# A PDE APPROACH TO CERTAIN LARGE DEVIATION PROBLEMS FOR SYSTEMS OF PARABOLIC EQUATIONS

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Abstract. We prove an exponential decay estimate for solutions of certain scaled systems of parabolic PDE. Our techniques employ purely PDE methods, mainly the theory of viscosity solutions of Hamilton-Jacobi equations, and provide therefore an alternate approach to the probabilistic large deviation methods of Freidlin-Wentzell, Donsker-Varadhan, etc.

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1. Introduction.

This paper extends certain techniques developed in Evans-Ishii [6], Fleming-Souganidis [9], Evans-Souganidis [8], etc. regarding a PDE approach to various questions concerning large deviations. The starting point for these studies was the observation that the action functions controling large deviations for various problems involving diffusions are, formally at least, solutions of certain Hamilton-Jacobi PDE; see, for example, Freidlin-Wentzell [10, p. 107, 159, 233, 237, 275, etc.]. Our new contribution has been to seize upon this fact and, utilizing the rigorous tools now available with the new theory of viscosity solutions of Hamilton-Jacobi equations introduced by Crandall-Lions [3], to recover many of the basic results heretofore derived only by purely probabilistic means. We argue that these new PDE tools are often simpler and more flexible than the probabilistic ones; the papers [2] and [3], in particular, demonstrate the ease with which we can now handle nonconvex Hamiltonians. (We realize of course that many important applications of large deviations have no connections with PDE's.)

This current paper continues the program above by undertaking to investigate the asymptotics of a system of coupled linear parabolic PDE. The underlying probabilistic mechanism here comprises a collection of diffusion processes among which the system switches at random times determined by a continuous time Markov chain. We rescale so that a small parameter $\varepsilon$ occurs multiplying the diffusion terms in the corresponding PDE, whereas a term $\frac{1}{\varepsilon}$ occurs multiplying the coupling terms. Then following Bensoussan- Lions-Papanicolaou [1] we seek a WKB-type estimate for the solution $u^{\varepsilon}$

of the PDE, this of the form

$$u_k^{\varepsilon}(x,t) = e^{\frac{-I(x,t)+o(1)}{\varepsilon}} \quad \text{as } \varepsilon \to 0 \ (k = 1,\dots,m) \tag{1.1}$$

where I, the action function, must be computed. We carry out a proof of (1.1) by performing a logarithmic change of variables (an idea introduced by Fleming), and showing that I solves in the viscosity sense a Hamilton-Jacobi PDE of the form

$$I_t + H(x,DI) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{1.2}$$

Using then routine PDE theory we can write down a representation formula for I.

The novelty in these purely PDE techniques is that we can calculate the Hamiltonian H occurring in (1.2) directly from the original system of coupled parabolic equations, with no recourse to probability or ergodic theory. This seems to us fairly interesting, as the structure of H, involving the principal eigenvalue of a certain matrix, is not at all obvious, even formally, from the system of PDE we start with. It is also worth noting that, although the Hamiltonian H is convex in its second argument, our analysis depends crucially upon max-min and min-max representation formulas. In any case, we hope that the techniques developed here and in [6], [8], etc., will make some of the probabilistic results more understandable to PDE experts.

We have organized this paper by presenting first in §2 a review of useful facts from the Perron-Frobenius theory of positive matrices. Then in §3 we state carefully our PDE results and provide some preliminary estimates. Finally, §§4-5 complete the proof of our main theorem.

We hope in future work to extend these ideas to certain systems of reaction-diffusion equations. (Freidlin has recently undertaken a probabilistic analysis of such problems.)

2. The principal eigenvalue of a positive matrix.

We briefly review in this section some consequences of the Perron-Frobenius theory of positive matrices, and derive also certain max-min and min-max characterizations for the principal eigenvalues of such matrices.

