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On the Blowup of Multidimensional Semilinear Heat Equations

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ABSTRACT. — This work is concerned with positive, blowing-up solutions of the semilinear heat equation $u_t - \Delta u = u^p$ in R^n. No symmetry assumptions are made. Working with the equation in similarity variables, we first prove a result suggested by center manifold theory. We then calculate the refined asymptotics for u in a backward space-time parabola near a blowup point, and we obtain some information about the local structure of the blowup set. Our results suggest that in space dimension n, among solutions that follow the center manifold, there are exactly n different blowup patterns.

Key words : Multidimensional semilinear heat equation, center manifold theory.

RÉSUMÉ. — On étudie les solutions positives explosant en temps fini de l'équation semilinéaire de la chaleur : $u_t - \Delta u = u^p$ dans R^n. On ne suppose aucune hypothèse de symétrie. On calcule le comportement asymptotique de la solution au voisinage d'un point d'explosion et on obtient certaines informations sur l'ensemble des points d'explosion.

Classification A.M.S. : 35 B 40, 35 K 55, 35 K 65.

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1. INTRODUCTION

This work is concerned with positive, blowing-up solutions of the semilinear heat equation

$$u_t - \Delta u = u^p \quad \text{in } \mathbb{R}^n \times (0, T), \qquad p > 1. \tag{1.1}$$

Working with the equation in similarity variables, we first prove a result suggested by center manifold theory. We then calculate the refined asymptotics for u in a backward space-time parabola near a blowup point, and we obtain some information about the local structure of the blowup set.

A lot of work has been done concerning the blowup of solutions of (1.1) and related equations such as $u_t - \Delta u = e^u$. For extensive discussions and bibliobraphies we refer to ([2], [3], [5], [8]-[15]).

When studying the local properties of the blowing-up solutions of (1.1), it is convenient to use the method developed by Giga and Kohn ([12], [13], [14]) based on similarity variables. This change of both dependent and independent variables is defined by

$$\begin{gathered} w(y, s) = (T - t)^{1/p-1} u(x, t), \\ y = (x - b)/\sqrt{T - t}, \qquad s = -\ln(T - t), \end{gathered} \tag{1.2}$$

where b is a blowup point and T is the blowup time. If u solves (1.1) then w exists for all time s and solves

$$w_s - \frac{1}{\rho} \nabla(\rho \nabla w) + \frac{1}{p-1} w = w^p, \tag{1.3}$$

where $\rho = e^{-|y|^2/4}$. Studying the behavior of u near blowup is equivalent to studying the large time behavior of w. Let us assume that u is nonnegative and p is "subcritical", *i.e.* $n = 1$, 2 and $p > 1$, or, if $n \geq 3$, $1 < p < (n+2)/(n-2)$. Then, it follows from [13], [14] that

$$w(y, s) \to \kappa, \quad \text{as } s \to \infty,$$

uniformly on bounded sets in y, where κ is the constant nonzero stationary solution of (1.3), *i.e.* $\kappa = (p-1)^{-1/(p-1)}$.

Recently, Filippas and Kohn [7] and independently and simultaneously Herrero and Velázquez [15], were able to obtain more information about the way w approaches κ. Since our analysis relies primarily on the ideas of [7] we recall certain facts from there.

To learn more about the way w approaches κ it is natural to linearize equation (1.3) about κ. Let

$$v(y, s) = w(y, s) - \kappa. \tag{1.4}$$

Then v is a solution of

$$v_s - \frac{1}{\rho} \nabla (\rho \nabla v) - v = f(v), \tag{1.5}$$

with

$$f(v) = \frac{p}{2\kappa} v^2 + g(v), \qquad |g(v)| < c |v|^3,$$

and $v \to 0$ uniformly on compact sets in y.

Let us at this point introduce some notation which we will keep throughout the rest of this work. We tenote by L^2_ρ the Hilbert space of functions $v(y)$ such that $\int v^2 \rho < \infty$ with $\rho = e^{-|y|^2/4}$ as usual. The linear operator

$$\mathscr{L} v \stackrel{\triangle}{=} \frac{1}{\rho} \nabla (\rho \nabla v) + v, \tag{1.6}$$

defines a self-adjoint operator in L^2_ρ with eigenvalues 1, 1/2, 0, $-1/2, \ldots$ We denote by $\{ e_j^+ (y) \}_{j=1}^k (k = n+1)$, the eigenfunctions of $\mathscr{L}$ corresponding to positive eigenvalues and similarly for $\{ e_j^0 (y) \}_{j=1}^m (m = n(n+1)/2)$ and $\{ e_j^- (y) \}_{j=1}^\infty$. We also expand $v(y, s)$ as:

$$v(y, s) = \sum_{j=1}^k \beta_j(s) e_j^+ (y) + \sum_{j=1}^m \alpha_j(s) e_j^0 (y) + \sum_{j=1}^\infty \gamma_i(s) e_j^- (y)$$

$$\stackrel{\triangle}{=} v_+ + v_0 + v_-.$$

The presence of a nontrivial null space in the linear operator of (1.5) suggests the use of center manifold theory: if a trajectory of (1.5) goes to zero it should generically do so like a trajectory on the center manifold. Formally speaking, if a trajectory is on the center manifold, then $\gamma = (\gamma_1, \gamma_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ are dominated by $\alpha = (\alpha_1, \alpha_2, \ldots)$ with

$$|\beta| + |\gamma| \leqq C |\alpha|^2. \tag{1.7}$$

Although this is what should happen generically, there are exceptional solutions of (1.5) which approach zero exponentially fast. In the context of center manifold theory these are trajectories lying on the stable manifold (for a rigorous and more extensive treatement of these ideas *see* e.g. [6]).

With these as motivation, it has been shown in [7] that in any space dimension .

THEOREM 1. — *Either v decays exponentially fast, or else for any $\varepsilon>0$ there exists an s_0 such that*

$$\|v_+\|_{L^2_\rho}+\|v_-\|_{L^2_\rho}\leqq\varepsilon\|v_0\|_{L^2_\rho},\qquad \text{for}\quad s\geqq s_0. \tag{1.8}$$

Moreover.

THEOREM 2. — *If v does not decay exponentially fast, then the neutral modes satisfy the following $m\times m$ nonlinear ODE system*

$$\dot{\alpha}_l=\frac{p}{2\kappa}\Pi_l^0(v_0^2)+o\left(\sum_{j=1}^{m}\alpha_l^2\right),\qquad l=1, 2, \ldots, m. \tag{1.9}$$

Where Π_l^0 denotes the orthogonal projection onto the neutral eigenfunction e_l^0. The above ODE system will play a crucial role in the analysis of the present work. As we show in Section 2 it can be put in a remarkably simple form, and we are eventually able to solve it explicitly.

In the one dimensional case it was shown in [7].

THEOREM 3. — *If v does not decay exponentially fast then for large enough time s, and any $C>0$*

$$\sup_{|y|<C}\left|v(y, s)-\frac{\kappa}{2ps}\left(1-\frac{1}{2}y^2\right)\right|=o\left(\frac{1}{s}\right). \tag{1.10}$$

Moreover, the center of scaling is an isolated blowup point.

Herrero and Velázquez ([15], [16], [17], [22]), following the ideas developed in [11], consider the one dimensional case where their results go far beyond the above Theorems. They obtain all possible large time profiles for v including the case where v decays exponentially fast. They also obtain refined asymptotics for $|y|\sim\sqrt{s}$, and they compute the blowup spatial profiles. Finally they show that in all cases the blow up points are isolated. Similar results are proved for the e^u nonlinearity.

More recently Liu [20] and Bebernes and Bricher [1] have extended most of the above results in arbitrary space dimension under the assumption that u is a radially symmetric solution of (1.1).

In this work our main concern is to understand what happens in higher dimensions without imposing any symmery assumptions.

Our results apply to any nonnegative solution of the Cauchy problem (1.1) satisfying

$$\|u\|_{L^\infty}(t)\leqq C(T-t)^{-1/(p-1)}, \tag{1.11}$$

and having the asymptotic behavior

$$(T-t)^{1/(p-1)}u(b+y\sqrt{T-t}, t)\to\kappa,\quad \text{as}\ t\uparrow T, \tag{1.12}$$

uniformy for $|y|\leqq C$.

Our first main result is an improvement of (1.8). Thus, in any space dimension and in complete agreement with the formal argument [*cf.* (1.7)] we show.

THEOREM A. – *Either* v *decays exponentially fast, or else there exists a constant* C *and a time* s_0 *such that:*

$$\|v_+\|_{L^2_\rho}+\|v_-\|_{L^2_\rho}\leqq C\|v_0\|^2_{L^2_\rho}, \qquad for \quad s\geqq s_0. \tag{1.13}$$

As in [7] we believe that the situation where v decays exponentially fast is in some sense exceptional, but (as in [7]) we are unable to prove it.

The rest of our results are the analogue of Theorem 3, stated above, in higher dimensions. Concerning the refined asymptotics of $v(y, s)$ we show.

