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Geodesics on product Lorentzian manifolds (*)

by

F. GIANNONI

Istituto di Matematiche Applicate,
Universita' di Pisa, Italy.

and

A. MASIELLO

Dipartimento di Matematica,
Politecnico di Bari, Italy.

ABSTRACT. – In this paper, using global variational methods, we prove existence and multiplicity results for geodesics joining two given events of a product Lorentzian manifold $\mathcal{M}_0 \times \mathbb{R}$, where $\mathcal{M}_0$ is a complete Riemannian manifold.

Key words: Lorentzian metrics, geodesics, critical point theory.

RÉSUMÉ. – Dans cet article, avec des méthodes variationnelles globales, on démontre des résultats d'existence et de multiplicité de géodésiques joignant deux points dans une variété de Lorentz $\mathcal{M}_0 \times \mathbb{R}$, où $\mathcal{M}_0$ est une variété de Riemann complète.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $\mathcal{M}$ be a Lorentzian manifold, *i.e.* a smooth manifold equipped with a non-degenerate symmetric (0,2)-tensor field $g(z)[\cdot, \cdot]$ $(z \in \mathcal{M})$ having index 1. (This means that every matricial representation of g has exactly

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one negative eigenvalue). The points of $\mathcal{M}$ are called events. A geodesic on $\mathcal{M}$ is a smooth curve $\gamma : [a, b] \rightarrow \mathcal{M}$ solving

$$D_s \dot{\gamma}(s) = 0 \quad \text{for all } s \in [a, b]$$

where $a < b \in \mathbb{R}$, $\dot{\gamma}(s)$ is the derivative of $\gamma(s)$ and $D_s \dot{\gamma}(s)$ is the covariant derivative of $\dot{\gamma}(s)$ with respect to the metric tensor g.

It is well known that a geodesic is a critical point of the "energy" functional

$$f(\gamma) = \int_a^b g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)]\, ds,$$

and, if γ is a geodesic on $\mathcal{M}$, there exists a constant $E_\gamma \in \mathbb{R}$ such that

$$E_\gamma = g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] \quad \text{for all } s \in [a, b].$$

A geodesic γ is called timelike, null or spacelike if E_γ is less, equal or greater than zero respectively. In General Relativity a timelike geodesic represents the trajectory of a free falling particle. Null geodesics represent the light rays, while spacelike geodesics have a more subtle interpretation: for a suitable local observer they represent "Riemannian" geodesics consisting of simultaneous events.

During the last few years, the existence of geodesics joining two given events in static and stationary Lorentzian manifolds, has been widely studied by global variational methods (*cf.* [2] and references therein).

The aim of this paper is to prove, always by global variational methods, existence and multiplicity results for timelike and spacelike geodesics joining two given events of a product Lorentzian manifold $\mathcal{M}_0 \times \mathbb{R}$, in general situations in which the Lorentzian metric of $\mathcal{M}$ depends on the time variable and has mixed terms.

The results of this paper are different from those of [3] (where the case without mixed terms is considered) and are obtained using different methods. In particular we get a priori estimates for the critical points of the energy functional, using suitable test functions in the weak equation of the critical points and, using the relative category (*see* [6]), we get also a multiplicity result for timelike geodesics.

One of the main difficulties of a global variational approach to the problem is the study of the Palais-Smale compactness condition (*cfr* Definition 2.1) for the energy integral f. We bypass this difficulty having a set of assumptions which allow to get a priori bounds for the critical points and the Palais-Smale condition, for a suitable penalization functional of f.

The proofs of the existence results use a Saddle Point Theorem which is a slight device of the classical Saddle Point Theorem of Rabinowitz (*cf.* [15]), while the multiplicity results use the topological concept of Relative Category (*cf.* [4], [6], [7], [17]).

Relative category gives a topological approach which seems simpler than the relative cohomology used in [1] to get infinitely many spacelike geodesics joining two given events, and it allows also to obtain a multiplicity result for timelike geodesics.

Let $T_z\,\mathcal{M} = T_x\,\mathcal{M}_0 \times \mathbb{R}$ be the tangent space to $\mathcal{M}$ at $z = (x, t)$. Assume that the metric tensor g on $\mathcal{M}$, for any $\zeta = (\xi, \tau) \in T_z\,\mathcal{M}$, has the form

$$g(z)[\zeta, \zeta] = \langle \alpha(z)\xi, \xi\rangle_x + 2\langle \delta(z), \xi\rangle_x \tau - \beta(z)\tau^2 \tag{1.1}$$

where $\langle\cdot, \cdot\rangle_x$ is a Riemannian metric on $\mathcal{M}_0$, $\alpha(z)[\cdot] = \alpha(x, t)[\cdot]$ is a smooth, symmetric, positive linear operator on $T_x\,\mathcal{M}_0$, δ is a smooth vector field and β is a smooth, positive scalar field. Denote by α_t, δ_t and β_t the derivative with respect to the variable t of α, δ, β respectively.

In order to get a priori bounds on the critical points of the energy functional we shall make assumptions on g (on the compact subsets of $\mathbb{R}$, globally and at infinity) as follows.

Define

$$A^+(t) = \sup\{\langle \alpha_t(x, t)\xi, \xi\rangle_x / \langle \alpha(x, t)\xi, \xi\rangle_x : x \in \mathcal{M}_0,\ \xi \in T\mathcal{M}_0\},$$

$$A^-(t) = \sup\{-\langle \alpha_t(x, t)\xi, \xi\rangle_x / \langle \alpha(x, t)\xi, \xi\rangle_x : x \in \mathcal{M}_0,\ \xi \in T\mathcal{M}_0\},$$

$$D(t) = \sup\{\|\delta_t(x, t)\| : x \in \mathcal{M}_0\},$$

$$B^+(t) = \sup\{\beta_t(x, t)/\beta(x, t) : x \in \mathcal{M}_0\},$$

$$\beta^-(t) = \sup\{-\beta_t(x, t)/\beta(x, t) : x \in \mathcal{M}_0\},$$

$$B(t) = \sup\{|\beta_t(x, t)| : x \in \mathcal{M}_0\}.$$

Moreover set

$$A^+_{+\infty} = \limsup_{t \to +\infty} A^+(t),$$

$$A^-_{-\infty} = \limsup_{t \to -\infty} A^-(t),$$

$$D^+ = \limsup_{t \to +\infty} D(t),$$

$$D^- = \limsup_{t \to -\infty} D(t),$$

$$B^-_{+\infty} = \limsup_{t \to +\infty} B^-(t),$$

$$B^+_{-\infty} = \limsup_{t \to -\infty} B^+(t).$$

The global assumptions on g are the following:

$$(\mathcal{M}_0,\ \langle\cdot\,,\,\cdot\rangle_x)\ \text{is a complete Riemannian manifold,} \tag{1.2}$$

$$\inf\{\langle\alpha(z)\,\xi,\ \xi\rangle_x/\langle\xi,\ \xi\rangle_x : z\in\mathcal{M},\ \xi\in T_x\mathcal{M}_0\}=\lambda>0, \tag{1.3}$$

$$\inf\{\beta(z) : z\in\mathcal{M}\}=b>0. \tag{1.4}$$

About the behavior of $g(x,\ t)$ on the compact subsets of $\mathbb{R}$ we make the following assumptions:

$$A^+,\ A^-,\ D\ \text{and}\ B\ \text{are bounded on the compact subsets of}\ \mathbb{R}, \tag{1.5}$$

$$\left.\begin{array}{c}\sup\{\|\,\delta(x,\ 0)\,\| : x\in\mathcal{M}_0\}\equiv\delta_0<+\infty;\\ \sup\{\beta(x,\ 0) : x\in\mathcal{M}_0\}\equiv\beta_0<+\infty.\end{array}\right\} \tag{1.6}$$

A first reasonable assumption about the behavior of g at infinity is the following:

$$A^+_{+\infty},\ A^-_{-\infty},\ D^+,\ D^-,\ B^-_{+\infty},\ B^+_{-\infty}\ \text{are finite.} \tag{1.7}$$

As observed in Remark 3.5, assumptions (1.2)-(1.6) are sufficient to prove a priori estimates about timelike geodesics. Unfortunately they are not sufficient to prove a priori estimates for spacelike geodesics having energy bounded from above, that we need for the study of the geodesic connectedness (also (1.2)-(1.7) are not sufficient). Indeed the a priori estimates (for spacelike geodesics) do not hold for the Anti-de Sitter space-time, *i.e.* $\mathbb{R}\times]-\pi/2,\ \pi/2[$ with the Lorentz metric

$$ds^2=\frac{dx^2-dt^2}{\cos^2 t}\,.$$

But, by a suitable change of variable, the Anti-de Sitter space-time becomes $\mathbb{R}^2$ with metric

$$ds^2=\cosh^2\vartheta\,dx^2-d\vartheta^2$$

which satisfies assumptions (1.2)-(1.7).

Therefore, in order to study the geodesic connectedness of $\mathcal{M}$ and the multiplicity of spacelike geodesics we reinforce (1.7) in the following way:

$$\left.\begin{array}{c}A^+_{+\infty},\ A^-_{-\infty},\ B^-_{+\infty},\ B^+_{-\infty}\leqq 0\\ D^+=D^-=0\end{array}\right\} \tag{1.8}$$

In order to prove existence and multiplicity results we also need some assumptions assuring the topological nontriviality of the sublevels of the energy functional. To this aim we assume that *there exist*

$b_0,\ b_1 \in C^0(\mathcal{M}_0,\ \mathbb{R}^+)$, $\gamma_0 \in [0,\ 2[$, $\gamma_1 \in [0,\ 1[$ *such that, for any* $(x,\ t) \in \mathcal{M}$,

$$\frac{1}{\beta(x,\ t)} \cdot \frac{\langle \alpha(x,\ t)\,\xi,\ \xi\rangle_x}{\langle \xi,\ \xi\rangle_x} \leqq (b_0(x) + b_1(x)\,|\,t\,|^{\gamma_0}) \quad \text{for any } \xi \in T_x\mathcal{M}_0,$$

$$\frac{1}{\beta(x,\ t)} \cdot \|\,\delta(x,\ t)\,\| \leqq (b_0(x) + b_1(x)\,|\,t\,|^{\gamma_1}). \tag{1.9}$$

The first result concerns the existence of a timelike geodesic joining two given events. Let $z_0 = (x_0,\ t_0)$, $z_1 = (x_1,\ t_1) \in \mathcal{M}$ and

$$C^1(x_0,\ x_1) = \{x \in C^1([0,\ 1],\ \mathcal{M}_0) : x(0) = x_0 \text{ and } x(1) = x_1\},$$

$$C^1(t_0,\ t_1) = \{t \in C^1([0,\ 1],\ \mathbb{R}) : t(0) = t_0 \text{ and } t(1) = t_1\}.$$

We shall prove the following:

THEOREM 1.1. – *Let* $(\mathcal{M},\ g)$ *satisfying* (1.2)-(1.7) *and* (1.9). *Assume that*

$$\sup_{t\in C^1(t_0,t_1)} \quad \inf_{x\in C^1(x_0,x_1)}$$

$$\int_0^1 [\langle \alpha(z)\,\dot{x},\ \dot{x}\rangle_x + 2\,\langle \delta(z),\ \dot{x}\rangle_x\,\dot{t} - \beta(z)\,\dot{t}^2]\,ds < 0, \tag{1.10}$$

where $z = (x,\ t)$.

