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On the Jäger–Kaul theorem concerning harmonic maps

by

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ABSTRACT. – In 1983, Jäger and Kaul proved that the equator map $u^*(x) = (\frac{x}{|x|}, 0) : B^n \to S^n$ is unstable for $3 \leqslant n \leqslant 6$ and a minimizer for the energy functional $E(u, B^n) = \int_{B^n} |\nabla u|^2 \,\mathrm{d}x$ in the class $H^{1,2}(B^n, S^n)$ with $u = u^*$ on ∂B^n when $n \geqslant 7$. In this paper, we give a new and elementary proof of this Jäger–Kaul result. We also generalize the Jäger–Kaul result to the case of p-harmonic maps.

RÉSUMÉ. – En 1983, Jäger et Kaul ont demontrés que l'application équateurielle $u^*(x) = (\frac{x}{|x|}, 0) : B^n \to S^n$ n'est pas stable si $3 \leqslant n \leqslant 6$ et que c'est une minimisateur pour la fonctionelle d'energie $E(u, B^n) = \int_{B^n} |\nabla u|^2 \,\mathrm{d}x$ dans la classe $H^{1,2}(B^n, S^n)$ avec $u = u^*$ sur ∂B^n si $n \geqslant 7$. Nous donnons une preuve nouvelle, élémentaire de ce résultat de Jäger–Kaul. En plus nous généralisons le résultat de Jäger–Kaul au cas des applications p-harmoniques.
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1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold with (possibly empty) boundary ∂M and let (N, h) be another compact Riemannian manifold without boundary. Let u be a map from M to N which belongs to $H^{1,2}(M, N)$. We define the energy of u by

$$E(u, M) = \int_M |\mathrm{d}u|^2 \, \mathrm{d}M, \tag{1.1}$$

where $|\mathrm{d}u|$ denotes the Hilbert–Schmidt norm of the differential $\mathrm{d}u(x)$. The critical point of E is called “harmonic”. In a fundamental paper [4], Eells and Sampson established existence of smoothly harmonic maps from M to N assuming N has nonpositive section curvature. Let $K \geqslant 0$ be an upper bound for the section curvature of N and $B_\rho(q)$ the open geodesic ball in N with center q and radius ρ. Assuming essentially the size restriction

$$u(\partial M) \subset B_\rho(q), \qquad \rho \leqslant \frac{\pi}{2\sqrt{K}}, \tag{1.2}$$

Hildebrandt et al. [10] showed existence of “small” smooth harmonic maps satisfying the condition (1.2) and also discovered that for $n \geqslant 3$ the equator map $u^* = (\frac{x}{|x|}, 0) : B^n \to S^n$ is a weakly harmonic maps. The uniqueness of harmonic maps in [10] was later proved by Jäger and Kaul [11]. Many important contributions on the regularity of minimizing harmonic maps have been made since then. Schoen and Uhlenbeck [15, 16] obtained that the minimizer of E in $H^{1,2}(M, N)$ is smooth except for a singular set, where the Hausdorff dimension of the singular set is less than or equal to $n - 3$. Meanwhile, Giaquinta and Giusti in [5,6] also proved this result for the case when the image lies in a coordinate chart (boundary regularity by Jost and Meier [13]). We would also like to mention Simon's deep works on the structure of singularity of minimizing harmonic maps (e.g., [18]).

Let B^n be the unit ball in $\mathbb{R}^n$ with boundary $\partial B^n = S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$.

A map $u : B^n \to S^n$ is called “weakly harmonic” if $u \in H^{1,2}(B^n, S^n)$ and it is a critical point of the energy $\int_{B^n} |\nabla u|^2 \, \mathrm{d}x$, i.e.,

$$\int_{B^n} \nabla u \cdot \nabla \phi \, \mathrm{d}x = \int_{B^n} |\nabla u|^2 u \cdot \phi \, \mathrm{d}x$$

for all $\phi \in H_0^{1,2}(B^n, \mathbb{R}^{n+1}) \cap L^\infty(B^n, \mathbb{R}^{n+1})$.

In 1983, Jäger and Kaul [12] proved the following result:

THEOREM (Jäger–Kaul). –

(i) *When* $3 \leqslant n \leqslant 6$, *the equator map* $u^* = (\frac{x}{|x|}, 0)$ *is unstable.*

(ii) *When* $n \geqslant 7$, *the equator map* u^* *is a minimizer of the energy functional* $E(u, B^n) = \int_{B^n} |\nabla u|^2$ *for all maps* $u \in H^{1,2}(B^n, S^n)$ *with* $u = u^*$ *on* ∂B^n.