THEOREM B. – *Assume that* v *does not decay exponentially. Then for large enough* s, *we have that*

$$v(y, s)-\frac{\kappa}{2ps}\left(\operatorname{tr} A_0-\frac{1}{2}y^T A_0 y\right)=O\left(\frac{1}{s^{1+\delta}}\right), \tag{1.14}$$

for some δ *positive, with*

$$A_0=Q\begin{pmatrix} I_{n-k} & 0 \\ 0 & 0_k \end{pmatrix}Q^{-1}, \tag{1.15}$$

for some $k\in\{0, 1, \ldots, n-1\}$, *where* Q *is an orthonormal matrix*, I_{n-k} *is the* $(n-k)\times(n-k)$ *indentity matrix and* 0_k *is the* $k\times k$ *zero matrix. (If* $k=0$, *then* $A_0=I_n$). *The convergence in* (1.14) *takes place in* $C^{m,\alpha}_{loc}$ *for any* $m\geqq 1$ *and some* $\alpha\in(0, 1)$.

If b is a blowup point, we can rewrite (1.14) in terms of the original variables as:

$$(T-t)^{1/p-1}u(x, t)\sim\kappa+\frac{\kappa}{2p|\ln(T-t)|}\left(1-\frac{(x-b)^T A_0(x-b)}{2(T-t)}\right),$$

in the sense that the difference is $o(|\ln(T-t)|^{-1})$ as $t\uparrow T$, in parabolas $|x-b|^2\leqq C(T-t)$.

If $n=1$, then necessarily $k=0$ and the asymptotic behavior of v is the same as that given by (1.10). For $n\geqq 2$ the above result suggests that there are n different possibilities depending on the rank of the matrix A_0. If u (or equivalently v) is radially symmetric and decreasing in $|x|$ then it has been proved recently in [1] and [20] that the large time behavior of

$v(y, s)$ is given by

$$v(y, s) \sim \frac{\kappa}{2ps}\left(n - \frac{1}{2}|y|^2\right). \tag{1.16}$$

This is also given by (1.14) with $k=0$.

Blowup along a continuum is known to exist. It has been proved in [14] that there are initial data in R^n for which the blowup set of (1.1) is exactly an $(n-1)$-dimensional sphere. We believe (but we have no proof so far) that the refined asymptotics of v, given by (1.14) for $k=n-1$, correspond to such a situation.

Our third main result in concerned with the local geometry of the blowup set:

THEOREM C. – *Assume that v does not decay exponentially fast and let b be the center of scaling. Then either b is an isolated blowup point or else there exists a k-dimensional linear subspace E_0^k of R^n for some $k \in \{1, 2, \ldots, n-1\}$ passing through b with the property: if $\mathbf{r}$ is a ray emanating from b and angle $(E_0^k, \mathbf{r}) = \varepsilon$ for some $\varepsilon \in \left(0, \frac{\pi}{2}\right]$, then there is no other blowup point along $\mathbf{r}$ within distance $r^*(\varepsilon) > 0$ from b. This $r^*(\varepsilon)$ is such that $r^*(\varepsilon) \to 0$ as $\varepsilon \to 0$. E_0^k coincides with the zero eigenspace of the matrix A_0 given by (1.15).*

If $n=1$, then necessarily the center of scaling is an isolated point. This is by now well known, since in the one dimensional case the blowup set is made of isolated points, *see* [5], [17]. If $n \geqq 2$ there are n different possibilities which strongly suggest that the blowup set should locally look like a k-dimensional submanifold of R^n for $k \leqq n-1$. (*See* Section 2 for a discussion on that.)

If u is radially symmetric and decreasing then it can blow up only at the origin, *see* [9]. In fact, the above theorem says that if the asymptotics of the solution is given by (1.16), that is, the solution is *asymptotically* radially summetric and decreasing, then the blowup point is an isolated one.

Center manifold ideas have also been used by Bressan [4] in the study of the blowing up solutions of $u_t = \Delta u + e^u$ in a bounded domain $\Omega \in R^n$. Working with $y/\sqrt{s}$ as the spatial variable he shows the existence of solutions which blow up at a single point according to a precise asymptotic pattern. He also conjectures that this behavior should be generic. In Section 2 we give a formal argument supporting the idea that the blowup set of (1.1) should generically be made of isolated points.

After this work was completed we learned that a result similar to our Theorem B was independently and simultaneously proved by

J. J. L. Velázquez [23]. He also obtains some information about the large time profile of v in the case where v decays exponentially fast.

2. THE FORMAL PICTURE

In this Section we first present a convenient way of dealing with the neutral component of v. We then present a formal argument which underlies the rigorous analysis of the other Sections.

We being by recalling the properties of the linear operator $\mathscr{L}$ defined by (1.6). It is easy to see that $\mathscr{L}$ is a self adjoint operator on L^2_ρ. Concerning its spectral properties we have the following

LEMMA 2.1. – *In* R^n, $n \geqq 1$, *the eigenvalues of* $\mathscr{L}$ *are given by*

$$\lambda_k = 1 - \frac{k}{2}, \qquad k = 0, 1, 2, \ldots$$

The corresponding normalized eigenfunctions are as follows

$$\begin{aligned} &\textit{for} \quad \lambda_0 = 1, \quad h_0^n, \\ &\textit{for} \quad \lambda_1 = \frac{1}{2}, \quad h_0^{n-1} k_1(y_i), \qquad i = 1, \ldots, n, \\ &\textit{for} \quad \lambda_2 = 0, \quad h_0^{n-1} h_2(y_i), \qquad i = 1, \ldots, n, \\ &\qquad\qquad\qquad h_0^{n-2} h_1(y_i) h_1(y_j), \qquad i \neq j, \quad i, j = 1, \ldots, n, \end{aligned}$$

and so forth, where $h_k(y) = d_k \mathrm{H}_k(y/2)$ *with* $d_k = (\pi^{1/2} 2^{k+1} k!)^{-1/2}$ *and* H_k *is the* k^{th} *Hermite polynomial for* $k = 0, 1, \ldots$ *In particular these eigenfunctions form an orthonormal basis for* L^2_ρ.

This can be found in [7]. We also remind the reader that the Hermite polynomials are defined by

$$\mathrm{H}_k(x) - (-1)^k e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}), \qquad k = 0, 1, \ldots$$

It follows from the above Lemma that the null space of $\mathscr{L}$ has dimension $n(n+1)/2 = m$ and the (normalized) neutral eigenfunctions are

$$\begin{aligned} h_{ii}(y) &= \frac{1}{\sqrt{2}} (4\pi)^{-n/4} \left(\frac{1}{2} y_i^2 - 1 \right), \qquad i = 1, \ldots, n, \\ h_{ij}(y) &= \frac{1}{2} (4\pi)^{-n/4} y_i y_i, \qquad i \neq j, \quad i, j = 1, \ldots, n. \end{aligned}$$

By its definition, $v_0(y, s)$ can be expanded in terms of these eigenfunctions. For reasons that will become apparent in the sequel, we express the neutral component of v as:

$$v_0(y, s) = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} a_{ii}(s) h_{ii}(y) + \frac{1}{2} \sum_{\substack{i, j=1 \\ i \neq j}}^{n} a_{ij}(s) h_{ij}(y)$$

$$= \frac{1}{2} (4\pi)^{-n/4} \left\{ \frac{1}{2} \sum_{i=1}^{n} (a_{ii}(s) y_i^2 - 2 a_{ii}(s)) + \frac{1}{2} \sum_{\substack{i, j=1 \\ i \neq j}}^{n} a_{ij}(s) y_i y_i \right\}, \quad (2.1)$$

with $a_{ij}(s) = a_{ji}(s)$. Define now the coefficient matrix-function $A(s)$ to be

$$A(s) = \{ a_{ij}(s) \}_{i, j=1}^{n}.$$

Clearly $A(s)$ is an $n \times n$ symmetric matrix for all s. Moreover we can rewrite (2.1) as

$$v_0(y, s) = \frac{1}{2} (4\pi)^{-n/4} \left(\frac{1}{2} y^T A(s) y - \operatorname{tr} A(s) \right). \quad (2.2)$$

We next tun our attention to the ODE satisfied by the neutral modes:

$$\dot{\alpha}_l = \frac{p}{2\kappa} \Pi_l^0 (v_0^2) + o\left(\sum_{i=1}^{m} \alpha_i^2 \right), \qquad 1 \leqq l \leqq m. \quad (2.2)$$

In view of (2.1), the α_l's are equal to either $a_{ii}/\sqrt{2}$ or $a_{ij} (i \neq j)$. Let us consider the first possibility. Then, (2.3) can be written as

$$\frac{1}{\sqrt{2}} \dot{a}_{ii} = \frac{p}{2\kappa} \int v_0^2 h_{ii} \rho + o\left(\sum_{i, j=1}^{n} a_{ij}^2 \right), \qquad 1 \leqq i \leqq n. \quad (2.4)$$

The integral in the right hand side can be computed by using (2.1, integration by parts, and the fact that the h_{ij}'s form an orthonormal family in L_ρ^2 (we omit the details); we finally get:

$$\dot{a}_{ii} = c_n \left(a_{ii}^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^2 \right) + o\left(\sum_{i, j=1}^{n} a_{ij}^2 \right), \qquad i = 1, \ldots, n, \quad (2.5)$$

with

$$c_n = \frac{p}{\kappa} (4\pi)^{-n/4}.$$

Similarly, we obtain

$$\dot{a}_{ij} = c_n \left((a_{ii} + a_{jj}) a_{ij} + \sum_{\substack{k=1 \\ k \neq i, j}}^{n} a_{ik} a_{jk} \right) + o\left(\sum_{i, j=1}^{n} a_{ij}^2 \right), \qquad i \neq j, \quad 1 \leqq i, j \leqq n. \quad (2.7)$$

One can easily verify that (2.5), (2.7) are equivalent to:

$$\dot{A}(s)=c_n A^2(s)+o(\|A(s)\|^2),$$

where $\|A\|$ denotes the L^2 norm of A, *i.e.*

$$\|A\|=\left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{1/2}.$$

We summarize:

LEMMA 2.2. – *Suppose that v does not decay exponentially fast, and let v_0 denote its neutral component. Then*

$$v_0(y, s)=\frac{1}{2}(4\pi)^{-n/4}\left(\frac{1}{2}y^T A(s)y-\operatorname{tr} A(s)\right), \tag{2.8}$$

where A (*s*) *satisfies*

$$\dot{A}(s)=c_n A^2(s)+o(\|A(s)\|^2), \tag{2.9}$$

with c_n as given in (2.6).