Then there exists a timelike geodesic in $\mathcal{M}$ *joining* z_0 *and* z_1.

About the geodesic connectedness of $\mathcal{M}$ we have

THEOREM 1.2. – *Let* $(\mathcal{M},\ g)$ *satisfying* (1.2)-(1.7), *and* (1.8)-(1.9). *Then* $\mathcal{M}$ *is geodesically connected,* i.e. *for every* $z_0,\ z_1 \in \mathcal{M}$ *there exists a geodesic in* $\mathcal{M}$ *joining* z_0 *and* z_1.

Theorem 1.2 does not generalize Theorem 1.1 of [3], only because of the assumptions of β_t. Indeed here we assume $B^-_{+\infty},\ B^+_{-\infty} \leqq 0$, while in [3] it is assumed β_t bounded.

Since $(\mathcal{M}_0, \langle\cdot\,,\cdot\rangle_x)$ is complete, the uniform estimates in x of assumptions (1.3), (1.5), and (1.6) imply that $\mathcal{M}$ is globally hyperbolic (for the definition *cf.* e.g. [12]). Then by a theorem of Geroch (*cf.* [8]), there exist coordinates such that $\mathcal{M}$ is a product manifold with metric tensor g in (1.1) having $\delta \equiv 0$. However in our proofs we can not use the Geroch result to reduce us to the case $\delta \equiv 0$, because it is not known how the coefficients α and β become after the change of coordinates.

Eventhough in our case $\mathcal{M}$ is globally hyperbolic, our results are motivated by the fact that, as far as we know, on globally hyperbolic Lorentz manifolds only existence results for time-like geodesics have been proved (*cf.* [2]), while here we deal also with multiplicity results about timelike geodesics and existence and multiplicity results about spacelike geodesics. Moreover it is physically relevant to consider product Lorentzian manifolds not satisfying (1.2) (*cf.* e.g. [9] for Lorentzian stationary product manifolds), and therefore not necessarily globally hyperbolic, and we hope that the techniques used here (in particular the a priori estimates for the critical points of f, based on the choice of suitable test functions) can be useful in many cases.

Whenever $\mathcal{M}_0$ has a rich topology multiplicity results for spacelike geodesics and timelike geodesics joining two given events hold.

THEOREM 1.3. – *Assume that* $(\mathcal{M}, g)$ *satisfies* (1.2)-(1.7), *and* (1.8)-(1.9) *and* $\mathcal{M}_0$ *is not contractible.*

Then there exists a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ *of spacelike geodesics joining* z_0 *and* z_1 *in* $\mathcal{M}$ *such that*

$$\lim_{k\to+\infty} E_{\gamma_k} = +\infty.$$

THEOREM 1.4. – *Assume that* $(\mathcal{M}, g)$ *satisfies* (1.2)-(1.7) *and* (1.9), *and denote by* $N(x_0, x_1, t_0, t_1)$ *the number of timelike geodesics joining* (x_0, t_0) *with* (x_1, t_1). *Moreover assume that* $\mathcal{M}_0$ *is not contractible. Then, for any* $x_0, x_1 \in \mathcal{M}_0$,

$$\lim_{|t_1-t_0|\to+\infty} N(x_0, x_1, t_0, t_1) = +\infty.$$

The paper is organized as follows. In section 2 we describe the functional tools used for the proofs of the theorems above. In section 3 we prove the a priori estimates for the critical points of the energy functional and the Palais-Smale compactness condition for a suitable penalization of such a functional. In section 4 we prove Theorems 1.1 and 1.2, while in section 5 we prove Theorems 1.3 and 1.4.

2. TECHNICAL PRELIMINARIES

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and g as in (1.1) with $(\mathcal{M}_0, \langle\cdot, \cdot\rangle_x)$ complete, connected Riemannian manifold. By the well known embedding Theorem of Nash (*see* [11]) we can assume that $\mathcal{M}_0$ is a submanifold of $\mathbb{R}^N$ (with N

sufficiently large) and $\langle \cdot, \cdot \rangle_x$ is the Riemannian structure on $\mathcal{M}_0$ inherited by the Euclidean metric of $\mathbb{R}^N$.

We shall prove Theorems 1.1-1.4 by looking for critical points of the energy functional

$$f(z) = \int_0^1 [\langle \alpha(z) \dot{x}, \dot{x} \rangle + 2 \langle \delta(x), \dot{x} \rangle \dot{t} - \beta(z) \dot{t}^2] \, ds \tag{2.1}$$

(where $\langle \cdot, \cdot \rangle$ denotes the Euclidean structure of $\mathbb{R}^{\mathrm{N}}$) on the set of the sufficiently smooth curves joining $z_0 = (x_0, t_0)$ and $z_1 = (x_1, t_1)$. More precisely set

$$\Omega^1 \equiv \Omega^1(\mathcal{M}_0, x_0, x_1) = \Big\{ x : [0, 1] \to \mathcal{M}_0 \text{ absolutely continuous} : \int_2^1 \langle \dot{x}, \dot{x} \rangle \, ds < +\infty, \, x(0) = x_0, \, x(1) = x_1 \Big\}, \tag{2.2}$$

$$W^1 \equiv W^1(\mathbb{R}, t_0, t_1) = \Big\{ t : [0, 1] \to \mathbb{R} \text{ absolutely continuous} : \int_2^1 \dot{t}^2 \, ds < +\infty, \, t(0) = t_0, \, t(1) = t_1 \Big\}. \tag{2.3}$$

It is well known that Ω^1 is a smooth, complete Hilbert manifold (*cfr* e.g. [16]) with Riemannian structure given by

$$\langle \xi, \xi \rangle_1 = \int_0^1 \langle \nabla_s \xi, \nabla_s \xi \rangle \, ds + \int_0^1 \langle \xi, \xi \rangle \, ds \tag{2.4}$$

(where ∇_s denotes the covariant derivative with respect to the Riemannian structure $\langle \cdot, \cdot \rangle$), while the tangent space to Ω^1 at x is given by

$$T_x \Omega^1 = \Big\{ \xi : [0, 1] \to T\mathcal{M}_0 \text{ absolutely continuous} : \xi(s) \in T_{x(s)} \mathcal{M}_0, \int_0^1 \langle \nabla_s \xi, \nabla_s \xi \rangle \, ds < +\infty, \xi(0) = \xi(1) = 0 \Big\}, \tag{2.5}$$

where $T\mathcal{M}_0$ is the tangent bundle of $\mathcal{M}_0$.

Moreover $W^1(\mathbb{R}, t_0, t_1)$ is an affine manifold, that we equip by the Riemannian structure

$$\langle t, t \rangle_1 = \int_0^1 \dot{t}^2 \, ds + \int_0^1 t^2 \, ds, \tag{2.6}$$

and whose tangent space is

$$W_0^{1,2} = \Big\{ \tau : [0, 1] \to \mathbb{R} \text{ absolutely continuous} : \int_0^1 \langle \dot\tau, \dot\tau \rangle\, ds < +\infty,\ \tau(0) = 0,\ \tau(1) = 0 \Big\}. \tag{2.7}$$

We shall look for critical points of the functional (2.1) on the Hilbert manifold

$$\mathcal{Z} = \Omega^1 \times W^1. \tag{2.8}$$

Using the Young inequality in $\mathbb{R}$ (to control the effects on the weak equation satisfied by the critical points of f due to the mixed term $\langle \delta(x), \dot x \rangle \dot t$), classical regularization arguments show that any critical point of f on $\mathcal{Z}$ is a (smooth) geodesic in $\mathcal{M}$ joining $z_0 = (x_0, t_0)$ with $z_1 = (x_1, t_1)$.

Consider now a functional $I \in C^1(X, \mathbb{R})$ where X is a Hilbert manifold. Let $c \in \mathbb{R}$. We recall the following

DEFINITION 2.1. – *I satisfies* $(\text{P.S.})_c$ *on X (the Palais-Smale condition at the level c) if any sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ such that*

$$I(x_k) \underset{k}{\to} c \qquad \textit{and} \qquad I'(x_k) \underset{k}{\to} 0$$

has a converging subsequence (in X).

The following classical Deformation Lemma (*cfr.* e.g. [13], [15]) is needed to prove the next Lemma 2.3.

LEMMA 2.2. – *Let X be an Hilbert manifold, $I \in C^2(X, \mathbb{R})$ and $c \in \mathbb{R}$. Assume that I satisfies* $(\text{P.S.})_c$ *on X.*

Then, if c is not a critical level of I (i.e. $I'(x) \neq 0$ *for any* $x \in I^{-1}(c)$), *there exist numbers $0 < \delta_1 < \delta_2 < \bar\delta$ and a homeomorphism $\eta : X \to X$ such that*

$$\eta(I^{-1}(]-\infty, c+\delta_1])) \subset I^{-1}(]-\infty, c-\delta_1]),$$

and

$$\eta(x) = x \quad \textit{for all } x \notin I^{-1}([c-\delta_2, c+\delta_2]).$$

The following Lemma on the existence of critical points (which will be used to prove Theorem 1.1) is a slight variant of a well known Saddle Point Theorem of Rabinowitz (*see* [14]). The proof is based on Lemma 2.2 and it is the same of Rabinowitz Theorem.

LEMMA 2.3. – *Let $X = \Omega \times H$ where Ω is a complete Hilbert manifold and H is an affine space with* $\dim H < +\infty$. *Let $I \in C^2(X, \mathbb{R})$. Assume that*

(i) *there exist* $\bar{x} \in \Omega$, $e \in H$ *and an open neighborhood* U *of* e *in* H *such that*

$$b_1 \equiv \sup_{(\bar{x},\, \partial U)} I < b_2 \equiv \inf_{(\Omega,\, e)} I.$$

(ii) I *satisfies* $(\text{P.S.})_c$ *on* X *for all* $c \in [b_1, b_2]$.

Then I *has a critical value* $c \in [b_2, \sup_{(\bar{x}, \bar{U})} I]$ *and* c *can be characterized as follows:*

Let $U_0 = \{\bar{x}\} \times U$, $\partial U_0 = \{\bar{x}\} \times \partial U$, $\bar{U}_0 = \{\bar{x}\} \times \bar{U}$ *and*

$$S = \{h \in C^0(\bar{U}_0, H) : h(u) = u \text{ for all } u \in \partial U_0\}.$$

Then

$$c = \inf_{h \in S} \sup_{z \in \bar{U}_0} I(h(z)).$$

Actually we do not know if f satisfies the Palais-Smale condition. For this reason, we consider, for any $\varepsilon \in]0, 1]$ a penalized functional defined as follows.

