEMMA 3. – *Assume that $f : Q^n \to R^k$ is a Borel mapping. Then the following two conditions are equivalent:*

1. *For almost all segments $I \subseteq Q^n$, parallel to one of the coordinate axes, $H^k(f(I)) = 0$.*

2. *For almost all $x \in R^k$, the set $f^{-1}(x)$ is disjoint from almost all segments parallel to one of the coordinate axes.*

Remark. – For generalizations and further results *see* [11], [13].

Proof. – Let $P : Q^n \to Q^{n-1}$, $P(x_1, \ldots, x_n) = (x_2, \ldots, x_n)$ be the orthogonal projection in the direction parallel to the first coordinate axis. It induces the projection $\overline{P} : Q^n \times R^k \to Q^{n-1} \times R^k$, $\overline{P}(x, y) = (P(x), y)$. By Lemma 2, the graph $G_f \subseteq Q^n \times R^k$ is a Borel set and hence by Theorem 5, the set $\overline{P}(G_f) \subseteq Q^{n-1} \times R^k$ is $H^{n-1} \otimes H^k$-measurable. Now it follows directly from Fubini's theorem that the following three conditions are equivalent.

1. $(H^{n-1} \otimes H^k)(\overline{P}(G_f)) = 0$.

2. For almost all segments I, parallel to the axis x_1, $H^k(f(I)) = 0$.

3. For almost all $x \in R^k$, $f^{-1}(x)$ is disjoint from almost all segments parallel to x_1.

Of course one can repeat the above construction with x_1 replaced by $x_2, x_3, \ldots, x_n$. Then the lemma follows easily.

The above lemma can be applied to $\mathrm{ACL}(Q^n, R^k)$ mappings as it follows from the following elementary observation.

LEMMA 4. – *If* $f : [0,1] \to R^k$, $k \geq 2$ *is absolutely continuous then* $H^k(f([0,1])) = 0$.

Proof. – Otherwise, applying Fubini's theorem, we would find the one dimensional slice of the set $f([0,1])$ with positive one dimensional measure, and such that this slice is the image of a certain subset of $[0,1]$ with the measure zero. This contradicts the following well known fact: If $g : [0,1] \to R$ is absolutely continuous and $E \subset [0,1]$, $H^1(E) = 0$ then $H^1(g(E)) = 0$ (*see* [27, Theorem 7.18]).

Lemmas 3 and 4 lead to the fact that if $f \in \mathrm{ACL}(Q^n, R^k)$, where $k \geq 2$, then for almost all $x \in R^k$, $f^{-1}(x)$ is disjoint from almost all lines parallel to an arbitrary coordinate axis. Hence $\pi_{x,\varepsilon} \circ f \in \mathrm{ACL}(Q^n, R^k)$ for almost all $x \in R^k$. Now we are in position to prove Lemma 1.

Proof of Lemma 1. – As we have already seen, for almost all $\overline{x} \in A_1 \times \cdots \times A_k$, $\pi_{\overline{x},\varepsilon} \circ u \in \mathrm{ACL}(\Omega)$; hence the gradient $D(\pi_{\overline{x},\varepsilon} \circ u)$ is defined almost everywhere and now in order to prove $\pi_{\overline{x},\varepsilon} \circ u \in W^{1,p}(\Omega)$ it suffices to prove that $|D(\pi_{\overline{x},\varepsilon} \circ u)| \in L^p(\Omega)$ (*see* Theorem 4).

Evidently $\pi_{\overline{x},\varepsilon} \circ u(z) = u(z)$ for $z \in \Omega \setminus \bigcup_{i=1}^{k} u^{-1}(B^n(x_i, \varepsilon))$ and hence $D(\pi_{\overline{x},\varepsilon} \circ u)(z) = Du(z)$ for almost all $z \in \Omega \setminus \bigcup_{i=1}^{k} u^{-1}(B^n(x_i, \varepsilon))$. Now it is clear that the lemma will follow if we prove the inequality

$$\int_{A_k} \cdots \int_{A_1} \int_{u^{-1}\left(\bigcup_{i=1}^{k} B^n(x_i,\varepsilon)\right)} |D(\pi_{\overline{x},\varepsilon} \circ u)(z)|^p \, dz \, dx_1 \ldots dx_k$$
$$\leq C(n,p)|A_1| \cdots |A_k| \int_\Omega |Du(z)|^p \, dz. \tag{2}$$