Thus, the large time behavior of v_0 (as well as of v itself) is encoded in the large time behavior of A (*s*), which in its turn satisfies the ODE (2.9).

If the error terms are neglected in (2.9), it is very easy to solve the ODE. So let us omit the error terms throughout the rest of this Section in order to (formally) complete the picture.

At first we show

LEMMA 2.3. – *Let* A (*s*) *be an $n\times n$ nonzero symmetric matrix-function which exists for all time and satisfies*

$$\dot{A}(s)=c_n A^2(s), \tag{2.10}$$

for some nonzero constant c_n. Then

$$A(s)=-\frac{c_n^{-1}}{s}A_0+O\left(\frac{1}{s^2}\right), \tag{2.11}$$

with

$$A_0=Q\begin{pmatrix} I_{n-k} & 0 \\ 0 & 0_k \end{pmatrix}Q^{-1}, \tag{2.12}$$

for some $k\in\{0, 1, \ldots, n-1\}$, where Q is a (constant) orthonormal matrix, I_{n-k} is the $(n-k)\times(n-k)$ identity matrix, and 0_k is the $k\times k$ zero matrix.

Proof. – By rescaling in time, we may assume c_n to be equal to 1. Consider the initial value problem

$$\dot{A}(s)=A^2(s), \qquad A(s_1)=A_1, \tag{2.13}$$

for some symmetric matrix A_1. Since A_1 is symmetric it can be diagonalized by an orthonormal matrix Q, *i.e.*

$$A_1 = Q \Lambda_1 Q^{-1},$$

where the diagonal elements of Λ_1 are the eigenvalues of A_1. In general, some of the eigenvalues of A_1 can be zero. So let us assume that k of them are equal to zero; *i.e.* $\lambda_i \neq 0$ for $1 \leqq i \leqq n-k$, whereas $\lambda_i = 0$ for $n-k+1 \leqq i \leqq n$. Consider the diagonal matrix

$$\Lambda(s) = \text{Diag}\,\{\underbrace{(\lambda_1^{-1}+s_1-s)^{-1}, \ldots, (\lambda_{n-k}^{-1}+s_1-s)^{-1}}_{n-k}, \underbrace{0, \ldots, 0}_{k}\},$$

and let

$$A(s) = Q\,\Lambda(s)\,Q^{-1}. \tag{2.14}$$

One can verify in a straightforward manner that the matrix $A(s)$ defined by (2.14) solves the initial value problem (2.13). From uniqueness considerations we conclude that (2.14) represents all solutions of (2.10). Moreover we have that for large s :

$$(\lambda_i^{-1}+s_1-s)^{-1} = -\frac{1}{s} + O\left(\frac{1}{s^2}\right), \qquad 1 \leqq i \leqq n-k.$$

Therefore for large s we also have

$$A(s) = -\frac{1}{s} A_0 + O\left(\frac{1}{s^2}\right),$$

with A_0 as described in (2.12). □

Remark 2.1. — An examination of the proof shows that each eigenvalue of $A(s)$ is either identically equal to zero, or else it is always different from zero. Since the generic $n \times n$ symmetric matrix is of full rank, we deduce that the case $k=0$ describes the generic solution of the ODE.

We know from (1.8) that v_0 is the dominant component of v (if it does not decay exponentially fast), therefore we expect that the large time behavior of v is described by that of v_0. Moreover from (2.8), (2.11) we also have that

$$v(y, s) \sim v_0(y, s) \sim \frac{\kappa}{2ps}\left(\text{tr}\,A_0 - \frac{1}{2} y^T A_0 y\right), \tag{2.15}$$

and this is the content of Theorem B.

As in [7] we expect that the large time profile of $v(y, s)$ should reflect the local geometric of the blow up set. More precisely, the blow up set should locally coincide with the region where $v(y, s)$ takes on its maximum values. Because of (2.15), $v(y, s)$ takes on its maximum values wherever the quadratic form $y^T A_0 y$ takes on its minima. At this point it is convenient to distinguish two cases:

(i) The matrix A_0 is of full rank; *i.e.* $k=0$. As we explained above (*see* Remark 2.1) this is what happens generically. From (2.12) we see that A_0 is equal to the identity matrix. Therefore from (2.15) we have that

$$v(y, s) \sim \frac{\kappa}{2ps}\left(n - \frac{1}{2}|y|^2\right).$$

In this case, $v(y, s)$ is asymptotically radially symmetric. It has an isolated maximum at $y=0$ and as we show (Theorem C) it corresponds to an isolated blow up point b (the center of scaling) for u. Thus, generically, the blow up set in space dimension n should be made of isolated points.

(ii) The matrix A_0 is defective; *i.e.* $k \geqq 1$. As can be seen from (2.12) A_0 is a nonnegative definite matrix and the minimum value the quadratic from can take on is zero. The region where $y^T A_0 y$ is zero, coincides with the eigenspace of A_0 corresponding to the zero eigenvalue. Clearly this is a subspace of R^n of dimension k; let us denote it by E_0^k. We have therefore that $v(y, s)$ takes on its maximum values along E_0^k. This strongly suggests that the blow up set should locally look like an k-dimensional submanifold of R^n, containing the center of scaling b, and tangent to E_0^k at b. (Notice that our rigorous result, *i.e.* Theorem C, is weaker than what the formal argument suggest. Namely, we prove that there are no other blow up points in a neighborhood of b, *except possibly* along the direction of E_0^k. The method we use – which is based on a criterion for characterizing non blowup points – is inconclusive when it comes to points along the direction of E_0^k.)

Thus, it is reasonable to expect, that in space dimension n, if v decays like $1/s$, there are n different blow up patterns. These patterns correspond to blowup at (isolated) points, blowup along curves, surfaces, and more generally k-dimensional submanifolds of R^n for $k \leqq n-1$.

Of course, to obtain a more complete picture about the blow up set, one has to take into consideration the case where v decays exponentially fast. As we explained in the introduction, we believe that this should be the exceptional case. Herrero and Velázquez ([15], [16], [17]) have given a complete characterisation of the exponentially decaying profiles in the special case where $n=1$. Moreover, they have shown that in all cases, either v decays at an algebraic rate or at an exponential one, the blow up set consists of isolated points. In analogy, one may think that in higher dimensions the blowup patterns are independent of the decay rate of v. But we have no solid evidence for that.

We close this Section by briefly commenting on the content of the rest of this work. It is evident that it is of great importance to know the behavior of the solutions of the ODE satisfied by $A(s)$. Unfortunately, the presence of the error terms introduces several technical difficulties and the easy argument used in Lemma 2.2 is not applicable. In Section 3 we obtain a preliminary estimate, namely that $\|A(s)\|$ (or equivalently v) decays like $1/s$. This information along with other ideas is then used in Section 4 where we prove Theorem A. Using Theorem A we rederive the ODE, but this time the error terms are of the order $O(1/s^3)$. In Section 5 we solve explicitly the ODE, there by proving Theorem B. Finally, in Section 6 we give the proof of Theorem C.

3. A PRELIMINARY ESTIMATE

Our aim in this section is to prove the following

PROPOSITION 3.1. — *If v does not decay at an exponential rate then for s large enough:*

$$\frac{c}{s} \leqq \|v(s)\|_{L^2_\rho} \leqq \frac{C}{s} \tag{3.1}$$

for some positive constants c and C.

Because of (1.8) we have that

$$\|v\|_{L^2_\rho} = (1+0(1))\|v_0\|_{L^2_\rho};$$

therefore it is enough to show that $\|v_0\|_{L^2_\rho}$ decays like $1/s$. It follows from the definition of the $A(s)$ that $\|v_0(s)\|_{L^2_\rho}$ is equivalent to $\|A(s)\|$. Consequently, Proposition 3.1 will be proved if we show

LEMMA 3.1. — *The coefficient matrix-function* $A(s)$ *satisfies for s large enough:*

$$\frac{c}{s} \leqq \|A(s)\| \leqq \frac{C}{s}, \tag{3.2}$$

for some positive constants c and C.

The above estimates have been proved in [20] for the special case $n=2$. Our proof folows these ideas, although at the technical level it is different. We prefer to use a rather general approach, which has the advantage that it simplifies the analysis of Section 5, where a more detailed study of the large time behavior of $A(s)$ is presented.

To show (3.2) we will use the ODE satisfied by $A(s)$. From the previous Section we have that

$$\dot{A}(s) = c_n A^2(s) + \mathscr{E}(s), \tag{3.3}$$

where $\mathscr{E}(s)$ is a symmetric matrix-function with the property that given an $\varepsilon > 0$, there exists an s^* such that for $s > s^*$:

$$\|\mathscr{E}(s)\| < \varepsilon \|\mathrm{A}(s)\|^2. \tag{3.4}$$

We begin by quoting a standard result from the perturbation theory of linear operators.

