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Connectivity properties of the range of a weak diffeomorphism (*)

by

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ABSTRACT. – We discuss connectivity properties of the range of Sobolev maps, and of weak diffeomorphisms or equivalently of elastic deformations.

Key words: Calculus of variations, finite elasticity.

RÉSUMÉ. – On présente des résultats sur la connexion de l'image d'un difféomorphisme faible.

In this paper we shall study some properties of the range $u(\Omega)$ of a weak diffeomorphism u, where Ω is an open connected domain in $\mathbf{R}^n$. We denote by $\widehat{\mathbf{R}}^n$ another copy of $\mathbf{R}^n$ and we recall, compare [3] [2], that a

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function $u \in W^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$ is said to belong to $\operatorname{cart}^1(\Omega, \widehat{\mathbf{R}}^n)$ if

$$\int_\Omega |M(Du)|\, dx \;<\; \infty \;,$$

where $M(Du)$ denotes the n–vector in $\Lambda_n \mathbf{R}^n \times \widehat{\mathbf{R}}^n$

$$M(Du) := (e_1 + D_1 u) \wedge (e_2 + D_2 u) \wedge \ldots \wedge (e_n + D_n u)$$

$e_1, \ldots, e_n$ being the canonical basis of $\widehat{\mathbf{R}}^n$, and moreover

$$\partial G_u \llcorner \Omega \times \widehat{\mathbf{R}}^n \;=\; 0 \;,$$

where G_u denotes the n–dimensional rectifiable current in $\mathbf{R}^n \times \widehat{\mathbf{R}}^n$ carried by the graph of u. If

$$\mathcal{L}_u := \{\, x \in \Omega \,|\, x \text{ is a Lebesgue point for } u \,\}$$

$$\mathcal{R}_u \;:=\; \mathcal{L}_u \cap \mathcal{L}_{Du},$$

and

$$\tilde{u}(x) := \begin{cases} \text{Lebesgue value of } u \text{ at } x, \text{ if } x \in \mathcal{R}_u \\ \qquad\qquad\qquad \text{constant, if } x \notin \mathcal{R}_u \end{cases}$$

then

$$G_u \;=\; \tau(\mathcal{M}, 1, \xi)$$

where $\mathcal{M} \;:=\; \{\, (x, \tilde{u}(x)) \,|\, x \in \mathcal{R}_u \,\}$ and $\xi(x, \tilde{u}(x))$ is the n–vector orienting $\mathcal{M}$

$$\xi(x, \tilde{u}(x)) \;:=\; \frac{M(Du(x))}{|M(Du(x))|} \,.$$

compare [5].

The class of *weak diffeomorphisms* is then defined as the class of $u \in \operatorname{cart}^1(\Omega, \widehat{\mathbf{R}}^n)$ such that

(i) $\det Du > 0$ a.e. in Ω,

(ii) u is *globally invertible* in the sense that

$$\int_\Omega \phi(x, u) \det Du \, dx \;\leq\; \int_{\widehat{\mathbf{R}}^n} (\sup_{x \in \Omega} \phi(x, y))\, dy$$

holds for all $\phi \in C^0_c(\Omega \times \widehat{\mathbf{R}}^n)$ with $\phi \geq 0$.

In the physical case $n = 3$ such a class arises naturally in finite elasticity, and in fact characterizes *elastic deformations* in the sense of [4].

In order to discuss properties of the range of u, of course we should first identify $u(\Omega)$. First we consider Sobolev maps $u \in W^{1,p}(\Omega, \widehat{\mathbf{R}}^n)$, $p \geq 1$, and we do it in terms of *approximate continuity*, compare e.g. [1]. Given a point $x \in \Omega$ at which Ω has positive density, $\theta(\Omega, x) > 0$, one says that $z \in \widehat{\mathbf{R}}^n$ is the approximate limit of $u(y)$ for $y \to x$,

$$z = \operatorname{ap} \lim_{y \to x} u(y) \ ,$$

if for every $r > 0$ the point x is a zero density point for the set $u^{-1}(\widehat{\mathbf{R}}^n \setminus B(z, r))$. Here the density is defined in terms of Lebesgue measure

$$\theta(A, x) \ := \ \lim_{\rho \to 0} \frac{|B(x, \rho) \cap A|}{|B(x, \rho)|} \ .$$

