LIMIT PROBLEMS IN OPTIMAL CONTROL THEORY

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1. INTRODUCTION

In this paper we deal with sequences of optimal control problems of the form

$$(P_h) \qquad \min \left\{ \int_0^1 f_h(t,y,u)\, dt \;:\; y'=g_h(t,y,u)\,,\; y(0)=y_h^0 \right\}$$

where the state variable y belong to the Sobolev space $W^{1,1}(0,1;\mathbf{R}^n)$ and the control variable u is in $L^1(0,1;\mathbf{R}^m)$. We are interested in the asymptotic behaviour (as $h \to +\infty$) of the optimal pairs (u_h,y_h) of (P_h); more precisely, we shall construct a new problem (P_∞) such that

> *if* (u_h,y_h) *is an optimal pair of* (P_h) *and if* (u_h,y_h) *tends to* (u_∞,y_∞) *in the topology* $wL^1(0,1;\mathbf{R}^m)\times L^\infty(0,1;\mathbf{R}^n)$, *then* (u_∞,y_∞) *is an optimal pair for* (P_∞).

The basic tool for treating the asymptotic problem above is the Γ-convergence theory which has been already used very fruitfully for many variational problems (see for istance[1],[2],[4],[5],[8],[9])

Here we use a more sophisticated version of the usual Γ-limits, because we shall consider our problems (P_h) as minimzation problems on the product space U×Y (U is the space of controls and Y the space of states) for the functionals

$$F_h(u,y) = \begin{cases} \int_0^1 f_h(t,y,u)\,dt & \text{if } y'=g_h(t,y,u)\,,\ y(0)=y_h^0 \\ +\infty & \text{otherwise}\,, \end{cases}$$

and the spaces U and Y will play a different role with respect to Γ-convergence.

In Section 2 we develope the abstract theory we shall need in the following; in Section3 we show the applications to problems (P_h) above, and we give an example showing that in some situations the domain of problem (P_∞) is not given by a state equation $y'=g_\infty(t,y,u)$ but coincides with the entire product space U×Y.

2. THE ABSTRACT FRAMEWORK

Let us denote by U and Y two topological spaces and let $F_h:U\times Y\to\overline{\mathbf{R}}$ be a sequence of functions; by Z(+) we shall denote the "sup" operator and by Z (–) the "inf" operator. For every $u\in U$ and $y\in Y$ we define

$$\Gamma_{seq}(N^\alpha,U^\beta,Y^\gamma)\lim_h F_h(u,y) = \underset{(u_h)\in S(u)}{Z(\beta)}\ \underset{(y_h)\in S(y)}{Z(\gamma)}\ \underset{k\in N}{Z(-\alpha)}\ \underset{h\geq k}{Z(\alpha)}\ F_h(u_h,y_h)$$

where α,β,γ are the signs + or –, and S(u) and S(y) respectively denote the set of all sequences $u_h\to u$ in U and $y_h\to y$ in Y. For example we have

$$\Gamma_{seq}(N^+,U^-,Y^+)\lim_h F_h(u,y) = \inf_{u_h\to u}\ \sup_{y_h\to y}\ \limsup_{h\to\infty} F_h(u_h,y_h)\ .$$

When a Γ-limit is independent of the sign + or – associated to one of the spaces N,U,Y this sign will be omitted. For example, if

$$\Gamma_{seq}(N^+,U^-,Y^+)\lim_h F_h(u,y) = \Gamma_{seq}(N^+,U^+,Y^+)\lim_h F_h(u,y)\ ,$$

then their common value will be indicated by $\Gamma_{seq}(N^+,U,Y^+)\lim_h F_h(u,y)$.

The following propositions are proved in [4].

PROPOSITION 2.1. *Let* (u_h,y_h) *be a minimum point for* F_h *, or simply a pair such that*

$$\lim_h F_h(u_h,y_h) = \lim_h \left[\inf_{U\times Y} F_h \right] .$$

Assume that (u_h,y_h) *converges to* (u_∞,y_∞) *in* $U\times Y$ *and that there exist*

$$F_\infty = \Gamma_{seq}(N,U^-,Y^-) \lim_h F_h .$$

Then we have

(i) (u_∞,y_∞) *is a minimum point for* F_∞ *on* $U\times Y$;

(ii) $\lim_h \left[\inf_{U\times Y} F_h \right] = \min_{U\times Y} F_\infty$.

