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Small solutions to nonlinear Schrödinger equations

by

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ABSTRACT. — It is shown that the initial value problem for the nonlinear Schrödinger equations

$$\partial_t u = i\Delta u + P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

where $P(.)$ is a polynomial having no constant or linear terms, is locally well posed for a class of "small" data u_0.

The main ingredients in the proof are new estimates describing the smoothing effect of Kato type for the group $\{e^{it\Delta}\}_{-\infty}^{\infty}$. This method extends to systems and other dispersive models.

Key words : Nonlinear Schrödinger equations, smoothing effects.

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RÉSUMÉ. – On montre que le problème de Cauchy pour l'équation de Schrödinger non linéaire

$$\partial_t u = i\Delta u + P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

où P(.) est un polynôme sans termes constants ou linéaires, est bien posé pour une classe de « petites » données u_0.

Les ingrédients principaux de la démonstration sont des nouvelles estimations qui décrivent l'effet régularisant de type de Kato pour le groupe $\{e^{it\Delta}\}$. La méthode admet des extensions aux systèmes et aux autres modèles dispersifs.

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1. INTRODUCTION

Consider the initial value problem (IVP) for nonlinear Schrödinger equations of the form

$$\left.\begin{aligned} \partial_t u = i\Delta u + P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(x, 0) = u_0(x) \end{aligned}\right\} \tag{1.1}$$

where $u = u(x, t)$ is a complex valued function,

$$P: \quad \mathbb{C}^{2n+2} \to \mathbb{C}$$

is a polynomial having no constant or linear terms and

$$\nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u).$$

The main purpose of this paper is to establish a local existence theory for the IVP (1.1) with "small" data $u_0(.)$.

In the semilinear case $P = f(|u|)u$ [with $f(.)$ a real valued function] the IVP (1.1) has extensively studied. In particular local and global theories and blow up results have been proven. These depend upon the regularity and size of the data, the degree and sign of $f(.)$ and the dimension n ([3], [8], [9], [12], [18], [34], [36], 37], for a complete list of references *see* [2]).

The general IVP (1.1) has been mainly treated when one of the following hypothesis hold:

– energy estimates are available

or

– analytic data (analytic solutions).

Energy estimates for the equation in (1.1) can be established when using integration by parts one can show that

$$\left|\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n}\partial_x^\alpha \mathrm{P}(u,\nabla_x u,\bar{u},\nabla_x\bar{u})\,\partial_x^\alpha u\,dx\right|\leqq c_s(1+\|u\|_{s,2}^{\rho})\|u\|_{s,2}^2 \quad (1.2)$$

for any $u\in \mathrm{H}^s(\mathbb{R}^n)$ with $s>n/2+1$ and $\rho=\rho(\mathrm{P})\in\mathbb{Z}^+$.

It is clear that the estimate (1.2) can only be guaranteed if $\mathrm{P}(.)$ exhibits an appropriate symmetry. For example:

$$n=1 \quad\text{and}\quad \mathrm{P}=\partial_x(|u|^k u),\qquad k\in\mathbb{Z}^+\ (\textit{see}\ [38],[39]),\qquad (1.3)$$

and

$$n\geqq 1 \quad\text{and}\quad \mathrm{D}_{\partial_{x_j}u}\mathrm{P},\ \mathrm{D}_{\partial_{x_j}\bar{u}}\mathrm{P}\qquad \text{for}\quad j=1,\ldots,n$$

are real valued functions (*see* [27]-29]).

When $\mathrm{P}(.)$ satisfies (1.2) the proof of the local existence theory in $\mathrm{H}^s(\mathbb{R}^n)$ with $s>n/2+1$ follows the argument used for quasilinear symmetric hyperbolic systems (*see* [17]). Indeed, for those P's the same proof works if one removes the term involving the Laplacian from the equation in (1.1). In other words, this local result does not use the dispersive structure of the equations.

It is interesting to remark that the case $k=2$ in (1.3) was treated by D. J. Kaup and A. C. Newell [21] using an appropriate version of the inverse scattering method.

The other approach uses analytic functions techniques to overcome the loss of derivatives introduced by the nonlinearly $\mathrm{P}(.)$. Thus under appropriate assumptions on the analytic data u_0 and the nonlinearity $\mathrm{P}(.)$ local and global analytic results have been obtained (*see* [13], [16], [30]).

Our approach here is quite different. It is based on the dispersive character of the associated linear equation. More precisely, it relies in a crucial manner on new estimates for the smoothing effects in the group $\{e^{it\Delta}\}_{-\infty}^{\infty}$.

In [19] T. Kato showed that solutions of the Korteweg-de Vries equation

$$\partial_t w+\partial_x^3 w+w\,\partial_x w=0,\qquad x,\ t\in\mathbb{R}\qquad (1.4)$$

satisfy that

$$\int_{-\mathrm{T}}^{\mathrm{T}}\int_{-\mathrm{R}}^{\mathrm{R}}|\partial_x w(x,t)|^2\,dx\,dt\leqq c(\mathrm{T},\mathrm{R},\|w(x,0)\|_2).$$

The corresponding version of the above estimate for the Schrödinger group $\{e^{it\Delta}\}_{-\infty}^{\infty}$

$$\int_{-\mathrm{T}}^{\mathrm{T}}\int_{-\mathrm{R}}^{\mathrm{R}}|(1-\Delta)^{1/4}e^{it\Delta}u_0|^2\,dx\,dt\leqq c(\mathrm{T},\mathrm{R},\|u_0\|_2)\qquad (1.5)$$

was simultaneously established by P. Constantin and J.-C. Saut [6], P. Sjölin [31] and L. Vega [41].

In [23] we show that in the one dimensional case (1.5) can be improved *i.e.*

$$\sup_x \int_{-\infty}^{\infty} |D_x^{1/2} e^{it\partial_x^2} u_0|^2 \, dt \leqq c \| u_0 \|_2^2 \tag{1.6}$$

where $D_x = (-\Delta)^{1/2}$ and that this estimate is sharp.

Our main new tool is the inhomogenenous version of (1.5)-(1.6). More precisely, it will be proven in section 2 (Theorem 2.3) that for $n=1$

$$\sup_x \left(\int_{-\infty}^{\infty} \left| \partial_x \left(\int_0^t e^{i(t-\tau)\partial_x^2} F(., \tau) \, d\tau \right) \right|^2 dt \right)^{1/2} \leqq c \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |F(x, t)|^2 \, dt \right)^{1/2} dx, \tag{1.7}$$

and for $n \geqq 2$

$$\sup_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} \left| \partial_x \left(\int_0^t e^{i(t-\tau)\Delta} F(., \tau) \, d\tau \right) \right|^2 dt \, dx \right)^{1/2} \leqq c R \sum_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} |F(x, t)|^2 \, dt \, dx \right)^{1/2} \tag{1.8}$$

where $\{Q_\alpha\}_{\alpha \in \mathbb{Z}^n}$ denotes a family of disjoint cubes of size R such that $\mathbb{R}^n = \bigcup_{\alpha \in \mathbb{Z}^n} Q_\alpha$.

Roughly speaking (1.7)-(1.8) tell us that the gain of derivatives in the inhomogeneous case is twice that obtained for the homogenous problem (1.5)-(1.6). For the global smoothing effect of Strichartz type [35] (*see* also [8]) this principle was already proven in [22] and [23]. In this case the gain of derivatives is only present when the equation is of order larger than two. In particular for the group $\{V(t)\}_{-\infty}^{\infty}$ associated with the linearized KdV equation (1.4)

$$\partial_t w + \partial_x^3 w = 0$$

it was shown ([22], Theorem 2.4) that

$$\left(\int_{-\infty}^{\infty} \| D_x^{1/4} V(t) u_0 \|_\infty^4 \, dt \right)^{1/4} \leqq c \| u_0 \|_2,$$

and

$$\left(\int_{-\infty}^{\infty} \left\| D_x^{1/2} \int_0^t V(t-\tau) F(., \tau) \, d\tau \right\|_\infty^4 dt \right)^{1/4} \leqq c \left(\int_{-\infty}^{\infty} \| F(., t) \|_1^{4/3} \, dt \right)^{3/4}.$$

The extension of (1.7)-(1.8) to general dispersive equations and applications to associated non-linear problem including the generalized KdV equation and other dispersive systems (*see* [11]) will be given elsewhere, (*see* [24], [25]).

To complement the inequalities (1.7)-(1.8) one needs to use estimates related with the maximal function: $\sup_{[0,T]} |e^{it\Delta}.|$, (*see* [1], [7], [26], [31], [40]-[41]). This idea is implicit in the splitting argument introduced by J. Ginibre and Y. Tsutsumi [10] in their work on uniqueness for the generalized KdV equation. Thus in section 3 we shall study boundedness properties of this maximal function.

Finally in section 4 we establish a local existence theory for the IVP (1.1). Four cases will be considered $n=1$ or $n\geqq 2$, and whether or not P has quadratic terms. In the later case [*i.e.* when P(.) does not have quadratic terms] we shall show that the IVP (1.1) is well posed in $\{u_0 \in H^s(\mathbb{R}^n)/\|u_0\|_{s,2}<\delta\}$ where $s\geqq s_0(n)$ and $\delta=\delta(P, n)>0$. When P(.) possesses quadratic terms we need to introduce weighted H^s-norms, (*see* Theorems 4.2, 4.4).

Our method of proof is based on the contraction mapping principle. Depending on the case considered it uses several norms: those introduced by the smoothing effect the maximal function and the persistence properties and if necessary some weighted Sobolev norms.

It should be mentioned that in those cases where both the energy estimate and the method provided below apply, our method seems to give better results. In particular in [24] it will be shown that the IVP the generalized Benjamin-Ono equation

$$\partial_t u + D_x \partial_x u + u^k \partial_x u = 0, \qquad x, t \in \mathbb{R}, \quad k \in \mathbb{Z}^+ \tag{1.9}$$

is locally well-posed in $H^s(\mathbb{R})$ with $\|u_0\|_{s,2}<\delta=\delta(k)$ and $s\geqq s_0(k)$ where $s_0(k)\leqq 3/2$ if $k\geqq 2$. In this case the energy method proves the result for $s>3/2$. A similar argument will allow us to improve some of the results in [38], [39] (*see* [24]). Observe that our techniques below work equally well for real or complex valued, therefore they apply to the equation (1.9). Also it shall be clear from the proof below that our method extends without any modification to systems and to non-linearities given by smooth functions $F(u, \nabla_x u, \bar{u}, \nabla_x \bar{u})$ with Taylor expansion at the origin having no constant or linear terms.

The results in section 4 present the following questions:

– Under what conditions do these local solutions extend globally?

– For "large" data u_0 does the IVP (1.1) have a local solution?

For particular nonlinearities P(.) our method combined with the global estimate (3.2 *b*) and its extensions to $\mathbb{R}^n$ gives some positive answers to the first question above (*see* [24]).

Finally we would like to point out that here we shall not attempt to obtaint the best results (Theorem 4.1-4.4) provided by our method since in any case is not clear that they would be optimal.

2. LOCAL SMOOTHING EFFECTS

This section is concerned with local smoothing effects of Kato type exhibited by the group $\{e^{it\Delta}\}_{-\infty}^{\infty}$.

In the homogeneous case these effects basically assert that if $u_0 \in L^2(\mathbb{R}^n)$ then $D_x^{1/2} e^{it\Delta} u_0 \in L^2_{loc}(\mathbb{R}^n \times \mathbb{R})$. As was mentioned in the introduction we shall deduce new estimates for the inhomogeneous case.

Before stating any results we need to introduce the following notation: $\{Q_\alpha\}_{\alpha \in \mathbb{Z}^n}$ is a familly of nonoverlapping cubes of size R such that $\mathbb{R}^n = \bigcup_{\alpha \in \mathbb{Z}^n} Q_\alpha$.