It turns out that for a.e. $x \in \Omega$ the approximate limit of $u(y)$ for $y \to x$ exists and agrees with Lebesgue's value of u. More precisely, setting

$$\mathcal{A}_u \ := \ \{x \in \Omega \ \mid \ \operatorname{ap} \lim_{y \to x} u(x) \text{ exists}\}$$

$$\overline{u}(x) \ := \ \operatorname{ap} \lim_{y \to x} u(x) \ , \qquad x \in \mathcal{A}_u$$

we have

$$\mathcal{R}_u \subset \mathcal{L}_u \subset \mathcal{A}_u \ , \qquad \mathcal{H}^n(\Omega \setminus \mathcal{R}_u) \ = \ 0$$

$$\overline{u}(x) \ = \ \lim_{r \to 0^+} \fint_{B(x,r)} u(y)\, dy \qquad \text{for } x \in \mathcal{L}_u \ ,$$

and finally, compare [7], [13]

$$\dim_{\mathcal{H}}(\Omega \setminus \mathcal{L}_u) \leq n - p, \qquad \mathcal{H}^{n-1}(\Omega \setminus \mathcal{L}_u) = 0$$

consequently

$$\dim_{\mathcal{H}}(\Omega \setminus \mathcal{A}_u) \leq \ n - p, \qquad \mathcal{H}^{n-1}(\Omega \setminus \mathcal{A}_u) = 0 \ .$$

We shall now work with the approximately continuous representative $\overline{u}$ on $\mathcal{A}_u$ of u and prove that the image $\overline{u}(\Omega)$ of a connected set Ω by a $W^{1,p}$ map is connected.

DEFINITION 1. – *We say that* $\Lambda \subset \mathbf{R}^n$ *is* d-open *if and only if* $\theta(\Lambda, x) = 1$ *for all* $x \in \Lambda$.

The collection of d-open sets defines the so-called d-topology; it is in fact easily seen that unions and finite intersections of d-open sets are d-open sets. Such a topology is studied for instance in [8] [9] and [10], where the notion of d_0–connectedness is introduced.

LEMMA 1. – *Let $\Omega \subset \mathbf{R}^n$ be open and connected and let $A \subset \Omega$ be a set of positive measure with finite perimeter in Ω for which $|A| > 0$ and $P(A, \Omega) = 0$. Then $|\Omega \setminus A| = 0$.*

Proof. – From

$$\begin{aligned} P(A,\Omega) &= |D\chi_A|(\Omega) \\ &= \sup\left\{ \int \chi_A \operatorname{div}\varphi \, dx \,\middle|\, \varphi \in C_c^1(\Omega, \mathbf{R}^n),\ |\varphi| \le 1 \right\} = 0 \end{aligned}$$

we infer that the distribution χ_A has zero derivatives in Ω, hence χ_A is constant in Ω, as Ω is connected. □

We can now state

THEOREM 1. – *Let Ω be an open connected set in $\mathbf{R}^n$ and let $u \in W^{1,p}(\Omega, \mathbf{R}^N)$, $p \ge 1$. Then the set $\overline{u}(\Omega \cap \mathcal{A}_u)$ is connected.*

Proof. – Suppose that $B := \overline{u}(A \cap \mathcal{A}_u)$ is not connected. Then there are disjoint open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathbf{R}^N$ such that

$$B \subset \mathcal{U}_1 \cup \mathcal{U}_2 \ , \quad B \cap \mathcal{U}_i \neq \emptyset \quad i = 1, 2 \ .$$

Since $\overline{u}$ is approximately continuous on $\mathcal{A}_u$ the sets

$$\Lambda_i \ := \ \overline{u}^{\,-1}(\mathcal{U}_i) \qquad i = 1, 2$$

are contained in $\mathcal{A}_u$ and d-open in $\mathbf{R}^n$.