PROPOSITION 2.2. *Let* $\{F_h\}$ *and* $\{G_h\}$ *be two sequences of function from* $U\times Y$ *into* $[0,+\infty]$, *and let* $(u,y)\in U\times Y$. *Assume there exist*

$$\Gamma_{seq}(N,U^-,Y) \lim_h F_h(u,y) \qquad \textit{and} \qquad \Gamma_{seq}(N,U,Y^-) \lim_h G_h(u,y) .$$

Then we have

$$\Gamma_{seq}(N,U^-,Y^-) \lim_h \left[F_h+G_h\right](u,y) = \Gamma_{seq}(N,U^-,Y) \lim_h F_h(u,y) + \Gamma_{seq}(N,U,Y^-) \lim_h G_h(u,y) .$$

In many applications, the introduction of a new auxiliary variable can be helpful; the following proposition shows the behaviour of Γ-limits with respect to this operation.

PROPOSITION 2.3. *Let* $F_h:U\times Y\to\overline{\mathbf{R}}$ *be a sequence of functions, let* V *be another topological space, and let* $\Xi_h:U\times Y\to \wp(V)$ *be a sequence of multimappings. Assume that the following compactness condition is satisfied:*

> *for every converging sequence* (u_h,y_h) *with* $F_h(u_h,y_h)$ *bounded, there exist a sequence* $v_h\in\Xi_h(u_h,y_h)$ *relatively compact in* V.

Then setting

$$\Phi_h(u,v,y) = \begin{cases} F_h(u,y) & \textit{if } v\in \Xi_h(u,y) \\ +\infty & \textit{otherwise}, \end{cases}$$

we have for every $(u,y)\in U\times Y$

$$\inf \left\{ \Gamma_{seq}(N^-,(U\times V)^-,Y^-) \lim_h \Phi_h(u,v,y) \ : \ v\in V \right\} \le$$

$$\le \Gamma_{seq}(N^-,U^-,Y^-) \lim_h F_h(u,y) \le \Gamma_{seq}(N^+,U^-,Y^-) \lim_h F_h(u,y) \le$$

$$\le \inf \left\{ \Gamma_{seq}(N^+,(U\times V)^-,Y^-) \lim_h \Phi_h(u,v,y) \ : \ v\in V \right\} .$$

Therefore, if for every $(u,v,y)\in U\times V\times Y$ *there exists*

$$\Gamma_{seq}(N,(U\times V)^-,Y^-) \lim_h \Phi_h(u,v,y) ,$$

we have

$$\Gamma_{seq}(N,U^-,Y^-) \lim_h F_h(u,y) = \inf \left\{ \Gamma_{seq}(N,(U\times V)^-,Y^-) \lim_h \Phi_h(u,v,y) \ : \ v\in V \right\} .$$

Proof. It is enough to repeat, with just some slight modifications, the proof of Proposition 2.4 of [3]. ∎

In the following, if A is a set we denote by χ_A the function

$$\chi_A(x) = \begin{cases} 0 & \text{if } x\in A \\ +\infty & \text{otherwise}. \end{cases}$$

3. APPLICATIONS TO CONTROL PROBLEMS

Let k,m,n be positive integers and let p>1. The space Y of states we consider is the Sobolev space $W^{1,1}(0,1;\mathbf{R}^n)$ endovedwith the $L^\infty(0,1;\mathbf{R}^n)$ topology, and the space U of controls is the space $L^p(0,1;\mathbf{R}^n)$ endowed with its weak topology (weak* if $p=+\infty$). The cost functions are of the form

$$(3.1) \qquad J_h(u,y) = \int_0^1 f_h(t,y,u)\,dt$$

where $f_h:[0,1]\times\mathbf{R}^n\times\mathbf{R}^m\to[0,+\infty]$ are Borel functions. Finally, the state equations are

$$(3.2) \qquad \begin{cases} y' \in a_h(t,y) + B_h(t,y)\,b_h(t,u) \\ y(0)=y_h^0 \end{cases}$$

where $a_h:[0,1]\times\mathbf{R}^n\to\mathbf{R}^n$ and $B_h:[0,1]\times\mathbf{R}^n\to\mathbf{R}^{nk}$ are Borel functions, and the multimappings $b_h:[0,1]\times\mathbf{R}^m\to\wp(\mathbf{R}^{nk})$ are Borel mesurable (i.e. the sets $\{(t,u,v)\in[0,1]\times\mathbf{R}^n\times\mathbf{R}^k : v\in b_h(t,u)\}$ are Borel sets).