Let $\chi \in C^2(\mathbb{R}, [0, 1])$ such that $\chi(\sigma) = 0$ for any $\sigma \leqq 0$, $\chi(\sigma) = 1$ for any $\sigma \geqq 1$ and $\chi'(\sigma) > 0$ for any $\sigma \in]0, 1[$. For any $\varepsilon > 0$ set

$$\chi_\varepsilon(\sigma) = \begin{cases} \chi(\sigma - 1/\varepsilon) & \text{if } \sigma \geqq 1/\varepsilon, \\ 0 & \text{if } -1/\varepsilon \leqq \sigma \leqq 1/\varepsilon, \\ \chi(-\sigma - 1/\varepsilon) & \text{if } \sigma \leqq -1/\varepsilon, \end{cases}$$

$$b_\varepsilon = \sup\{\beta(x, t) : x \in \mathcal{M}_0,\, t \in [-(1/\varepsilon) - 1, (1/\varepsilon)] \cup [(1/\varepsilon), (1/\varepsilon) + 1]\}$$

(which is finite by (1.5) and (1.6)), and

$$\beta_\varepsilon(x, t) = (1 - \chi_\varepsilon(t))\, \beta(t, x) + \chi_\varepsilon(t)\, b_\varepsilon. \tag{2.9}$$

Moreover consider

$$\psi(\sigma) = e^\sigma - (1 + \sigma + \sigma^2/2)$$

and for any $\varepsilon > 0$ set

$$\psi_\varepsilon(\sigma) = \left\{ \begin{matrix} \psi(s - 1/\varepsilon) & \text{if } \sigma \geqq 1/\varepsilon \\ 0 & \text{if } \sigma \leqq 1/\varepsilon. \end{matrix} \right\} \tag{2.10}$$

Finally, for any $\varepsilon > 0$ define the penalized functional

$$f_\varepsilon(z) = \int_0^1 [\langle \alpha(z)\, \dot{x}, \dot{x} \rangle + 2 \langle \delta(x), \dot{x} \rangle \dot{t} - \beta_\varepsilon(z)\, \dot{t}^2]\, ds - \psi_\varepsilon\left(\int_0^1 \dot{t}^2\, ds\right). \tag{2.11}$$

In section 3, the Palais-Smale condition for the functional (2.11) and a priori estimates (independently of ε) on its critical points will be proved. The following properties of the function $\psi(\sigma)$ will be used:

$$\psi(\sigma) \text{ is positive;} \tag{2.12}$$

$$\psi(\sigma) \text{ is convex;} \tag{2.13}$$

$$\psi(\sigma) \leqq \psi'(\sigma), \quad \text{for every } \sigma \in \mathbb{R}; \tag{2.14}$$

for every $p > 0$, there exists two positive constants a_p and b_p, such that for every $\sigma \geqq 0$:

$$\psi(\sigma) \geqq a_p\, s^p - b_p; \tag{2.15}$$

for every $s \in \mathbb{R}_+$:

$$s\,\psi'(s) \geqq \psi(s). \tag{2.16}$$

3. PALAIS-SMALE CONDITION AND A PRIORI ESTIMATES FOR THE FUNCTIONAL (2.11)

We begin this section studying the Palais-Smale sequences for the functional f_ε.

PROPOSITION 3.1. – *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and g as in* (1.1). *Assume that* (1.3)-(1.7), *are satisfied. Let $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ such that*

$$f_\varepsilon(z_n) \leqq C \tag{3.1}$$

and

$$-\varepsilon_n \| \zeta \| \leqq f'_\varepsilon(z_n)(\zeta) \quad \textit{for any } \zeta = (\xi, \tau) \in T_{z_n}\mathcal{Z}, \tag{3.2}$$

where $\| \zeta \|^2 = \langle \xi, \xi \rangle_1 + \langle t, t \rangle_1$, $\varepsilon_n \underset{n}{\rightarrow} 0^+$ and C is a constant independent of n.

Then, $\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle\, ds$ and $\int_0^1 \dot{t}_n^2\, ds$ are bounded independently of n.

Proof. – By (2.11), for any $z = (x, t) \in \mathcal{Z}$ and $\tau \in W_0^{1,2}$,

$$
\begin{aligned}
f'_\varepsilon(z)(0,\tau) = \int_0^1 & [\langle \alpha_t(z)\dot{x},\, \dot{x}\rangle \tau + 2\langle \delta_t(x,\,t),\, \dot{x}\rangle \dot{t}\,\tau + 2\langle \delta(x,\,t),\, \dot{x}\rangle \dot{\tau} \\
& - (\beta_\varepsilon)_t(z)\,\dot{t}^2\,\tau - 2\,\beta_\varepsilon(z)\,\dot{t}\dot{\tau}]\,ds \\
& - 2\,\psi'_\varepsilon\left(\int_0^1 \dot{t}^2\,ds\right)\int_0^1 \dot{t}\dot{\tau}\,ds. \qquad (3.3)
\end{aligned}
$$

Let

$$t_*(s) = (1-s)\,t_0 + s t_1, \qquad (3.4)$$

and

$$\tau_n = t_n - t_*. \qquad (3.5)$$

By (3.2)-(3.5), since $\tau_n \in W_0^{1,\,2}$,

$$
\begin{aligned}
& - \varepsilon_n \,\|\, \dot{\tau}_n \,\|_{L^2} \\
& \leqq \int_0^1 [\langle \alpha_t(z_n)\,\dot{x}_n,\, \dot{x}_n\rangle\,(t_n - t_*) \\
& \quad 2\langle \delta_t(x_n,\,t_n),\, \dot{x}_n\rangle\,\dot{t}_n(t_n - t_*) + 2\langle \delta(x_n,\,t_n),\, \dot{x}_n\rangle\,(\dot{t}_n - \dot{t}_*) \\
& \quad - (\beta_\varepsilon)_t(z_n)\,\dot{t}_n^2\,(t_n - t_*)]\,ds \\
& \quad - \int_0^1 2\,\beta_\varepsilon(z_n)\,\dot{t}_n^2\;ds + 2\,(t_1 - t_0)\int_0^1 \beta_\varepsilon(z_n)\,\dot{t}_n\,ds \\
& \quad - \left(\int_0^1 \dot{t}_n^2\,ds\right) 2\,\psi'_\varepsilon\left(\int_0^1 \dot{t}_n^2\,ds\right) \\
& \quad + (t_1 - t_0)^2\, 2\,\psi'_\varepsilon\left(\int_0^1 \dot{t}_n^2\,ds\right).
\end{aligned}
$$

Now by (1.5) and (1.7) there exists $A > 0$ (independent of n) such that

$$\langle \alpha_t(z_n)\,\dot{x}_n,\, \dot{x}_n\rangle\,(t_n - t_*) \leqq A\,\langle \alpha(z_n)\,\dot{x}_n,\, \dot{x}_n\rangle\,|\,t_n - t_*\,|.$$

Moreover, by (1.5)-(1.7), there exists $D > 0$ such that

$$\|\,\delta_t(x,\,t)\,\| \leqq D \quad \text{and} \quad \|\,\delta(x,\,t)\,\| \leqq \delta_0 + D\,|\,t\,|, \quad \text{for any } (x,\,t) \in \mathcal{M},$$

where δ_0 is defined at (1.6), while by (1.5) and (1.6), there exist m_ε and M_ε such that

$$|\,(\beta_\varepsilon)_t(z)\,| \leqq M_\varepsilon \quad \text{and} \quad \beta_\varepsilon(z) \leqq m_\varepsilon \quad \text{for any } z \in \mathcal{M}.$$

Therefore

$$
\begin{aligned}
&- \varepsilon_n \| \dot{t}_n - \dot{t}_* \|_{L^2} \\
&\leqq \int_0^1 [A \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle | t_n - t_* | + 2D \| \dot{x}_n \| | \dot{t}_n | | t_n - t_* | \\
&\quad + 2(\delta_0 + D | t_n |) \| \dot{x}_n \| | \dot{t}_n - \dot{t}_* | + M_\varepsilon | \dot{t}_n^2 | | t_n - t_* |] \, ds \\
&\quad - \int_0^1 2 \beta_\varepsilon(z_n) \dot{t}_n^2 \, ds + 2 m_\varepsilon | t_1 - t_0 | \int_0^1 | \dot{t}_n | \, ds \\
&\quad - \left(\int_0^1 \dot{t}_n^2 \, ds \right) 2 \psi_\varepsilon' \left(\int_0^1 \dot{t}_n^2 \, ds \right) + (t_1 - t_0)^2 \, 2 \psi_\varepsilon' \left(\int_0^1 \dot{t}_n^2 \, ds \right) \\
&\leqq \| t_n - t_* \|_{L^\infty} \int_0^1 [A \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle \\
&\quad + 2D \| \dot{x}_n \| | \dot{t}_n | + M_\varepsilon | \dot{t}_n^2 |] \, ds \\
&\quad + \int_0^1 2(\delta_0 + D | t_n |) \| \dot{x}_n \| [| \dot{t}_n | + | t_1 - t_0 |] \, ds \\
&\quad - \int_0^1 2 \beta_\varepsilon(z_n) \dot{t}_n^2 \, ds + 2 m_\varepsilon | t_1 - t_0 | \int_0^1 | \dot{t}_n | \, ds \\
&\quad - \left(\int_0^1 \dot{t}_n^2 \, ds \right) 2 \psi_\varepsilon' \left(\int_0^1 \dot{t}_n^2 \, ds \right) \\
&\quad + (t_1 - t_0)^2 \, 2 \psi_\varepsilon' \left(\int_0^1 \dot{t}_n^2 \, ds \right).
\end{aligned}
$$

Now, by (1.3) and the Young inequality,

$$
\begin{aligned}
&\| t_n - t_* \|_{L^\infty} \int_0^1 2D \| \dot{x}_n \| | \dot{t}_n | \, ds \\
&\quad \leqq \| t_n - t_* \|_{L^\infty} \left[D \int_0^1 \| \dot{x}_n \|^2 \, ds + D \int_0^1 | \dot{t}_n |^2 \, ds \right] \\
&\quad \leqq \| t_n - t_* \|_{L^\infty} \left[D \int_0^1 \frac{1}{\lambda} \langle \alpha(z_n) \dot{x}_n, \dot{x}_n \rangle \, ds \right. \\
&\qquad\qquad \left. + D \int_0^1 | \dot{t}_n |^2 \, ds \right],
\end{aligned}
\tag{3.7}
$$

and

$$\begin{aligned}\int_0^1 & 2\,(\delta_0 + D\,|\,t_n\,|)\,\|\,\dot{x}_n\,\|\,[|\,\dot{t}_n\,| + |\,t_1 - t_0\,|]\\ &\leqq \int_0^1 \|\,\dot{x}_n\,\|^2\,ds\\ &\quad + \int_0^1 (\delta_0 + D\,|\,t_n\,|^2)\,[\,|\,\dot{t}_n\,| + |\,t_1 - t_0\,|]^2\,ds\\ &\leqq \int_0^1 \frac{1}{\lambda}\,\langle \alpha\,(z_n)\,\dot{x}_n,\,\dot{x}_n\rangle\,ds\\ &\quad + \int_0^1 (\delta_0 + D\,|\,t_n\,|)^2\,[\,|\,\dot{t}_n\,| + |\,t_1 - t_0\,|\,]^2\,ds;\end{aligned}$$

hence there exists a positive constant N_1 (independent of n) such that

$$\begin{aligned}\int_0^1 & 2\,(\delta_0 + D\,|\,t_n\,|)\,\|\,\dot{x}_n\,\|\,[\,|\,\dot{t}_n\,| + |\,t_1 - t_0\,|]\\ &\leqq \int_0^1 \frac{1}{\lambda}\,\langle \alpha\,(z_n)\,\dot{x}_n,\,\dot{x}_n\,\rangle\,ds + N_1\left[1 + \left(\int_0^1 |\,\dot{t}_n\,|^2\,ds\right)^2\right]. \qquad (3.8)\end{aligned}$$