From $B \subset \mathcal{U}_1 \cup \mathcal{U}_2$ we then deduce that $\mathcal{A}_u = \Lambda_1 \cup \Lambda_2$, hence

$$\mathcal{H}^{n-1}(\Omega \setminus (\Lambda_1 \cup \Lambda_2)) \ = \ 0 \ .$$

As $B \cap \mathcal{U}_i \neq \emptyset$ we have $\Lambda_i \neq \emptyset$, hence $|\Lambda_i| > 0$, since the Λ_i's are d–open. Also

$$\widehat{\Lambda}_2 \ := \ \{x \in \Omega \ \mid \ \theta(\Lambda_1, x) = 0\} \ \supset \ \Lambda_2 \ ,$$

Consider now the reduced boundary $\partial_\Omega^- \Lambda_1$ of Λ_1 in Ω. We have

$$\partial_\Omega^- \Lambda_1 \ \subset \ \{x \in \Omega \ \mid \ \text{neither } \theta(\Lambda_1, x) = 1 \text{ nor } \theta(\Lambda_1, x) = 0\}$$

hence

$$\partial^-_{\Omega}\Lambda_1 \subset \Omega \setminus (\Lambda_1 \cup \widehat{\Lambda}_2) \subset \Omega \setminus (\Lambda_1 \cup \Lambda_2) \ ,$$

so that

$$\mathcal{H}^{n-1}(\partial^-_{\Omega}\Lambda_1) \ = \ 0 \ .$$

From Lemma 1 we therefore deduce $|\Omega \setminus \Lambda_1| = 0$ which contradicts $|\Lambda_2| > 0$. □

Remark 1. – It is easily seen that in fact Theorem 1 holds for any representative v of u which is approximately continuous in $\mathcal{A}_v$, with $\mathcal{L}^n(\Omega \setminus \mathcal{A}_v) = 0$, replacing $\mathcal{A}_u$ and $\overline{u}$ respectively with $\mathcal{A}_v$ and v.

Remark 2. – The choice of a representative of u in Theorem 1 is essential. For example consider the function

$$u \ : \ B(0,1) \subset \mathbf{R}^2 \ \longrightarrow \ \mathbf{R}$$
$$u(x_1,x_2) \ := \ \begin{cases} x_1 & \text{if } x_1 \neq 0 \\ 2 & \text{if } x_1 = 0 \ . \end{cases}$$

Then $u(B(0,1)) = (-1,0) \cup (0,1) \cup \{2\}$, *i.e.* $u(B(0,1))$ is not connected. Only carefully redefining u on $\mathcal{H}^{n-1}$-a.e. point of the line $\{0\} \times (-1,1) \subset B(0,1)$ in terms of an approximately continuous representative v we can infer that $v(B(0,1))$ is connected.

We are now ready to discuss connectivity properties of the range af weak diffeomorphisms $u \in \widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. Since $\tilde{u}_{|\mathcal{R}_u}$ has the double Lusin's property, from [4] we known that there exists a null set $N_u \subset \Omega$, $|N_u| = 0$, such that $\tilde{u}$ is one to one on $\mathcal{R}_u \setminus N_u$ and

$$\tilde{u}(\mathcal{R}_u \setminus N_u) \cap \tilde{u}(N_u) \ = \ \emptyset \ .$$

In fact this holds for any Lusin's representative of u, and, moreover, we can assume that $\tilde{u}$ be defined in all of Ω.

We shall now introduce a variant of the notion of connectivity which is suited for our purposes. Given two subsets A and B of $\mathbf{R}^n$ we define their *essential distance* by

$$\mathrm{d}_{\mathrm{ess}}(A,B) \ := \ \sup\{\mathrm{dist}\,(A \setminus N_1, B \setminus N_2) \ \mid \ |N_1| = |N_2| = 0\} \ . \quad (1)$$

We then set

DEFINITION 2. – *A set $A \subset \mathbf{R}^n$ is said to be* essentially connected, *ess*-connected, *iff*

$$A = A_1 \cup A_2 \text{ with } |A_1| > 0 \text{ and } |A_2| > 0 \text{ implies } \mathrm{d}_{\mathrm{ess}}(A_1,A_2) = 0 \ . \quad (2)$$

We have

THEOREM 2. – *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded connected domain and let* $u \in \widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. *Then* $\tilde{u}(\Omega)$ *is* ess-*connected for any Lusin representative* $\tilde{u}$ *of* u.