After this theorem, Giaquinta and Soucek [7] and Schoen and Uhlenbeck [17] proved that the Hausdorff dimension of the singular set of minimizing harmonic maps into a hemisphere is less than or equal to $n - 7$.

For any $p \in \mathbb{R}$ with $n > p \geqslant 2$, we define the p-energy of maps in $H^{1,p}(B^n, S^n)$ by

$$E_p(u, B^n) = \int_{B^n} |\nabla u|^p \,\mathrm{d}x.$$

A map $u \in H^{1,p}(B^n, S^n)$ is called “weakly p-harmonic” if u satisfies

$$\int_{B^n} |\nabla u|^{p-2} \nabla u \cdot \nabla\phi \,\mathrm{d}x = \int_{B^n} |\nabla u|^p u \cdot \phi \,\mathrm{d}x$$

for all $\phi \in H_0^{1,p}(B^n, \mathbb{R}^{n+1}) \cap L^\infty(B^n, \mathbb{R}^{n+1})$. It is also easy to check that the equator map u^* is a weakly p-harmonic map from B^n to S^n for $2 \leqslant p < n$.

In this paper, we first generalize the Jäger–Kaul theorem to the p-energy by the following:

THEOREM A. – *Assume that* $n > p \geqslant 2$.

(i) *For* $3 \leqslant n < 2 + p + 2\sqrt{p}$, *the equator map* $u^* = (\frac{x}{|x|}, 0)$ *is an unstable* p*-harmonic map from* B^n *to* S^n.

(ii) *When* $n \geqslant 2 + p + 2\sqrt{p}$, *then* u^* *is a minimizer of the* p*-energy* E_p *in the class* $H^{1,p}(B^n, S^n)$ *with boundary value* u^* *on* ∂B^n. *Moreover, if* $n \geqslant 2 + p + 2\sqrt{p}$, u^* *is the unique minimizer.*

We present a new proof of Theorem A(ii) and also point out that our proof is different from and much simpler than the orginal proof by Jäger and Kaul.

Remark a. – When $p = 2$, $n \geqslant 7 > 4 + 2\sqrt{2}$. Thus the proof of Theorem A also extends the Helein theorem in [9, Theorem 3.1] for $n \geqslant 9$

to the case when $n \geqslant 7$, i.e., for $n \geqslant 7$, we have

$$E(u) - E(u^*) \geqslant K_n \|u - u^*\|^2_{H^{1,2}(B^n)},$$

where u is a map in $H^{1,2}(B^n, S^n)$ which agree with u^* on ∂B^n, and K_n is a strict positive constant.

Remark b. – Consider the case of maps from B^n into S^{n-1}. Then $u^* = \frac{x}{|x|}$ is a minimizer of $E(u; B^n)$ for maps from B^n into S^{n-1}. This result has been proved by Brezis et al. [2] for $n = 3$, by Jäger and Kaul [12] for $n \geqslant 7$ and by Lin [14] for $n \geqslant 3$. Moreover, Coron and Gulliver proved that $u^* = \frac{x}{|x|}$ is a minimizer of the p-energy functional $E_p(u; B^n)$ for maps from B^n into S^{n-1} for $p \leqslant n - 1$. Theorem A recovers the partial result of Coron and Gulliver [3] for $n \geqslant 2 + p + 2\sqrt{p}$. Perhaps, the uniqueness of the minimizer of p-energy functional of maps from B^n into S^{n-1} for $n \geqslant 2 + p + 2\sqrt{p}$ is also a new result.

Let $M = B^n$ again and let N be an ellipsoid of $\mathbb{R}^{n+1}$, i.e.,

$$N := \left\{ u = (v, z)\colon\ |v|^2 + \frac{z^2}{a^2} = 1 \right\} \subset \mathbb{R}^{n+1},$$

where $v \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $a > 0$ is a constant.

Baldes [1] in 1984 and Helein [8] in 1988 generalized the work of Jäger and Kaul when N is an ellipsoid:

THEOREM. –

(i) (Baldes) *When* $a^2 \geqslant 1$ *and* $n \geqslant 7$, *the equator map* u^* *is the unique minimizer of the energy functional* E *in* $H^{1,2}(B^n; N)$ *with boundary value* $(x, 0)$.

(ii) (Helein) *If* $0 < a < 1$ *and* $a^2 \geqslant 4(n-1)/(n-2)^2$, *the equator map* u^* *is the unique minimizer of the energy functional* E *in* $H^{1,2}(B^n; N)$ *with boundary value* $(x, 0)$.