Note that $\pi_{x_j,\varepsilon}|_{B^n(x_i,\varepsilon)} = \mathrm{id}$ for $j \neq i$ and hence for almost all $z \in u^{-1}(B^n(x_i, \varepsilon))$

$$\begin{aligned} |D(\pi_{\overline{x},\varepsilon} \circ u)(z)| &\leq |D(\pi_{x_i,\varepsilon})(u(z))||Du(z)| \\ &\leq C(n) \frac{\varepsilon}{|u(z) - x_i|} |Du(z)|. \end{aligned}$$

Denoting the left hand side of (2) by I we have

$$\begin{aligned} I &\leq C(n,p)\sum_{i=1}^{k}\int_{A_k} \\ &\cdots\int_{A_1}\int_{u^{-1}(B^n(x_i,\varepsilon))}|Du(z)|^p\left(\frac{\varepsilon}{|u(z)-x_i|}\right)^p dz\,dx_1\ldots dx_k \\ &\leq C(n,p)\sum_{i=1}^{k}\int_{u^{-1}(\mathcal{O}_\varepsilon(A_i))}|Du(z)|^p(|A_1|\cdots|A_k|(\varepsilon^p|A_i|^{-p/n}))\,dz \\ &\leq C(n,p)|A_1|\cdots|A_k|\int_\Omega |Du(z)|^p\,dz. \end{aligned}$$

In the last but one step we used the estimate $\int_A |x|^{-p} \leq C(n,p)|A|^{1-\frac{p}{n}}$. This ends the proof of (2) and hence that of the lemma.

Proof of Theorem 1. – First we will prove the theorem under the additional assumption that the image of u is bounded *i.e.* $u(x) \in Q^n$ for almost all $x \in \Omega$, where Q^n is a certain cube. Then, as we will see, the general case will follow from the density of bounded maps in the strong $A_{p,q}$ topology.

Divide the cube Q^n in a standard way, into ν^n identical, small cubes. Denote these cubes by $Q_i^{n,\nu}$, $i = 1,2,\ldots,\nu^n$. Let $\frac{1}{2}Q_i^{n,\nu}$ be a cube with the same centre as $Q_i^{n,\nu}$ and half the edgelength. Let $x \in \frac{1}{2}Q_i^{n,\nu}$. We define the retraction

$$\pi_{i,x}^{\nu} : Q^n \setminus \{x\} \longrightarrow Q^n \setminus Q_i^{n,\nu}$$

as follows

$$\pi_{i,x}^{\nu}(z) = \begin{cases} z & \text{when } z \in Q^n \setminus Q_i^{n,\nu} \\ (z-x)\theta(x,z)+x & \text{when } z \in Q_i^{n,\nu}, \end{cases}$$

where $\theta(x,z) > 0$ is such that $\pi_{i,x}^{\nu}(z) \in \partial Q_i^{n,\nu}$. In the other words, inside $Q_i^{n,\nu}$, $\pi_{i,x}^{\nu}$ is a retraction along radii onto $\partial Q_i^{n,\nu}$, with "source" at x. Evidently $\pi_{i,x}^{\nu}$ is bilipschitz equivalent with $\pi_{x,\varepsilon}$ type mapping (with $\varepsilon = C\nu^{-1}$), so we can repeate the proof of Lemma 1, replacing A_i by $\dfrac{1}{2}Q_i^{n,\nu}$. This leads to the following corollary: There exists $x_i \in \dfrac{1}{2}Q_i^{n,\nu}$ for $i = 1,2,\ldots,\nu^n$, such that

$$u^{(\nu)} = \pi_{1,x_1}^{\nu} \circ \cdots \circ \pi_{\nu^n,x_{\nu^n}}^{\nu} \circ u \in W^{1,p}(\Omega) \tag{3}$$

and

$$\int_\Omega |Du^{(\nu)}(z)|^p\,dz \leq C(n,p)\int_\Omega |Du(z)|^p\,dz. \tag{4}$$