Proof. – Suppose $\tilde{u}(\Omega)$ is not *ess*-connected. Then we can find $B_1, B_2 \subset \tilde{u}(\Omega)$ such that

$$|B_1|, |B_2| > 0 \ , \quad |\tilde{u}(\Omega) \setminus (B_1 \cup B_2)| = 0 \ , \quad \operatorname{dist}(B_1, B_2) > 0 \ . \qquad (3)$$

Consider now an open set $\mathcal{U} \supset B_1$ with smooth boundary $\partial\mathcal{U}$ such that $\operatorname{dist}(\mathcal{U}, B_2) > 0$, and the function

$$\lambda(y) \ := \ \operatorname{dist}(y, \mathcal{U}) \qquad y \in \widehat{\mathbf{R}}^n \ .$$

$\lambda(y)$ is Lipschitz-continuous and for some $\delta_0 > 0$

$$0 < \lambda(y) < \delta_0 \qquad \text{implies} \qquad y \in \widehat{\mathbf{R}}^n \setminus (\overline{\mathcal{U}} \cup \overline{B}_2) \ . \qquad (4)$$

Finally we extend λ to all of $\mathbf{R}^n \times \widehat{\mathbf{R}}^n$ by setting

$$\lambda(x, y) \ = \ \lambda(y) \qquad \text{for} \ \ (x, y) \in \mathbf{R}^n \times \widehat{\mathbf{R}}^n \ . \qquad (5)$$

Setting

$$A_i \ := \ \tilde{u}^{-1}(B_i) \qquad i = 1, 2 \qquad (6)$$

we see as consequence of the double Lusin's property of $\tilde{u}$ (in fact from $|\tilde{u}(A)| = 0$ if $|A| = 0$) that

$$|A_1|, |A_2| > 0 \quad \text{and} \quad |\Omega \setminus (A_1 \cup A_2)| = 0 \ . \qquad (7)$$

Now we slice the current G_u by λ, compare [11]. For a.e. δ, $0 < \delta < \delta_0$ the slice $< G_u, \lambda, \delta >$ exists as a $(n-1)$–rectifiable current and

$$< G_u, \lambda, \delta > = \ \partial(G_u \llcorner \{(x, y) \mid \lambda(x, y) < \delta\}$$

on $(n-1)$–forms with compact support in $\Omega \times \widehat{\mathbf{R}}^n$. On the other hand

$$G_u \llcorner \{(x, y) \mid \lambda(x, y) < \delta\} \ = \ G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n$$

and

$$G_u \llcorner \{(x, y) \mid \lambda(x, y) > \delta_0\} = \ G_u \llcorner A_2 \times \widehat{\mathbf{R}}^n$$

so that

$$\operatorname{spt}\partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) \subset \overline{\Omega} \times \{(x,y) \mid \lambda(x,y) \leq \delta\}$$
$$\operatorname{spt}\partial(G_u \llcorner A_2 \times \widehat{\mathbf{R}}^n) \subset \overline{\Omega} \times \{(x,y) \mid \lambda(x,y) \geq \delta_0\}$$

Since $\partial G_u \llcorner \Omega \times \widehat{\mathbf{R}}^n = 0$, we get

$$\partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) = -\partial(G_u \llcorner A_2 \times \widehat{\mathbf{R}}^n)$$

and therefore

$$\partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) = 0 \ .$$

This implies

$$P(A_1, \Omega) = \mathbf{M}_\Omega(\partial[\![A_1]\!]) = 0 \tag{8}$$

as $\partial[\![A_1]\!] = \pi_\# \partial(G_u \llcorner A_1 \times \widehat{\mathbf{R}}^n) = 0$, where π is the map $\pi : (x,y) \to x$. This gives a contradiction since by Lemma 1 we then have $|A_2| = 0$. □

We shall now show that maps in $\widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$ do *not* cavitate. Results of these type are relatively simple if one has some control on the boundary. Indeed we have