THEOREM 2.1. (*Local smoothing effect: homogeneous case*). – *For* $n=1$

$$\sup_x \left(\int_{-\infty}^{\infty} | D_x^{1/2} e^{it\partial_x^2} u_0(x) |^2 \, dt \right)^{1/2} \leqq c \| u_0 \|_2, \tag{2.1}$$

and for $n \geqq 2$

$$\sup_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} | D_x^{1/2} e^{it\Delta} u_0(x) |^2 \, dt \, dx \right)^{1/2} \leqq c \, R \| u_0 \|_2 \tag{2.2}$$

where $D_x^\gamma f(x) = c_\gamma \, |\xi|^\gamma \hat{f}(\xi))^\vee (x)$, i.e. *the homogeneous derivative of order* γ *of* f.

Proof. – (2.2) was established in [6], [31], [41]. The fact that its right hand side grows linearly in R independently of the dimension n was proven in [23] (section 4).

The improvement in the one dimensional case (2.1) was given in [23] (section 4). There, it was shown that this estimate is sharp in the sense that there exists a class of initial data u_0 in which (2.1) becomes an identity. □

The dual version of (2.1)-(2.2) is given by

COROLLARY 2.2. – *For* $n=1$

$$\left\| D_x^{1/2} \int_0^T e^{-i\tau\partial_x^2} f(.,\tau) \, d\tau \right\|_2 \leqq c \int_{-\infty}^{\infty} \left(\int_0^T | f(x,t) |^2 \, dt \right)^{1/2} dx, \tag{2.3}$$

and for $n \geqq 2$

$$\left\| D_x^{1/2} \int_0^T e^{-i\tau\Delta} f(.,\tau)\,d\tau \right\|_2 \leqq c\,R \sum_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_0^T |f(x,t)|^2\,dt\,dx \right)^{1/2}. \quad (2.4)$$

In section 4 the estimates (2.3)-(2.4) will be used in connection with Duhamel's principle. More precisely, for $t \in [0, T]$ (2.3)-(2.4) imply that

$$\left\| D_x^{1/2} \int_0^t e^{i(t-\tau)\partial_x^2} f(.,\tau)\,d\tau \right\|_2 \leqq c \int_{-\infty}^{\infty} \left(\int_0^T |f(x,t)|^2\,dt \right)^{1/2} dx, \quad (2.5)$$

for $n=1$ and

$$\left\| D_x^{1/2} \int_0^t e^{i(t-\tau)\Delta} f(.,\tau)\,d\tau \right\|_2 \leqq c\,R \sum_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_0^T |f(x,t)|^2\,dt\,dx \right)^{1/2} \quad (2.6)$$

for $n \geqq 2$ respectively.

Now we consider the inhomogeneous IVP

$$\left.\begin{aligned} \partial_t u = i\Delta u + F(x,t), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(x,0)=0 \end{aligned}\right\} \quad (2.7)$$

with $F \in S(\mathbb{R}^n \times \mathbb{R})$.

Our main result in this section is

THEOREM 2.3. (*Local smoothing effect: inhomogeneous case*). – (*a*) *When* $n=1$ *the solution* $u(x,t)$ *of the IVP* (2.7) *satisfies*

$$\sup_x \left(\int_{-\infty}^{\infty} |\partial_x u(x,t)|^2\,dt \right)^{1/2} \leqq c \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |F(x,t)|^2\,dt \right)^{1/2} dx. \quad (2.8)$$

(*b*) *When* $n \geqq 2$ *the solution* $u(u,t)$ *of the IVP* (2.7) *satisfies*

$$\sup_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} |\nabla_x u(x,t)|^2\,dt\,dx \right)^{1/2} \leqq c\,R \sum_{\alpha \in \mathbb{Z}^n} \left(\int_{Q_\alpha} \int_{-\infty}^{\infty} |F(x,t)|^2\,dt\,dx \right)^{1/2}. \quad (2.9)$$

Proof of Theorem 2.3 (*a*) (estimate (2.8)). – Formally taking Fourier transform in both variables in the equation in (2.7) we get

$$\hat{u}(\xi,\tau) = c\,\frac{\hat{F}(\xi,\tau)}{\tau - \xi^2},$$

and consequently

$$\partial_x u(x,t) = c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it\tau} e^{x\xi} \frac{\xi}{\tau - \xi^2} \hat{F}(\xi,\tau)\,d\xi\,d\tau.$$

By Plancherel's theorem (in the time variable)

$$\left(\int_{-\infty}^{\infty} |\partial_x u(x, t)|^2 dt\right)^{1/2}$$
$$= c\left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{ix\xi}\,\frac{\xi}{\tau-\xi^2}\,\hat{\mathrm{F}}(\xi, \tau)\,d\xi\right|^2 d\tau\right)^{1/2} \tag{2.10}$$
$$= c\left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} \mathrm{K}(x-y, \tau)\,\hat{\mathrm{F}}^{(t)}(y, \tau)\,dy\right|^2 d\tau\right)^{1/2}$$

$\hat{\mathrm{F}}^{(t)}$ denoting the Fourier transform of F in the time variable and

$$\mathrm{K}(l, \tau) = \int_{-\infty}^{\infty} e^{il\xi}\,\frac{\xi}{\tau-\xi^2}\,d\xi$$

where the integral is understood in the principal value sense.

We claim:

$$\mathrm{K} \in \mathrm{L}^{\infty}(\mathbb{R}^2), \qquad \text{with norm} \quad \mathrm{M}. \tag{2.11}$$

The claim (2.11) combined with Minkowski's integral inequality and Plancherel's theorem in the time variable shows that the terms in (2.10) can be bounded as follows

$$c\,\mathrm{M}\int_{-\infty}^{\infty} \|\hat{\mathrm{F}}^{(t)}(y, .)\|_2\,dy = c\,\mathrm{M}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |\mathrm{F}(y, s)|^2 ds\right)^{1/2} dy,$$

which yields (2.8).

It remains to prove the claim (2.11). For $\tau > 0$

$$\mathrm{K}(l, \tau) = \int_{-\infty}^{\infty} e^{il\xi}\,\frac{\xi}{\tau-\xi^2}\,d\xi = \int_{-\infty}^{\infty} e^{il\sqrt{\tau}\,\eta}\,\frac{\eta}{1-\eta^2}\,d\eta.$$

Observe that in a neighborhood of the singular points $\eta = \pm 1$ and $\eta = \infty$ the function $\eta(1-\eta^2)^{-1}$ behaves like the kernel of the Hilbert transform (*i.e.* $1/\eta$) (or its translated) whose Fourier transform equals $-i$ sgn (y). Hence a simple comparison argument yields (2.11) when $\tau > 0$. The proof of the case $\tau < 0$ is similar and therefore it will be omitted.

The above formal process can be justified by applying it to the equation

$$\partial_t u = i\Delta u + \varepsilon u + \mathrm{F}(x, t), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad \varepsilon > 0.$$

In this case the estimate (2.8) holds uniformly in $\varepsilon > 0$. Then taking the limit we obtain the desired result.

Thus we have proven that there exists a solution $u(x, t)$ of the equation in (2.7) satisfying the estimate (2.8). In general, this solution may not vanish at $t=0$. However by using Parseval's identity one sees that

$$u(x, 0)=c\int_{-\infty}^{\infty} e^{ix\xi}\left(\int_{-\infty}^{\infty} \frac{1}{\tau-\xi^2}\hat{F}(\xi, \tau)\,d\tau\right)d\xi$$
$$=c\int_{-\infty}^{\infty} e^{ix\xi}\left(\int_{-\infty}^{\infty} \hat{F}^{(x)}(\xi, s)\,\mathrm{sgn}(s)\,e^{-is\xi^2}\,ds\right)d\xi$$
$$=c\int_{-\infty}^{\infty} e^{is\partial_x^2}\,\mathrm{sgn}(s)\,F(x, s)\,ds.$$

Now from (2.3) it follows that $D_x^{1/2}u(x, 0)\in L^2(\mathbb{R})$, which combined with (2.1) shows that

$$u(x, t)-e^{it\partial_x^2}u(x, 0)$$

is the solution of (2.7) and satisfies the estimate (2.8). □

Proof of Theorem 2.3 (*b*) (estimate (2.9)). – Define F_α's and u_α's such that

$$F=\sum_{\alpha\in\mathbb{Z}^n} F\chi_{Q_\alpha}=\sum_{\alpha\in\mathbb{Z}^n} F_\alpha,$$

and

$$u=\sum_{\alpha\in\mathbb{Z}^n} u_\alpha$$

where $u_\alpha=u_\alpha(x, t)$ is the corresponding solution of the IVP (2.7) with inhomogenous term F_α instead of F.

Taking Fourier transform in both variables one finds that

$$\hat{u}_\alpha(\xi, \tau)=\frac{c}{|\xi|^2-\tau}\hat{F}_\alpha(\xi, \tau)\quad\text{for each } \alpha\in\mathbb{Z}^n.$$

Using Plancherel's theorem in the t-variable the left hand side of (2.9) with $u_\beta(.)$ instead of $u(.)$ can be written as

$$\sup_{\alpha\in\mathbb{Z}^n}\left(\int_{Q_\alpha}\int_{-\infty}^{\infty}\left|\int_{\mathbb{R}^n} e^{ix\xi}\frac{c\,\xi}{|\xi|^2-\tau}\hat{F}_\beta(\xi, \tau)\,d\xi\right|^2 d\tau\,dx\right)^{1/2}. \tag{2.12}$$

The following result (whose proof will be given later) provides the key estimate.

LEMMA 2.4. – *If* $(\mathrm{T}h)^\wedge(\xi)=\dfrac{c\,\xi}{|\xi|^2-1}\hat{h}(\xi)=m(\xi)\,\hat{h}(\xi)$ (*here* $\wedge$ *denotes the Fourier transform in x only*) *then*

$$\sup_{\alpha\in\mathbb{Z}^n}\left(\int_{Q_\alpha}|T(g\chi_{Q_\beta})|^2\,dx\right)^{1/2}\leq c\,R\left(\int_{Q_\beta}|g|^2\,dx\right)^{1/2}.\ \square \tag{2.13}$$

We shall restrict ourselves to consider only the term

$$\sup_\alpha \left(\int_{Q_\alpha} \int_0^\infty \left| \int_{\mathbb{R}^n} e^{ix\xi} \frac{\xi}{|\xi|^2 - \tau} \hat{F}_\beta(\xi, \tau)\, d\xi \right|^2 d\tau\, dx \right)^{1/2} \qquad (2.14)$$

since the other portion [*ie.* $\tau \in (-\infty, 0)$] corresponding to the symbol $\frac{\eta}{|\eta|^2+1}$ in the argument below is easier to handle.