LEMMA 3.2. – *Suppose that* $\mathrm{A}(s)$ *in an* $n \times n$ *symmetric and continuously differentiable matrix-function in some interval I. Then, there exist continuously differentiable functions* $\lambda_1(s), \ldots, \lambda_n(s)$ *in* I, *such that*

$$\mathrm{A}(s)\,\phi^{(j)}(s) = \lambda_j(s)\,\phi^{(j)}(s), \qquad j = 1, \ldots, n,$$

for some (properly chosen) orthonormal system of vector-functions $\phi^{(1)}(s), \ldots, \phi^{(n)}(s)$.

The proof of this Lemma can be found for instance in [18] or [21]. Roughly speaking, the above Lemma says that a smooth symmetric matrix-function has smooth eigenvalues. (This is not true in general for the eigenvectors.)

Let $\lambda_1(s), \ldots, \lambda_n(s)$, denote the eigenvalues of $\mathrm{A}(s)$. It is a well known fact that $\sum_{i=1}^{n} |\lambda_i|$ is a norm in the space of symmetric matrices. Therefore, in order to prove (3.2) it is enough to prove that $\sum_{i=1}^{n} |\lambda_i(s)|$ decays like $1/s$. Working in this direction we first show

LEMMA 3.3. – *The eigenvalues of the coefficient matrix-function* $\mathrm{A}(s)$ *satisfy:*

$$\dot{\lambda}_i(s) = c_n \lambda_i^2(s) + o\left(\sum_{i=1}^{n} \lambda_i^2\right), \qquad i = 1, \ldots, n. \tag{3.5}$$

Proof. – By rescaling in time we may set $c_n = 1$. Fix a time s_0 (large enough) and consider any arbitrary eigenvalue $\lambda_j(s_0)$. Let us assume that it has algebraic multiplicity h. We denote by $\mathscr{M}$ the (h-dimensional) eigenspace corresponding to $\lambda_j(s_0)$, and by $\mathrm{P}(s_0)$ the projector of R^n onto $\mathscr{M}$.

We claim that $\dot{\lambda}_j(s_0)$ is an eigenvalue of the operator $\mathrm{P}(s_0)\,\dot{\mathrm{A}}(s_0)$. (This can also be found in [21] but we include it here for the sake of completeness.) To show this, let us first denote by $\phi^{(1)}(s_0), \ldots, \phi^{(h)}(s_0)$ the (orthonormal) eigenvectors corresponding to $\lambda_j(s_0)$. Next, we pick up a sequence τ_m of non-zero real numbers tending to zero, for which $\lim_{m \to \infty} \phi^{(i)}(s_0 + \tau_m) = \psi^{(i)}(s_0)$, exists for $i = 1, \ldots, h$. This is always possible since the $\phi^{(l)}(s)$ are unit vectors for all s. Since $(\phi^{(i)}(s_0 + \tau_m), \phi^{(l)}(s_0 + \tau_m)) = \delta_{il}$, we also have that $(\psi^{(i)}(s_0), \psi^{(l)}(s_0)) = \delta_{il}$. In particular, the ψ^{i}'s form an orthonormal basis for $\mathcal{M}$ and

$$A(s_0)\,\psi^{(i)}(s_0)=\lambda_j(s_0)\,\psi^{(i)}(s_0), \qquad i=1, \ldots, h. \tag{3.6}$$

From the identity

$$(A(s_0)-\lambda_j(s_0))\,\phi^{(i)}(s_0+\tau_m)+(A(s_0+\tau_m)-A(s_0))\,\phi^{(i)}(s_0+\tau_m) \\ -(\lambda_j(s_0+\tau_m)-\lambda_j(s_0))\,\phi^{(i)}(s_0+\tau_m)=0,$$

it follows that

$$\left(\psi, \frac{A(s_0+\tau_m)-A(s_0)}{\tau_m}\phi^{(i)}(s_0+\tau_m)\right) \\ -\frac{\lambda_j(s_0+\tau_m)-\lambda_j(s_0)}{\tau_m}(\psi, \phi^{(i)}(s_0+\tau_m))=0,$$

for any element ψ in $\mathcal{M}$. Letting $m \to \infty$, we obtain

$$(\psi, \dot{A}(s_0)\,\psi^{(i)}(s_0))-\dot{\lambda}_j(s_0)\,(\psi, \psi^{(i)}(s_0))=0$$

or, since this is true for any ψ in $\mathcal{M}$,

$$P(s_0)\,\dot{A}(s_0)\,\psi^{(i)}(s_0)=\dot{\lambda}_j(s_0)\,\psi^{(i)}(s_0); \tag{3.7}$$

and this proves our claim.

Next, it is clear from (3.6) that we also have:

$$P(s_0)\,A^2(s_0)\,\psi^{(i)}(s_0)=\lambda_j^2(s_0)\,\psi^{(i)}(s_0). \tag{3.8}$$

Subtracting (3.8) from (3.7) and using equation (3.3) we obtain:

$$P(s_0)\,\mathcal{E}(s_0)\,\psi^{(i)}(s_0)=(\dot{\lambda}_j(s_0)-\lambda_j^2(s_0))\,\psi^{(i)}(s_0), \tag{3.9}$$

from which we conclude that $\dot{\lambda}_j(s_0)-\lambda_j^2(s_0)$ is an eigenvalue of $P(s_0)\,\mathcal{E}(s_0)$. Hence

$$|\dot{\lambda}_j(s_0)-\lambda_j^2(s_0)| \leqq \|P(s_0)\,\mathcal{E}(s_0)\| \leqq \|\mathcal{E}(s_0)\| \leqq \varepsilon \|A(s_0)\|^2 \leqq \varepsilon n^2 \sum_{i=1}^{n} \lambda_i^2(s_0),$$

where we used (3.4) and standard inequalities. Since the above estimate is true independently of s_0, the proof of the Lemma is complete. □

We intend to use the ODE's (3.5) to show that

$$\frac{c}{s} \leqq \sum_{i=1}^{n} |\lambda_i(s)| \leqq \frac{C}{s}, \tag{3.10}$$

for some positive constants c. C. These estimates, of course, are equivalent to (3.2).

Proof of (3.10). – At first we note that $\sum_{i=1}^{n} |\lambda_i(s)|$ is always different from zero. Indeed, if it is zero at some time s_0 then $A(s_0)=0$. But then, by the uniqueness of the initial value problem, $A(s)$ would be zero for all times, and this forces v_0 as well as v to be identically zero.

To show the left inequality of (3.10) we use (3.5) to obtain

$$\left(\sum_{i=1}^{n} \lambda_i^2\right)^{\cdot} = 2 \sum_{i=1}^{n} \lambda_i \dot{\lambda}_i \geqq -c \left(\sum_{i=1}^{n} \lambda_i^2\right)^{3/2}.$$

for some positive constant c, and the desired estimate follows by integration.

The right inequality of (3.10) requires more work. We first obtain some preliminary estimates. From (3.5) it is easy to see that

$$\left(\sum_{i=1}^{n} \lambda_i\right)^{\cdot} \geqq \frac{c_n}{2} \sum_{i=1}^{n} \lambda_i^2 > 0, \tag{3.11}$$

for s sufficiently large. Since $\sum_{i=1}^{n} \lambda_i(s) \to 0$ as $s \to \infty$, we must have $\sum_{i=1}^{n} \lambda_i(s) < 0$ for s large. Furthermore, by (3.11)

$$\left(\sum_{i=1}^{n} \lambda_i\right)^{\cdot} \geqq c \left(\sum_{i=1}^{n} \lambda_i\right)^2,$$

for some $c > 0$, from which we conclude that

$$\left|\sum_{i=1}^{n} \lambda_i(s)\right| \leqq \frac{C}{s}. \tag{3.12}$$

To complete the proof, we now use an argument by contradiction. Suppose that

$$\limsup_{s \to \infty} s \sum_{i=1}^{n} |\lambda_i(s)| = \infty. \tag{3.13}$$

We will show that there exists a sequence $s_j \to \infty$ such that

$$s_j \left|\sum_{i=1}^{n} \lambda_i(s_j)\right| \to \infty, \tag{3.14}$$

which is a contradiction to (3.12).

By (3.13), there exists a sequence $s_j \to \infty$ such that

$$\lim_{j \to \infty} s_j \sum_{i=1}^{n} |\lambda_i(s_j)| = \infty. \tag{3.15}$$

We shall show that (3.15) implies (3.14). Since

$$\frac{|\lambda_i(s)|}{\sum_{k=1}^{n} |\lambda_k(s)|} \leq 1, \qquad i=1, \ldots, n,$$

we can find a subsequence of $\{s_j\}$ (again labeled as s_j) such that

$$\frac{\lambda_i(s_j)}{\sum_{k=1}^{n} |\lambda_k(s_j)|} \to \gamma_i, \qquad i=1, \ldots, n.$$

It is clear that at least one $\gamma_i \neq 0$. Set

$$\begin{aligned} I^+ &= \{ i : \gamma_i > 0 \}, \\ I^- &= \{ i : \gamma_i < 0 \}, \\ \lambda^+(s) &= \sum_{i \in I^+} \lambda_i(s), \\ \lambda^-(s) &= \sum_{i \in I^-} \lambda_i(s). \end{aligned}$$

We know that $I^+ \cup I^- \neq \varnothing$. If we show that $I^+ = \varnothing$, then (3.14) will follow from (3.15). Therefore, the proof of (3.10) reduces to the proof of the fact that $I^+ = \varnothing$.