PROPOSITION 1. – *Let Ω be a bounded open connected set in $\mathbf{R}^n$ and let $u \in \widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. Suppose that*

$$K := \operatorname{spt} \widehat{\pi}_\# \partial G_u$$

is compact and let

$$\widehat{\mathbf{R}}^n \setminus K = \bigcup_{i=0}^{m} \mathcal{U}_i \qquad m \in \mathbf{N}, \text{ or } m = \infty$$

the decomposition of the complement of K into connected components, where $\mathcal{U}_0$ is the unbounded component. Then

$$|\tilde{u}(\Omega) \cap \mathcal{U}_0| = 0$$

and for each $i = 1, \ldots, m$ we have either $|\tilde{u}(\Omega) \cap \mathcal{U}_i| = 0$ or $|\mathcal{U}_i \setminus \tilde{u}(\Omega)| = 0$. In particular, if $m = 1$, then

$$|\tilde{u}(\Omega) \Delta \mathcal{U}_1| = 0 \ .$$

Proof. – It is an easy application of the constancy theorem, see e.g. [11]. In fact

$$\widehat{\pi}_{\#} G_u = [\![\tilde{u}(\Omega)]\!]$$

is an n–dimensional rectifiable current in $\widehat{\mathbf{R}}^n$ with multiplicity one and $[\![\tilde{u}(\Omega) \cap \mathcal{U}_i]\!]$ is boundaryless in $\mathcal{U}_i$. The last statement follows then at once as $|\tilde{u}(\Omega)| > 0$ because of $\det Du > 0$ a.e. □

Remark 3. – A simple consequence of Proposition 1 is the result in [12]. In fact one easily sees that the class $\mathcal{A}_{p,q}$ introduced in [12] is a subclass of $\widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$ and, if ∂G_u is the current carried by the graph of a continuous map u, then $K \subset \tilde{u}(\partial\Omega)$.

Our last theorem concerns the regularity, in the sense of non-cavitation, of weak diffeomorphisms without any control on the boundary. We consider a bounded and simply connected domain Ω in $\mathbf{R}^n$. We denote by $d : \Omega \to (0, \infty)$ a *Whitney's regularized distance* function, *i.e.*, a smooth function $d(x)$ which is equivalent to $\operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. We then assume that

(S) *For some* $\delta_0 > 0$ *the open sets*

$$\Omega_\delta := \{x \in \Omega \mid d(x) > \delta\} \qquad \text{and} \qquad \Omega \setminus \Omega_\delta$$

are connected for all δ *with* $0 < \delta < \delta_0$.

THEOREM 3. – *Let* $\Omega \subset \mathbf{R}^n$ *be a connected domain satisfying* (S) *and let* $u \in \widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. *Then for almost every* δ, $0 < \delta < \delta_0$, *the image* $\tilde{u}(\Omega_\delta)$ *of* Ω_δ *and* $\widehat{\mathbf{R}}^n \setminus \tilde{u}(\Omega_\delta)$ *are both essentially connected.*

Proof. – Set $\Delta_\delta := \tilde{u}(\Omega_\delta)$, $0 < \delta < \delta_0$. We know that $\tilde{u}(\Omega_\delta)$ is *ess*-connected. Assume on the contrary that $\widehat{\mathbf{R}}^n \setminus \Delta_\delta$ is not *ess*–connected for a non zero measure set of δ's. For almost every such δ's

$$\partial(G_u \llcorner \Omega_\delta \times \widehat{\mathbf{R}}^n)$$

is of course a rectifiable current with finite mass, consequently Δ_δ is a Caccioppoli set, as

$$[\![\Delta_\delta]\!] = \widehat{\pi}_{\#}(G_u \llcorner \Omega_\delta \times \widehat{\mathbf{R}}^n) ,$$

$\widehat{\pi}$ being the map $(x, y) \to y$. Fix now one such a δ. We claim that for almost every δ_1 with $0 < \delta_1 < \delta$ we have

$$\partial(G_u \llcorner \Omega_{\delta_1} \times \widehat{\mathbf{R}}^n) \qquad \text{is a rectifiable current} \tag{9}$$

and moreover

$$\mathcal{H}^{n-1}(\partial^-\Delta_{\delta_1} \cap \partial^-\Delta_\delta) = 0 . \tag{10}$$