We would point out that all proofs of Baldes [1] and Helein [8] are variants of the proof of Jäger and Kaul [12]. After the first version of this paper (CMA-Preprint, September 1996), the author was asked whether one could recover and generalize the results of Baldes and Helein using our proof of Theorem A. Here we generalize their results to p-harmonic maps with $n > p \geqslant 2$ by the following:

THEOREM B. –
(i) *When* $a^2 \geqslant 1$ *and* $n \geqslant 2 + p + 2\sqrt{p}$, *the equator map* u^* *is the unique minimizer of the* p*-energy functional* E_p *in* $H^{1,p}(B^n; N)$ *with boundary value* $(x, 0)$. .
(ii) *If* $0 < a < 1$ *and* $a \geqslant 4(n-1)/(n-p)^2$, *the equator map* u^* *is the unique minimizer of the* p*-energy functional* E_p *in* $H^{1,p}(B^n; N)$ *with boundary value* $(x, 0)$.

2. PROOF OF THEOREM A

LEMMA 1. – *For any* $p < n$, *we have*

$$(n-p)^2 \int_{B^n} \left|\phi(x)\right|^2 \frac{1}{|x|^p} \,\mathrm{d}x \leqslant 4 \int_{B^n} \frac{1}{|x|^{p-2}} \left|\frac{\partial \phi}{\partial r}\right|^2 \mathrm{d}x$$

for all $\phi \in H_0^{1,p} \cap L^\infty(B^n, \mathbb{R})$ *with* $r = |x|$ *where that equality occurs only if* $\phi = 0$.

Proof. – The case of $p = 2$ was proved in [1]. Integrating by parts and using Cauchy's inequlaity, we have

$$\begin{aligned}\int_{B^n} |\phi|^2 \frac{1}{|x|^p} \,\mathrm{d}x &= \int_{|\omega|=1} \int_0^1 \phi^2 r^{n-p-1} \,\mathrm{d}r \,\mathrm{d}\omega \\ &= \frac{2}{n-p} \int_{|\omega|=1} \int_0^1 \phi \frac{\partial \phi}{\partial r} r^{n-p} \,\mathrm{d}r \,\mathrm{d}\omega \\ &\leqslant \frac{1}{2} \int_{|\omega|=1} \int_0^1 \phi^2 r^{n-p-1} \,\mathrm{d}r \,\mathrm{d}\omega \\ &\quad + \frac{2}{(n-p)^2} \int_{|\omega|=1} \int_0^1 \left(\frac{\partial \phi}{\partial r}\right)^2 r^{n-p+1} \,\mathrm{d}r \,\mathrm{d}\omega\end{aligned}$$

for all $\phi \in H_0^{1,p} \cap L^\infty(B^n, \mathbb{R})$. The above inequality becomes equality iff

$$\phi = \frac{2}{(n-p)} \frac{\partial \phi}{\partial r},$$

this is psossible ony if $\phi = 0$. This proves our claim. □

Remark c. – Lemma 1 can be also proved in following way:

$$\begin{aligned}
&\inf_{\phi\not\equiv 0,\ \operatorname{supp}\phi\subset\bar B^n\setminus\{0\}} \frac{\int_{B^n}|\nabla\phi|^2 r^{-(p-2)}\,\mathrm{d}x}{\int_{B^n} r^{-p}|\phi|^2\,\mathrm{d}x}\\
&\quad=\inf_{\tilde\phi\not\equiv 0,\ \operatorname{supp}\tilde\phi\subset(0,1]} \frac{\int_0^1 |\frac{\partial\tilde\phi}{\partial r}|^2 r^{n+1-p}\,\mathrm{d}r}{\int_0^1 |\tilde\phi|^2 r^{n-1-p}\,\mathrm{d}r}\\
&\quad\geqslant\inf_{\tilde\phi\not\equiv 0,\ \operatorname{supp}\tilde\phi\subset(0,\infty)} \frac{\int_0^\infty |\frac{\partial\tilde\phi}{\partial r}|^2 r^{n+1-p}\,\mathrm{d}r}{\int_0^\infty |\tilde\phi|^2 r^{n-1-p}\,\mathrm{d}r}=\frac{(n-p)^2}{4}.
\end{aligned}$$

This can be done by modifying a lemma taken from [17, Lemma 1.3].