Suppose that $I^+ \neq \varnothing$. Notice that $I^- \neq \varnothing$ since $\sum_{i=1}^{n} \lambda_i(s) < 0$. Clearly, for sufficiently large j

$$\lambda^+(s_j) > 0, \qquad \lambda^-(s_j) < 0. \tag{3.16}$$

Moreover

$$0 < \lim_{j \to \infty} \frac{|\lambda^-(s_j)|}{|\lambda^+(s_j)|} = \frac{|\Sigma_{I^-} \gamma_i|}{\Sigma_{I^+} \gamma_i} < \infty. \tag{3.17}$$

On the other hand, using (3.5) we get that

$$\frac{d}{ds} \lambda^+ = c_n \sum_{i \in I^+} \lambda_i^2 + o\left(\sum_{i=1}^{n} \lambda_i^2 \right), \tag{3.18}$$

$$\frac{d}{ds} \lambda^- = c_n \sum_{i \in I^-} \lambda_i^2 + o\left(\sum_{i=1}^{n} \lambda_i^2 \right). \tag{3.19}$$

It follows from (3.17)-(3.19) that

$$\frac{d}{ds} \lambda^+(s_j) > 0, \qquad \frac{d}{ds} \lambda^-(s_j) > 0,$$

for s_j large enough.

Fix an s_j. Since $\lambda^+(s_j)>0, \frac{d}{ds}\lambda^+(s_j)>0$ and $\lambda^+(s_j)\to 0$, there exists at least one $s>s_j$, such that $\frac{d}{ds}\lambda^+(s)=0$. Let us denote by t_j the first such s. Clearly $|\lambda^+|(s)=\lambda^+(s)$ is increasing in (s_j, t_j). Because of (3.20), either $\frac{d}{ds}\lambda^-(s)>0$ throughout the interval (s_j, t_j), or else there exists a time s_0 in (s_j, t_j) at which $\frac{d}{ds}\lambda^-(s_0)=0$. In both case we will reach a contradiction.

(i) Consider first the case where there exists an $s_0\in(s_j, t_j)$ such that $\frac{d}{ds}\lambda^-(s_0)=0$. Clearly, $\lambda^+(s_0)>0$. On the other hand, it follows from (3.19) that

$$\sum_{i\in I^-}\lambda_i^2(s_0)=o\Big(\sum_{i\in I^+}\lambda_i^2(s_0)\Big).$$

Consequently, for any $\delta>0$,

$$\left|\frac{\lambda^-(s_0)}{\lambda^+(s_0)}\right|\leqq\delta,$$

for s_j sufficiently large. But this is a contradiction to the fact that $\sum_{i=1}^{n}\lambda_i(s)<0$.

(ii) Now assume that $\frac{d}{ds}\lambda^-(s)>0$ in (s_j, t_j). Then $\lambda^+(s)>0$ is increasing while $|\lambda^-|(s)>0$ is decreasing in (s_j, t_j). Consequently,

$$\left|\frac{\lambda^+(t_j)}{\lambda^-(t_j)}\right|\geqq\left|\frac{\lambda^+(s_j)}{\lambda^-(s_j)}\right|\geqq c>0, \tag{3.21}$$

by (3.17). On the other hand, since $\frac{d}{ds}\lambda^+(t_j)=0$, the same argument as before shows that for any $\delta>0$,

$$\left|\frac{\lambda^+(t_j)}{\lambda^-(t_j)}\right|\leqq\delta,$$

for s_j large, which contradicts (3.21). The proof of (3.10) is now complete.

4. CENTER MANIFOLD ANALYSIS IN $L^2_p(R^n)$

In this Section we will give the proof of Theorem A. We recall that we are studying a nonnegative, blowing up solution of the semilinear heat equation (1.1). We assume that when written in similarity variables [*see* (1.2)], the solution satisfies (i) w is nonnegative, (ii) w is uniformly bounded in space-time and (iii) it tends to κ as $s \to \infty$ uniformly on compact sets in y. These conditions are known to be valid for any nonnegative solution of the Cauchy problem (1.1) provided that (*a*) u is uniformly bounded at infinity (*e.g.* $u \to 0$ at infinity); (*b*) $n \leqq 2$ or, if $n \geqq 3$, $p < \frac{n+2}{n-2}$; and (*c*) the center of scaling a is a blowup point (*cf.* [12], [13], [14]).

In giving the proof of Theorem A two Lemmas will be used. The first one is an elementary ODE Lemma similar to the one used in [7]. In fact it is a slight modification of it, which allows us to deal with quadratic terms. It basically contains all the center manifold ideas that we use in the present work.

LEMMA 4.1. – *Let $x(t)$, $y(t)$, $z(t)$ be absolutely continuous, real valued functions which are nonnegative and satisfy:*

$$\dot{z} \geqq c_0 z - c_1 (x+y)^2 \tag{4.1}$$

$$|\dot{x}| \leqq c_1 (x+y+z)^2 \tag{4.2}$$

$$\dot{y} \leqq -c_0 y + c_1 (x+z)^2 \tag{4.3}$$

$$x, y, z \to 0, \quad \text{as } t \to \infty \tag{4.4}$$

where c_0, c_1 are positive constants. Then

either (i) *x, y, $z \to 0$ exponentially fast*

or (ii) *there exists a time t_0 after which $z+y<bx^2$, where b is a positive constant depending only on c_0, c_1.*

Proof. – By rescaling in time we may assume $c_1 = 1$. We divide the proof into five steps:

Step 1: Unless x, y, $z \to 0$ exponentially fast, there will be a time at which $c_0 y < 2(x+z)^2$.

Indeed, if $c_0 y \geqq 2(x+z)^2$ for all time then from (4.3) we would have:

$$\dot{y} \leqq -c_0 y + (x+z)^2 \leqq -c_0 y + \frac{c_0}{2} y = -\frac{c_0}{2} y.$$

This implies that $y \to 0$ exponentially fast and that forces x, z to decay exponentially fast as well.

Step 2: Let $\alpha(t) = c_0 y - 2(x+z)^2$. Once $\alpha(t)$ becomes negative, it will stay nonpositive thereafter, for large times.

The key observation here is that:

$$\alpha(t) \geqq 0 \quad \Rightarrow \quad \dot{\alpha}(t) \leqq 0 \tag{4.5}$$

provided $t \geqq t_0$ for some t_0 large enough. Indeed, using (4.1)-(4.3) we write:

$$\begin{aligned}\dot{\alpha}(t) &= c_0 \dot{y} - 4(x+z)(\dot{x}+\dot{z}) \\ &\leqq -c_0 y^2 + c_0 (x+z)^2 + 4(x+z)[(x+y+z)^2 - c_0 z + (x+y)^2] \\ &= -c_0^2 y [1 - \frac{8}{c_0^2}(x+z)(2x+y+z)] + c_0 (x+z)^2 \\ &\qquad - 4(x+z)[c_0 z - (x+z)^2 - x^2].\end{aligned}$$

By choosing t_0 large enough, we can arrange, [because of (4.4)] so that:

$$1 - \frac{8}{c_0^2}(x+z)(2x+y+z) \geqq \frac{2}{3}. \tag{4.6}$$

Using (4.6) and the fact that $c_0 y \geqq 2(x+z)^2$ [which is equivalent to $\alpha(t) \geqq 0$] we end up with:

$$\dot{\alpha}(t) = -4 c_0 (x+z) z - (x+z)\left[\frac{c_0}{3} x + \frac{c_0}{3} z - 4(x+z)^2 - 4x^2\right] \leqq 0,$$

for t_0 large enough, and (4.5) has been proved.

Let $\alpha_+(t)$ be the positive part of $\alpha(t)$, *i.e.* $\alpha_+(t) = \max\{\alpha(t), 0\}$. From (4.5) we have that

$$\dot{\alpha}_+(t) \leqq 0,$$

for all $t > t_0$. Suppose now, that $\alpha(t') < 0$ for some $t' (> t_0)$. From the fundamental theorem of calculus, we have that for any $t > t'$:

$$\alpha_+(t) = \alpha_+(t) - \alpha_+(t') = \int_{t'}^{t} \dot{\alpha}_+(s)\, ds \leqq 0,$$

from which we conclude that $\alpha(t)$ stays nonpositive for all $t > t'$; *i.e.*

$$c_0 y \leqq (x+z)^2 \tag{4.7}$$

for t large enough.

Step 3: There exists some time after t_0 at which $c_0 z < 2(x+y)^2$.

If not, (4.1) would force z to grow exponentially fast contradicting (4.4).

Step 4: Let $\beta(t) = c_0 z - 2(x+y)^2$. Once this quantity becomes negative it will stay nonpositive thereafter.

Working as in step 2 we first show that so long as $\beta(t) \geqq 0$ then:

$$\dot{\beta}(t) \geqq 4 c_0 (x+y) y + (x+y)\left[\frac{c_0}{3} x + \frac{c_0}{3} y - 4(x+y)^2 - 4x^2\right] \geqq 0,$$

for t large enough. But then, if $\beta(t)$ ever becomes positive it should have a nonnegative slope, contradicting the fact that $\beta(t) \to 0$. We conclude:

$$c_0 z \leqq 2(x+y)^2, \tag{4.8}$$

for t large enough.