In order to prove (10) we first observe that

$$\mathcal{H}^n(E) = 0$$

where

$$E := \{ x \in (\Omega \setminus \Omega_\delta) \cap \mathcal{R}_u \,|\, \tilde{u}(x) \in \partial^-\Delta_\delta \}$$

because $\tilde{u}$ satisfies the double Lusin property. From this we easily infer

$$\mathcal{H}^{n-1}(E_{\delta_1}) = 0 \qquad \text{for a.e. } \delta_1 \tag{11}$$

where

$$E_{\delta_1} := \{ x \in \partial\Omega_{\delta_1} \cap \mathcal{R}_u \,|\, \tilde{u}(x) \in \partial^-\Delta_\delta \} .$$

On the other hand the trace $\overline{u}$ of u on $\partial\Omega_{\delta_1}$ belongs for a.e. δ_1 to $W^{1,1}(\partial\Omega_{\delta_1}, \widehat{\mathbf{R}}^n)$, $\mathcal{H}^{n-1}$ almost every point in $\partial\Omega_{\delta_1}$ is a regular point for u and $\overline{u}$, and

$$\overline{u}(x) = \tilde{u}(x) \qquad \mathcal{H}^{n-1}\text{– a.e. } x \in \partial\Omega_{\delta_1} ;$$

moreover

$$\partial(G_u \llcorner \Omega_{\delta_1} \times \widehat{\mathbf{R}}^n) = G_{\overline{u}_{|\partial\Omega_{\delta_1}}} ,$$

compare e.g. [6]. Denoting by $\tilde{\overline{u}}$ the Lusin representative of $\overline{u}$, we then infer that

$$\partial^-\Delta_{\delta_1} \subset \tilde{\overline{u}}(\partial\Omega_{\delta_1}) \qquad \mathcal{H}^{n-1}\text{– a.e.}$$

consequently

$$\partial^-\Delta_{\delta_1} \cap \partial^-\Delta_\delta \subset \tilde{\overline{u}}(\{x \in \partial\Omega_{\delta_1} \cap \mathcal{R}_{\overline{u}} \,|\, \tilde{\overline{u}}(x) \in \partial^-\Delta_\delta\})$$

Since the set $\{x \in \partial\Omega_{\delta_1} \cap \mathcal{R}_{\overline{u}} \,|\, \tilde{\overline{u}}(x) \in \partial^-\Delta_\delta\}$ differs form E_{δ_1} by a null set and $\tilde{\overline{u}}$ has the H^{n-1} Lusin property on $\partial\Omega_{\delta_1}$, we finally infer (10) from (11).

We now choose δ_1 in such a way that (9) and (10) hold. Setting

$$B := \tilde{u}(\Omega_\delta) = \Delta_\delta$$
$$C := \tilde{u}(\Omega_{\delta_1} \setminus \Omega_\delta) = \Delta_{\delta_1} \setminus \Delta_\delta$$

(10) yields

$$\partial^- \mathcal{B} \subset \partial^- C \qquad \mathcal{H}^{n-1} \text{ a.e.} \tag{12}$$

On account or our assumption, $R := \widehat{\mathbf{R}}^n \setminus \tilde{u}(\Omega_\delta)$ is not *ess*-connected. We can then find disjoint open sets $\mathcal{U}_1, \mathcal{U}_2 \subset \widehat{\mathbf{R}}^n$ such that

$$R \subset \mathcal{U}_1 \cup \mathcal{U}_2 \text{ a.e.}, |R \cap \mathcal{U}_1| > 0, |R \cap \mathcal{U}_2| > 0, \operatorname{dist}(\mathcal{U}_1, \mathcal{U}_2) > 0 . \tag{13}$$

Being $\tilde{u}$ one-to-one in Ω we have

$$|\mathcal{B} \cap C| = 0 \tag{14}$$

and

$$C = (C \cap \mathcal{U}_1) \cup (C \cap \mathcal{U}_2) \qquad \text{a.e.} \tag{15}$$

if we take into account (13) and (14).