Combining (3.6), (3.7) and (3.8) gives

$$\begin{aligned}-\,&\varepsilon_n\,\|\,\dot{t}_n - \dot{t}_*\,\|_{L^2}\\ &\leqq [(A + (D/\lambda))\,\|\,t_n - t_*\,\|_{L^\infty} + (1/\lambda)]\int_0^1 \langle\,\alpha\,(z_n)\,\dot{x}_n,\,\dot{x}_n\,\rangle\,ds\\ &\quad + [(D + M_\varepsilon)\,\|\,t_n - t_*\,\|_{L^\infty}]\int_0^1 |\,\dot{t}_n^2\,|\,ds + N_1\left[1 + \left(\int_0^1 |\,\dot{t}_n\,|^2\,ds\right)^2\right]\\ &\quad - \int_0^1 2\,\beta_\varepsilon\,(z_n)\,\dot{t}_n^2\,ds + 2\,m_\varepsilon\,|\,t_1 - t_0\,|\int_0^1 |\,\dot{t}_n\,|\,ds\\ &\quad - \left(\int_0^1 \dot{t}_n^2\,ds\right)2\,\psi_\varepsilon'\left(\int_0^1 \dot{t}_n^2\,ds\right)\\ &\quad + (t_1 - t_0)^2\,2\,\psi_\varepsilon'\left(\int_0^1 \dot{t}_n^2\,ds\right). \qquad (3.9)\end{aligned}$$

Now, by (2.11),

$$\begin{aligned}
&\int_0^1 \langle \alpha(z_n)\,\dot{x}_n,\ \dot{x}_n\rangle\, ds \\
&\quad = f_\varepsilon(z_n) - 2\int_0^1 \langle \delta(x_n,\ t_n),\ \dot{x}_n\rangle\, \dot{t}_n\, ds \\
&\qquad + \int_0^1 \beta_\varepsilon(z_n)\,\dot{t}_n^2\, ds + \psi_\varepsilon\left(\int_0^1 \dot{t}_n^2\, ds\right) \\
&\quad \leqq f_\varepsilon(z_n) + \int_0^1 2\,(\delta_0 + D\,|\,t_n\,|)\,\|\,\dot{x}_n\,\|\,|\,\dot{t}_n\,|\, ds \\
&\qquad + \int_0^1 \beta_\varepsilon(z_n)\ \dot{t}_n^2\, ds + \psi_\varepsilon\left(\int_0^1 \dot{t}_n^2\, ds\right) \\
&\quad \leqq f_\varepsilon(z_n) + \eta\int_0^1 \langle \dot{x}_n,\ \dot{x}_n\rangle\, ds + \eta^{-1}\int_0^1 (\delta_0 + D\,|\,t_n\,|)^2\, \dot{t}_n^2\, ds \\
&\qquad + \int_0^1 \beta_\varepsilon(z_n)\ \dot{t}_n^2\, ds + \psi_\varepsilon\left(\int_0^1 \dot{t}_n^2\, ds\right) \\
&\quad \leqq f_\varepsilon(z_n) + \eta\int_0^1 \langle \dot{x}_n,\ \dot{x}_n\rangle\, ds + \eta^{-1} N_2\left[1 + \left(\int_0^1 |\,\dot{t}_n\,|^2\, ds\right)^2\right] \\
&\qquad + \int_0^1 \beta_\varepsilon(z_n)\,\dot{t}_n^2\, ds + \psi_\varepsilon\left(\int_0^1 \dot{t}_n^2\, ds\right),
\end{aligned} \tag{3.10}$$

for any positive real number η and for a suitable positive constant N_2 independent of n and η.

Therefore, by (3.10), (1.3) and (3.1), choosing η such that

$$1 - \eta/\lambda = 1/2$$

gives,

$$\begin{aligned}
\int_0^1 \langle \alpha(z_n)\,\dot{x}_n,\ \dot{x}_n\rangle\, ds &\leqq 2\,C + 2\,N_2\,\eta^{-1}\left[1 + \left(\int_0^1 |\,\dot{t}_n\,|^2\, ds\right)^2\right] \\
&\quad + 2\int_0^1 \beta_\varepsilon(z_n)\,\dot{t}_n^2\, ds + 2\,\psi_\varepsilon\left(\int_0^1 \dot{t}_n^2\, ds\right).
\end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11) gives

$$
\begin{aligned}
&- \varepsilon_n \| \dot{t}_n - \dot{t}_* \|_{L^2} \\
&\leqq [(A + (D/\lambda)) \| t_n - t_* \|_{L^\infty} + (1/\lambda)] \\
&\quad \times \left[2\, C + 2\, N_2\, \eta^{-1} \left[1 + \left(\int_0^1 | \dot{t}_n |^2 \, ds \right)^2 \right] \right. \\
&\qquad \left. + 2 \int_0^1 \beta_\varepsilon (z_n)\, \dot{t}_n^2 \, ds + 2\, \psi_\varepsilon \left(\int_0^1 \dot{t}_n^2 \, ds \right) \right] \\
&\quad + [(D + M_\varepsilon) \| t_n - t_* \|_{L^\infty}] \int_0^1 | \dot{t}_n^2 | \, ds \\
&\quad + N_1 \left[1 + \left(\int_0^1 | \dot{t}_n |^2 \, ds \right)^2 \right] \\
&\quad - \int_0^1 2\, \beta_\varepsilon (z_n)\, \dot{t}_n^2 \, ds + 2\, m_\varepsilon\, | t_1 - t_0 | \int_0^1 | \dot{t}_n | \, ds \\
&\quad - \left(\int_0^1 \dot{t}_n^2 \, ds \right) 2\, \psi_\varepsilon' \left(\int_0^1 \dot{t}_n^2 \, ds \right) \\
&\quad + (t_1 - t_0)^2\, 2\, \psi_\varepsilon' \left(\int_0^1 \dot{t}_n^2 \, ds \right).
\end{aligned}
\tag{3.12}
$$

Since

$$
\| t_n - t_* \|_{L^\infty} \leqq \left(\int_0^1 | \dot{t}_n - \dot{t}_* |^2 \, ds \right)^{1/2},
$$

and β_ε is bounded, (3.12), (2.14) and (2.15) imply that $\int_0^1 \dot{t}_n^2 \, ds$ is bounded independently of n. Moreover by (1.5), (3.1), (2.11), and (1.3) also $\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds$ is bounded independently of n. □

Proposition 3.1, and standard arguments (*cf.* e.g. [1], [9]) allows to get the following

PROPOSITION 3.2. – *Under the assumptions of Proposition* 3.1, *if* (1.2) *holds, then* f_ε *satisfies* $(\text{P.S.})_c$ *for any* $c \in \mathbb{R}$ *and for any* $\varepsilon > 0$.

Now we shall prove the *a priori* estimates on the critical points of f_ε starting with an *a priori* estimate in L^∞ for the time variable.

PROPOSITION 3.3. – *Let* $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ *and* g *as in* (1.1). *Assume that* (1.2)-(1.6) *and* (1.8) *are satisfied. Let* $z = (x, t) \equiv z_\varepsilon = (x_\varepsilon, t_\varepsilon)$ *be a* *critical point of* f_ε *such that*

$$f_\varepsilon(z) \leqq C \tag{3.13}$$

where C *is a constant independent of* ε.

Then $t(s) \equiv t_\varepsilon(s)$ *is uniformly bounded on* $\varepsilon > 0$ *and* $s \in [0, 1]$.

In order to prove Proposition 3.3 the following Lemma is needed.

LEMMA 3.4. – *Let* $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *and* $L > 0$ *such that*

$$0 \leqq \varphi'(\sigma) \leqq L\,\varphi(\sigma) \quad \textit{for any } \sigma \in \mathbb{R}^+ \tag{3.14}$$

and

$$\lim_{\sigma \to +\infty} \varphi(\sigma) = +\infty. \tag{3.15}$$

Let $\bar{t} \in \mathbb{R}^+$ *and consider*

$$\Gamma(\vartheta, \varphi, \bar{t}) = \Big\{ t \in W^{1,2}([0, 1], \mathbb{R}) : \quad \textit{for any } [a, b] \subset [0, 1] \textit{ satisfying}$$
$$t(a) = \bar{t},\ t(b) = \bar{t},\ \textit{and } t(s) > \bar{t} \textit{ for any } s \in]a, b[,$$
$$\textit{the following inequality holds :}$$
$$\int_a^b \dot{t}^2(s)\,\varphi(t(s))\,ds \leqq \vartheta \int_a^b \varphi(t(s))\,ds \Big\}, \tag{3.16}$$

where ϑ *is a positive number such that*

$$L\sqrt{\vartheta} < 2. \tag{3.17}$$

Then there exists a constant $S = S(\varphi, L, \vartheta, \bar{t})$ *such that*

$$\sup\{t(s) : s \in [0, 1], \quad t \in \Gamma(\vartheta, \varphi, \bar{t})\} \leqq S. \tag{3.18}$$

Proof. – Set, for any $t \in \Gamma(\vartheta, \varphi, \bar{t})$,

$$S(t) = \sup\{t(s) : s \in [0, 1]\}.$$

Assume, by contradiction, that there exists $\{t_n\}_{n \in \mathbb{N}} \subset \Gamma(\vartheta, \varphi, \bar{t})$ such that

$$S(t_n) \underset{n}{\to} +\infty, \tag{3.19}$$

and let $[a_n, b_n]$ be an interval satisfying the properties in (3.16). By (3.14), Hölder inequality and (3.16),

$$\begin{aligned}
\sqrt{\varphi(S(t_n))} - \sqrt{\varphi(\bar{t})} &= \sqrt{\varphi(S(t_n))} - \sqrt{\varphi((t_n(a_n))} \\
&\leqq \frac{1}{2} \int_{a_n}^{b_n} \varphi^{-1/2}(t_n(s))(\varphi_n'(t_n(s)))\,|\,\dot{t}_n(s)\,|\,ds \\
&\leqq \frac{1}{2} L \int_{a_n}^{b_n} \varphi^{1/2}(t_n(s))\,|\,\dot{t}_n(s)\,|\,ds \\
&\leqq \frac{1}{2} L(b_n - a_n)^{1/2} \left(\int_{a_n}^{b_n} \varphi(t_n(s))\,|\,\dot{t}_n(s)\,|^2\,ds\right)^{1/2} \\
&\leqq \frac{1}{2} L \left(\int_{a_n}^{b_n} \varphi(t_n(s))\,|\,\dot{t}_n(s)\,|^2\,ds\right)^{1/2} \\
&\leqq \frac{1}{2} L\sqrt{\vartheta} \left(\int_{a_n}^{b_n} \varphi(t_n(s))\,ds\right)^{1/2} \leqq \frac{1}{2} L\sqrt{\vartheta}\cdot\sqrt{\varphi(S(t_n))}.
\end{aligned}$$

Therefore, by (3.19) and (3.17) we get a contradiction, proving (3.18). □

Proof of Proposition 3.3. – If $z = (x, t) \equiv z_\varepsilon = (x_\varepsilon, t_\varepsilon)$ is a critical point of f_ε, it satisfies the differential equation

$$D_s\,\dot{z} - \psi_\varepsilon'\left(\int_0^1 \dot{t}^2\,ds\right)(0,\ \ddot{t}) = 0, \tag{3.20}$$

where 0 is the null vector in $T_x\,\mathcal{M}_0$ and $\ddot{t}$ is the second derivative of t. Multiplying by $\dot{z}$ and integrating give the existence of $E_z \in \mathbb{R}$ such that

$$\langle \dot{z},\ \dot{z}\rangle - \dot{t}^2\,\psi_\varepsilon'\left(\int_0^1 \dot{t}^2\,ds\right) = E_z, \tag{3.21}$$

where

$$\langle \dot{z},\ \dot{z}\rangle = \langle \alpha(x,\ t)\,\dot{x},\ \dot{x}\rangle + 2\,\langle \delta(x,\ t),\ \dot{x}\rangle\,\dot{t} - \beta_\varepsilon(x,\ t)\,\dot{t}^2.$$