Step 5: The desired result follows from (4.7) and (4.8). □

The second Lemma is concerned with an *a priori* estimate of solutions of:

$$v_s = \Delta v - \frac{y}{2} . \nabla v + v + f(v), \quad \text{in } \mathbf{R}^n \tag{4.9}$$

with $f(x) = O(x^2)$ as $x \to 0$. This is due to Herrero and Velázquez [15]; although they proved it in the one dimensional case the same proof works for any space dimension.

LEMMA 4.2. – *Assume that v solves* (4.9) *and* $|v| \leqq M < \infty$. *Then for any $r > 1$, $q > 1$ and $L > 0$ there exists $s_0^* = s_0^*(q, r)$ and $C = C(r, q, L) > 0$ such that*

$$\left(\int_{\mathbf{R}^n} v^r(., s+s^*) \rho \right)^{1/r} \leqq C \left(\int_{\mathbf{R}^n} v^q(., s) \rho \right)^{1/q}, \tag{4.10}$$

for any $s > 0$ and any $s^* \in [s_0^*, s_0^* + L]$.

Using the above Lemma and Proposition 3.1 we write (for large s)

$$\left(\int_{\mathbf{R}^n} v^r(., s) \rho \right)^{1/r} \leqq \left(\int_{\mathbf{R}^n} v^2(., s - s_0^*) \rho \right)^{1/2}$$
$$\leqq \frac{C_1}{s - s_0^*} < \frac{C_2}{s} \leqq C_3 \left(\int_{\mathbf{R}^n} v^2(., s) \rho \right)^{1/2},$$

for suitable constants C_i. That is, if we know the decay rate of v, we can eliminate the "delay constant" s^* in (4.10); in particular we have

COROLLARY 4.1. – *If $\| v \|_{L^2_\rho}$ decays like $1/s$ then for every $r > 1$, $q > 1$ there exists a $C = C(r, q) > 0$ such that*

$$\left(\int_{\mathbf{R}^n} v^r(., s) \rho \right)^{1/r} \leqq C \left(\int_{\mathbf{R}^n} v^q(., s) \rho \right)^{1/q}, \tag{4.11}$$

for s large enough.

We now give the proof of Theorem A.

Proof of Theorem A. – Let, as usual, v_+ denote the projection of v onto the eigenfunctions of $\mathscr{L}$ corresponding to the positive eigenvalues and similarly for v_0 and v_-. We also set $x = \| v_0 \|_{L^2_\rho}$, $y = \| v_- \|_{L^2_\rho}$, $z = \| v_+ \|_{L^2_\rho}$ and $N = \| v^2 \|_{L^2_\rho}$. It has been shown in [7] that these quantities satisfy the following differential inequalities:

$$\begin{aligned} \dot{z} &\geqq \frac{1}{2} z - \mathrm{CN} \\ |\dot{x}| &\leqq \mathrm{CN} \\ \dot{y} &\leqq -\frac{1}{2} y + \mathrm{CN}, \end{aligned} \tag{4.12}$$

with some positive constant C. We already know from (1.8) and Proposition 3.1, that either x, y, z they all go to zero exponentially fast, or else $\|v\|_{L^2_\rho}$ decays like $1/s$. Assuming we are in the second case and using (4.11) we have that

$$\mathrm{N}(s) = \left(\int v^4(.,s)\rho\right)^{1/2} \leqq c' \int v^2(.,s)\rho = c'(x(s)+y(s)+z(s))^2,$$

for some c' positive. From the first inequality of system (4.12) we get:

$$\dot{z} \geqq \left[\frac{1}{2} - cz - 2c(x+y).\right] z - c(x+y)^2.$$

with $c = c'\mathrm{C}$. Since x, y, z they all go to zero with s, we get for large enough time:

$$\dot{z} \geqq \frac{1}{3} z - c(x+y)^2.$$

The last inequality of (4.12) can be dealt with similarly. Hence, from system (4.12) we have:

$$\begin{aligned} \dot{z} &\geqq \frac{1}{3} z - c(x+y)^2 \\ |\dot{x}| &\leqq c(x+y+z)^2 \\ \dot{y} &\leqq \frac{1}{3} y + c(x+z)^2. \end{aligned} \tag{4.13}$$

We now use lema 4.1 to conclude that there exists a time s_0 after which

$$y + z < bx^2,$$

and the Theorem has been proved. □

We next rederive the ODE satisfied by the neutral modes. As we explained in Section 2 we do that in order to obtain a better estimate on the error terms.

PROPOSITION 4.1. — *Assume that v does not approach* 0 *exponentially fast. Then the neutral modes* $\{\alpha_j\}_{j=1}^m$ *satisfy*

$$\dot{\alpha}_j = \frac{p}{2\kappa} \Pi_j^0(v_0^2) + O\left(\left(\sum_{j=1}^m \alpha_j^2\right)^{3/2}\right). \tag{4.14}$$

Proof. – We first recall the equation (1.5) satisfied by v

$$v_s = \mathscr{L}\, v + \frac{p}{2\kappa}\, v^2 + g(v), \tag{4.15}$$

with $|g(v)| \leqq c\,|v|^3$ for some positive constant c. Let $e_j^0(y)$, $j = 1, 2, \ldots, m$ be the neutral eigenfunctions of $\mathscr{L}$ as described in Section 2. Projecting (4.15) onto e_j^0 we get:

$$\dot{\alpha}_j = \frac{p}{2\kappa}\, \Pi_j^0(v^2) + \Pi_j^0(g(v)),$$

or

$$\dot{\alpha}_j = \frac{\rho}{2\kappa}\, \Pi_j^0(v_0^2) + \mathscr{E},$$

with

$$\mathscr{E} = \Pi_j^0(v^2 - v_0^2) + \Pi_j^0(g(v)) \overset{\triangle}{=} \mathscr{E}_1 + \mathscr{E}_2.$$

We will show that $|\mathscr{E}| \leqq C\left(\sum_{i=1}^{m} \alpha_j^2\right)^{3/2}$. We first estimate $\mathscr{E}_1$. Recalling that $v = v_+ + v_0 + v_-$ we write:

$$\begin{aligned} |\mathscr{E}_1| &\leqq \int |v^2 - v_0^2|\,|e_j^0|\,\rho = \int |v_+ + v_-|\,.\,|v + v_0|\,|e_j^0|\,\rho \\ &\leqq \left(\int |v_+ + v_-|^2\,\rho\right)^{1/2} \left(\int |v + v_0|^2\,|e_j^0|^2\,\rho\right)^{1/2} \\ &\leqq C\left(\int v_0^2\,\rho\right).\left\{\left(\int v^2\,|e_j^0|^2\,\rho\right)^{1/2} + \left(\int v_0^2\,|e_j^0|^2\,\rho\right)^{1/2}\right\}, \end{aligned}$$

where we used theorem A and standard inequalities. Moreover we have:

$$\left(\int v^2\,|e_j^0|^2\,\rho\right)^{1/2} \leqq \left(\int v^4\,\rho\right)^{1/4} \left(\int |e_j^0|^4\,\rho\right)^{1/4} \leqq C\left(\int v^2\,\rho\right)^{1/2},$$

using (4.11). Since

$$\int v^2\,\rho = \int v_+^2\,\rho + \int v_0^2\,\rho + \int v_-^2\,\rho \leqq C \int v_0^2\,\rho, \tag{4.16}$$

we get that

$$\left(\int v^2\,|e_j^0|^2\,\rho\right)^{1/2} \leqq C\left(\int v_0^2\,\rho\right)^{1/2}.$$

Next we note that v_0 belongs to a finite dimensional space (the neutral subspace of $\mathscr{L}$) and therefore all norms of it are equivalent:

$$\left(\int v_0^2 |e_j^0|^2 \rho\right)^{1/2} \leqq C\left(\int v_0^2 \rho\right)^{1/2}.$$

Putting everything together we have:

$$|\mathscr{E}_1| \leqq C\left(\int v_0^2 \rho\right)^{3/2} = C\left(\sum_{j=1}^{m} \alpha_j^2\right)^{3/2}.$$

We finally estimate $\mathscr{E}_2$:

$$|\mathscr{E}_2| \leqq \int |g(v)||e_j^0| \rho \leqq C \int |v|^3 |e_j^0| \rho \leqq C\left(\int v^6 \rho\right)^{1/2} \leqq C\left(\int v^2 \rho\right)^{3/2},$$

where in the last inequality we used (4.11). Using now (4.16) we get

$$|\mathscr{E}_2| \leqq C\left(\int v_0^2\right)^{3/2},$$

and the proof of Proposition 4.1 is complete. □

As a consequence of the above Proposition we have the following

Corollary 4.2. — *The coefficient matrix-function satisfies for s large enough*

$$\dot{A}(s) = c_n A^2(s) + O\left(\frac{1}{s^3}\right), \qquad (4.17)$$

with $c_n = \frac{p}{\kappa}(4\pi)^{-n/4}$.

This follows from (4.14), (3.2) and the definition of A (*s*) (*see* Section 2).

5. REFINED ASYMPTOTICS

Our task in this section is to give the proof of Theorem B. To do this we first have to study in more details the large time behaviour of A (*s*).