From Theorem 2 we know that C is *ess*-connected, hence one of the addenda of (15), say $C \cap \mathcal{U}_1$, must be a null set

$$|C \cap \mathcal{U}_1| = 0 , \qquad i.e., \qquad C \subset \mathcal{U}_2 \text{ a.e.},$$

consequently

$$\partial^- C \subset \overline{\mathcal{U}}_2 \qquad \mathcal{H}^{n-1} \text{ a.e. .}$$

From (12) we therefore obtain

$$\partial^- \mathcal{B} \subset \overline{\mathcal{U}}_2 \qquad \mathcal{H}^{n-1} \text{ a.e. .} \tag{16}$$

Now we define

$$\widetilde{\mathcal{B}} := \mathcal{B} \cup \overline{\mathcal{U}}_2 \tag{17}$$

and observing that B, $\mathcal{U}_1$, $\mathcal{U}_2$ is a covering of $\widehat{\mathbf{R}}^n$, we get

$$\partial^- \widetilde{\mathcal{B}} = \partial^- \mathcal{B} \cap \overline{\mathcal{U}}_1 ,$$

consequently $\partial^- \widetilde{\mathcal{B}} = \emptyset$, if we take into account (16). We therefore conclude that either $\widetilde{\mathcal{B}} = \widehat{\mathbf{R}}^n$ a.e. or $|\widetilde{\mathcal{B}}| = 0$. Because of (13) we finally infer $\widetilde{\mathcal{B}} = \widehat{\mathbf{R}}^n$ a.e. and, because of (17), that

$$\mathcal{B} \supset \widehat{\mathbf{R}}^n \setminus \overline{\mathcal{U}}_2 \supset \overline{\mathcal{U}}_1$$

which contradicts $|R \cap \mathcal{U}_1| > 0$ in (13). □

We shall now show by means of a few examples that Theorems 2 and 3 are in some sense optimal. Our first example shows that $\tilde{u}(\Omega)$ may be disconnected.

Example 1. – Let $\Omega = (-1,1)^2 \subset \mathbf{R}^2$ and let $\alpha \in (0,1)$. Consider the map $u : \Omega \to \mathbf{R}^2$ defined by

$$u^1(x) := x_1 , \qquad u^2(x) := x_2 \, |x_1|^\alpha .$$

We have $u \in \widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^n)$. In fact $G_{u_\varepsilon} \rightharpoonup G_u$ where u_ε is the family of maps defined by

$$u_\varepsilon^1 := u^1 \quad \text{in } \Omega$$
$$u_\varepsilon^2 := \begin{cases} u^2 & \text{in } \Omega \cap \{x \mid |x_1| \geq \varepsilon\} \\ x_2 \, \varepsilon^\alpha & \text{in } \Omega \cap \{x \mid |x_1| < \varepsilon\} . \end{cases}$$

It is immediately seen that

$$\tilde{u}(\Omega \cap \mathcal{R}_u) = \{y \in \widehat{\mathbf{R}}^2 \mid |y_2| < |y_1|^\alpha , \quad |y_1| < 1\}$$

is *not* connected, but it *is* *ess*-connected. Notice that instead for the approximately continuous representative $\overline{u}$ of u we have

$$\overline{u}(\Omega \cap \mathcal{A}_u) = \{y \mid |y_2| < |y_1|^\alpha , \quad |y_1| < 1\} \cup \{(0,0)\} .$$

In fact $\{0\} \times (-1,1)$ is mapped to zero and by the Hausdorff estimates of the singular set of u we cannot have $\{0\} \times (-1,1) \subset \Omega \setminus \mathcal{L}_u$. We therefore see that $\overline{u}(\Omega)$ is connected according to Theorem 1.