Moreover, integrating (3.21) in the interval [0, 1], give

$$E_z = f_\varepsilon(z) - \left(\int_0^1 \dot{t}^2\,ds\right)\psi_\varepsilon'\left(\int_0^1 \dot{t}^2\,ds\right) + \psi_\varepsilon\left(\int_0^1 \dot{t}^2\,ds\right). \tag{3.22}$$

By (1.8), for any $\mu > 0$, there exists $t_+ = t_+(\mu) > \max(t_0, t_1)$ such that

$$\begin{aligned}
\langle \alpha_t(x,\ t)\ \xi,\ \xi\rangle_x \leqq \mu\,\langle \alpha(x,\ t)\,\xi,\ \xi\rangle_x \quad & \text{for any } x \in \mathcal{M}_0,\ \xi \in T_x\,\mathcal{M}_0,\ t \geqq t_+, \\
\|\,\delta_t(x,\ t)\,\| \leqq \mu \quad & \text{for any } x \in \mathcal{M}_0,\ t \geqq t_+, \\
-\beta_t(x,\ t) \leqq \mu\beta(x,\ t) \quad & \text{for any } x \in \mathcal{M}_0,\ t \geqq t_+.
\end{aligned} \tag{3.23}$$

If, by contradiction, t_ε is not uniformly bounded from above, there exist a critical point $(x, t) \equiv (x_\varepsilon, t_\varepsilon)$ of f_ε and an interval $I \subset [0, 1]$, such that

$$t(s) \geqq t_+ \quad \text{for any } s \in I, \quad \text{and} \quad t_{|\partial I} = t_+. \tag{3.24}$$

Now, by (2.9),

$$\begin{aligned}(\beta_\varepsilon)_t(x, t) &= \chi'_\varepsilon(t)(b_\varepsilon - \beta(x, t)) + (1 - \chi_\varepsilon(t))\beta_t(x, t)\\ &\geqq (1 - \chi_\varepsilon(t))\beta_t(x, t).\end{aligned}$$

Then putting, in (3.3),

$$\tau(s) = \begin{Bmatrix} \sinh(\omega(t(s) - t_+)) & \text{if} \quad s \in I \\ 0 & \text{if} \quad s \notin I, \end{Bmatrix} \tag{3.25}$$

(where $\omega \in \mathbb{R}^+ \backslash \{0\}$), (3.23) give

$$\begin{aligned}0 \leqq & \int_I [\mu\langle\alpha(z)\dot{x}, \dot{x}\rangle|\tau| + 2\mu\|\dot{x}\||\dot{t}||\tau| + \mu\beta_\varepsilon\dot{t}^2|\tau|]\,ds\\ &+ \omega\int_I 2\langle\delta(z), \dot{x}\rangle\dot{t}\cosh(\omega(t - t_+))\,ds\\ &- 2\omega\int_I \beta_\varepsilon(z)\dot{t}^2\cosh(\omega(t - t_+))\,ds\\ &- 2\omega\left(\int_I \dot{t}^2\cosh(\omega(t - t_+))\,ds\right)\psi'_\varepsilon\left(\int_0^1 \dot{t}^2\,ds\right).\end{aligned} \tag{3.26}$$

Now, by (3.21) and (3.22),

$$\begin{aligned}2\langle\delta(z), \dot{x}\rangle\dot{t} = f_\varepsilon(z) &- \left(\int_0^1 \dot{t}^2\,ds\right)\psi'_\varepsilon\left(\int_0^1 \dot{t}^2\,ds\right)\\ &+ \psi_\varepsilon\left(\int_0^1 \dot{t}^2\,ds\right) + \dot{t}^2\psi'_\varepsilon\left(\int_0^1 \dot{t}^2\,ds\right)\\ &- \langle\alpha(z)\dot{x}, \dot{x}\rangle + \beta_\varepsilon(z)\dot{t}^2.\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27) gives

$$
\begin{aligned}
0 \leqq & \int_I [\mu \langle \alpha(z)\dot{x},\, \dot{x}\rangle \,|\tau| + 2\mu \|\dot{x}\|\,|\dot{t}|\,|\tau| + \mu \beta_\varepsilon \dot{t}^2 |\tau|]\, ds \\
& + \omega f_\varepsilon(z) \int_I \cosh(\omega(t - t_+))\, ds \\
& + \omega \left[-\left(\int_0^1 \dot{t}^2\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right) \right. \\
& \qquad \left. + \psi_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right)\right] \int_I \cosh(\omega(t - t_+))\, ds \\
& + \omega \psi'_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right) \left(\int_I \dot{t}^2 \cosh(\omega(t - t_+))\, ds\right) \\
& - \omega \int_I \langle \alpha(z)\dot{x},\, \dot{x}\rangle \cosh(\omega(t - t_+)) \\
& + \omega \int_I \beta_\varepsilon(z)\, \dot{t}^2 \cosh(\omega(t - t_+)) \\
& - 2\omega \int_0^1 \beta_\varepsilon(z)\, \dot{t}^2 \cosh(\omega(t - t_+))\, ds \\
& - 2\omega \left(\int_I \dot{t}^2 \cosh(\omega(t - t_+))\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right) \\
= & \int_I [\mu \langle \alpha(z)\dot{x},\, \dot{x}\rangle \,|\tau| + 2\mu \|\dot{x}\|\,|\dot{t}|\,|\tau| + \mu \beta_\varepsilon \dot{t}^2 |\tau|]\, ds \\
& + \omega f_\varepsilon(z) \int_I \cosh(\omega(t - t_+))\, ds \\
& + \omega \left[-\left(\int_0^1 \dot{t}^2\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right) \right. \\
& \qquad \left. + \psi_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right)\right] \int_I \cosh(\omega(t - t_+))\, ds \\
& + \omega \psi'_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right) \left(\int_I \dot{t}^2 \cosh(\omega(t - t_+))\, ds\right), \\
& - \omega \int_I \langle \alpha(z)\dot{x},\, \dot{x}\rangle \cosh(\omega(t - t_+)) \\
& - \omega \int_I \beta_\varepsilon(z)\, \dot{t}^2 \cosh(\omega(t - t_+))\, ds \\
& - 2\omega \left(\int_I \dot{t}^2 \cosh(\omega(t - t_+))\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot{t}^2\, ds\right).
\end{aligned}
$$

Therefore, by the choice of ψ_ε [*cf.* (2.14)-(2.16)],

$$\begin{aligned}0 \leqq \int_I &[\mu \langle \alpha(z)\dot{x},\, \dot{x}\rangle \,|\,\tau\,| + 2\,\mu\,\|\dot{x}\|\,|\,\dot{t}\,|\,|\,\tau\,| + \mu\,\beta_\varepsilon\,\dot{t}^2\,|\,\tau\,|]\,ds \\ &+ \omega\, f_\varepsilon(z) \int_I \cosh(\omega(t - t_+))\,ds \\ &- \omega \int_I \langle \alpha(z)\dot{x},\, \dot{x}\rangle \cosh(\omega(t - t_+)) \\ &- \omega \int_I \beta_\varepsilon(z)\,\dot{t}^2 \cosh(\omega(t - t_+))\,ds. \end{aligned} \tag{3.28}$$

Moreover, by (1.3)

$$\begin{aligned}\int_I &2\,\mu\,\|\,\dot{x}_n\,\|\,|\,\dot{t}_n\,|\,|\,\tau\,|\,ds \\ &\leqq \mu \int_I \frac{1}{\lambda}\langle \alpha(z_n)\dot{x}_n,\, \dot{x}_n\rangle\,|\,\tau\,|\,ds + \mu \int_I |\,\dot{t}_n\,|^2\,|\,\tau\,|\,ds. \end{aligned} \tag{3.29}$$

Then, since

$$|\,\tau(s)\,| \leqq \cosh(\omega(t(s) - t_+(s))) \quad \text{for any } s \in [0,\, 1],$$

choosing ω such that

$$\omega \geqq (\mu + \mu/\lambda),$$

(3.28), and (3.29) give

$$\begin{aligned}0 \leqq \int_I &\mu\,\dot{t}^2 \cosh(\omega(t - t_+))\,ds \\ &+ \omega\, f_\varepsilon(z) \int_I \cosh(\omega(t - t_+))\,ds \\ &- \int_I [\omega - \mu]\,\beta_\varepsilon(z)\,\dot{t}^2 \cosh(\omega(t - t_+))\,ds,\end{aligned}$$

hence by (3.13), (1.4), and the choice of β_ε [*cf.* (2.9)], if

$$\omega/2 \geqq \mu \qquad \text{and} \qquad b\,\omega/4 \geqq \mu,$$

we have

$$\int_I \dot{t}^2 \cosh(\omega(t - t_+)) \leqq (4\,C/b) \int_I \cosh(\omega(t - t_+))\,ds. \tag{3.30}$$

Finally by (1.8), μ and ω can be chosen so small that

$$\omega\,(4\,C/b)^{1/2} < 2,$$

hence, by Lemma 3.4, $t \equiv t_\varepsilon$ is uniformly bounded from above (independently of ε). Analogously the uniformly boundeness from below can be obtained using the analogous of Lemma 3.4 in $\mathbb{R}^-$, concluding the proof of Proposition 3.3. □

Remark 3.5. – By the proof of Proposition 3.3 it turns out that the *a priori* estimate in L^∞ for the timelike geodesics requires only assumptions (1.7) instead of (1.8), because, in this case, $C < 0$.

Moreover, about the geodesic connectedness, it is clear that (1.8) can be weakened, asking that the constant in (1.8) are small with respect to the smallest critical level C of the energy functional.

PROPOSITION 3.6. – *Let* $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ *and* g *as in* (1.1). *Assume that* (1.2)-(1.6) *are satisfied. Let* $z = (x,\, t) \equiv z_\varepsilon = (x_\varepsilon,\, t_\varepsilon)$ *be a critical point of* f_ε *such that*

$$f_\varepsilon\,(z_\varepsilon) \leqq C \tag{3.31}$$

where C *is a constant independent of* ε, *and*

$$t_\varepsilon \text{ is uniformly bounded (independently of } \varepsilon). \tag{3.32}$$

Then $\int_0^1 \langle \dot{x}_\varepsilon,\, \dot{x}_\varepsilon \rangle\, ds$, *and* $\int_0^1 \dot{t}_\varepsilon^2\, ds$ *are bounded independently of* ε.