Then the control problems we are concerned are

$(P_h) \qquad \inf\{J_h(u,y) \ : \ (u,y)\in\Lambda_h\}$

or equivalently

$(P_h) \qquad \inf\{F_h(u,y) \ : \ (u,y)\in U\times Y\}$

where

$$(3.3) \qquad \Lambda_h = \{(u,y)\in U\times Y \ : \ y'\in a_h(t,y) + B_h(t,y)\,b_h(t,u)\,,\ y(0)=y_h^0\}$$

$$(3.4) \qquad F_h = J_h + \chi_{\Lambda_h}\,.$$

We introduce now an auxiliary variable $v\in L^q(0,1;\mathbf{R}^k)$ with $q>1$ and define a new sequence of functionals by setting

$$(3.5) \qquad \Phi_h(u,v,y) = \begin{cases} F_h(u,y) & \text{if } v\in b_h(t,u) \\ +\infty & \text{otherwise}\,. \end{cases}$$

In this way the problems (P_h) take the form

$$\inf\Big\{\int_0^1 [f_h(t,y,u) + \chi_{\{v\in b_h(t,u)\}}]\,dt \ : \ y'=a_h(t,y) + B_h(t,y)\,v\,,\ y(0)=y_h^0\Big\}\,.$$

In order to apply the abstract theory presented in Section 2 (more precisely Proposition 2.1), we have to calculate the $\Gamma_{seq}(N,U^-,Y^-)$ limit of the sequence F_h. To do this, we make some hypotheses on f_h , a_h , B_h , b_h .

(3.6) For every $t\in[0,1]$, $r\geq 0$, $y\in\mathbf{R}^n$ with $|y|\leq r$ we have (if $1/q + 1/q' = 1$)

$$|a_h(t,y)| \leq M_h(t,r) \quad \text{with} \quad \|M_h(\cdot,r)\|_{L^1(0,1)} \leq M(r) < +\infty$$
$$|B_h(t,y)| \leq N_h(t,r) \quad \text{with} \quad \|N_h(\cdot,r)\|_{L^{q'}(0,1)} \leq N(r) < +\infty .$$

(3.7) For every $t\in[0,1]$, $r\geq 0$, $y_1,y_2\in\mathbf{R}^n$ with $|y_1|,|y_2|\leq r$ we have

$$|a_h(t,y_1) - a_h(t,y_2)| \leq \alpha_h(t,r)\,|y_1-y_2| \quad \text{with} \quad \|\alpha_h(\cdot,r)\|_{L^1(0,1)} \leq \alpha(r) < +\infty$$
$$|B_h(t,y_1) - B_h(t,y_2)| \leq \beta_h(t,r)\,|y_1-y_2| \quad \text{with} \quad \|\beta_h(\cdot,r)\|_{L^{q'}(0,1)} \leq \beta(r) < +\infty .$$

(3.8) There exist $\lambda>0$ and $a\in L^1(0,1)$ such that

$$\lambda\big(|u|^p + |v|^q\big) - a(t) \leq f_h(t,0,u) + \chi_{\{v\in b_h(t,u)\}}$$

for every $t\in[0,1]$, $u\in\mathbf{R}^m$, $v\in\mathbf{R}^k$. When $p=+\infty$ or $q=+\infty$ the quantities $|u|^p$ and $|v|^q$ in the left-hand side have to be substituted by $\chi_{\{u\in H\}}$ and $\chi_{\{v\in K\}}$ respectively, where $H\subset\mathbf{R}^m$ and $K\subset\mathbf{R}^k$ are bounded sets.

(3.9) For every $t\in[0,1]$, $r\geq 0$, $u\in\mathbf{R}^m$, $y_1,y_2\in\mathbf{R}^n$ with $|y_1|,|y_2|\leq r$ we have

$$f_h(t,y_1,u) \leq f_h(t,y_2,u) + \rho_r(t,|y_1-y_2|) + \sigma_r(t,|y_1-y_2|)\,\big|f_h(t,y_2,u)\big|^{(\alpha-1)/\alpha}$$

for a suitable $a\in[1,+\infty]$ and functions $\rho_r(t,s)$, $\sigma_r(t,s)$ from $[0,1]\times[0,+\infty[$ into $[0,+\infty[$ measurable in t, increasing and continuous in s, with $\rho_r(t,0)=\sigma_r(t,0)=0$, and such that $z\to\rho_r(t,|z(t)|)$, $z\to\sigma_r(t,|z(t)|)$ are continuous operators from Y into $L^1(0,1)$, $L^\alpha(0,1)$ respectively.