2.1. Proof of Theorem A(ii)

Let $u^*=(\frac{x}{|x|},0)$ be the "equator map" from $B^n\to S^n$. It is easy to see

$$|\nabla u^*|^2=\frac{n-1}{r^2}.$$

Let $w\in H^{1,2}(B^n,S^n)$ be any function with boundary value $w=u^*$ on ∂B^n.

By Lemma 1, we obtain

$$\int_{B^n}|\nabla u^*|^{p-2}|\nabla\phi|^2\,\mathrm{d}x\geqslant\frac{(n-p)^2}{4(n-1)}\int_{B^n}|\nabla u^*|^p\phi^2\,\mathrm{d}x.$$

When $n\geqslant 2+p+\sqrt{4p}$, we have

$$\frac{(n-p)^2}{4(n-1)}\geqslant 1.$$

When $n\geqslant 2+p+\sqrt{4p}$, we get

$$\int_{B^n}|\nabla u^*|^{p-2}\big|\nabla(u^*-w)\big|^2\,\mathrm{d}x\geqslant\int_{B^n}|\nabla u^*|^p(u^*-w)^2\,\mathrm{d}x \tag{2.1}$$

for all $w\in H^{1,p}(B^n,S^n)$ with $w=u^*$ on ∂B^n. Moreover, we know

$$\int_{B^n}|\nabla u^*|^{p-2}|\nabla(u^*-w)|^2\,\mathrm{d}x$$

$$= \int_{B^n} |\nabla u^*|^p \, \mathrm{d}x - 2 \int_{B^n} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla w \, \mathrm{d}x$$
$$+ \int_{B^n} |\nabla u^*|^{p-2} |\nabla w|^2 \, \mathrm{d}x \tag{2.2}$$

and

$$\int_{B^n} |\nabla u^*|^p (u^* - w)^2 \, \mathrm{d}x = \int_{B^n} |\nabla u^*|^p (2 - 2u^* \cdot w) \, \mathrm{d}x. \tag{2.3}$$

Notice that u^* is a weakly p-harmonic map, i.e.,

$$\int_{B^n} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla \phi \, \mathrm{d}x = \int_{B^n} |\nabla u^*|^p u^* \cdot \phi \, \mathrm{d}x$$

for all $\phi \in H_0^{1,p}(B^n, \mathbb{R}^{n+1})$. By taking $\phi = u^* - w$, we have

$$\int_{B^n} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla w \, \mathrm{d}x = \int_{B^n} |\nabla u^*|^p u^* \cdot w \, \mathrm{d}x. \tag{2.4}$$

From (2.1)–(2.4), we get for $n \geqslant 2 + p + \sqrt{4p}$

$$\int_{B^n} |\nabla u^*|^{p-2} |\nabla w|^2 \, \mathrm{d}x \geqslant \int_{B^n} |\nabla u^*|^p \, \mathrm{d}x.$$

By the Hölder inequaity, we have

$$\int_{B^n} |\nabla u^*|^{p-2} |\nabla w|^2 \, \mathrm{d}x \leqslant \left(\int_{B^n} |\nabla u^*|^p \, \mathrm{d}x \right)^{(p-2)/p} \left(\int_{B^n} |\nabla w|^p \, \mathrm{d}x \right)^{2/p}.$$

Combing two above inequalities, we have

$$\int_{B^n} |\nabla u^*|^p \, \mathrm{d}x \leqslant \int_{B^n} |\nabla w|^p \, \mathrm{d}x$$

for all $w \in H^{1,p}(B^n, S^n)$ with boundary value $w = u^*$ on ∂B^n when $n \geqslant 2 + p + 2\sqrt{p}$.

From Lemma 1, we know that (2.1) become equality only if $w = u^*$. If $n \geqslant 2 + p + 2\sqrt{p}$, we have

$$\int_{B^n} |\nabla u^*|^p \,\mathrm{d}x < \int_{B^n} |\nabla w|^p \,\mathrm{d}x$$

for all $w \in H^{1,p}(B^n, S^n)$ with boundary value $w = u^*$ on ∂B^n and $w \not\equiv u^*$. It means that u^* is the unique minimizer for $n \geqslant 2 + p + 2\sqrt{p}$. This proves Theorem A(ii).