Proof. – Consider

$$\tau = \sinh\,(\omega\,(t - t_*)), \tag{3.33}$$

where t_* is defined by (3.4) and $\omega \in \mathbb{R}^+\backslash\{0\}$ will be chosen later. By (3.32), (1.5), (1.6), and the definition of β_ε [*cf.* (2.9)], choosing in formula (3.3) τ as defined in (3.33), give the existence of A, D_1, $B_1 \in \mathbb{R}^+$ such that

$$\begin{aligned} 0 \leqq & \int_0^1 [A\,\langle \alpha\,(z)\,\dot{x},\, \dot{x}\rangle\,|\,\tau\,| + 2\,D_1\,\|\,\dot{x}\,\|\,|\,\dot{t}\,|\,|\tau\,| + B_1\,\dot{t}^2\,\tau]\,ds \\ & + \omega \int_0^1 2\,\langle \delta\,(z),\, \dot{x}\rangle\,\dot{t}\cosh\,(\omega\,(t - t_*))\,ds \\ & - 2\,\omega\,(t_1 - t_0) \int_0^1 \langle \delta\,(z),\, \dot{x}\rangle \cosh\,(\omega\,(t - t_*))\,ds \end{aligned}$$

$$
\begin{aligned}
&- 2\,\omega \int_0^1 \beta(z)\, \dot t^2 \cosh(\omega(t - t_*))\, ds \\
&+ 2\,\omega\,(t_1 - t_0) \int_0^1 \beta(z)\, \dot t \cosh(\omega(t - t_*))\, ds \\
&- 2\,\omega \left(\int_0^1 \dot t^2 \cosh(\omega(t - t_*))\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot t^2\, ds\right) \\
&+ 2\,\omega\,(t_1 - t_0) \left(\int_0^1 \dot t \cosh(\omega(t - t_*))\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot t^2\, ds\right). \qquad (3.34)
\end{aligned}
$$

Combining (3.27) and (3.34) gives

$$
\begin{aligned}
0 \leqq & \int_0^1 [A \langle \alpha(z)\, \dot x,\, \dot x\rangle\, |\tau| + 2\, D_1\, \|\dot x\|\, |\dot t|\, |\tau| + B_1\, \dot t^2\, \tau]\, ds \\
&+ \omega f_\varepsilon(z) \int_0^1 \cosh(\omega(t - t_*))\, ds \\
&+ \omega \left[-\left(\int_0^1 \dot t^2\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot t^2\, ds\right) \right. \\
&\qquad \left. + \psi_\varepsilon \left(\int_0^1 \dot t^2\, ds\right)\right] \int_0^1 \cosh(\omega(t - t_*))\, ds \\
&+ \omega \psi'_\varepsilon \left(\int_0^1 \dot t^2\, ds\right) \left(\int_0^1 \dot t^2 \cosh(\omega(t - t_*))\, ds\right) \\
&- \omega \int_0^1 \langle \alpha(z)\, \dot x,\, \dot x\rangle \cosh(\omega(t - t_*)) \\
&- 2\,\omega\,(t_1 - t_0) \int_0^1 \langle \delta(z),\, \dot x\rangle \cosh(\omega(t - t_*))\, ds \\
&- \omega \int_0^1 \beta(z)\, \dot t^2 \cosh(\omega(t - t_*))\, ds \\
&+ 2\,\omega\,(t_1 - t_0) \int_0^1 \beta(z)\, \dot t \cosh(\omega(t - t_*))\, ds \\
&- 2\,\omega \left(\int_0^1 \dot t^2 \cosh(\omega(t - t_*))\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot t^2\, ds\right) \\
&+ 2\,\omega\,(t_1 - t_0) \left(\int_0^1 \dot t \cosh(\omega(t - t_*))\, ds\right) \psi'_\varepsilon \left(\int_0^1 \dot t^2\, ds\right). \qquad (3.35)
\end{aligned}
$$

Since $\dot{t} - \dot{t}_* = \dot{t} - (t_1 - t_0)$,

$$\int_0^1 \dot{t}\cosh(\omega(t-t_*))\,ds = \int_0^1 (\dot{t}-\dot{t}_*)\cosh(\omega(t-t_*))\,ds + (t_1-t_0)\int_0^1 \cosh(\omega(t-t_*))\,ds = (t_1-t_0)\int_0^1 \cosh(\omega(t-t_*))\,ds. \quad (3.36)$$

Obviously we can assume that $\int_0^1 \dot{t}^2\,ds$ is large as we want. Then, (3.14), (3.35) and (3.36) give

$$\begin{aligned} 0 \leqq & \int_0^1 [A\langle\alpha(z)\dot{x},\,\dot{x}\rangle\,|\tau| + 2D_1\|\dot{x}\|\,|\dot{t}|\,|\tau| + B_1\dot{t}^2\tau]\,ds \\ & + \omega f_\varepsilon(z)\int_0^1 \cosh(\omega(t-t_*))\,ds \\ & - \omega\int_0^1 \langle\alpha(z)\dot{x},\,\dot{x}\rangle\cosh(\omega(t-t_*))\,ds \\ & - 2\omega(t_1-t_0)\int_0^1 \langle\delta(z),\,\dot{x}\rangle\cosh(\omega(t-t_*))\,ds \\ & - \omega\int_0^1 \beta(z)\,\dot{t}^2\cosh(\omega(t-t_*))\,ds \\ & + 2\omega(t_1-t_0)\int_0^1 \beta(z)\,\dot{t}^2\cosh(\omega(t-t_*))\,ds. \end{aligned} \quad (3.37)$$

Moreover by (3.32), (1.5), (1.6) and (1.3), there exists a positive constant D_2 such that

$$\|\delta(x,\,t)\| \leqq D_2,$$

$$\begin{aligned} & \left| 2\omega(t_1-t_0)\int_0^1 \langle\delta(z),\,\dot{x}\rangle\cosh(\omega(t-t_*))\,ds \right| \\ & \leqq 2\omega\,|t_1-t_0|\,D_2\int_0^1 \langle\dot{x},\,\dot{x}\rangle^{1/2}\cosh(\omega(t-t_*))\,ds \\ & \leqq \omega\,|t_1-t_0|\,D_2(\eta_1/\lambda)\int_0^1 \langle\alpha(z)\dot{x},\,\dot{x}\rangle\cosh(\omega(t-t_*))\,ds \\ & \quad + \omega\,|t_1-t_0|\,D_2\,\eta_1^{-1}\int_0^1 \cosh(\omega(t-t_*))\,ds, \end{aligned} \quad (3.38)$$

for any $\eta_1 > 0$, while by (1.3)

$$\int_0^1 2\,D_1\,\|\dot{x}_n\|\,|\,\dot{t}_n\,|\,|\,\tau\,|\,ds$$
$$\leqq D_1 \int_0^1 \frac{1}{\lambda}\langle\alpha(z_n)\,\dot{x}_n,\,\dot{x}_n\rangle\,|\,\tau\,|\,ds + D_1 \int_0^1 |\,\dot{t}_n\,|^2\,|\,\tau\,|\,ds. \quad (3.39)$$

Then, since

$$|\,\tau(s)\,| \leqq \cosh(\omega(t(s) - t_*(s))) \quad \text{for any } s \in [0,\,1],$$

choosing η_1 such that

$$|\,t_1 - t_0\,|\,D_2(\eta_1/\lambda) = 1/4$$

and ω such that

$$\omega \geqq 4(D_1/\lambda + A),$$

(3.37), (3.38) and (3.39) give

$$\begin{aligned} 0 \leqq & \int_0^1 (D_1 + B_1)\,\dot{t}^2 \cosh(\omega(t - t_*))\,ds \\ & + \omega f_\varepsilon(z) \int_0^1 \cosh(\omega(t - t_*))\,ds \\ & + \omega\,|\,t_1 - t_0\,|\,D_2\,\eta_1^{-1} \int_0^1 \cosh(\omega(t - t_*))\,ds \\ & - \omega \int_0^1 \beta(z)\,\dot{t}^2 \cosh(\omega(t - t_*))\,ds \\ & + 2\,\omega(t_1 - t_0) \int_0^1 \beta(z)\,\dot{t}^2 \cosh(\omega(t - t_*))\,ds. \end{aligned} \quad (3.40)$$

Now, by (3.32), (1.5) and (1.6) there exists a positive constant B_2 such that

$$\beta(x,\,t) \leqq B_2 \quad \text{for any } (x,\,t),$$

therefore,

$$\left|\, 2\,\omega(t_1 - t_0) \int_0^1 \beta(z)\,\dot{t} \cosh(\omega(t - t_*))\,ds \,\right|$$
$$\leqq \omega\,|\,t_1 - t_0\,|\,B_2 \int_0^1 (\eta_2\,\dot{t}^2 + \eta_2^{-1}) \cosh(\omega(t - t_*))\,ds, \quad (3.41)$$

for any $\eta_2 > 0$, and, choosing η_2 such that

$$|t_1 - t_0| B_2 \eta_2 = b/4,$$

[*cf.* (1.4)], and ω such that

$$\omega \geqq 4(D_1 + B_1)/b$$

combining (1.4), (3.40) and (3.41), gives

$$\begin{aligned} 0 \leqq \omega f_\varepsilon(z) & \int_0^1 \cosh(\omega(t - t_*))\, ds \\ & + \omega |t_1 - t_0| D_2 \eta_1^{-1} \int_0^1 \cosh(\omega(t - t_*))\, ds \\ & - (\omega b)/2 \int_0^1 \dot{t}^2 \cosh(\omega(t - t_*))\, ds \\ & + \omega |t_1 - t_0| B_2 \eta_2^{-1} \int_0^1 \cosh(\omega(t - t_*))\, ds. \end{aligned}$$

Therefore by (3.31) there exists a real constant

$$K = K(C, |t_1 - t_0|, A, D_1, D_2, B_1, B_2, b, \lambda) > 0$$

such that

$$\int_0^1 \dot{t}^2 \cosh(\omega(t - t_*))\, ds \leqq K \int_0^1 \cosh(\omega(t - t_*))\, ds. \tag{3.42}$$

Finally, by (3.32) and (3.42), since $\cosh(\sigma) \geqq 1$ for any σ, there exist a constant H independently of ε such that

$$\int_0^1 \dot{t}^2\, ds \leqq \int_0^1 \dot{t}^2 \cosh(\omega(t - t_*))\, ds \leqq H. \tag{3.43}$$

Moreover, (2.11), (3.32), (1.3), (1.5), (1.6), (3.33) and (3.43) imply also that $\int_0^1 \langle \dot{x}, \dot{x} \rangle\, ds$ is bounded independently of ε. □

Remark 3.7. – By remark 3.5 and Proposition 3.6, the a priori estimates in $H^{1,2}$ for timelike geodesics can be obtained assuming only (1.7) instead of (1.8).

Remark 3.8. – By the choice of ψ_ε [*cf.* (2.10)], under the assumptions of Proposition 3.6, if z_ε is a critical point of f_ε and ε is sufficiently small, then z_ε is a critical point of f.

4. PROOF OF THEOREMS 1.1 AND 1.2

Since the functional f_ε is unbounded both from below and from above, to overcome these difficulties we shall use a finite dimensional approximation on the space of the time variable.

For $k \in \mathbb{N}$ we set

$$\mathcal{Z}_k = \Omega^1 \times W_k, \qquad W_k = (t_* + W_{k,0}), \tag{4.1}$$

where $t_*(s) = t_0 + (t_1 - t_0)\, s$ and $W_{k,0} = \text{span}\,\{\sin(\pi q s),\ q = 1, \ldots, k\}$.

Following the proof of Proposition 3.1 and the ideas of the proof of Lemma (3.4) of [1] give the following result which allows us to look for critical points of f_ε on a manifold which is finite dimensional in the variable t.