Evidently the image of $u^{(\nu)}$ is $(n-1)$-dimensional. Since for $\varphi \in C^1$, $|\operatorname{adj} D(\varphi \circ u)(z)| \leq |D\varphi(u(z))|^{n-1}|\operatorname{adj} Du(z)|$, then it is easy to see repeating arguments from the proof of Lemma 1 that we can choose $x_i \in \frac{1}{2}Q_i^{n,\nu}$ for $i = 1, 2, \ldots, \nu^n$ in a way, such that in addition to (4)

$$\int_\Omega |\operatorname{adj} Du^{(\nu)}(z)|^q \, dz \leq C(n,q) \int_\Omega |\operatorname{adj} Du(z)|^q \, dz. \tag{5}$$

Now it suffices to prove that, after extracting a subsequence, $u^{(\nu)} \rightharpoonup u$ in $W^{1,p}$ and $\operatorname{adj} Du^{(\nu)} \rightharpoonup \operatorname{adj} Du$ in L^q. We have $|u(z) - u^{(\nu)}(z)| \leq C\nu^{-1}$ a.e., so $u^{(\nu)} \to u$ a.e. and hence in L^p (since u and $u^{(\nu)}$ are bounded).

The estimations (4), (5) show that sequences $Du^{(\nu)}$ and $\operatorname{adj} Du^{(\nu)}$ are bounded in L^p and L^q respectively. Hence $u^{(\nu)} \rightharpoonup u$ in $W^{1,p}$ and moreover after taking a subsequence $\operatorname{adj} Du^{(\nu)} \rightharpoonup H$ in L^q to a certain $H \in L^q$. By the theorem of Ball and Reshetnyak [1], [26], [5, Chapter 4, Thm. 2.6], [16], $\operatorname{adj} Du^{(\nu)} \overset{*}{\rightharpoonup} \operatorname{adj} Du$ in measures, and hence $\operatorname{adj} Du = H$. This completes the proof in the case of bounded maps.

Since the constants in the inequalities (4) and (5) do not depend on the size of the cube Q^n, then the theorem will follow if we prove the density of bounded maps in the strong topology of $A_{p,q}$. To this end, let $P_R : R^n \to [-R, R]^n$ be a retraction with a Lipschitz constant 1; now it is easy to see that $P_R \circ u \overset{R\to\infty}{\longrightarrow} u$ in $W^{1,p}$ and $\operatorname{adj} D(P_R \circ u) \overset{R\to\infty}{\longrightarrow} \operatorname{adj} Du$ in L^q (in the proof of the second convergence we use an elementary inequality $|\operatorname{adj} D(\varphi \circ u)| \leq |\operatorname{Lip} \varphi|^{n-1}|\operatorname{adj} Du|$). This completes the proof.

Proof of Theorem 3. – By the same reason as in the proof of Theorem 1, we can assume that u is bounded, *i.e.* $u(x) \in Q^n$ for a.e. x, where Q^n is a fixed cube.

At the beginning we prove the first part of the theorem, *i.e.* we will be concerned with the weak convergence of $u^{(\nu)}$, without taking care of minors $\operatorname{adj}_s Du^{(\nu)}$.

In the first step, as in the proof of Theorem 1, we compose the mapping u with the discontinuous retraction onto $(n-1)$-dimensional set which is the union of the boundaries of all cubes $Q_i^{n,\nu}$, $i = 1, 2, \ldots, \nu^n$.