The expression in (2.14) is majorized by

$$\left(\int_0^\infty \sup_\alpha \int_{Q_\alpha} \left| \int_{\mathbb{R}^n} e^{ix\xi} \frac{\xi}{|\xi|^2 - \tau} \hat{F}_\beta(\xi, \tau)\, d\xi \right|^2 dx\, d\tau \right)^{1/2}. \qquad (2.15)$$

Successive changes of variable $\xi = \tau^{1/2} \eta$ and $y = \tau^{1/2} x$ give

$$\sup_\alpha \int_{Q_\alpha} \left| \int e^{ix\xi} \frac{\xi}{|\xi|^2 - \tau} \hat{F}_\beta(\xi, \tau)\, d\xi \right|^2 dx$$

$$\int e^{i\tau^{1/2} x\eta} \frac{\eta}{|\eta|^2 - 1} = \tau^{n-1} \sup_x \int_{Q_\alpha} \left| \int e^{i\tau^{1/2} x\eta} \frac{\eta}{|\eta|^2 - 1} \hat{F}_\beta(\tau^{1/2}\eta, \tau)\, d\eta \right|^2 dx$$

$$= \tau^{n-1} \tau^{-n/2} \sup_x \int_{\tau^{1/2} Q_\alpha} \left| \int e^{iy\eta} \frac{\eta}{|\eta|^2 - 1} \hat{F}_\beta(\tau^{1/2}\eta, \tau)\, d\eta \right|^2 dy. \qquad (2.16)$$

Observe that by taking inverse Fourier transform in the space variable one has that

$$(\hat{F}_\beta(\tau^{1/2}\eta, \tau))^\vee (x, \tau) = \tau^{-n/2} \hat{F}_\beta^{(t)}(\tau^{-1/2} x, \tau).$$

Thus

$$\text{support } \hat{F}_\beta^{(t)}(\tau^{-1/2}., \tau) \subseteq \tau^{1/2} Q_\beta \quad \text{for any } \tau \in \mathbb{R}.$$

Now using lemma 2.4 for the family $\tau^{1/2} Q_\beta$ and a change of variable the terms in (2.16) can be bounded by

$$\tau^{n-1} \tau^{n/2} \tau R^2 \int_{\tau^{1/2} Q_\beta} |\tau^{-n/2} \hat{F}_\beta^{(t)}(\tau^{-1/2} x, \tau)|^2 dx$$

$$= R^2 \int_{Q_\beta} |\hat{F}_\beta^{(t)}(y, \tau)|^2 dy. \qquad (2.17)$$

Combining (2.12)-(2.17) with Plancherel's theorem we have that

$$\sup_{\alpha \in \mathbb{Z}^n} \left(\int_{-\infty}^{\infty} \int_{Q_\alpha} |\nabla_x u_\beta(x, t)|^2 \, dx \, dt \right)^{1/2}$$
$$\leqq c \mathrm{R} \left(\int_{Q_\beta} \int_{-\infty}^{\infty} |\hat{\mathrm{F}}_\beta^{(t)}(x, \tau)|^2 \, d\tau \, dx \right)^{1/2}$$
$$\leqq c \mathrm{R} \left(\int_{Q_\beta} \int_{-\infty}^{\infty} |\mathrm{F}_\beta(x, t)|^2 \, dt \, dx \right)^{1/2},$$

which implies the desired result (2.9). □

We now turn to the proof of lemma 2.4.

Proof of lemma 2.4 – Let $\varphi \in C_0^\infty(\mathbb{R})$ with supp $\varphi \subseteqq [-1, 1]$, $\varphi \equiv 1$ in $[-1/2, 1/2]$ and $0 \leqq \varphi \leqq 1$. Let $\varphi_1(\xi) = \varphi(2(|\xi| - 1))$ and $\varphi_2(\xi)$ such that $\varphi_1(\xi) + \varphi_2(\xi) = 1$.

Define

$$\mathrm{T}_i h(x) = (m_i(\xi) \hat{h}(\xi))^\vee(x), \qquad i = 1, 2$$

where $m_i(\xi) = \varphi_i(\xi) m(\xi)$.

First, we shall establish (2.16) for the operator T_2 whose symbol $m_2(\xi)$ has no singularities. For p, p' such that

$$\frac{1}{p} + \frac{1}{p'} = 1 \qquad \text{and} \qquad \frac{1}{p'} - \frac{1}{p} = \frac{1}{n},$$

by Hölder's inequality, Sobolev's lemma and Mihlin's theorem it follows that

$$\left(\int_{Q_\alpha} |\mathrm{T}_2(g \chi_{Q_\beta})|^2 \, dx \right)^{1/2} \leqq c \mathrm{R}^{n((1/2) - (1/p))} \| \mathrm{T}_2(g \chi_{Q_\beta}) \|_p$$
$$\leqq c \mathrm{R}^{n((1/2) - (1/p))} \| \nabla_x \mathrm{T}_2(g \chi_{Q_\beta}) \|_{p'}$$
$$\leqq c \mathrm{R}^{n(1/2) - (1/p))} \| g \chi_{Q_\beta} \|_{p'} \leqq c \mathrm{R} \| g \chi_{Q_\beta} \|_2.$$

To estimate T_1 we split its symbol $m_1(\xi)$ into a finite number of pieces (depending only on the dimension n). Let $\theta \in C_0^\infty(\mathbb{R})$ with supp $\theta \subseteqq (-1, 1)$, define

$$m_{1,1}(\xi) = \varphi(2(|\xi| - 1)) m(\xi) \theta(4|\bar{\xi}|) = m_1(\xi) \theta(4|\bar{\xi}|) \qquad (2.18)$$

where $\xi = (\bar{\xi}, \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

$m_1(.)$ can be expressed as a finite sum of $m_{1,1}$'s (by a rotation argument). Notice that

$$\text{supp } m_{1,1} \subseteqq \{ \xi = (\bar{\xi}, \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} / |\bar{\xi}| < 1/4 \text{ and } 1/2 < |\xi| < 3/2 \}$$
$$Q_\alpha \subseteqq \mathbb{R}^{n-1} \times [a_\alpha, a_\alpha + \mathrm{R}]$$

and

$$Q_\alpha \subseteq \mathbb{R}^{n-1} \times [a_\beta, a_\beta + R]$$

for appropriate constants a_α, a_β.

Thus Plancherel's theorem in the $\bar{x}$ variable leads to

$$\begin{aligned}
\int_{Q_\alpha} |T_{1,1}(g\chi_{Q_\beta})|^2\,dx &\leq \int_{a_\alpha}^{a_\alpha+R}\int_{\mathbb{R}^{n-1}} |T_{1,1}(g\chi_{Q_\beta})|^2\,d\bar{x}\,dx_n \\
&= \int_{a_\alpha}^{a_\alpha+R}\int_{\mathbb{R}^{n-1}} \left|\int e^{ix_n\xi_n} m_{1,1}(\xi)(g\chi_{Q_\beta})\hat{}(\bar{\xi},\xi_n)\,d\xi_n\right|^2 d\bar{\xi}\,dx_n \\
&= \int_{a_\alpha}^{a_\alpha+R}\int_{\mathbb{R}^{n-1}} \left|\int_{a_\beta}^{a_\beta+R}\int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}}(g\chi_{Q_\beta})(\bar{y},y_n)\,a(x_n,y_n,\bar{\xi})\,d\bar{y}\,dy_n\right|^2 d\bar{\xi}\,dx_n \\
&\equiv E
\end{aligned}$$

where

$$a(x_n, y_n, \bar{\xi}) = \int_{-\infty}^{\infty} e^{i(x_n - y_n)\xi_n} m_{1,1}(\xi)\,d\xi_n.$$

CLAIM. — *There exists* $C>0$ *such that for any* $(x_n, y_n, \bar{\xi}) \in \mathbb{R}^{n+1}$

$$|a(x_n, y_n, \bar{\xi})|^2 \leq C.$$

Combining the claim (to be proven below) with Schwarz' inequality and the theorems of Fubini and Plancherel we find that

$$\begin{aligned}
E &\leq R\int_{a_\alpha}^{a_\alpha+R}\int_{\mathbb{R}^{n-1}}\int_{a_\beta}^{a_\beta+R}\left|\int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}}(g\chi_{Q_\beta})(\bar{y},y_n)\,a(x_n,\xi_n,\bar{\xi})\,d\bar{y}\right|^2 dy_n\,d\bar{\xi}\,dx_n \\
&\leq CR\int_{a_\alpha}^{a_\alpha+R}\int_{\mathbb{R}^{n-1}}\int_{a_\beta}^{a_\beta+R}\left|\int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}}(g\chi_{Q_\beta})(\bar{y},y_n)\,d\bar{y}\right|^2 dy_n\,d\bar{\xi}\,dx_n \\
&= CR\int_{a_\alpha}^{a_\alpha+R}\int_{a_\beta}^{a_\beta+R}\int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}^{n-1}} e^{-i\bar{y}\bar{\xi}}(g\chi_{Q_\beta})(\bar{y},y_n)\,d\bar{y}\right|^2 d\bar{\xi}\,dy_n\,dx_n \\
&= CR\int_{a_\alpha}^{a_\alpha+R}\int_{a_\beta}^{a_\beta+R}\int_{\mathbb{R}^{n-1}} |(g\chi_{Q_\beta})(\bar{y},y_n)|^2\,d\bar{y}\,dy_n\,dx_n \\
&= CR^2\int_{\mathbb{R}^n} |(g\chi_{Q_\beta})(y)|^2\,dy
\end{aligned}$$

which yields (2.13)

It remains to establish the claim. We have that

$$\begin{aligned}
a(x_n, y_n, \bar{\xi}) &= \int_{-\infty}^{\infty} e^{i(x_n-y_n)\xi_n}\frac{\xi}{|\xi|^2-1}\,\varphi(2(|\xi|-1))\,\theta(4|\bar{\xi}|)\,d\xi_n \\
&= \theta(4|\bar{\xi}|)\int_{-\infty}^{\infty} e^{i\lambda\xi_n}\frac{\xi}{|\xi|^2-1}\,\varphi(2(|\xi|-1))\,d\xi_n = K(\lambda, \bar{\xi})
\end{aligned}$$

where the support of the integrand is contained in

$$A=\{(\bar{\xi}, \xi_n)/|\bar{\xi}|<1/4 \text{ and } 1/2\leqq|\xi_n|\leqq 3/2\}.$$

It suffices to consider only the case where $\xi_n>0$.

For $(\bar{\xi}, \xi_n)\in A$ it follows that $|\xi|^2-1=\xi_n^2+|\bar{\xi}|^2-1=\xi_n^2-\mu^2$ with $\mu>1/2$. Then $K=K^+ + K^-$ with

$$\begin{aligned} K^+(\lambda, \bar{\xi}) &= \theta(4|\bar{\xi}|)\int_0^\infty e^{i\lambda\xi_n}\frac{\xi}{(\xi_n-\mu)(\xi_n+\mu)}\psi(\bar{\xi}, \xi_n)\,d\xi_n \\ &= \theta(4|\bar{\xi}|)\int_{-\infty}^\infty e^{i\lambda\xi_n}\frac{1}{\xi_n-\mu}\tilde{\psi}(\bar{\xi}, \xi_n)\,d\xi_n \end{aligned}$$

where $\tilde{\psi}\in C_0^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^+)$ with support and uniform estimates in

$\{\bar{\xi}\in\mathbb{R}^{n-1}/|\bar{\xi}|\leqq 1/4\}$, which proves the claim.

To complete the proof of (2.9) one just needs to use the same argument given at the end of the proof of the estimate (2.8). □

ESTIMATES FOR THE MAXIMAL FUNCTION

We begin by stating an L^2-continuity result for the maximal function $\sup_{[0,T]}|e^{it\partial_x^2}\,.\,|$

THEOREM 3.1. — *Let $n=1$. Then for any $s>1/2$ and any $\rho>1/4$*

$$\left(\sum_{j=-\infty}^{\infty}\sup_{[0,T]}\sup_{j\leqq x<j+1}|e^{it\partial_x^2}u_0(x)|^2\right)^{1/2}\leqq c(1+T)^\rho\|u_0\|_{s,2}. \quad \square \tag{3.1}$$

Proof (*see* [22], Corollary 2.8 and [40]). —
Observe that (3.1) implies that

$$\left(\int_{-\infty}^{\infty}\sup_{[0,T]}|e^{it\partial_x^2}u_0(x)|^2\,dx\right)^{1/2}\leqq c(1+T)^\rho\|u_0\|_{s,2}. \tag{3.2a}$$

The following global estimate was established in [26]

$$\left(\int_{-\infty}^{\infty}\sup_{(-\infty,\infty)}|e^{it\partial_x^2}u_0(x)|^4\,dx\right)^{1/4}\leqq c\|D_x^{1/4}u_0\|_2. \tag{3.2b}$$

In section 4 and for future references we shall need the following extension of Theorem 3.1 to the n-dimensional case.