Assume that $p = 2$. When $n \geqslant 7 > 4 + 2\sqrt{2}$, one sees $(n-2)^2/(4(n-1)) > 1$. By Lemma 1, we know

$$\int_{B^n} |\nabla\phi|^2 \,\mathrm{d}x \geqslant \left(1 - \frac{4(n-1)}{(n-2)^2}\right) \int_{B^n} |\nabla\phi|^2 \,\mathrm{d}x + \int_{B^n} |\nabla u^*|^2 \phi^2 \,\mathrm{d}x$$

for all $\phi \in H_0^{1,2}(B^n, \mathbb{R}^{n+1})$. Taking $\phi = w - u^*$, we have

$$\int_{B^n} |\nabla w|^2 \,\mathrm{d}x - \int_{B^n} |\nabla u^*|^2 \,\mathrm{d}x \geqslant \left(1 - \frac{4(n-1)}{(n-2)^2}\right) \int_{B^n} \left|\nabla(w - u^*)\right|^2 \mathrm{d}x,$$

where $w \in H^{1,2}(B^n, S^n)$ agrees with u^* on ∂B^n. This proves our claim in Remark 1.

For t small we define $u_t : B^n \to S^n$ by setting

$$u_t(x) = \frac{\left(\frac{x}{|x|}, t\phi(x)\right)}{(1 + t^2\phi^2)^{1/2}}$$

for a smooth function ϕ on B^n vanishing near 0 and ∂B^n. A simple calculation gives

$$\left.\frac{\partial \nabla u_t(x)}{\partial t}\right|_{t=0} = \big(0, \ldots, 0, \nabla\phi(x)\big), \qquad \left.\frac{\partial}{\partial t}|\nabla u_t|^2\right|_{t=0} = 0$$

and

$$\left.\frac{\partial^2 \nabla u_t(x)}{\partial^2 t}\right|_{t=0} = \phi^2\left(\nabla\frac{x}{|x|}, 0\right).$$

Now

$$\left.\frac{1}{2}\frac{\partial^2}{\partial^2 t}|\nabla u_t|^2\right|_{t=0} = \left|\frac{\partial \nabla u_t(x)}{\partial t}\right|^2_{t=0} + \left(\frac{\partial^2 \nabla u_t(x)}{\partial^2 t}, \nabla u_t\right)_{t=0}.$$

Then

$$\frac{d^2}{d^2t}E_p(u_t)\bigg|_{t=0} = \int_{B^n} |\nabla u^*|^{p-2}\big[|\nabla\phi|^2 - \phi^2|\nabla u^*|^2\big]\,dx.$$

It is easy to check that $u^* = (\frac{x}{|x|}, 0)$ is a weakly p-harmonic map from B^n into S^n. If u^* is stable, we have

$$\int_{B^n} |\nabla u^*|^{p-2}\big[|\nabla\phi|^2 - \phi^2|\nabla u^*|^2\big]\,dx \geqslant 0$$

for all smooth ϕ vanishing near 0 and ∂B^n.

2.2. Proof of Theorem A(i)

Let us consider the following equation:

$$\begin{cases} \phi''(r) + \dfrac{n-p+1}{r}\phi'(r) + \dfrac{n-1-\varepsilon}{r^2}\phi(r) = 0, \\ \phi(r_0) = \phi(1) = 0. \end{cases} \tag{2.5}$$

for $0 < r_0 < 1$.

By setting $\xi(t) = \phi(e^t)$, Eq. (2.5) becomes

$$\xi''(t) + (n-p)\xi'(t) + (n-1-\varepsilon)\xi(t) = 0.$$

Let

$$\nu := \nu(\varepsilon) = \tfrac{1}{4}\big[n^2 - 2(p+2)n + p^2\big] + 1 + \varepsilon.$$

When $3 \leqslant n < 2 + p + 2\sqrt{p}$, there exists a small ε such that $\nu < 0$ and we choose r_0: $0 < r_0 < 1$ such that $\sqrt{-\nu}\ln r_0$ is a multiple of 2π. Then it is easily checked (see [12]) that the function

$$\phi(r) = \begin{cases} r^{(p-n)/2}\sin(\sqrt{-\nu}\cdot\ln r), & \text{for } r_0 < r \leqslant 1, \\ 0, & \text{for } r \leqslant r_0. \end{cases}$$

solves Eq. (2.5). This means that for $3 \leqslant n < 2 + p + 2\sqrt{p}$, there exists ε small and a non-zero $\phi(r)$ on $[r_0, 1]$, $\phi(1) = \phi(r_0) = 0$, such that

$$\int_0^1 r^{2-p}\left[\phi'(r)^2 - \frac{(n-1)}{r^2}\phi^2\right] r^{n-1}\,\mathrm{d}r = -\int_0^1 \frac{\varepsilon}{r^2} r^{2-p}\phi^2 r^{n-1}\,\mathrm{d}r < 0.$$

In other words, we see that u^* is unstable for $3 \leqslant n < 2 + p + 2\sqrt{p}$. This proves Theorem A(i).