(3.10) There exist $u_h\in L^p(0,1;\mathbf{R}^m)$ and $v_h\in L^q(0,1;\mathbf{R}^k)$ such that $v_h(t)\in b_h(t,u_h(t))$ for a.e. $t\in(0,1)$, and the sequence $f_h(t,0,u_h(t))$ is weakly compact in $L^1(0,1)$.

LEMMA 3.1. *Under the previous assumptions, the following compactness condition is satisfied: for every converging sequence* (u_h,y_h) *with* $F_h(u_h,y_h)$ *bounded, there exists a sequence* $\{v_h\}$ *relatively compact in* V *such that for a.e.* $t\in[0,1]$

$$v_h(t)\in b_h(t,u_h(t)) \qquad \textit{and} \qquad y_h'(t)=a_h(t,y_h(t)) + B_h(t,y_h(t))\,v_h(t) .$$

Proof. Let (u_h,y_h) be converging in $U\times Y$ with $F_h(u_h,y_h)$ bounded; then we have $(u_h,y_h)\in\Lambda_h$, so that we can find measurable functions $v_h(t)$ with

$$v_h(t)\in b_h(t,u_h(t)) \qquad \text{and} \qquad y_h'(t) = a_h(t,y_h(t)) + B_h(t,y_h(t))\, v_h(t)$$

for a.e. $t\in[0,1]$. It remains to prove that the sequence v_h is bounded in $L^q(0,1;\mathbf{R}^k)$. Since y_h is uniformly bounded, by (3.8) and (3.9) we have for a suitable $r>0$

$$\lambda\left(|u_h|^p + |v_h|^q\right) - a(t) \le f_h(t,0,u_h) \le$$

$$\le f_h(t,y_h,u_h) + \rho_r(t,r) + \sigma_r(t,r)\, \left|f_h(t,y_h,u_h)\right|^{(\alpha-1)/\alpha} \le$$

$$\le c\, f_h(t,y_h,u_h) + \gamma(t)$$

where $c>0$ is a constant and $\gamma\in L^1(0,1)$. Then, from the boundedness of $J_h(u_h,y_h)$ we get that v_h is bounded in $L^q(0,1;\mathbf{R}^k)$. ■

By Lemma 3.1, Proposition 2.3 applies, so that we have reduced our problem to the characterization of the $\Gamma_{seq}(N,(U\times Y)^-,Y^-)$ limit of the sequence $\Phi_h(u,v,y)$ defined in (3.5). Set now

$$\overline{f_h}(t,y,u,v) = f_h(t,y,u) + \chi_{\{v\in b_h(t,u)\}}$$

$$\overline{J_h}(u,v,y) = \int_0^1 \overline{f_h}(t,y,u,v)\, dt$$

$$\overline{\Lambda_h} = \left\{ (u,v,y)\in U\times V\times Y \;:\; y' = a_h(t,y) + B_h(t,y)\, v \text{ a.e. on } [0,1],\ y(0)=y_h^0 \right\}.$$

Therefore

$$\Phi_h = \overline{J_h} + \chi_{\overline{\Lambda_h}}$$

and, by Proposition 2.2 we may split the $\Gamma_{seq}(N,(U\times Y)^-,Y^-)$ limit of Φ_h into the sum

$$\Gamma_{seq}(N,(U\times V)^-,Y) \lim_h \overline{J_h} + \Gamma_{seq}(N,U\times V,Y^-) \lim_h \chi_{\overline{\Lambda_h}}. \tag{3.11}$$

The two terms in the sum above can be computed by using Lemma 3.1 and Theorem 3.4 of [4]. More precisely we have

PROPOSITION 3.2. *Assume that (3.8),(3.9),(3.10) hold and that for every* $y\in\mathbf{R}^n$, $\xi\in\mathbf{R}^n$, $\eta\in\mathbf{R}^k$

(3.12) $$\overline{f_h}^*(\cdot,y,\xi,\eta) \to \varphi(\cdot,y,\xi,\eta) \quad \textit{weakly in } L^1(0,1)$$

where $\overline{f_h}^*$ *are the polar functions of* $\overline{f_h}$ *defined by*

$$\overline{f_h}^*(t,y,\xi,\eta) = \sup\{\xi u + \eta v - \overline{f_h}(t,y,u,v) \;:\; u\in\mathbf{R}^m,\, v\in\mathbf{R}^k\}\,.$$

Then, for every $(u,v,y)\in U\times V\times Y$ *we have*

$$\Gamma_{seq}(N,(U\times V)^-,Y)\lim_h \overline{J_h}(u,v,y) = \int_0^1 \varphi^*(t,y,u,v)\,dt$$

where φ^* *is the polar function of* φ.