This way we obtain (*cf.* the proof of Theorem 1) a mapping $u_1^{(\nu)} \in W^{1,p}(\Omega, \bigcup_{i=1}^{\nu^n} \partial Q_i^{n,\nu})$ with

$$\int_\Omega |Du_1^{(\nu)}|^p \, dx \leq C(n,p) \int_\Omega |Du|^p \, dx. \tag{6}$$

($u_1^{(\nu)}$ is the composition of u with a discontinuous retraction described above.) Since $u_1^{(\nu)} \stackrel{\nu\to\infty}{\longrightarrow} u$ a.e., then it follows from (6) that $u_1^{(\nu)} \stackrel{\nu\to\infty}{\rightharpoonup} u$ in $W^{1,p}$. If $p \geq n-1$, then the proof is completed. Hence we can assume that $p < n-1$. The set $\partial Q_i^{n,\nu}$ consists of $2n$, $(n-1)$-dimensional cubes. Now we divide each such $(n-1)$-dimensional cube into a family of ν^{n-1} very small cubes (with edgelength $C\nu^{-2}$). This leads to the decomposition $\partial Q_i^{n,\nu} = \bigcup_{j=1}^{2n\nu^{(n-1)}} Q_{i,j}^{n-1,\nu}$ and hence to the decomposition of $\bigcup_{i=1}^{\nu^n} \partial Q_i^{n,\nu}$.

Now almost the same arguments as in the proof of (6) show that we can compose the mapping $u_1^{(\nu)}$ with the discontinuous retraction from $\bigcup_{i=1}^{\nu^n} \partial Q_i^{n,\nu}$ onto $(n-2)$-dimensional set $\bigcup_{i,j} \partial Q_{i,j}^{n-1,\nu}$. This way we obtain the mapping $u_2^{(\nu)} \in W^{1,p}(\Omega, \bigcup_{i,j} \partial Q_{i,j}^{n-1,\nu})$, with

$$\int_\Omega |Du_2^{(\nu)}|^p\, dx \leq C(n,p) \int_\Omega |Du_1^{(\nu)}|^p\, dx \leq C'(n,p) \int_\Omega |Du|^p\, dx.$$

As above $u_2^{(\nu)} \rightharpoonup u$ in $W^{1,p}$. If $p \geq n-2$, then the proof is completed. If $p < n-2$, then we can of course continue this construction and compose $u_2^{(\nu)}$ with retraction onto $(n-3)$-dimensional set. By induction, we can continue this construction up to the moment, we compose with retraction onto $[p]$-dimensional set. This way we get $u_{n-[p]}^{(\nu)} \in W^{1,p}(\Omega)$, with values in $[p]$-dimensional set. Moreover

$$\int_\Omega |Du_{n-[p]}^{(\nu)}|^p\, dx \leq C(n,p) \int_\Omega |Du|^p\, dx. \tag{7}$$

Since $u_{n-[p]}^{(\nu)} \stackrel{\nu\to\infty}{\longrightarrow} u$ a.e., then by (7), $u_{n-[p]}^{(\nu)} \stackrel{\nu\to\infty}{\rightharpoonup} u$ in $W^{1,p}$. This ends the proof of the first part of Theorem 3.

If we know additionally that $\operatorname{adj}_s Du \in L^{q_s}$ for some $1 < q_s < ([p]+1)/s$, then as in the proof of Theorem 1, we can assume that additionally to (7), the following inequality holds

$$\int_\Omega |\operatorname{adj}_s Du_{n-[p]}^{(\nu)}(x)|^{q_s}\, dx \leq C(n,s,q_s) \int_\Omega |\operatorname{adj}_s Du(x)|^{q_s}\, dx.$$

Hence up to a subsequence, $\operatorname{adj}_s Du_{n-[p]}^{(\nu)} \rightharpoonup H$ in L^{q_s} to a certain $H \in L^{q_s}$. The inequality $1 < q_s < ([p]+1)/s$ implies that $p \geq s$, hence by the theorem of Ball-Reshetnyak (*cf.* the proof of Theorem 1), $\operatorname{adj}_s Du_{n-[p]}^{(\nu)} \stackrel{*}{\rightharpoonup} \operatorname{adj}_s Du$ in measures and hence $H = \operatorname{adj}_s Du$. This ends the proof.