Let $\{Q_\alpha\}_{\alpha\in\mathbb{Z}^n}$ denote the mesh of dyadic cubes of unit size.

THEOREM 3.2. — *With the above notation, for any s,* $\rho > n/2$

$$\Big(\sum_{\alpha \in \mathbb{Z}^n} \sup_{[0,T]} \sup_{x \in Q_\alpha} |e^{it\Delta} u_0(x)|^2\Big)^{1/2} \leq c(1+T)^\rho \|u_0\|_{s,2}. \tag{3.3}$$

In particular,

$$\left(\int_{\mathbb{R}^n} \sup_{[0,T]} |e^{it\Delta} u_0(x)|^2 \, dx\right)^{1/2} \leq c(1+T)^\rho \|u_0\|_{s,2}. \quad \square \tag{3.4}$$

Observe that if one is only interested in the inequality (3.4), this was proved for $s > 1$ in all dimensions in [1]. It suffices to prove (3.3) in the time interval [0, 1] since the general case follows by using a simple homogeneity argument (*see* [22], section 2).

First we shall prove the following result,

LEMMA 3.3. — *Let* $\psi_k \in C_0^\infty([2^{k-1}, 2^{k+1}])$ *such that* $0 \leq \psi_k(x) \leq 1$. *Then for* $k \geq 4$ *and* $t \in [0, 2]$

$$\left|\int_{\mathbb{R}^n} e^{i(t|\xi|^2 + x\xi)} \psi_k(|\xi|) \, d\xi\right| \leq c \, H_k(|x|) \tag{3.5}$$

where $H_k(|\,.\,|)$ *is decreasing,*

$$\int_{\mathbb{R}^n} H_k(|x|) \, dx \leq c \, 2^{kn}, \tag{3.6}$$

and $H_k(r) \leq c \, 2^{kn}$ *for* $r \in (0, 10)$. *Also a similar result holds for* $\psi \in C_0^\infty([-10, 10])$ *with* 2^{kn} *replaced by c.* $\square$

Proof. — We divide the proof in three steps:

Sept 1. — One dimensional version of Lemma 3.3.

PROPOSITION 3.4. — *For* $k \in \mathbb{Z}^+$, $k \geq 4$ *and* $t \in [0, 2]$ *define*

$$I(t, r) = \int_{-\infty}^{\infty} e^{i(ts^2 - rs)} \psi_k(s) \, ds.$$

Then

$$|I(t, r)| \leq F(t, r) = \begin{cases} c \, 2^k & \text{for } r \in (0, 1) \\ c\left(\dfrac{2^k}{r}\right)^{1/2} & \text{for } r \in [1, c\,2^k] \\ c_N r^{-N} & \text{for } r > c\,2^k \end{cases}$$

for any $N \in \mathbb{Z}^+$. $\square$

First we observe that a related result was established in [22]. To prove Proposition 3.4 we need the following version of the classical Van de Corput lemma.

LEMMA 3.5. – *Let* $\varphi \in C_0^\infty(\mathbb{R})$ *and* $\phi \in C^2(\mathbb{R})$ *such that* $\phi''(\xi) > \lambda > 0$ *on the support of* φ. *Then*

$$\left| \int e^{i\phi(\xi)} \varphi(\xi)\, d\xi \right| \leqq 10\, \lambda^{-1/2} \{ \|\varphi\|_\infty + \|\varphi'\|_1 \}.$$

Proof (*see* [32], pp. 309-311).

Proof of Proposition 3.4. – Define

$$\Omega = \left\{ s \in \mathbb{R}^+ \Big/ \left| s - \frac{r}{2t} \right| < \frac{r}{4t} \right\}$$

and

$$I_k = [2^{k-1}, 2^{k+1}]$$

For $r \in (0, 1)$ is clear that $|I(t, r)| \leqq c\, 2^k$.

For $r > 1$ we consider three cases:

(i) Ω to the left of I_k

(ii) $\Omega \cap I_k \neq \phi$

(iii) Ω to the right of I_k.

In cases (i)-(ii) we have $t \geqq c\, \frac{r}{2^k}$. Since the phase function $\phi_r(s) = ts^2 - rs$ verifies $\phi_r''(s) = 2t \geqq c\, \frac{r}{2^k}$ by Van der Corput's lemma (lemma 3.5) one has $|I(t, r)| \leqq c \left(\frac{2^k}{r} \right)^{1/2}$.

In case (iii) we have $\frac{r}{2t} \geqq c\, 2^k$ with $|\phi_r'(s)| = |2ts - r| \geqq r/2$.

Hence integration by parts shows that

$$I(t, r) = \int_0^\infty e^{i\phi_r(s)} \frac{d}{ds} \left(\frac{1}{\phi_r'(s)} \psi_k(s) \right) ds$$
$$= \int_0^\infty e^{i\phi_r(s)} \left\{ - \frac{\phi_r''(s)}{(\phi_r'(s))^2} \psi_k(s) + \frac{1}{\phi_r'(s)} \psi_k'(s) \right\} ds \leqq \frac{c}{r}.$$

Since $r > 1$ it follows that [case (iii)]

$$\frac{c}{r} < c\, \frac{2^{k/2}}{r^{1/2}}.$$

Finally, when $r \geqq c\, 2^k$ we always have Ω to the right of I_k, *i.e.* $r/2t \geqq c\, 2^k$. Therefore integrating by parts as in (iii) N-times we obtain the bound d_N / r^N. □

Step 2. – The Fourier transform of a radial function $f(|x|)=f(s)$ is given by the formula

$$\hat{f}(r)=\hat{f}(|\xi|)=r^{-(n-2)/2}\int_0^\infty f(s)\,J_{(n-2)/2}(rs)\,s^{n/2}\,ds. \tag{3.7}$$

Thus to extend the results in step 1 to higher dimension we need the following asymptotic estimates for the Bessel fnctions

$$J_m(r)=\frac{(r/2)^m}{\Gamma(2m+1/2)\Gamma(1/2)}\int_{-1}^{1} e^{irs}(1-s^2)^{(2m-1)/2}\,ds.$$

LEMMA 3.6

$$J_m(r)=0(r^m)\quad \text{as } r\to 0 \tag{3.8a}$$

$$J_m(r)=e^{-ir}\sum_{j=0}^{N}\alpha_{m,j}r^{-(j+1/2)}+e^{ir}\sum_{j=0}^{N}\alpha_{m,j}r^{-(j+1/2)}+e^{ir}\sum_{j=0}^{N}\beta_{m,j}r^{-(j+1/2)}$$
$$+0(r^{-(N+3/2)})\quad \text{as } r\to\infty \tag{3.8b}$$

for each $N\in\mathbb{Z}^+$. –

Proof. – (3.8*a*) (*see* [33], p. 158).

(3.8*b*) It follows by combining the identity

$$\int_{-1}^{1} e^{irs}(1-s^2)^{(sm-1)/2}\,ds=e^{-ir}\int_0^\infty e^{-ry}(y^2+2(y))^{m-1/2}\,dy$$
$$-e^{ir}\int_0^\infty e^{-ry}(y^2-2iy)^{m-1/2}\,dy=e^{-ir}I_1-e^{ir}I_2$$

with the Taylor expansion for $(y^2\pm 2iy)^{m-1/2}$ and the estimates

$$\int_1^\infty e^{-yr}y^l\,dy=0(e^{ir/2})\quad \text{as } r\to\infty,$$

$$\int_1^0 e^{-ry}y^l\,dy=0(r^{-(l+1)})\quad \text{as } r\to\infty.$$

Step 3. – Proof of Lemma 3.3.

Using (3.7) we write

$$\tilde{I}(t,r)\equiv\left|\int_{\mathbb{R}^n} e^{it|\xi|^2}e^{ix\xi}\psi_k(|\xi|)\,d\xi)\right|$$
$$=\frac{1}{r^{(n-2)/2}}\left|\int_0^\infty e^{its^2}J_{(n-2)/2}(rs)\,\psi_k(s)\,s^{n/2}\,ds\right|$$

where $r=|x|$.

When $r\in(0,1)$ the second term in (3.9) shows that $\tilde{I}(t,r)\leq c\,2^{nk}$.

For $r>1$, we first plug in (3.9) the remainder term in the development of $J_{(n/2)/2}$ (3.8 *b*) to obtain the bound

$$\frac{2^{kn/2}}{r^{(n-2)/2}}\,\frac{1}{r^{N+3/2}}\,\frac{1}{2^{k(N+1/2)}}.$$

We shall fix N so large that

$$\frac{n-2}{2}+N+\frac{3}{2}>n.$$

Next we deal with the *j*-term in (3.8 *b*) with $0\leqq j\leqq N$

$$\frac{1}{r^{(n-2)/2}}\int_0^\infty e^{it^2 s}e^{isr}\frac{1}{(sr)^{j+1/2}}s^{n/2}\psi_k(s)\,ds$$
$$=\frac{1}{r^{(n-2)/2}}\,\frac{1}{r^{j+1/2}}\,2^{kn/2}\,2^{-(j+1/2)k}\int_0^\infty e^{it^2 s}e^{-isr}\tilde{\psi}_k(s)\,ds.$$

By the step 1 the last integral is bounded by

$$\begin{cases} c\left(\dfrac{2^k}{r}\right)^{1/2} & \text{if } r\in[1,\,c\,2^k] \\ c_m r^{-m} & \text{if } r>c\,2^k. \end{cases}$$

To finish the proof let us compute the L^1 norm. For $r\in[0,\,1]$ we obtain the bound $c\,2^{kn}$. In the annulus $r\in[1,\,c\,2^k]$ we have (after some computations) that

$$2^{kn/2}\,2^{k/2}\,2^{-(j+1/2)k}\int_1^{c2^k}\frac{r^{n-1}}{r^{(n-2)/2}\,r^{j+1/2}\,r^{3/2}}\,dr\leqq c\,2^{kn}\,2^{-2jk}.$$

Finally, the integral in $r\geqq c\,2^k$ is clearly bounded by

$$2^{kn/2}\,2^{-(j+1/2)k}\leqq 2^{kn}.\quad\square$$

Proof of Theorem 3.2. – Let $\{\psi_k\}_{k=0}^\infty$ be a smooth partition of unity in $\mathbb{R}^n$ such that the ψ_k's are radial with $\operatorname{supp}\psi_0\subseteqq\{|\xi|\leqq 1\}$ and $\operatorname{supp}\psi_k\leqq\{2^{k-1}\leqq|\xi|\leqq 2^k\}$ $k=1,\ldots$ Let

$$(W_k(t)\,u_0)^\wedge(\xi)=e^{it|\xi|^2}\psi_k(|\xi|)\,\hat{u}_0(\xi).$$

To prove (3.3) with $T=1$ it suffices to show that

$$\left(\sum_{\gamma\in\mathbb{Z}^n}\sup_{|t|<1}\sup_{x\in Q_\gamma}\left|\int_{-1}^1 W_k(t-\tau)\,g(.,\,\tau)\,d\tau\right|^2\right)^{1/2}\qquad(3.10)$$

As in the one dimensional case (*see* [22]) the proof of (3.10) is obtained by showing that

$$\left(\sum_{\gamma\in\mathbb{Z}^n}\sup_{|t|<1}\sup_{x\in Q_\gamma}\left|\int_{-1}^{1}W_k(t-\tau)g(.,\tau)\,d\tau\right|^2\right)^{1/2} \leqq c\,2^{nk}\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\int_{-1}^{1}\int_{Q_\gamma}|g(x,t)|\,dx\,dt\right)^2\right)^{1/2}. \quad (3.11)$$