Our main conclusion concerning the asymptotics of A (*s*) is the following

Proposition 5.1. — *For large enough times we have that*

$$A(s) = -\frac{c_n^{-1}}{s} A_0 + O\left(\frac{1}{s^{1+\delta}}\right), \qquad (5.1)$$

for some $\delta > 0$, *with*

$$A_0 = Q\begin{pmatrix} I_{n-k} & 0 \\ 0 & 0_k \end{pmatrix} Q^{-1}, \qquad (5.2)$$

for some $k \in \{0, 1, \ldots, n-1\}$, *where* Q *is a (constant) orthonormal matrix,* I_{n-k} *is the* $(n-k) \times (n-k)$ *identity matrix and* 0_k *is the* $k \times k$ *zero matrix.*

Proof. – As usual we set $c_n = 1$. We being by analyzing the asymptotics of the eigenvalues of $A(s)$. In view of (4.17), an argument identical to that of Lemma 3.2 shows that they satisfy:

$$\dot{\lambda}_i = \lambda_i^2 + O\left(\frac{1}{s^3}\right), \qquad i = 1, \ldots, n. \tag{5.3}$$

One can show that the λ_i's are either equal to $-\frac{1}{s} + O\left(\frac{1}{s^{1+\delta}}\right)$, or else they are of the order $O\left(\frac{1}{s^{1+\delta}}\right)$, for some δ positive. (To keep the flow of our argument we postpone the proof of this statement for later on.) Let us therefore assume that k of the eigenvalues of the matrix $A(s)$ follow the second pattern; *i. e.*

$$\begin{aligned} \lambda_i(s) &= -\frac{1}{s} + O\left(\frac{1}{s^{1+\delta}}\right), \qquad i = 1 \ldots n-k, \\ \lambda_i(s) &= O\left(\frac{1}{s^{1+\delta}}\right), \qquad i = n-k+1, \ldots, n, \end{aligned} \tag{5.4}$$

for some $k \geqq 0$. In view of Lemma 3.1 we also have that $k \leqq n-1$.

The knowledge of the asymptotics of the λ_i' does not suffice to conclude (5.2). We also need to know that Q is a constant matrix (independent of s). To this end, we will show that the elements of $A(s)$ are of the form:

$$a_{ij}(s) = \frac{a_{0ij}}{s} + O\left(\frac{1}{s^{1+\delta}}\right), \qquad 1 \leqq i, j \leqq n, \tag{5.5}$$

for suitable constants a_{0ij} (some of them possibly zero). Working in this direction we set $\gamma_l = \{i_1, \ldots, i_l\} \subset \{1, \ldots, n\}$, and let A_{λ_l} be the principal minor of order l of the matrix A, which is formed as follows:

$$A_{\gamma_l} = \det \begin{pmatrix} a_{i_1 i_1} \ldots a_{i_1 i_1} \\ \vdots \qquad \vdots \\ a_{i_1 i_1} \ldots a_{i_1 i_l} \end{pmatrix}.$$

(Warning: A_{γ_l} is a scalar, not a matrix.) Clearly,

$$|A_{\gamma_l}| \leqq \frac{C}{s^l}, \qquad l = 1, 2, \ldots, n, \tag{5.6}$$

for some positive constant C. Moreover, by a straightforward calculation, one can verify that

$$\dot{A}_{\gamma_l} = A_{\gamma_l} \operatorname{tr} A + \sum_{m \notin \gamma_l} A_{\gamma_l \cup \{m\}} + O\left(\frac{1}{s^{l+2}}\right), \qquad l = 1, 2, \ldots, n-1. \quad (5.7)$$

We intend to derive (5.5) by studying the system (5.7). At first we note that because of (5.4)

$$\operatorname{tr} A = -\frac{n-k}{s} + O\left(\frac{1}{s^{1+\delta}}\right),$$

and

$$A_{\gamma_n} = \det A = \begin{cases} O\left(\frac{1}{s^{n+\delta}}\right) & \text{if } k \geqq 1 \\ (-1)^n \frac{1}{s^n} + O\left(\frac{1}{s^{n+\delta}}\right) & \text{if } k = 0. \end{cases} \quad (5.8)$$

Assume $k \geqq 1$. The case $k = 0$ can be dealt with similarly. For $l = n-1$, from (5.7), (5.8) we get:

$$\dot{A}_{\gamma_{n-1}} = A_{\gamma_{n-1}} \operatorname{tr} A + \det A + O\left(\frac{1}{s^{n+1}}\right) = -\frac{n-k}{s} A_{\gamma_{n-1}} + O\left(\frac{1}{s^{n+\delta}}\right). \quad (5.9)$$

Solving (5.9), we get

$$A_{\gamma_{n-1}} = \frac{C_{\gamma_{n-1}}}{s^{n-k}} + O\left(\frac{1}{s^{n-1+\delta}}\right), \quad (5.10)$$

for some (undetermined) constant $C_{\gamma_{n-1}}$. In view of (5.6) we also have that if $k > 1$ then necessarily $C_{\gamma_{n-1}} = 0$. In any case, (5.10) describes the decay rate of $A_{\gamma_{n-1}}$. Clearly, once we know the asymptotic behavior of A_{γ_l} we can use (5.7) to obtain the same information about $A_{\gamma_{l-1}}$, for any $l = n-1, \ldots, 1$. Thus, by repeating the previous argument we eventually end up with:

$$\begin{aligned} A_{\gamma_l} &= O\left(\frac{1}{s^{l+\delta}}\right), && \text{if } n-k+1 \leqq l \leqq n, \\ A_{\gamma_l} &= \frac{C_{\gamma_l}}{s^l} + O\left(\frac{1}{s^{l+\delta}}\right), && \text{if } 1 \leqq l \leqq n-k. \end{aligned} \quad (5.11)$$

Notice that for $l = 1$, A_{γ_1} is equal to a_{ii} (if we choose $\gamma_1 = \{i\}$) and therefore (5.5) has been proved for the diagonal elements of A. For the off diagonal elements $a_{ij} (i \neq j)$ we observe that:

$$a_{ij}^2 = a_{ii} a_{jj} - A_{ij},$$

where the behavior of A_{ij} is given by (5.11) for $l=2$. Thus, (5.5) has been established.

We can now write

$$A(s)=-\frac{1}{s}A_0+O\left(\frac{1}{s^{1+\delta}}\right),$$

where A_0 is a constant symmetric matrix. Since the eigenvalues of $sA(s)$ are

$$\underbrace{-1+O\left(\frac{1}{s^{\delta}}\right),\ \ldots,\ -1+O\left(\frac{1}{s^{1+\delta}}\right)}_{n-k},\quad \underbrace{O\left(\frac{1}{s^{1+\delta}}\right),\ \ldots,\ O\left(\frac{1}{s^{1+\delta}}\right)}_{k},$$

and $sA(s)\to -A_0$, the eigenvalues of A_0 are

$$\underbrace{1,\ \ldots,\ 1}_{n-k},\quad \underbrace{0,\ \ldots,\ 0}_{k}.$$

Therefore, there exists an orthonormal matrix Q sucht that

$$A_0=Q\begin{pmatrix} I_{n-k} & 0 \\ 0 & 0_k \end{pmatrix}Q^{-1}.$$

This completes the proof of Proposition 5.1. □

It remains to give the proof of (5.4). This is a direct consequence of the following elementary Lemma.

LEMMA 5.1 *Let $x(s)$ be a solution of*

$$\dot{x}=x^2+O\left(\frac{1}{s^3}\right) \tag{5.12}$$

which exists for all time. Then either

$$x(s)=-\frac{1}{s}+O\left(\frac{1}{s^{1+\delta}}\right), \tag{5.13}$$

or else,

$$x(s)=O\left(\frac{1}{s^{1+\delta}}\right), \tag{5.14}$$

for some $\delta\in(0, 1/2)$.

Proof. – Fix an s_0 large enough and let β be some number in (0, 1/2). If

$$|x(s)|\leq\frac{1}{s^{1+\beta}},\qquad \text{for}\quad s\geq s_0, \tag{5.15}$$

we are done. If (5.15) is not true then there exists a time $s_1 > s_0$, such that $|x(s_1)| > s_1^{-1-\beta}$. There are two possibilities: either

$$|x(s)| \geqq \frac{1}{s^{1+\beta}}, \qquad \text{for} \quad s \geqq s_1, \tag{5.16}$$

or else, there exists an $s_2 > s_1$ such that

$$|x(s_2)| = \frac{1}{s_2^{1+\beta}}, \qquad |x(s)| < \frac{1}{s^{1+\beta}}, \tag{5.17}$$

in $(s_2, s_2 + \varepsilon)$ for some $\varepsilon > 0$. If (5.16) is the case then, from (5.12) we get

$$-\left(\frac{1}{x}\right)^{\cdot} = 1 + O\left(\frac{1}{s^{1-2\beta}}\right), \tag{5.18}$$

and (5.13) follows by integration. We next show that (5.17) cannot happen. For definiteness let us assume that $x(s_2) < 0$ (the case $x(s_2) > 0$ is easier). From (5.17) we have that

$$\dot{x}(s_2) \geqq \left(-\frac{1}{s^{1+\beta}}\right)^{\cdot}\Bigg|_{s=s_2} = \frac{1+\beta}{s_2^{2+\beta}}. \tag{5.19}$$

On the other hand, from (5.12) we have that

$$\dot{x}(s_2) = \frac{1}{s_2^{2+2\beta}} + O\left(\frac{1}{s_2^3}\right),$$

which contradicts (5.19), if s_0 is chosen sufficiently large. □

It follows from Lemma 2.2 and Proposition 5.1 that

$$v_0(y, s) = v^*(y, s) + O\left(\frac{|y|^2}{s^{1+\delta}}\right). \tag{5.20}$$

with

$$v^*(y, s) = \frac{\kappa}{2ps}\left(\operatorname{tr} A_0 - \frac{1}{2} y^T A_0 y\right).$$

We are now ready to give the proof of theorem B.