REFERENCES

[1] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rat. Mech. Anal.*, Vol. **63**, 1977, pp. 337-403.
[2] F. Bethuel, The approximation problem for Sobolev maps between two manifolds, *Acta Math.*, Vol. **167**, 1991, pp. 153-206.
[3] F. Bethuel and X. Zheng, Density of smooth functions between two manifolds in Sobolev spaces, *J. Funct. Anal.*, Vol. **80**, 1988, pp. 60-75.
[4] B. Bojarski, Geometric properties of Sobolev mappings, in: Pitman Res. Notes in Math., Vol. **211**, 1989, pp. 225-241.
[5] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag 1989.
[6] M. Esteban, A direct variational approach to Skyrme's model for meson fields, *Comm. Math. Phys.*, Vol. **105**, 1986, pp. 571-591.
[7] M. Esteban, A new setting for Skyrme's problem, in: Proc. Colloque problèmes variationnels (Paris, June 1988), Boston-Basel-Stuttgart 1990.
[8] M. Esteban and S. Müller, Sobolev maps with the integer degree and applications to Skyrme's problem, *Proc. R. Soc. Lond.*, Vol. **436**, 1992, pp. 197-201.
[9] H. Federer, *Geometric Measure Theory*, Springer-Verlag 1969.
[10] M. Giaquinta, G. Modica and J. Souček, Cartesian currents and variational problems into spheres, *Annali Sc. Norm. Sup. Pisa*, Vol. **16**, 1989, pp. 393-485.
[11] P. Hajłasz, A Sard type theorem for Borel mappings, *Colloq. Math.*, Vol. **67**, 1994, pp. 217-221.
[12] P. Hajłasz, Approximation of Sobolev mappings, *Nonlinear Anal.*, Vol. **22**, 1994, pp. 1579-1591.
[13] P. Hajłasz, Sobolev mappings, co-area formula and related topics (preprint).
[14] P. Hajłasz, A note on approximation of Sobolev maps, (in preparation).
[15] F. Hélein, Approximation of Sobolev maps between an open set and Euclidean sphere, boundary data, and singularities, *Math. Ann.*, Vol. **285**, 1989, pp. 125-140.
[16] T. Iwaniec and A. Lutoborski, Integral estimates for null lagrangians, *Arch. Rat. Mech. Anal.*, Vol. **125**, 1993, pp. 25-80.
[17] N. Lusin, Sur les ensembles analytiques, *Fund. Math.*, Vol. **10**, 1927, pp. 1-95.
[18] N. Lusin and W. Sierpiński, Sur quelques propriétés des ensembles (A), *Bull. de l'Acad. de Cracovie*, 1918, p. 44.
[19] J. Malý, L^p-approximation of Jacobians, *Comm. Math. Univ. Carolinae*, Vol. **32**, 1991, pp. 659-666.
[20] V. Maz'ya, *Sobolev Spaces*, Springer-Verlag 1985.
[21] S. Müller, Weak continuity of determinants and nonlinear elasticity, *C. R. Acad. Sci. Paris*, Vol. **307**, Série I, 1988, pp. 501-506.
[22] S. Müller, Higher integrability of determinants and weak convergence in L^1, *J. Reine Angew. Math.*, Vol. **412**, 1990, pp. 20-34.
[23] S. Müller, Det=det. A remark on the distributional determinant, *C. R. Acad. Sci. Paris*, Vol. **313**, 1990, pp. 13-17.
[24] S. Müller, T. Qi and B. S. Yan, On a new class of elastic deformations not allowing for cavitation, *Ann. I.H.P. Anal. Nonl.*, Vol. **11**, 1994, pp. 217-243.
[25] O. Nikodym, Sur une classe de fonctions considérées le problème de Dirichlet, *Fund. Math.*, Vol. **21**, 1933, pp. 129-150.
[26] Y. G. Reshetnyak, On the stability of conformal mappings in multidimensional space, *Siberian Math. J.*, Vol. **8**, 1967, pp. 65-85.
[27] W. Rudin, *Real and Complex Analysis*, Third edition, Mc Graw-Hill, New York 1987.
[28] R. Schoen and K. Uhlenbeck, Approximation theorems for Sobolev mappings, (preprint).
[29] V. Šverak, Regularity properties of deformations with finite energy, *Arch. Rat. Mech. Anal.*, Vol. **100**, 1988, pp. 105-127.

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