By using lemma 3.5 we find that

$$\left|\int_{-1}^{1}W_k(t-\tau)g(.,\tau)\,d\tau\right| \leqq \int H_k(|y|)\int_{-1}^{1}|g(x-y,\tau)|\,d\tau\,dy \leqq \sum_{\gamma\in\mathbb{Z}^n}\left(\sup_{y\in Q_\gamma}H_k(y)\right)\int_{-1}^{1}\int_{Q_\gamma}|g(x-y,\tau)|\,d\tau\,dy.$$

Thus the left hand side of (3.11) is bounded by

$$\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\sum_{\alpha\in\mathbb{Z}^n}(\sup_{y\in Q_\alpha}H_k(|y|))\ \sup_{x\in Q_\gamma}\int_{-1}^{1}\int_{Q_\alpha}|g(x-y,\tau)|\,d\tau\,dy\right)^2\right)^{1/2} \quad (3.12)$$

Let $E_{\alpha,\gamma}=2^n Q_\alpha - x_\gamma$, where x_γ denotes the center of Q_γ. Then

$$\sup_{x\in Q_\gamma}\int_{-1}^{1}\int_{Q_\alpha}|g(x-y,\tau)|\,d\tau\,dy \leqq \int_{-1}^{1}\int_{E_{\alpha,\gamma}}|g(z,\tau)|\,d\tau\,dz,$$

and consequently using Minkowski's inequality we estimate the expression in (3.12) by

$$\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\sum_{\alpha\in\mathbb{Z}^n}(\sup_{y\in Q_\alpha}H_k(|y|))\int_{-1}^{1}\int_{E_{\alpha,\gamma}}g(z,\tau)|\,d\tau\,dz\right)^2\right)^{1/2} \leqq \sum_{\alpha\in\mathbb{Z}^n}(\sup_{y\in Q_\alpha}H_k(|y|))\left(\sum_{\gamma\in\mathbb{Z}^n}\int_{-1}^{1}\int_{E_{\alpha,\gamma}}|g(z,\tau)\,d\,d\tau\,dz\right)^2\right)^{1/2}.$$

But

$$\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\int_{-1}^{1}\int_{E_{\alpha,\gamma}}|g(z,\tau)\,d\tau\,dz\right)^2\right)^{1/2} \leqq c\left(\sum_{\gamma\in\mathbb{Z}^n}\left(\int_{-1}^{1}\int_{Q_\gamma}|g(z,\tau)|\,d\tau\,dz\right)^2\right)^{1/2}$$

with c independent of α, since $E_{\alpha,\gamma}$ overlaps a finite number (independent of α) of Q_γ's.

Hence the proof reduces to show that

$$\sum_{\alpha\in\mathbb{Z}^n}\sup_{y\in Q_\alpha}|H_k(|y|)|\leqq c\,2^{kn}.$$

Let

$$\mathcal{F}_m = \{ Q_\alpha / Q_\alpha \cap \{ 2^m \leqq |y| \leqq 2^{m+1} \} \neq \phi \}, \qquad m = 1, \ldots$$

$$\mathcal{F}_0 = \{ Q_\alpha / Q_\alpha \cap \{ |y| < 2 \} = \phi \}.$$

Notice that if $m \geqq 4$ and $Q_{\alpha_0} \in \mathcal{F}_m$ then $Q_{\alpha_0} \subseteqq \{ 2^{m-2} \leqq |y| \leqq 2^{m+3} \}$. Therefore we see that

$$\begin{aligned}\sum_{\alpha \in \mathbb{Z}^n} \sup_{y \in Q_\alpha} H_k(|y|) &\leqq \sum_{m=0}^{\infty} \sum_{q_\alpha \in \mathcal{F}_m} \sup_{y \in Q_\alpha} H_k(|y|) \\ &\leqq \sup_{|y| \leqq 10} H_k(|y|) + \sum_{m \geqq 4} H_k(2^{m-2}) |\{ 2^{m-2} \leqq |y| \leqq 2^{m+3} \}| \\ &\leqq c\, 2^{nk} + c \int H_k(|y|)\, dy \leqq c\, 2^{nk}\end{aligned}$$

as desired. □

To estimate the maximal function $\sup_{[0, T]} |e^{it\Delta} u_0|(x)$ in the L^1, l^1-norms we shall use the following weighted inequalities.

PROPOSITION 3.7. – (i) *For* $n = 1$

$$\int_{-\infty}^{\infty} \sup_{[0, T]} |e^{it\partial_x^2} u_0(x)|\, dx \leqq c(1+T)^2 (\| u_0 \|_{3, 2} + \| u_0 \|_{2, 2, 2}). \tag{3.13}$$

(i) *For* $n \geqq 2$

$$\sum_{\alpha \in \mathbb{Z}^n} \sup_{[0, T]} \sup_{x \in Q_\alpha} |e^{it\Delta} u_0(x)| \leqq c(1+T)^{n+3} \{ \| u_0 \|_{2n+2, 2} + \| u_0 \|_{2n+2, 2, 2n+2} \} \tag{3.14}$$

where

$$\| f \|_{l, 2, j} = \sum_{|\gamma| \leqq l} \left(\int_{\mathbb{R}^n} |\partial_x^\gamma f(x)|^2 |x|^j\, dx \right)^{1/2}. \quad \square$$

Proof. – (i) By Sobolev's and Fubini's theorems we see that

$$\begin{aligned}\int_{-\infty}^{\infty} \sup_{[0, T]} |e^{it\partial_x^2} u_0(x)\, dx &\leqq \frac{c}{T} \int_{-\infty}^{\infty} \int_0^T |e^{it\partial_x^2} u_0|\, dt\, dx + c \int_{-\infty}^{\infty} \int_0^T |\partial_t e^{it\partial_x^2} u_0|\, dt\, dx \\ &= \frac{c}{T} \int_0^T \int_{-\infty}^{\infty} |e^{it\partial_x^2} u_0|\, dx\, dt + c \int_0^T \int_{-\infty}^{\infty} |e^{it\partial_x^2} \partial_x^2 u_0|\, dx\, dt. \end{aligned} \tag{3.15}$$

Inserting the formulas

$$\int_{-\infty}^{\infty} |g(x)|\, dx \leqq c \| g \|_2 + c \| xg \|_2,$$

and

$$x e^{it\partial_x^2} u_0 = e^{it\partial_x^2}(x u_0) - 2\, it\, e^{it\partial_x^2} \partial_x u_0$$

(*see* [8], [14]) in (3.15) we obtain the desired estimate (3.13).

(ii) As in the previous proof we have by using Sobolev's inequality in the cylinder $Q_\alpha \times [0, T]$ that

$$\sum_{\alpha \in \mathbb{Z}^n} \sup_{[0,\, T]} \sup_{x \in Q_\alpha} |e^{it\Delta} u_0(x)|$$
$$\leqq c \left\{ \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} |e^{it\Delta} u_0|\, dx\, dt + \sum_{|\beta| \leqq n+2} \int_0^T \int_{\mathbb{R}^n} |e^{it\Delta} \partial_x^\beta u_0|\, dx\, dt \right\}.$$

To estimate the L^1-norm above we combine the formulas

$$\| f \|_1 \leqq c \| f \|_2 + c \| \, |x|^{\bar{n}} f \|_2$$

where $\bar{n} = n+1$ if n is odd and $\bar{n} = n+2$ of n is even,

$$x_j e^{it\Delta} f(x, t) \equiv e^{it\Delta}(x_j f) - 2\, it\, e^{it\Delta}(\partial_{x_j} f),$$

and

$$x_k x_j e^{it\Delta} f(x, t) \equiv e^{it\Delta}(x_k x_j f) - 2\, it\, e^{it\Delta}(\delta_{kj} f + x_j \partial_{x_k} f) - 2\, it e^{it\Delta}(x_k \partial_{x_j} f) - 4\, t^2 e^{it\Delta}(\partial_{x_k} \partial_{x_j} f),$$

(*see* [15]) to infer that

$$\sum_{\alpha \in \mathbb{Z}^n} \sup_{[0,\, T]} \sup_{x \in Q_\alpha} |e^{it\Delta} u_0(x)| \leqq c(1+T)^{n+3} (\| u_0 \|_{2n+2,\, 2} + \| u_0 \|_{2n+2,\, 2,\, 2n+2}).$$

which completes the proof. □

4. THE NONLINEAR SCHRÖDINGER EQUATION

Consider the nonlinear IVP.

$$\left.\begin{aligned} \partial_t u = i \Delta u + P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\ u(x, 0) = u_0(x) \end{aligned}\right\} \tag{4.1}$$

where P denotes a complex valued polynomial defined in $\mathbb{C}^{2n+2}$ such that

$$P(\vec{z}) = P(z_1, \ldots, z_{2n+2}) = \sum_{\substack{d \leqq |\alpha| \leqq \rho \\ \alpha \in \mathbb{Z}^{2n+2}}} a_\alpha z^\alpha. \tag{4.2}$$

Assuming that for some $\alpha_0 \in \mathbb{Z}^{2n+2}$ with $|\alpha_0| = d$ there exists $a_{\alpha_0} \neq 0$ we shall consider four cases $d = 2$ or $d \geqq 3$ and $n = 1$ or $n \geqq 2$.

THEOREM 4.1 (Case $n=1$ and $d \geqq 3$). — *Let* $n=1$. *Then given any polynomial* P *as in* (4.2) *with* $d \geqq 3$ *there exists* $\delta = \delta(P) > 0$ *such that for* *any* $u_0 \in H^s(\mathbb{R})$ *with* $s \geqq 7/2$ *and* $\|u_0\|_{7/2,2} < \delta$ *the* IVP (4.1) *has a unique solution* $u(.)$ *defined in the time interval* $[0, T]$, $T = T(\|u_0\|_{7/2,2}) > 0$ *with* $T(\theta) \to \infty$ *as* $\theta \to 0$ *satisfying*

$$u \in C([0, T] : H^s(\mathbb{R})) \equiv X_T^s \tag{4.3}$$

and

$$u \in Y_T^s \equiv \{ u : \mathbb{R}^n \times [0, T] \to \mathbb{C} / D_x^{s+1/2} u \in L^\infty(\mathbb{R} : L^2([0, T])) \}. \tag{4.4}$$

Moreover, for any $T' \in (0, T)$ *there exists* $\varepsilon > 0$ *such that the map* $\tilde{u}_0 \to \tilde{u}(t)$ *from* $\{ \tilde{u}_0 \in H^s(\mathbb{R}) / \| \tilde{u}_0 - u_0 \|_{s,2} < \varepsilon \}$ *into* $X_{T'}^s \cap Y_{T'}^s$ *is Lipschitz.* □

Proof. – For simplicity in the exposition we shall only consider the most interesting case $s = 3 + 1/2$. The general case follows by combining this result with the fact that the highest derivatives involved in that proof always appear linearly and some commutator estimates (*see* [4], [5], [20]) for the cases where $s \neq k + 1/2$, $k \in \mathbb{Z}^+$.