LEMMA 4.1. – *For any $k \in \mathbb{N}$ let $z_k \in \mathcal{Z}_k$ be a critical point of $f_{\varepsilon|\mathcal{Z}_k}$. Assume that there exists $c_1, c_2 \in \mathbb{R}$, independent of k such that*

$$c_1 \leqq f_\varepsilon(z_k) \leqq c_2.$$

Then $\{z_k\}_{k\in\mathbb{N}}$ contains a subsequence which converges in $\mathcal{Z}$ to a critical point z of f_ε and

$$c_1 \leqq f_\varepsilon(z) \leqq c_2.$$

Remark 4.2. – By the same proof of Propositions 3.1 and 3.2 we see that for all $c \in \mathbb{R}$ and for all $k \in \mathbb{N}$, $f_{\varepsilon|\mathcal{Z}_k}$ satisfies $(\text{P.S.})_c$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. – Fix $k \in \mathbb{N}$ and put $f_\varepsilon^k = f_{\varepsilon|\mathcal{Z}_k}$ where $\mathcal{Z}_k$ is defined by (4.1). By assumption (1.10) there exists $\bar{x} \in \Omega^1$, $\lambda_1, \lambda_2 > 0$ (independent of $k \in \mathbb{N}$ and $\varepsilon \in]0, \varepsilon_0]$, $\varepsilon_0 > 0$) such that

$$\sup_{t \in W_k} f_\varepsilon^k(\bar{x}, t) \leqq -\lambda_1 < 0, \tag{4.2}$$

while

$$\inf_{x \in \Omega^1} f_\varepsilon^k(x, t_*) \geqq -\lambda_2. \tag{4.3}$$

Moreover by (2.11) and assumption (1.9), there exist an open subset Λ_k of W_k such that $t_* \in \Lambda_k$ and

$$\sup_{t \in \partial\Lambda_k} f_\varepsilon(\bar{x}, t) < -\lambda_2 \tag{4.4}$$

where λ_2 is defined at (4.3).

Then assumption (i) of Lemma 2.3 is satisfied with $e = t_*$, $b_1 < -\lambda_2$, $b_2 \geqq -\lambda_2$ and $U = \Lambda_k$.

Moreover, by Remark 4.2, also assumption (ii) of Lemma 2.3 is satisfied. Then, by Lemma 2.3, for all $k \in \mathbb{N}$, f_ε^k has a critical point $z_k = (x_k, t_k)$ in $\mathcal{Z}_k$ such that

$$-\lambda_2 \leqq f_\varepsilon (x_k, t_k) \leqq -\lambda_1 < 0$$

with λ_1 and λ_2 independent of k and ε.

Therefore, by Lemma 4.1, f_ε has as critical point z_ε in $\mathcal{Z}$ such that

$$-\lambda_2 \leqq f_\varepsilon (z_\varepsilon) \leqq -\lambda_1 < 0.$$

Finally, Remarks 3.7 and 3.8 gives that, for ε small enough, z_ε is a critical point of f, *i.e.* a geodesic joining the given events z_0 and z_1, such that

$$f (z_\varepsilon) \leqq -\lambda_1 < 0.$$

This concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. – The proof of Theorem 1.2 is the same of Theorem 1.1 taking account of Propositions 3.3, 3.6 and Remark 3.8. □

5. PROOF OF THEOREMS 1.3 AND 1.4

In order to get the multiplicity results we use the concept of Relative Category (*cf.* [4], [6], [7], [17]), which is an extension of the classical Lusternik and Schnirelmann category (*cf.* e.g. [16]). We recall the definition for the convenience of the reader.

DÉFINITION 5.1. – *Let X be a topological space, A, Y subsets of X, $A \neq \emptyset$. The relative category of A in X, with respect to Y [denoted by $\mathrm{cat}_{X,Y}(A)$] is the smallest integer p (possibly $+\infty$) such that $A \subset \bigcup_{i=0}^{p} A_i$ where the A_i's have the following properties:*

$$for\ any\ i = 1, \ldots, p, \quad A_i\ is\ closed\ and\ contractible\ in\ X; \tag{5.1}$$

$$A_0 \supset Y \cap A\ and\ A_0\ is\ closed; \tag{5.2}$$

there exists $h \in C^0([0, 1] \times A_0, X)$ *such that*

$$\text{(i)} \qquad h(0, x) = x \quad \textit{for any } x \in A_0,$$

$$\text{(ii)} \qquad h(1, x) \in Y \quad \textit{for any } x \in A_0,$$

$$\text{(iii)} \qquad h(s, y) = y \quad \textit{for any } s \in [0, 1], \quad y \in Y \cap A_0. \qquad (5.3)$$

Remark 5.2. – If Y is a strong deformation retract of A, then $\text{cat}_{X, Y}(A) = 0$.

Remark 5.3. – If $Y = \emptyset$, $\text{cat}_{X, Y}(A)$ is the Lusternik and Schnirelmann category of A in X.

Remark 5.4. – The use of Relative Category, together with a Galerkin approximation argument is used in [6] to get interesting results about multiplicity results of critical points for strongly indefinite functionals. Unfortunately, to prove Theorem 1.3, we can not directly apply Theorem 6.3 of [6] for four reasons. First in our case assumption 6.3 *b*) does not hold. Secondly the presence of the penalization term in (2.11) requires estimates independently of ε on the critical levels. Moreover we also need a sequence of critical values of the functional (2.1) going to $+\infty$ and we can not, in general, use the cuplenght of Ω^1 to estimate the relative category. Indeed, whenever $\mathcal{M}_0$ is not contractible it is not known, in general, if the cuplenght of Ω^1 is infinite.

For any $k \in \mathbb{N}$, we use the relative category on the manifold

$$\mathcal{Z}_k = \Omega^1 \times W_k$$

[*cf.* (4.1)], to get multiple critical levels for the functional

$$f_\varepsilon^k = f_{\varepsilon|\mathcal{Z}_k}. \qquad (5.4)$$

Fix $k \in \mathbb{N}$ and, for any $R > 0$, put

$$B_k(R) = \{t \in W_k : \|t - t_*\|_{W^{1,2}} \geqq R\},$$

[where t_* is defined at (4.1)]. Moreover, for any $\mu \in \mathbb{R}$, put

$$(f_\varepsilon^k)^\mu = \{z \in \mathcal{Z}_k : f_\varepsilon^k(z) \leqq \mu\}.$$

Assumptions (1.3), (1.5) and (1.6) imply that

$$d \equiv \inf\{f_\varepsilon^k(x, t_*) : x \in \Omega^1,\ \varepsilon \in]0, 1],\ k \in \mathbb{N}\} > -\infty. \qquad (5.5)$$

The constant d plays a crucial role in the proofs of Theorems 1.3 and 1.4, and, in particular in the definition of the following class of subsets of $\mathcal{Z}_k$.

For any $m \in \mathbb{N}$, $k \in \mathbb{N}$, $R > 0$ and $\varepsilon \in]0, 1]$, put

$$\Gamma_m^k \equiv \Gamma_m^k (R, \varepsilon) = \{B \subset \mathcal{Z}_k : B \text{ is closed},$$
$$B \cap (\Omega^1 \times B_k (R)) \neq \emptyset,$$
$$\sup f_\varepsilon^k (B \cap (\Omega^1 \times B_k (R))) \leqq d - 1, \tag{5.6}$$
$$\operatorname{cat}_{\mathcal{Z}_n, \Omega^1 \times B_k (R)} (B) \geqq m\}. \tag{5.7}$$

The following Lemma holds:

LEMMA 5.5. – *Assume $\mathcal{M}_0$ non contractible and 1-connected. Then for any $m \in \mathbb{N}$, there exists a compact subset K_m of $\mathcal{Z}_k$ and $R_m > 0$ such that*

$$K_m \in \Gamma_m^k (R, \varepsilon) \quad \textit{for any } R \geqq R_m, \qquad k \in \mathbb{N}, \quad \varepsilon \in]0, 1]. \tag{5.8}$$

Proof. – Fix $m \in \mathbb{N}$. Let D^k and S^k be the unit disk in W_k and its boundary respectively. Since $\mathcal{M}_0$ is not contractible and 1-connected, by a recent of Fadell and Husseini (*cf.* [5]), there exists C_m, compact subset of $\Omega^1 \times D^k$ such that

$$\operatorname{cat}_{\Omega^1 \times D^k, \Omega^1 \times S^k} (C_m) \geqq m. \tag{5.9}$$

Now let $K_m = K_m (R) = \{(x, t_* + R (t - t_*)) : (x, t) \in C_m\}$. By (5.9)

$$\operatorname{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k (R)} (K_m) \geqq m. \tag{5.10}$$

Moreover, by assumptions (1.4) and (1.9) and the compactness of C_m, there exists R_m (independent of k and ε) such that

$$\sup f_\varepsilon^k (K_m (R) \cap (\Omega^1 \times B_k (R))) \leqq d - 1$$

$$\text{for any } R \geqq R_m, \qquad k \in \mathbb{N}, \quad \varepsilon \in]0, 1],$$

concluding the proof of lemma 5.5. □

We shall need the following Lemma about the invariance of the class $\Gamma_m^k (R, \varepsilon)$ with respect to the flow generated by the curves of maximal slope of the functional f_ε.

LEMMA 5.6. – *Let $\Gamma_m^k (R, \varepsilon) \neq \emptyset$ and c be a regular value for f, $c \geqq d$* [*cf.* (5.5)].

Then there exists $\sigma_0 = \sigma_0(c) > 0$ *such that for any* $\sigma \in [0, \sigma_0]$ *there exists a homeomorphism* $\Phi_\sigma : \mathcal{Z}_k \to \mathcal{Z}_k$, *satisfying*

$$\Phi_\sigma(\{f_\varepsilon(z) \leqq c + \sigma\}) \subset \{f_\varepsilon(z) \leqq c - \sigma\}, \tag{5.11}$$

$$\Phi_\sigma(z) = z \quad \text{if} \quad f_\varepsilon(z) \notin [c - 2\sigma, c + 2\sigma], \tag{5.12}$$

and,

$$\Phi_\sigma(x) = x \quad \text{for any } x \in \{f_\varepsilon(z) \leqq d - 1\} \tag{5.13}$$

Moreover, for any $B \in \Gamma_m^k(R, \varepsilon)$,

$$\operatorname{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(\Phi_\sigma(B)) \geqq \operatorname{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(B). \tag{5.14}$$

Proof. – Since c is a regular value for f_ε and $(\mathrm{P.S.})_c$ holds, there exists $\sigma_0 \in]0, 1/4[$, such that $[c - \sigma_0, c + \sigma_0]$ consists of regular values for f_ε.

Now let $\sigma \in]0, \sigma_0[$ and let Φ_σ be the homeomorphism given by the solution (at the instant 2σ) of the Cauchy problem

$$\begin{cases} \dot{\eta}_\sigma = -\chi_\sigma(f_\varepsilon(\eta_\sigma))\, f'_\varepsilon(\eta_\sigma)/\| f'_\varepsilon(\eta_\sigma)\|^2 \\ \eta_\sigma(0) = x, \end{cases} \tag{5.15}$$

[*i.e.* $\Phi_\sigma(z) = \eta_\sigma(2\sigma, z)$], where χ_σ is a Lipschitz continuous real function with values on [0, 1], such that $\chi_\sigma(s) = 1$ if $s \in [c-\sigma, c+\sigma]$ and $\chi_\sigma(s) = 0$ if $s \notin [c - 2\sigma, c + 2\sigma]$ (*cf.* [15]). Clearly Φ_σ satisfies (5.11)-(5.13).