Proof of Theorem B. – At first we show that convergence takes place in the L^2 norm in every ball $B_R \subset R^n$ of radius R $(0 < R < \infty)$. Let C denote a positive constant, not necessarily the same in each occurrence.

$$\begin{aligned}\| v-v^* \|_{L^2(B_R)} &\leqq e^{R^2/8} \| v-v^* \|_{L^2_\rho(R^n)} \\ &\leqq e^{R^2/8} (\| v_+ \|_{L^2_\rho(R^n)} + \| v_- \|_{L^2_\rho(R^n)} + \| v_0 - v^* \|_{L^2_\rho(R^n)} \\ &\leqq C e^{R^2/8} \| v_0 \|^2_{L^2_\rho(R^n)} + C e^{R^2/8} \frac{1}{s^{1+\delta}} \\ &\leqq \frac{C}{s^{1+\delta}},\end{aligned}$$

where we used theorem A and (5.20).

We next use standard parabolic estimates to improve the mode of convergence. Since the arguments here are well known, we simply sketch them.

Set $U = v - v^*$. Then U solves the parabolic PDE:

$$U_s - \mathscr{L} U = F(U), \tag{5.22}$$

where

$$F(U) = f(U + v^*) + \frac{1}{s} v^*,$$

with $|f(U+v^*)| \leqq C |U+v^*|^2$. Let us denote by Q_1, Q_2 two space-time parabolic cylinders located inside the strip $(s_0 - 1, s_0)$. We also assume that Q_2 is strictly contained in Q_1. From interior regularity theory (*see* [19], Chap. IV, § 10, p. 355) we have that for $p > 1$,

$$\| U \|_{W^{2,1}_p(Q_2)} \leqq \| F \|_{L^p(Q_1)} + C \| U \|_{L^p(Q_1)}. \tag{5.23}$$

Set $p = 2$ to start with. It is easy to check that (for s_0 large enough):

$$\| F \|_{L^2(Q_1)}, \ \| U \|_{L^2(Q_1)} \leqq \frac{C}{s_0^{1+\delta}}.$$

Therefore from (5.23) we have that $\| U \|_{W^1_2(Q_2)} \leqq C s_0^{-(1+\delta)}$. By the Sobolev embedding theorem we also that $\| U \|_{L^p(Q_2)} \leqq C_0^{-(1+\delta)}$, for some $p > 2$.

We iterate this scheme until we get $\| U \|_{W^{2,1}_p(Q)} \leqq C s_0^{-(1+\delta)}$, for some $p > n+1$, in some (smaller) cylinder $Q = B \times (s'_0, s'_1)$. Then, again from the Sobolev embedding theorem, we conclude that

$$\| U \|_{C^\lambda(Q)} \leqq \frac{C}{s_0^{1+\delta}}, \tag{5.24}$$

for some $\lambda>0$. Since (5.24) holds true independently of s_0, we have in particular that

$$\sup_{y\in B}|U|\leqq\frac{C}{s^{1+\delta}},\qquad\text{for}\quad s\geqq s_0'.$$

Finally, we use Schauder estimates in a similar fashion to obtain convergence in higher norms. The details are omitted.

6. ON THE LOCAL STRUCTURE OF THE BLOW UP SET

As explained in section 2 we expect that the asymptotic profile of $v(y, s)$ should reflect the local geometry of the blow up set. It has been proved in [7] that this indeed is hat happens, in the special case where $n=1$ and v decays at an algebraic rate. We can now use the same method, to extend this result to arbitrary space dimension n. Roughly speaking we will show that if the center of scaling b is a blow up point, there is no other blow up point in a neighborhood of b except possibly along the region where $v(y, s)$ takes on its maximum values. Our precise statement is Theorem C.

We briefly recall the method of [7]. We denote by $E[w_a](s)$ the "energy" functional, defined for all solutions of (1.1) rescaled about any point (a, T) by:

$$E[w_a](s)=\int_{\mathbf{R}^n}e^{-y^2/4}\times\left(\frac{1}{2}|\nabla w_a(y, s)|^2+\frac{1}{2(p-1)}w_a^2(y, s)-\frac{1}{p+1}w_a^{p+1}(y, s)\right)dy.\qquad(6.1)$$

At the heart of the method is the following result due to Giga and Kohn [14]: if

$$E[w_a](s_0)<E[\kappa],\qquad(6.2)$$

for some time s_0, then a is not a blow up point, where $\kappa=\left(\frac{1}{p-1}\right)^{1/(p-1)}$.

This result can be used in the following way. Assuming that b is a blowup point, and using the similarity change of variables and the refined asymptotics of w_b, one can calculate the "energy" functional corresponding to points a in a neighborhood of b. Those point a for which (6.2) is true are then excluded from the blowup set.

We now give the proof of Theorem C.

Proof of Theorem C. — Assume that 0 is a blow up point and let a be a point near it. Then, using the usual transformation (1.2) we have:

$$w_a(\tilde{y}, s)=w_0(y, s),$$

and

$$\tilde{y}=y-\frac{a}{\sqrt{T-t}}=y-\gamma,$$

with $\gamma=a/\sqrt{T-t}$. Moreover $w_0(y,s)=v(y,s)+\kappa$ and we know from Theorem B that for large times $v(y,s)$ behaves likes

$$v^*(y,s)=\frac{\kappa}{2ps}\left(\operatorname{tr} A_0-\frac{1}{2}y^T A_0 y\right).$$

It has been shown in [7] that for large times we can rewrite (6.1) as:

$$E[w_a]=E[\kappa]+\frac{1}{2}\int(\nabla v^*|^2-|v^*|^2)\rho_\gamma+o\left(\frac{1}{s^2}\right), \tag{6.3}$$

where

$$\rho_\gamma=e^{-(|y-\gamma|^2/4)}.$$

Although this was proved for $n=1$, the proof works for any space dimension with trivial changes. (In fact, all arguments in [7] are n-dimensional, except Lemma 7.2. This Lemma is a direct consequence of Lemma 7.3 there and the fact that v decays like $1/s$).

We next turn our attention to the second term of the right hand side of (6.3). It is convenient at this point to change variables by $y'=Q^T y$ where Q is the same orthonormal matrix as in Theorem B. In the new coordinate system (we drop the prime for simplicity) we have:

$$\int_{R^n}(|\nabla v^*|^2-|v^*|^2)\rho_\gamma=\frac{\kappa^2}{4p^2s^2}\int_{R^n}\left\{\sum_{i=1}^{n-k}y_i^2-\left(\sum_{i=1}^{n-k}\left(1-\frac{1}{2}y_i^2\right)\right)^2\right\}\rho_\gamma.$$

Let us now introduce some notation. We denote by E_I^{n-k} the span of $(y_1, y_2, \ldots, y_{n-k})$ and by E_0^k the span of $(y_{n-k+1}, \ldots, y_n)$. We note that E_I^{n-k} coincides with the eigenspace of A_0 corresponding to the eigenvalue 1, whereas E_0^k (as previously) is the eigenspace associated with the zero eigenvalue. We also write $\gamma=\gamma_I+\gamma_0$ where γ_I is the projection of γ onto E_I^{n-k}, and similarly for γ_0. We can now rewrite (6.4) as:

$$\int_{R^n}(|\nabla v^*|^2-|v^*|^2)\rho_\gamma$$

$$=-\frac{\kappa^2}{4p^2s^2}\int_{R^k}\rho_{\gamma 0}\int_{R^{n-k}}\left\{\sum_{i=1}^{n-k}y_i^2-\left(\sum_{i=1}^{n-k}\left(1-\frac{1}{2}y_i^2\right)\right)^2\right\}\rho_{\gamma I}. \tag{6.5}$$

The first integral of the right hand side is easily found to be equal to $(4\pi)^k/2$. For the second integral we expand the square, change variables $x\to y-\gamma$ and after several integration by parts we can compute it, so that finally we get:

$$\int_{\mathbb{R}^n} (|\nabla v^*|^2 - |v^*|^2)\rho_\gamma = -\frac{\kappa^2}{4p^2 s^2}(4\pi)^{n/2}|\gamma_1|^2\left(1+\frac{1}{4}|\gamma_1|^2\right). \quad (6.6)$$

Hence, we can rewrite (6.3) as:

$$E[w_a] = E[\kappa] - \frac{F(\gamma)}{s^2} + o\left(\frac{1}{s^2}\right), \quad (6.7)$$

with

$$F(\gamma) = \frac{\kappa^2 (4\pi)^{n/2}}{8p^2}|\gamma_1|^2\left(1+\frac{1}{4}|\gamma_1|^2\right).$$

From (6.7), (6.8) it becomes obvious that if $|\gamma_1| \neq 0$, there exists some time s_* after which

$$E[w_a](s) < E[\kappa].$$

Since $a = \gamma\sqrt{T-t} = \gamma e^{-s/2} \to 0$ as $s \to \infty$, this shows that there are no blow up points (except zero of course) along the ray $\mathbf{a}$ within distance $|\gamma| e^{-s_*/2}$ from zero. It is also clear that $|\gamma_1| \to 0$ if and only if angle $(E_0^k, \gamma) \to 0$, and this completes the proof of Theorem C.

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