For $u_0 \in H^{7/2}(\mathbb{R})$ with $\|u_0\|_{7/2,2} \leqq \delta$ (to be determined below) we denote by $\Phi(v) = \Phi_{u_0}(v) = u$ the solution of the linear inhomogenous IVP

$$\left.\begin{aligned} \partial_t u &= i \partial_x^2 u + P(v, \partial_x v, \bar{v}, \partial_x \bar{v}) \\ u(x, 0) &= u_0(x) \end{aligned}\right\} \tag{4.5}$$

where

$$v \in Z_T^a = \left\{ v : \mathbb{R} \times [0, T] \to \mathbb{C} / \sup_{[0,T]} \|v(t)\|_{7/2,2} \leqq a, \right.$$
$$\sup_x \left(\int_0^T |\partial_x^4 v(x, t)|^2 \, dt \right)^{1/2} \leqq a,$$

and

$$\left. (1+T)^{-1} \left(\int_{-\infty}^{\infty} \sup_{[0,T]} (|v(x, t)|^2 + |\partial_x v(x, t)|^2) \, dx \right)^{1/2} \leqq a \right\}.$$

It will be established that for appropriate a and T (depending only on $\|u_0\|_{7/2,2}$ in the appropriate manner and $P(.)$) if $v \in Z_T^a$ then the solution $u = \Phi(v)$ of the IVP (4.5) belongs to Z_T^a and

$$\Phi : \quad Z_T^a \to Z_T^a$$

is a contraction.

For this purpose we use the integral equation

$$u(t) = \Phi(v)(t) = e^{it\partial_x^2} u_0 + \int_0^t e^{i(t-\tau)\partial_x^2} P(v, \partial_x v, \bar{v}, \partial_x \bar{v})(\tau) \, d\tau. \tag{4.6}$$

First we notice that

$$\partial_x^3 P(v, \partial_x v, \bar{v}, \partial_x \bar{v}) = \partial_x^4 v R_1(.) + \partial_x^4 \bar{v} R_2(.) + S(.) \tag{4.7}$$

where

$$R_j(.)=R_j(v,\bar{v},\partial_x v,\partial_x\bar{v})\,j=1,2 \quad\text{and}\quad S(.)=S((\partial_x^\alpha v)_{|\alpha|\leq 3};(\partial_x^\alpha\bar{v})_{|\alpha|\leq 3}).$$

Thus combining the estimates (2.8) and (2.1) for the terms in (4.7) involving R's and S respectively it follows that

$$\begin{aligned}
&\sup_x\left(\int_0^T|\partial_x^4 u(x,t)|^2\,dt\right)^{1/2}\leqq c\,\|u_0\|_{7/2,2}\\
&\quad+\sum_{j=1}^{2}c\int_{-\infty}^{\infty}\left(\int_0^T|\partial_x^4 v\,R_j(.)|^2(x,t)\,dt\right)^{1/2}dx+c\int_0^T\|D_x^{1/2}S(.)\|_2\,dt\\
&\quad\leqq c\,\|u_0\|_{7/2,2}+c\left(\sup_x\left(\int_0^T|\partial_x^4 v(x,t)|^2\,dt\right)^{1/2}\right)\\
&\quad\times\left(\int_{-\infty}^{\infty}\sup_{[0,T]}(|v|^2+|v_x|^2)(x,t)\,dx\right)\\
&\quad\times\left\{1+\sup_{[0,T]}\sup_x((|v|+|\partial_x v|)(x,t))^{\rho-3}\right\}\\
&\quad+c\,T\sup_{[0,T]}(\|v(t)\|_{7/2,2}^3\{1+\|v(t)\|_{7/2,2}^{\rho-3}\})\\
&\quad\leqq c\,\|u_0\|_{7/2,2}+c\left(\sup_x\left(\int_0^T|\partial_x^4 v(x,t)|^2\,dt\right)^{1/2}\right)\\
&\quad\times\left(\int_{-\infty}^{\infty}\sup_{[0,T]}(|v|^2+|v_x|^2)(x,t)\,dx\right)\{1+\sup_{[0,T]}\|v(t)\|_{7/2,2}^{\rho-3}\}\\
&\quad+c\,T\sup_{[0,T]}(\|v(t)\|_{7/2,2}^3\{1+\|v(t)\|_{7/2,2}^{\rho-3}\})\equiv D_1. \qquad (4.8)
\end{aligned}$$

Above we have used the commutator estimates obtained in the appendix of [20]. Next we combine the estimate (2.5), the group properties and the integral equations (4.6) to find that

$$\sup_{[0,T]}\|u(t)\|_{7/2,2}\leqq D_1. \qquad (4.9)$$

From the estimate (3.2 *a*) for the maximal function $\sup_{[0,T]}|e^{it\partial_x^2}\cdot|$ we obtain

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty}\sup_{[0,T]}(|u|^2+|\partial_x u|^2)(x,t)\,dx\right)^{1/2}\leqq c\,(1+T)\\
&\qquad\times\{\|u_0\|_{7/2,2}+c\,T^{1/2}\sup_{[0,T]}(\|v(t)\|_{7/2,2}^3\{1+\|v(t)\|_{7/2,2}^{\rho-3}\})\}. \qquad (4.10)
\end{aligned}$$

Inserting the notation

$$\Lambda_T(w)=\max\left\{\sup_x\left(\int_0^T|\partial_x^4 w(x,t)|^2\,dt\right)^{1/2};\ \sup_{[0,T]}\|w(t)\|_{7/2,2};\right.$$
$$\left.(1+T)^{-1}\left(\int_{-\infty}^{\infty}\sup_x(|w|^2+|\partial_x w|^2)(x,t)\,dx\right)^{1/2}\right\} \quad (4.11)$$

in (4.8)-(4.10) we see that

$$\Lambda_T(u)\leqq c\,\delta_0+c\,(1+T)\,((\Lambda_T(v))^3+(\Lambda_T(v))^\rho)$$

with $\delta_0=\|u_0\|_{7/2,2}<\delta$ (to be determined) and where the constant $c>1$ depends only on P(.) and on the linear estimates (2.1), (2.5), (2.8) (3.2 *a*).

First we fix δ such that

$$(4c)^3\,\delta=1 \quad (4.12)$$

and then choose any $a=a(\|u_0\|_{7/2,2})>0$ such that

$$a\in(2c\,\delta_0,\ 4c\,\delta_0). \quad (4.13)$$

Thus

$$\Lambda_T(u)\leqq c\,\delta_0+2c\,(1+T)\,(4c\,\delta_0)^3\leqq 2c\,\delta_0 \quad (4.14)$$

for any T satisfying

$$2\,(4c)^2\,(1+T)\,\delta_0^2\leqq 1. \quad (4.15)$$

Hence, fixing an a as in (4.13) and then a T as in (4.15) we obtain that the map

$$\Phi=\Phi_{u_0}: Z_T^a\to Z_T^a$$

is well defined.

To prove that $\Phi(.)$ is a contraction we apply the estimates described in (4.8)-(4.10) to the integral equation

$$\Phi(v)(t)-\Phi(\tilde v)(t)=\int_0^t e^{i(t-\tau)\partial_x^2}(P(v,\ldots)-P(\tilde v,\ldots))(\tau)\,d\tau.$$

to obtain [*see* notation in (4.11)]

$$\begin{aligned}\Lambda_T(\Phi(v)-\Phi(\tilde v))&\leqq c\,(1+T)\,\Lambda_T(v-\tilde v)\\&\quad\times((\Lambda_T(v))^2+((\Lambda_T(v))^{\rho-1}+((\Lambda_T(\tilde v))^2+((\Lambda_T(\tilde v))^{\rho-1})\\&\leqq 4c\,(1+T)\,a^2\,\Lambda_T(v-\tilde v)\leqq 4c\,(1+T)\,(4c)^2\,\delta_0^2\,\Lambda_T(v-\tilde v)\end{aligned} \quad (4.16)$$

since $v,\ \tilde v\in Z_T^a$.

From (4.15) it is clear that T can be chosen such that

$$4c\,(1+T)\,(4c)^2\,\delta_0^2\leqq 1/2. \quad (4.17)$$

Therefore for those T's the map $\Phi_{u_0}(.)$ is a contraction in Z_T^a. Consequently, there exists a unique $u \in Z_T^a$ with $\Phi_{u_0}(u) = u$ which due its regularity solves the IVP (4.1).

Since $u \in Z_T^a$ it follows that

$$u \in L^\infty([0, T] : H^{7/2}) \tag{4.18}$$

which combined with the integral equation $\Phi_{u_0}(u) = u$ shows that

$$u \in C([0, T] : H^{5/2}). \tag{4.19}$$

From (4.18)-(4-19) one can only conclude that for any $\varepsilon > 0$

$$u \in C([0, T] : H^{7/2-\varepsilon}).$$

To establish the persistence property of $u(t)$ in $H^{7/2}$, *i.e.*

$$u \in C([0, T] : H^{7/2}) \tag{4.20}$$

one needs to use the following argument:

(i) Since (4.2) described a local property it suffices to prove it at $t = 0$.

(ii) Inserting the estimates (4.12)-(4.14), (4.19) in the integral equation $\Phi_{u_0}(u) = u$ as in (4.8) one sees that for T_0 sufficiently small

$$\sup_x \left(\int_0^{T_0} |\partial_x^4 u(x, t)|^2 \, dt \right)^{1/2} \leqq 2C \sup_x \left(\int_0^{T_0} |\partial_x^4 e^{it\partial_x^2} u_0|^2 \, dt \right)^{1/2} + o(1) \tag{4.21}$$

as T_0 tends to zero.

(iii) For $\varepsilon > 0$ let $u_0^\varepsilon \in H^\infty(\mathbb{R})$ such that $\| u_0 - u_0^\varepsilon \|_{7/2, 2} \leqq \varepsilon$, that it follows from (2.8) that

$$\begin{aligned} \sup_x \left(\int_0^{T_0} |\partial_x^4 e^{it\partial_x^2} u|^2 \, dt \right)^{1/2} & \leqq c \| u_0 - u_0^\varepsilon \|_{7/2, 2} + c \sup_x \left(\int_0^{T_0} |\partial_x^4 e^{it\partial_x^4} u_0^\varepsilon|^2 \, dt \right)^{1/2} \\ & \leqq c\varepsilon + c \left(\int_0^{T_0} \sup_x |e^{it\partial_x^4} \partial_x^4 u_0^\varepsilon|^2 \, dt \right)^{1/2} \leqq c\varepsilon + c T_0 \| u_0^\varepsilon \|_{5, 2}. \end{aligned} \tag{4.22}$$

Therefore from (4.21)-(4.22) one can conclude that

$$\sup_x \left(\int_0^{T_0} |\partial_x^4 u(x, t)|^2 \, dt \right)^{1/2} = o(1) \tag{4.23}$$

as T_0 tends to zero.

(iv) Finally combining the integral equation

$$u(t) - u_0 = e^{it\partial_x^4} u_0 - u_0 + \int_0^t e_x^{i(t-\tau)\partial^2} P(u, \ldots)(\tau) \, d\tau$$

with the estimate (2.5) and the group properties as in (4.8)-(4.9) together with (4.23) one obtains the desired result.

Using a similar argument we shall establish the uniqueness result in a class larger than Z_T^a. Let $\tilde{u}$ be a solution of the IVP (4.1) in the time interval $[0, T_1]$ with $T_1 < T$. Moreover assume that $\tilde{u} \in Z_{T_1}^{a_1}$ for some $a_1 > a$, with $\tilde{u} \in C([0, T_1] : H^{7/2})$. Thus $\tilde{u}$ satisfies the integral equation form of (4.1). By continuity there exists $T_2 \in (0, T_1)$ such that

$$\sup_{[0, T_2]} \| \tilde{u}(t) \|_{7/2, 2} \leqq a.$$

Combining this estimate with (3.2*a*) as in (4.9) one finds that there exists $T_3 \in (0, T_2)$ such that

$$\left(\int_{-\infty}^{\infty} \sup_{[0, T_3]} (|\tilde{u}|^2 + |\partial_x \tilde{u}|^2)\, dx \right)^{1/2} \leqq a.$$

A similar technique shows that for $T_4 \in (0, T_3)$

$$\sup_x \left(\int_0^{T_4} |\partial_x^4 \tilde{u}(x, t)|^2\, dt \right)^{1/2} \leqq a.$$

Hence $\tilde{u} \in Z_{T_4}^a$ and consequently $u \equiv \tilde{u}$ for $(x, t) \in \mathbb{R} \times [0, T_4]$. Reapplying this proccess we extend the uniqueness result to the interval $[0, T]$.