It remains to prove (5.14). To this aim let $A_0, \ldots, A_k$ be closed sets covering $\Phi_\sigma(B)$, satisfying (5.1)-(5.3), and consider $B_0 = \Phi_\sigma^{-1}(A_0) \cap B, \ldots, B_k = \Phi_\sigma^{-1}(A_k) \cap B$. Clearly, $B_0, \ldots, B_k$ are closed subsets covering B and satisfying (5.1)-(5.2). Then it remains to prove that B_0 satisfies (5.3).

Towards this goal, note that by (5.13) and (5.6), (setting $Y = \Omega^1 \times B_k(R)$),

$$\begin{aligned} \Phi_\sigma^{-1}(A_0) \cap B \cap Y &= \Phi_\sigma^{-1}(A_0) \cap \Phi_\sigma^{-1}(B \cap Y) \\ &= \Phi_\sigma^{-1}(A_0 \cap B \cap Y) = A_0 \cap B \cap Y. \end{aligned} \tag{5.16}$$

Then, if $h : [0, 1] \times A_0 \to \mathcal{Z}_k$ is the homotopy satisfying (5.3) relatively to A_0, $\tilde{h} : [0, 1] \times B_0 \to \mathcal{Z}_k$, defined by

$$\tilde{h}(s, z) := \begin{cases} \eta_\sigma(4\sigma s, z) & \text{if} \quad s \leqq 1/2 \\ h(2s - 1, \Phi_\sigma(z)) & \text{if} \quad s \geqq 1/2 \end{cases},$$

satisfies (5.3) relatively to B_0. [*cf.* (5.11)-(5.13)]. □

Now for any $\varepsilon \in]0, 1]$, $1 \leqq k \in \mathbb{N}$, $1 \leqq m \in \mathbb{N}$, define

$$c_\varepsilon^k(m) = \inf_{B \in \Gamma_m^k} \sup_{z \in B} f_\varepsilon^k(z), \tag{5.17}$$

where $\Gamma_m^k = \Gamma_m^k(R_m)$.

The following Lemma holds.

LEMMA 5.7. – *Assume $\mathcal{M}_0$ non contractible and 1–connected. Then*

(i) $c_\varepsilon^k(m) \geqq d$, *for any* ε, k, m, *where* d *is defined at* (5.5),

(ii) *there exists two positive real constant* a_1, a_2 *such that, for any* $c \in \mathbb{R}$, *there exists* $m_c \in \mathbb{N}$, *such that*

$$c_\varepsilon^k(m) \geqq \lambda c - a_1\sqrt{c} - a_2 \quad \text{for any } m > m_c, \qquad k \geqq 1, \quad \varepsilon \in]0, 1],$$

where λ *is defined at* (1.3).

(iii) *there exists* $c(m)$, *independent of* k *and* ε *such that*

$$c_\varepsilon^k(m) \leqq c(m),$$

(iv) $c_\varepsilon^k(1) \leqq c_\varepsilon^k(2) \leqq \ldots$,

(v) $c_\varepsilon^k(m)$ *is a critical value of* f_ε^k.

Proof. – If, by contradiction, $c_\varepsilon^k(m) < d$, by (5.17) and (5.5) there exists $B \in \Gamma_m^k(R, \varepsilon)$ such that $B \cap \{(x, t_*) : x \in \Omega^1\} = \varnothing$. Then

$$\operatorname{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(B) = 0,$$

proving (i).

Fix $c \in \mathbb{R}$ and put

$$E^c = \left\{x \in \Omega^1 : \int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle\, ds \leqq c\right\},$$

and

$$E_c = \left\{x \in \Omega^1 : \int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle\, ds \geqq c\right\}.$$

Let $B \in \Gamma_m^k(R, \varepsilon)$. Suppose

$$B \cap (E_c \times \{t_*\}) = \varnothing. \tag{5.18}$$

Then

$$B \subset (E^c \times \{t_*\}) \cup (\mathcal{Z}_k \backslash \{(x, t_*),\, x \in \Omega^1\}),$$

and, since $\operatorname{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(\mathcal{Z}_k \backslash \{(x, t_*),\, x \in \Omega^1\}) = 0$, if B satisfies (5.18),

$$\operatorname{cat}_{\mathcal{Z}_k, \Omega^1 \times B_k(R)}(B) \leqq \operatorname{cat}_{\Omega^1}(E^c). \tag{5.19}$$

Moreover by a well known result E^c is a strong deformation retract of a finite dimensional manifold whose dimension depends on c (*cf.* [10]). Therefore by the properties of the Lusternik and Schnirelmann category (*cf.* e.g. [16]) there exists m_c such that

$$\mathrm{cat}_{\Omega^1}(E^c) \leqq m_c.$$

Then by (5.18) and (5.19), if $m > m_c$,

$$B \cap (E_c \times \{t_*\}) \neq \emptyset,$$

hence, [by assumptions (1.3), (1.5), (1.6)], there exists two positive real constant a_1 and a_2 such that

$$\begin{aligned}\sup f_\varepsilon^k(B) &\geqq \inf f_\varepsilon^k(E_c \times \{t_*\})\\ &\geqq \inf_{x \in \Omega^1} \left(\lambda \int_0^1 \langle \dot{x}, \dot{x} \rangle\, ds - a_1 \left(\int_0^1 \langle \dot{x}, \dot{x} \rangle\, ds\right)^{1/2} - a_2\right).\end{aligned}$$

Therefore, for any $B \in \Gamma_m^k(R, \varepsilon)$,

$$\sup f_\varepsilon^k(B) \geqq \lambda\, c - a_1 \sqrt{c} - a_2,$$

proving (ii).

In order to prove (iii) choose $B = K_m$ and C_m as in Lemma 5.5. Then by (5.17) and (5.4)

$$\begin{aligned}c_\varepsilon^k(m) &\leqq \sup f_\varepsilon^k(K_m))\\ &\leqq \sup \left\{ \int_0^1 \left(\frac{1}{\beta_\varepsilon(x, t)} \cdot \langle \alpha(x, t)\, \dot{x}, \dot{x} \rangle \right.\right.\\ &\qquad \left. + \frac{2}{\beta_\varepsilon(x, t)} \cdot \|\delta(x, t)\| \cdot \|\dot{x}\|\, |\dot{t}| - \dot{t}^2\right)\\ &\qquad \times \beta_\varepsilon(x, t)\, ds - \psi_\varepsilon\left(\int_0^1 \dot{t}^2\, ds\right) :\\ &\qquad \left. x \in \pi_1(C_m), \quad t \in W^1, \quad \|t - t_*\|_{W^{1,2}} \leqq R \right\},\end{aligned}$$

where π_1 is the projection on Ω^1. Therefore, (since C_m is compact) by (1.4) and (1.9) we get (iii), because $t - t_* \in W_0^{1,2}$ and, in (1.9), $\gamma_0 < 2$, $\gamma_1 < 1$.

Since $\Gamma_{m+1}^k(R, \varepsilon) \subset \Gamma_m^k(R, \varepsilon)$, (iv) follows immediately. In order to prove (v) assume by contradiction that $c_\varepsilon^k(m)$ is not a critical value of f_ε^k. Then, since $c_\varepsilon^k(m) \geqq d$, we can use Lemma 5.6 with $c = c_\varepsilon^k(m)$.

Now let $B \in \Gamma_m^k(R, \varepsilon)$ such that

$$\sup f_\varepsilon^k(B) \leqq c_\varepsilon^k(m) + \sigma.$$

If σ is sufficiently small, by Lemma 5.6, $\Phi_\sigma(B) \in \Gamma_m^k(R, \varepsilon)$, in contradiction with (5.27), because $\sup f_\varepsilon(\Phi_\sigma(B)) < c_\varepsilon^k(m)$. $\square$

Now we are finally ready to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. – Assume $\mathcal{M}_0$ 1-connected. Choose c such that

$$\lambda c - a_1 \sqrt{c} - a_2 > d,$$

and m_c as in (ii) of Lemma 5.7.

Then, by (v) of Lemma 5.7, $c_\varepsilon^k(m_c)$ is a critical value of f_ε^k. Moreover by (iii) of Lemma 5.7, and Lemma 4.1, there exists a critical point z_ε of f_ε such that

$$\lambda c - a_1 \sqrt{c} - a_2 \leqq f_\varepsilon(z_\varepsilon) \leqq c(m_c).$$

Finally by Proposition 3.6, if ε is sufficiently small z_ε is a critical point of f_ε (*cf.* also Remark 3.8), such that

$$f_\varepsilon(z_\varepsilon) \geqq \lambda c - a_1 \sqrt{c} - a_2,$$

giving the proof of Theorem 1.3 when $\mathcal{M}_0$ is simply connected. If the fundamental group of $\mathcal{M}_0$ is finite, the proof of Theorem 1.3 can be got using the universal covering, while if it is infinite the proof can be got working on the connected components of Ω^1 which are infinite. $\square$

Proof of Theorem 1.4. – Recalling the conclusion of the proof of Theorem 1.3 we can reduce ourselves to prove Theorem 1.4 whenever $\mathcal{M}_0$ is 1–connected. The same proofs of Propositions 3.1 and 3.2 and Lemma 4.1 show that f_ε^k satisfies the following condition of Palais and Smale (uniformly with respect to k) at every level $c \in \mathbb{R}$ (*cf.* [6]):

$$\begin{gathered}\text{any } \{z_k\}_{k\in\mathbb{N}} \subset \mathcal{Z}_k \text{ such that}\\ f_\varepsilon^k(z_k) \underset{k}{\rightarrow} c, \qquad df_\varepsilon^k(z_k) \underset{k}{\rightarrow} 0\\ \text{possesses a subsequence which converges in } \mathcal{Z}\\ \text{to a critical point of } f_\varepsilon.\end{gathered} \tag{5.20}$$

Fix $m \in \mathbb{N}$ and let K_m as in Lemma 5.5. The proof of (iii) of Lemma 5.7 shows that, if $|t_1 - t_0|$ is sufficiently large (depending only by m) the critical values $c_\varepsilon^k(1) \leqq c_\varepsilon^k(2) \leqq \ldots c_\varepsilon^k(m)$ are $\leqq -1$.

Using (5.20) it is possible to show (*cf.* [6]) that, if there exists a sequence $\{k_p\}_{p\in\mathbb{N}}$ such that

$$c_\varepsilon^{k_p}(i) \underset{p}{\rightarrow} c_\varepsilon(i), \qquad c_\varepsilon^{k_p}(i) - c_\varepsilon^{k_p}(j) \underset{p}{\rightarrow} 0,$$

for some $i \neq j$, then $c_\varepsilon(i)$ is a critical value of f_ε reached by infinitely many critical points of f_ε.

Therefore there exists $c_\varepsilon(1), c_\varepsilon(2), \ldots, c_\varepsilon(m) \in]-\infty, -1]$, critical values of f_ε such that, if $c_\varepsilon(i) = c_\varepsilon(j)$ for some $i \neq j$, then f_ε has infinitely many critical points at the level $c_\varepsilon(i)$. Then, by (2.11) and Remark 3.7, for any fixed m there exists Δ_m such that, if

$$|t_1 - t_0| \geqq \Delta_m,$$

and ε is sufficiently small, f has at least m critical points $z_\varepsilon^1, \ldots, z_\varepsilon^m$, where f is negative, *i.e.* m timelike geodesics joining (x_0, t_0) *with* (x_1, t_1). □

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