From the proof of Theorem A, we have

COROLLARY 2. – *Assume that u is a stable p-harmonic map from B^n into S^n and the values of u are on the equator $(S^{n-1}, 0)$ of S^n. Then u is a local minimizer of the energy functional E_p in $H^{1,p}(B^n, S^n)$.*

3. PROOF OF THEOREM B

In this section, let N be the ellipsoid of $\mathbb{R}^{n+1}$ defined in Section 1 and suppose that $p \in \mathbb{R}$ with $n > p \geqslant 2$. We define the p-energy of maps in $H^{1,p}(B^n, N)$ by

$$E_p(v, z; B^n) = \int_{B^n} \left(|\nabla v|^2 + |\nabla z|^2\right)^{p/2}\,\mathrm{d}x.$$

We write $u = (v, z)$ with $v \in \mathbb{R}^n$, $z \in \mathbb{R}$. A map $u = (v, z) \in H^{1,p}(B^n, N)$ is called "weakly p-harmonic" if u satisfies in the sense of distributions the following equations:

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla v\right) + |\nabla u|^{p-2}\lambda v = 0,$$
$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla z\right) + |\nabla u|^{p-2}\lambda \frac{z}{a^2} = 0,$$

where

$$\lambda = \left(|\nabla v|^2 + \frac{|\nabla z|^2}{a^2}\right)\frac{a^4}{a^4 v^2 + z^2}.$$

Proof of Theorem B. – Let $u = (v, z)$ be any function in $H^{1,p}(B^n, N)$ with boundary values $(x, 0)$ on ∂B^n. Using Lemma 1 again, we have

$$\int_{B^n} \left|\nabla \frac{x}{|x|}\right|^{p-2} \left|\nabla\left(\frac{x}{|x|} - v\right)\right|^2 \mathrm{d}x$$

$$\geqslant \frac{(n-p)^2}{4(n-1)} \int_{B^n} \left|\nabla \frac{x}{|x|}\right|^p \left|\frac{x}{|x|} - v\right|^2 \mathrm{d}x \tag{3.1}$$

and

$$\int_{B^n} \left|\nabla \frac{x}{|x|}\right|^{p-2} |\nabla z|^2 \,\mathrm{d}x \geqslant \frac{(n-p)^2 a^2}{4(n-1)} \int_{B^n} \left|\nabla \frac{x}{|x|}\right|^p \frac{|z|^2}{a^2} \,\mathrm{d}x. \tag{3.2}$$

(i) Assume that $a \geqslant 1$ and $n \geqslant 2 + p + 2\sqrt{p}$. Note the fact $v^2 + \frac{z^2}{a^2} = 1$. Thus using (3.1) and (3.2), the proof of Theorem A(ii) yields

$$\int_{B^n} \left|\nabla \frac{x}{|x|}\right|^{p-2} |\nabla u|^2 \,\mathrm{d}x \geqslant \frac{(n-p)^2}{4(n-1)} \int_{B^n} \left|\nabla \frac{x}{|x|}\right|^p \mathrm{d}x$$

for all $u = (v, z)$ with same boundary values $(x, 0)$. The same argument in the proof of Theorem A(ii) gives that $(\frac{x}{|x|}, 0)$ is the unique minimizer of E_p if $n \geqslant 2 + p + 2\sqrt{p}$.

(ii) Assume that $0 < a < 1$ and $a^2 \geqslant 4(n-1)/(n-p)^2$. Thus using (3.1) and (3.2) again, we get

$$\int_{B^n} \left|\nabla \frac{x}{|x|}\right|^{p-2} |\nabla u|^2 \,\mathrm{d}x > \frac{(n-p)^2 a^2}{4(n-1)} \int_{B^n} \left|\nabla \frac{x}{|x|}\right|^p \mathrm{d}x$$

for all $u = (v, z) \not\equiv (\frac{x}{|x|}, 0)$ with same boundary values. The same argument in the proof of Theorem A(ii) gives that $(\frac{x}{|x|}, 0)$ is the unique minimizer of E_p. This proves Theorem B. □

Remark d. – It is obvious from the proof of Theorem A(i) to see that $(\frac{x}{|x|}, 0)$ is unstable for E_p when $a^2 < 4(n-p)/(n-2)^2$.

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