Finally we notice that Theorem 2 does not hold for Cartesian maps as it is shown by the following variant of the previous example. Set

$$\Omega = (-2,2) \times (-1,1) \subset \mathbf{R}^2$$

and

$$u^1(x) = x_1 , \qquad u^2(x) = x_2 \, (\max(|x_1| - 1, 0))^\alpha .$$

Then it is easily seen that

$$\begin{aligned} \overline{u}(\Omega \cap \mathcal{A}_u) &= \tilde{u}(\Omega \cap \mathcal{R}_u) \\ &= \{y \mid |y_2| < (|y_1| - 1)^2 , \ 1 < |y_1| < 2\} \cup ([-1,1] \times \{0\}) \end{aligned}$$

is *not* *ess*-connected while it *is* connected in the usual sense.

Example 2. – We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$, we set

$$\Omega := \left\{ z \in \mathbf{C} \,\middle|\, \frac{1}{2} < |z| < 1,\ \operatorname{Im} z > 0 \right\},$$

and we consider $u(z) := z^2$ for $z \in \Omega$. In the a.e. sense we have

$$\tilde{u}(\Omega \cap \mathcal{R}_u) = \widehat{\Omega} := \left\{ z \in \mathbf{C} \,\middle|\, \frac{1}{4} < |z| < 1 \right\}.$$

Clearly $\mathbf{R}^2 \setminus \tilde{u}(\Omega \cap \mathcal{R}_u)$ is not essentially connected, while $\mathbf{R}^2 \setminus \tilde{u}(\Omega_\delta \cap \mathcal{R}_u)$ is essentially connected for all $\delta > 0$.

The map u is of course of class C^1 in Ω and its *exact image* (not the a.e. image) is given by

$$\overline{u}(\Omega \cap \mathcal{A}_u) = \widehat{\Omega}' := \left\{ z \in \mathbf{C} \,\middle|\, \frac{1}{4} < |z| < 1 \right\} \setminus \{ z \in \mathbf{C} \,|\, \operatorname{Im} z = 0, \operatorname{Re} z > 0 \} \quad (18)$$

which is simply connected in the usual sense.

The next example of a discontinuous weak deffeomorphism shows that in general $\mathbf{R}^2 \setminus \overline{u}(\Omega \cap \mathcal{A}_u)$ need not be connected (in the usual sense).

Example 3. – Let $\Omega = B(0,2) \subset \mathbf{R}^2$, $Q = \{(x_1, x_2) \,|\, |x_1| + |x_2| < 1\}$, and $u = (u^1, u^2) : B(0,2) \to \widehat{\mathbf{R}}^2$ given by

$$u^1(x) := \begin{cases} \dfrac{x_1}{|x_1| + |x_2|}, & \text{if } x \in Q \\ x_1, & \text{if } x \in \Omega \setminus Q \end{cases} \qquad u^2(x) := x_2, \quad \text{if } x \in \Omega .$$

One can show that $u \in \widetilde{\operatorname{dif}}^{1,1}(\Omega, \widehat{\mathbf{R}}^2)$ (in fact $u \in \widetilde{\operatorname{dif}}^{p,q}(\Omega, \widehat{\mathbf{R}}^2)$ for all p, $q < 2$), by approximating u by a sequence of smooth maps u_ε, for instance

$$u^1_\varepsilon(x) := \begin{cases} \dfrac{x_1}{\varepsilon}, & \text{if } |x_1| + |x_2| < \varepsilon \\ u^1, & \text{otherwise} \end{cases}, \qquad u^2(x) := u^2 .$$

One then sees that

$$\begin{aligned} \mathcal{L}_u &= B(0,2) \setminus \{0\}) \\ \overline{u}(x_1, 0) &= (1,0) \qquad \text{for} \quad 0 < x_1 < 1 \\ \overline{u}(x_1, 0) &= (-1,0) \qquad \text{for} \quad -1 < x_1 < 0 \,; \end{aligned}$$

while u is not approximately continuous at $(0,0)$. Thus we have

$$\overline{u}(\Omega \cap \mathcal{L}_u) = B(0,2) \setminus ((-1,1) \times \{0\})$$

which is not simply connected, but $\overline{u}(\Omega \cap \mathcal{L}_u)$ and $\mathbf{R}^2 \setminus \overline{u}(\Omega \cap \mathcal{L}_u)$ are both essentially connected.

Example 3 shows that weak diffeomorphisms may create holes of zero measure.

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