To complete the proof of Theorem 4.1 we need to establish the continuous dependence. Denoting by $u(t)$, $v(t)$ the corresponding solution of the IVP (4.1) with initial value u_0, v_0 respectively we write

$$u(t) - v(t) = e^{it\partial_x^2}(u_0 - v_0) + \int_0^t e^{i(t-\tau)\partial_x^2}(P(u...) - P(v...))\, d\tau.$$

The same proof of the contraction property of $\Phi_{u_0}(.)$, (4.16), (4.17), shows that

$$\Lambda_{T_0}(u - v) \leqq \| u_0 - v_0 \|_{7/2, 2} + K\, \Lambda_{T_0}(u - v)$$

where the constant K depends only on P, $\| u_0 \|_{7/2}$, $\| v_0 \|_{7/2}$ and $T_0 \in (0, T)$. Indeed, this argument shows that the constant K can be taken smaller than 1/2 if

$$\| u_0 - v_0 \|_{7/2, 2} < \delta$$

with $\delta = \delta(T - T_0) > 0$ is sufficiently small. □

THEOREM 4.2 (case $n=1$ and $d=2$). – *Let $n=1$. Then given any polynomial* P *as in* (4.2) *with $d=2$ there exists $\delta = \delta(P) > 0$ such that for any* $u_0 \in H^s(\mathbb{R}) \cap H^3(\mathbb{R} : x^2\, dx) \equiv G_s(\mathbb{R})$ with $s \geqq 5 + 1/2$ *and*

$$\delta_0 = \| u_0 \|_{11/2, 2} + \| u_0 \|_{3, 2, 2} \leqq \delta$$

the IVP (4.1) *has a unique solution* $u(.)$ *defined in the interval* $[0, T]$, $T=T(\delta_0)>0$ with $T(\theta)\to\infty$ *as* $\theta\to 0$ *satisfying*

$$u\in C([0, T]: G_s(\mathbb{R}))\equiv\tilde{X}_T^s$$

and

$$u\in Y_T^s,$$

where Y_T^s *was defined in the statement of Theorem* 4.1. *Moreover, for any* $T'\in(0, T)$ *there exists a neighborhood* V_{u_0} *of* u_0 *in* $G_s(\mathbb{R})$ *such that the map* $\tilde{u}_0\to\tilde{u}(t)$ *from* V_{u_0} *into* $\tilde{X}_{T'}^s\cap\tilde{Y}_{T'}^s$ *is Lipschitz.* □

We recall the notation $\|\cdot\|_{k,2,m}$

$$\|f\|_{k,2,m}=\sum_{|\beta|\leqq k}\left(\int_{\mathbb{R}^n}|\partial_x^\beta f(x)|^2|x|^m\,dx\right)^{1/2} \tag{4.24}$$

Proof. – For simplicity in the exposition we shall assume

$$P(u, \partial_x u, \bar{u}, \partial_x\bar{u})=(\partial_x u)^2.$$

It will be clear from the argument presented below that this does not represent any loss of generality.

As in the proof of Theorem 4.1 we consider the most interesting case $s=5+1/2$.

For $u_0\in H^{11/2}(\mathbb{R})\cap H^3(\mathbb{R}:x^2\,dx)$ with $\|u_0\|_{11/2,2}+\|u_0\|_{3,2,2}<\delta$ (to be determined below) we denote by $\Phi(v)=\Phi_{u_0}(v)=u$ the solution of the linear problem

$$\left.\begin{aligned}\partial_t u&=i\,\partial_x^2 u+(\partial_x v)^2\\ u(x, 0)&=u_0(x)\end{aligned}\right\} \tag{4.25}$$

where

$$v\in E_T^a=\Big\{v:\mathbb{R}\times[0, T]\to\mathbb{C}/$$

$$\sup_{[0,T]}\|v(t)\|_{11/2,2}\leqq a,\ \sup_{[0,T]}\|v(t)\|_{3,2,2}\leqq a,\ \sup_x\left(\int_0^T|\partial_x^6 v(x,t)|^2\,dt\right)^{1/2}\leqq a$$

$$\text{and } (1+T)^{-2}\int_{-\infty}^{\infty}\sup_{[0,T]}|\partial_x v(x,t)|\,dx\leqq a\Big\}.$$

It shall be established that for appropriate a and T (depending only on $\|u_0\|_{11/2,2}+\|u_0\|_{3,2,2}$ in the appropiate manner) the map

$$\Phi:\ E_T^a\to E_T^a$$

is a contraction.

We will use that for $t \in [0, T]$ $u = \Phi(v)$ the solution of (4.25) safisfies the integral equation

$$u(t) = \Phi(v)(t) = e^{it\partial_x^4} u_0 + \int_0^t e^{i(t-\tau)\partial_x^2} (\partial_x v)^2(\tau)\, d\tau. \qquad (4.26)$$

Since

$$\partial_x^5((\partial_x v)^2) = c_1\, \partial_x v\, \partial_x^6 v + S((\partial^\alpha v)_{|\alpha| \leq 5})$$

combining (2.8), (2.1) as in (4.8) it follows that

$$\begin{aligned}
\sup_x \left(\int_0^T |\partial_x^6 u(x, t)|^2 dt \right)^{1/2} &\leq c\|u_0\|_{11/2, 2} + c\int_{-\infty}^{\infty} \left(\int_0^T |(\partial_x v\, \partial_x^6 v)(x, t)|^2 dt \right)^{1/2} dx \\
&+ c\int_0^T \|D_x^{1/2} S(.)\|_2\, dt \leq c\|u_0\|_{11/2, 2} \\
&+ c\left(\sup_x \left(\int_0^T |\partial_x^6 v(x, t)|^2 dt \right)^{1/2} \right) \\
&\times \left(\int_{-\infty}^{\infty} \sup_{[0, T]} |\partial_x v(x, t)|\, dt \right) + cT \sup_{[0, T]} \|v(t)\|_{11/2, 2}^2 \equiv D_2. \qquad (4.27)
\end{aligned}$$

As in the previous proof above we have used the commutator estimates obtained in [20].

Writing the equation (4.6) as

$$u(t) = e^{it\partial_x^2}\left(u_0 + \int_0^t e^{-it\partial_x^2} (\partial_x v)^2(\tau)\, d\tau \right) \qquad (4.28)$$

from (3.13) we have that

$$\begin{aligned}
(1+T)^{-2} \int_{-\infty}^{\infty} \sup_{[0, T]} |\partial_x u(x, t)|\, dx &\leq c(\|u_0\|_{4, 2} + \|u_0\|_{3, 2, 2}) + cT \sup_{[0, T]} \|v(t)\|_{5, 2}^2 \\
+ \sum_{j=0}^{2} \int_0^T \|xe^{-it\partial_x^2} \partial_x^{j+1}((\partial_x v)^2)(\tau)\|_2\, d\tau &\leq c\delta_0 + cT(1+T) \sup_{[0, T]} \|v(t)\|_{5, 2}^2 \\
+ \sum_{j=0}^{2} \int_0^T \|x\partial_x^{j+1}((\partial_x v)^2)(\tau)\|_2\, d\tau &\leq c\delta_0 + cT(1+T) \sup_{[0, T]} \|v(t)\|_{5, 2}^2 \\
&+ T(\sup_{[0, T]} \|v(t)\|_{5, 2})(\sup_{[0, T]} \|v(t)\|_{3, 2, 2}), \qquad (4.29)
\end{aligned}$$

since $\Gamma(x, \tau) = x + 2i\tau\partial_x = e^{i\tau\partial_x^2} x e^{-i\tau\partial_x^2}$ and

$$\|xe^{-i\tau\partial_x^2} f\|_2 = \|\Gamma(x, \tau) f\|_2 \leq \|xf\|_2 + 2\tau\|\partial_x f\|_2,$$

(*see* [14]). The same estimate shows that

$$\sup_{[0,T]} \|u(t)\|_{3,2,2} \leqq c\|u_0\|_{3,2,2} + c\mathrm{T}\|u_0\|_{4,2} + c(1+\mathrm{T}^2) \sup_{[0,T]} \|v(t)\|_{5,2}$$
$$+ c\int_0^{\mathrm{T}} \|(\partial_x v)^2(\tau)\|_{3,2,2}\, d\tau \leqq c(1+\mathrm{T})\delta_0 + (1+\mathrm{T}^2) \sup_{[0,T]} \|v(t)\|_{5,2}$$
$$+ c\mathrm{T}(\sup_{[0,T]} \|v(t)\|_{5,2})(\sup_{[0,T]} \|v(t)\|_{3,2,2}). \quad (4.30)$$

The inequality (2.3) and the group properties used in the integral equation lead to

$$\sup_{[0,T]} \|u(t)\|_{11/2,2} \leqq \mathrm{D}_2 \quad (4.31)$$

where D_2 was defined in (4.27).

Introducing the notation

$$\Omega_{\mathrm{T}}(w) = \max\Big\{\sup_{[0,T]} \|w(t)\|_{11/2,2};\ \sup_{[0,T]} \|w(t)\|_{3,2,2};$$
$$\sup_x \left(\int_0^{\mathrm{T}} |\partial_x^6 w(x,t)|^2\, dt\right)^{1/2};\ (1+\mathrm{T})^{-2} \int_{-\infty}^{\infty} \sup_{[0,T]} |\partial_x w(x,t)|\, dx\Big\}$$

one easily see that the estimates (4.27), (4.29)-(4.31) yield the expression

$$\Omega_{\mathrm{T}}(u) \leqq c(1+\mathrm{T})\delta_0 + c(1+\mathrm{T})^2(\Omega_{\mathrm{T}}(v))^2, \quad (4.32)$$

with $\delta_0 = \|u_0\|_{11/2,2} + \|xu_0\|_{3,2} < \delta$ to be fixed.

At this point the rest of the proof follows by the method given in details in the previous proof, therefore it will be omitted. □

THEOREM 4.3 (Case $n \geqq 2$ and $d \geqq 3$). – *Let $n \geqq 2$. Then given any polynomial* P *as in* (4.2) *with $d \geqq 3$ there exists $\delta = \delta(\mathrm{P}) > 0$ such that for any $u_0 \in \mathrm{H}^s(\mathbb{R}^n)$ with $s \geqq s_0 = n + 2 + 1/2$ and $\|u_0\|_{s_0,2} \leqq \delta$ the IVP* (4.1) *has a unique solution $u(.)$ defined in the time interval* $[0, \mathrm{T}]$, $\mathrm{T} = \mathrm{T}(\|u_0\|_{s_0,2}) > 0$ *with $\mathrm{T}(\theta) \to \infty$ as $\theta \to 0$ satisfying*

$$u \in \mathrm{C}([0,\mathrm{T}] : \mathrm{H}^s(\mathbb{R}^n)) \equiv \mathrm{X}_{\mathrm{T}}^s$$

and

$$u \in \mathrm{W}_{\mathrm{T}}^s$$

where

$$\mathrm{W}_{\mathrm{T}}^s \equiv \Big\{ w : \mathbb{R}^n \times [0,\mathrm{T}] \to \mathbb{C} / \exists\, \mathrm{M} > 0$$
$$\text{s.t. for all cube Q of side one} \int_0^{\mathrm{T}} \int_{\mathrm{Q}} |\mathrm{D}_x^{s+1/2} w|^2\, dx\, dt \leqq \mathrm{M} \Big\}.$$

with its norm described in (4.34). *Moreover, for any $\mathrm{T}' \in (0, \mathrm{T})$ there exists a neighborhood V_{u_0} of u_0 in $\mathrm{H}^s(\mathbb{R}^n)$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$ from V_{u_0} into $\mathrm{X}_{\mathrm{T}'}^s \cap \mathrm{W}_{\mathrm{T}'}^s$ is Lipschitz.* –

Proof. – For simplicity of the exposition we shall assume

$$P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}) = \partial_{x_l} u \, \partial_{x_j} u \, \partial_{x_k} u$$

with $l, j, k \in \{1, \ldots, n\}$ fixed.