PROPOSITION 3.3. *Assume that (3.6),(3.7) hold and that*

(3.13) *for every* $y\in\mathbf{R}^n$ $\quad a_h(\cdot,y)\to a(\cdot,y)$ *weakly in* $L^1(0,1;\mathbf{R}^n)$;

(3.14) *for every* $y\in\mathbf{R}^n$ $\quad B_h(\cdot,y)\to B(\cdot,y)$ *strongly in* $L^{q'}(0,1;\mathbf{R}^{nk})$;

(3.15) $\quad y_h^0 \to y^0$ *in* $\mathbf{R}^n$.

Then we have

$$\Gamma_{seq}(N,U\times V,Y^-)\lim_h \chi_{\overline{\Lambda_h}} = \chi_{\overline{\Lambda}}$$

where

$$\overline{\Lambda} = \{(u,v,y)\in U\times V\times Y \;:\; y' = a(t,y) + B(t,y)\,v\,,\; y(0)=y^0\}\,.$$

Finally, we are in a position to compute the $\Gamma_{seq}(N,U^-,Y^-)$ limit of F_h . In fact, by Propositions 2.3, 3.2, and 3.3, and by (3.11) we get for every $(u,y)\in U\times Y$

$$\Gamma_{seq}(N,U^-,Y^-)\lim_h F_h(u,y) =$$

$$= \inf\left\{\int_0^1 \varphi^*(t,y,u,v)\,dt \;:\; y' = a(t,y) + B(t,y)\,v\,,\; y(0)=y^0\right\} =$$

$$= \int_0^1 f(t,y,u,y')\,dt + \chi_{\{y(0)=y^0\}}$$

where the function f is defined by

$$f(t,y,u,w) = \inf\{j(t,y,u,v) : w = a(t,y) + B(t,y)\,v\}.$$

We conclude with an example showing that in general the domain of the limit functional

$$F(u,y) = \int_0^1 f(t,y,u,y')\,dt + \chi_{\{y(0)=y^0\}}$$

is not given by a differential equation of the form $y'=g(t,y,u)$ but may be the whole space $U\times Y$.

EXAMPLE 3.4. Consider the sequence of optimal control problems

$$(P_h) \qquad \min\left\{\int_0^1 [u^2 + |y - y_0(t)|^2]\,dt \ : \ y' = a_h(t)\,y + b_h(t)\,u\,,\ y(0)=\xi\right\}$$

where u varies in $U=L^2(0,1)$, y varies in $Y=W^{1,1}(0,1)$, and $y_0\in L^2(0,1)$, $\xi\in \mathbf{R}$ are given. About the functions a_h and b_h we assume that

$$\begin{cases} a_h \to a & \text{weakly in } L^1(0,1) \\ b_h \to b & \text{weakly}^* \text{ in } L^\infty(0,1) \\ b_h^2 \to \beta^2 & \text{weakly}^* \text{ in } L^\infty(0,1)\,. \end{cases}$$

It is not difficult to check that all hypotheses (3.6),...,(3.10) and (3.12),...,(3.15) are satisfied, and after some standard calculations we find that the limit problem (P_∞) has the form

$$(P_\infty) \qquad \min\left\{\int_0^1 \left[u^2 + |y - y_0(t)|^2 + \frac{|y' - a(t)\,y - b(t)\,u|^2}{\beta^2(t) - b^2(t)}\right] dt \ : \ y(0)=\xi\right\}.$$

Note that it is $\beta^2(t)\geq b^2(t)$ for a.e. $t\in[0,1]$, and

$$\beta^2=b^2 \text{ a.e. on } [0,1] \iff b_h\to b \text{ a.e. on } [0,1]\,.$$

In this last situation, problem (P_∞) takes the usual form

$$(P_\infty) \qquad \min\left\{\int_0^1 [u^2 + |y - y_0(t)|^2]\,dt \ : \ y' = a(t)\,y + b(t)\,u\,,\ y(0)=\xi\right\},$$

but this does not arrive in the general case. Take for instance

$$b_h(t) = \sin(ht)$$

and we get $b \equiv 0$ and $\beta^2 \equiv {}^1/_2$, so that the limit problem is

$$\min \left\{ \int_0^1 \left[u^2 + |y - y_0(t)|^2 + 2|y' - a(t)y|^2 \right] dt \ : \ y(0) = \xi \right\} .$$

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