We also restrict ourselves to the most interesting case $s = s_0 = n + 2 + 1/2$.

As in previous proofs, for $u \in H^{s_0}(\mathbb{R}^n)$ with $\|u_0\|_{s_0, 2} = \delta_0 < \delta$ (to be determined below) we consider the linear IVP

$$\left.\begin{aligned} \partial_t u &= i \Delta u + \partial_{x_l} v \, \partial_{x_j} v \, \partial_{x_k} v \\ u(x, 0) &= u_0(x) \end{aligned}\right\} \tag{4.33}$$

for $v \in \tilde{Z}_T^a = \{ v : \mathbb{R}^n \times [0, T] \to C / \lambda_j^T(v) \leq a \text{ for } j = 1, 2, 3 \}$ where the λ_j^T's are defined

$$\lambda_1^T(v) = \sup_{[0, T]} \|v(t)\|_{s_0, 2},$$

$$\lambda_2^T(v) = \sum_{|\beta| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} |\partial_x^\beta v(x, t)|^2 \, dx \, dt \right)^{1/2}, \tag{4.34}$$

and

$$\lambda_3^T(v) = (1 + T)^{-n} \Big(\sum_{\alpha \in \mathbb{Z}^n} \sup_{[0, T]} \sup_{x \in Q_\alpha} |\nabla_x v(x, t)|^2 \Big)^{1/2},$$

with the Q_α's forming a family of disjoint cubes of side one such that $\mathbb{R}^n = \bigcup_{\alpha \in \mathbb{Z}^n} Q_\alpha$.

It will be shown that for appropriate positive constants $\delta = \delta(P)$, $a = a(\delta)$ and $T = T(\delta_0) > 0$ if $v \in \tilde{Z}_T^a$ so does the solution $u(.)$ of (4.33), and that the map $\Phi(v) = u$ is a contraction.

We shall rely on the integral equation

$$u(t) = \Phi(v)(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-\tau)\Delta} (\partial_{x_l} v \, \partial_{x_j} v \, \partial_{x_k} v)(\tau) \, d\tau. \tag{4.35}$$

Our first estimate deals with the local smoothing effect. Observe that for any $\beta \in Z^n$ with $|\beta| = s_0 - 1/2$

$$\begin{aligned} \partial_x^\beta (\partial_{x_l} v \, \partial_{x_j} v \, \partial_{x_k} v) = {} & \partial_x^\beta \partial_{x_l} v \, \partial_{x_j} v \, \partial_{x_k} v + \partial_{x_l} v \, \partial_x^\beta \partial_{x_j} v \, \partial_{x_k} v \\ & + \partial_{x_l} v \, \partial_{x_l} v \, \partial_{x_j} v \, \partial_x^\beta \partial_{x_k} v + S((\partial_x^\gamma v)_{1 \leq |\gamma| \leq s_0 - 1/2}). \end{aligned} \tag{4.36}$$

Thus the estimates (2.2) and (2.9) allow us to write

$$\lambda_2^T(u) \leqq c \| u_0 \|_{s_0, 2}$$

$$+ c \sum_{|\beta| = s_0 - 1/2} \sum_{p, m, r = 1}^{n} \sum_{\alpha \in \mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} | \partial_x^\beta \partial_{x_p} v \, \partial_{x_m} v \, \partial_{x_r} v |^2 \, dx \, dt \right)^{1/2}$$

$$+ c \int_0^T \| D_x^{1/2} S(.) \|_2 \, dt \leqq c \| u_0 \|_{s_0, 2}$$

$$+ c \left(\sum_{|\beta| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} | \partial_x^\beta v |^2 \, dx \, dt \right)^{1/2} \right)$$

$$\times \Big(\sum_{\alpha \in \mathbb{Z}^n} \sup_{[0, T]} \sup_{x \in Q_\alpha} | \nabla_x v(x, t) |^2 \Big) + c T \big(\sup_{[0, T]} \| v(t) \|_{s_0, 2} \big)^3$$

$$\leqq c \delta_0 + c (1 + T)^{2n} \lambda_2^T(v) (\lambda_3^T(v))^2 + c T (\lambda_1^T(v))^3 \equiv D_3. \quad (4.37)$$

where the commutator estimates deduced in [20] have been used. Combining the estimate (2.4) with the group properties we obtain

$$\lambda_1^T(u) \leqq D_3. \quad (4.38)$$

To estimate $\lambda_3^T(u)$ we insert (3.3) in (4.35), thus

$$\lambda_3^T(u) \leqq c \| u_0 \|_{n/2 + 2, 2} + c T^{1/2} (\lambda_1^T(v))^3. \quad (4.39)$$

Hence defining

$$\Upsilon^T(w) = \max_{k = 1, 2, 3} \{ \lambda_k^T(w) \}$$

from (4.36)-(4.39) it follows that

$$\Upsilon^T(u) \leqq c \delta_0 + c (1 + T)^{2n} (\Upsilon^T(v))^3.$$

At this point the rest of the proof follows the argument given for Theorem 4.1, therefore it will be omitte. □

Finally we have

THEOREM 4.4 (Case $n \geqq 2$ and $d = 2$). — *Let $n = 2$. Then given any polynomial* P *as in* (4.2) *with $d = 2$ there exists $\delta = \delta(P) > 0$ such that for any $u_0 \in H^s(\mathbb{R}^n) \cap H^{2n+3}(\mathbb{R}^n : |x|^{2n+2} dx) \equiv \tilde{G}_s$ with $s \geqq s_0 = 3n + 4 + 1/2$ and*

$$\delta_0 = \| u_0 \|_{s_0, 2} + \| u_0 \|_{2n+3, 2n+2} < \delta$$

the IVP (4.1) *has a unique solution $u(.)$ defined in the time interval* [0, T], $T = T(\delta_0) > 0$ *with $T(\theta) \to \infty$ as $\theta \to 0$ satisfying*

$$u \in C([0, T] : \tilde{G}_s) \equiv R_T^s$$

and

$$u \in W_T^s$$

where W_T^s *and its norm were defined in the statement of Theorem* 4.3 *and in* (4.34) *respectively. Moreover, for any* $T' \in (0, T)$ *there exists a neighborhood* $\tilde{V}_{u_0}$ *of* u_0 *in* $\tilde{G}_s$ *such that the map* $\tilde{u}_0 \to \tilde{u}(t)$ *from* $\tilde{V}_{u_0}$ *into* $R_{T'}^s \cap W_{T'}^s$ *is Lipschitz.* —

Proof. — We shall follow the argument presented in the previous proofs, therefore a sketch will suffice.

Since our method does not rely on any special structure of the nonlinear term, besides its quadratic character, we restrict ourselves to consider the case

$$P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}) = |\nabla u|^2.$$

Also for simplicity in the exposition we assume $s = s_0 = 3n + 4 + 1/2$.

Define

$$\tilde{E}_T^a = \{ w : \mathbb{R}^n \times [0, T] \to \mathbb{C} / \mu_j^T(w) \leq a, j = 1, \ldots, 4 \}$$

where

$$\mu_1^T(w) = \sup_{[0, T]} \| w(t) \|_{s_0, 2},$$

$$\mu_2^T(w) = \sum_{|\beta| = s_0 + 1/2} \sup_{\alpha \in \mathbb{Z}^n} \left(\int_0^T \int_{Q_\alpha} |\partial_x^\beta w(x, t)|^2 \, dx \, dt \right)^{1/2},$$

$$\mu_3^T(w) = (1 + T)^{-(n+3)} \sum_{\alpha \in \mathbb{Z}^n} \sup_{[0, T]} \sup_{x \in Q_\alpha} |\nabla_x w(x, t)|$$

and

$$\mu_4^T(w) = \sup_{[0, T]} \| w(t) \|_{2n+2, 2, 2n+2}$$

with the Q_α's and the $\| \cdot \|_{k, 2, m}$-norm described as in the previous proof and in (4.24) respectively.

For a given $v \in \tilde{E}_T^a$ denote by $u = \Phi(v)$ the solution of the linear problem

$$\left. \begin{aligned} \partial_t u &= i \Delta u + |\nabla v|^2 \\ u(x, 0) &= u_0(x) \end{aligned} \right\} \tag{4.40}$$

with $u_0 \in \tilde{G}_{s_0}$ such that $\| u_0 \|_{3n+4+1/2, 2} + \| u_0 \|_{2n+2, 2, 2n+2} = \delta_0 < \delta$ (to be determined).

It is clear that $u(.)$ satisfies the integral equation

$$u(t) = \Phi(v)(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-\tau)\Delta} |\nabla v(\tau)|^2 \, d\tau. \tag{4.41}$$

Thus an argument similar to that used in (4.36)-(4.37) shows that

$$\mu_2^T(u) \leq c\,\delta_0 + c(1+T)^{n+3} \mu_2^T(v)\, \mu_3^T(v) + c\,T(\mu_1^T(v))^2 \equiv D_4 \tag{4.42}$$

Also, as in (4.38), (2.6) and the group properties yield the inequality

$$\mu_1^T(u) \leq D_4. \tag{4.43}$$

Next by combining (3.14) with successive applications of the rule

$$x_k x_j e^{-i\tau\Delta} f(x,\tau) = e^{-i\tau\Delta}(\Gamma_k(\tau)\Gamma_j(\tau) f(x,\tau)) = e^{-i\tau\Delta}(x_k + 2it\,\partial_{x_k})(x_j + 2it\,\partial_{x_j}) f(x,\tau)$$

(*see* [15]) and the equation in (4.41) written as

$$u(t) = e^{it\Delta}\left(u_0 + \int_0^t e^{-i\tau\Delta} |\nabla v(\tau)|^2 \, d\tau\right)$$

we can infer that

$$\mu_3^T(u) \leqq c\,\delta_0 + \int_0^T \|(\nabla v(\tau))^2\|_{2n+2,\,2}\, d\tau + \sum_{|\beta| \leqq 2n+2} \int \| |x|^{n+1} e^{-i\tau\Delta} \partial_x^\beta (|\nabla v|^2(\tau)) \|_2 \, d\tau$$
$$\leqq c\,\delta_0 + c\,T(\mu_1^T(v))^2 + c(1+T)^{n+2}(\mu_1^T(v))(\mu_1^T(v) + \mu_4^T(v)). \quad (4.44)$$

To estimate $\mu_4^T(u)$ we use the formula

$$x_j e^{it\Delta} f(x,t) = (x_j + 2it\,\partial_{x_j}) e^{it\Delta} f - 2it\, e^{it\Delta} \partial_{x_j} f = e^{it\Delta}(x_j f) - 2\, it\, e^{it\Delta}(\partial_{x_j} f).$$

(*see* [15]) several times to conclude that

$$\mu_4^T(u) \leqq c(1+T)^{n+1}\delta_0 + c(1+T)^{n+1}(\mu_1^T(v))(\mu_1^T(v) + \mu_4^T(v)). \quad (4.45)$$

From (4.42)-(4.45) we find that if

$$\Psi^T(w) = \max_{i=1,\ldots,4} \{\mu_k^T(w)\}$$

then

$$\Psi^T(u) \leqq c(1+T)^{n+1}\delta_0 + c(1+T)^{n+3}(\Psi^T(v))^2.$$

At this point we remark that the rest of the proof follows in the same manner as that of previous theorems. Therefore it will be omitted. □

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