Notation. Given $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, let us write

$$x > 0$$

provided

$$x_i > 0 \quad (1 \le i \le m),$$

and

$$x \ge 0$$

whenever

$$x_i \ge 0 \quad (1 \le i \le m).$$

Similarly, if $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$, we write

$$x \ge y$$

to mean

$$x_i \ge y_i \quad (1 \le i \le m).$$

Notation. If $A = ((a_{ij}))$ is an $m \times m$ matrix and $x = (x_1,\ldots,x_m) \in \mathbb{R}^m$, set

$$(Ax)_i \equiv i^{th} \text{ component of } Ax \equiv \sum_{j=1}^{m} a_{ij}x_j \quad (1 \le i \le m).$$

Definition. Let $A = ((a_{ij}))$ be a real $m \times m$ matrix. We say that $A$ is strongly positive, written

$$A > 0,$$

provided

$$a_{ij} > 0 \quad (1 \le i,j \le m).$$

Theorem 2.1 (Perron-Frobenius). Assume $A > 0$ and define

$$\lambda^0 = \lambda^0(A) \equiv \sup\{\lambda \in \mathbb{R} \mid \text{there exists } x \ge 0 \text{ such that } Ax \ge \lambda x\}. \tag{2.1}$$

(i) There then exists a vector $x^0 > 0$ satisfying

$$Ax^0 = \lambda^0 x^0.$$

(ii) If $\lambda \in \mathbb{C}$ is any other eigenvalue of $A$, then $\operatorname{Re}\lambda < \lambda^0$.

(iii) Furthermore

$$\lambda^0 = \max_{x>0} \min_{1\le i\le m} \frac{(Ax)_i}{x_i} \tag{2.2}$$

and

$$\lambda^0 = \min_{x>0} \max_{1\le i\le m} \frac{(Ax)_i}{x_i}. \tag{2.3}$$

(iv) Finally,

$$\lambda^0 = \sup_{p\in P} \inf_{x>0} \sum_{i=1}^{m} \frac{p_i (Ax)_i}{x_i}, \tag{2.4}$$

for

$$P \equiv \{p \in \mathbb{R}^m \mid p > 0, \ \sum_{i=1}^{m} p_i = 1\}.$$

Proof. See Gantmacher [11, Chapter XIII] or Karlin-Taylor [12, Appendix 2] for proofs of (i), (ii). Assertion (iii) is also found in Gantmacher [11, p. 65], but as it is important for the calculations in §4, we provide the following simple proof.

Since $A^T$ has the same spectrum as $A$ and since assertion

(i) applies as well to $A^T > 0$, there exists a vector $y^0 > 0$ satisfying

$$A^T y^0 = \lambda^0 y^0.$$

Then for each $x > 0$

$$0 = x\cdot(A^T y^0 - \lambda^0 y^0) = (Ax - \lambda^0 x)\cdot y^0;$$

and consequently

$$(Ax - \lambda^0 x)_j \le 0$$

for some index $1 \le j \le m$. Hence

$$\lambda^0 \ge \frac{(Ax)_j}{x_j} \ge \min_{1\le i\le m} \frac{(Ax)_i}{x_i},$$

and so

$$\lambda^0 \ge \max_{x>0} \min_{1\le i\le m} \frac{(Ax)_i}{x_i}.$$

On the other hand

$$Ax^0 = \lambda^0 x^0;$$

whence

$$\lambda^0 = \min_{1\le i\le m} \frac{(Ax^0)_i}{x_i^0} \le \max_{x>0} \min_{1\le i\le m} \frac{(Ax)_i}{x_i}.$$

This proves (2.2), and the proof of (2.3) is similar.

Lastly, assertion (iv) is from Ellis [5, Problem IX.6.8] and is a special case of Donsker-Varadhan [4]. A direct proof is this:

$$\lambda^0 = \min_{x>0} \max_{1\le i\le m} \frac{(Ax)_i}{x_i} = \min_{x>0} \sup_{p\in P} \sum_{i=1}^{m} \frac{p_i(Ax)_i}{x_i}$$

$$= \min_{q\in\mathbb{R}^m} \sup_{p\in P} \sum_{i,j=1}^{m} a_{ij} p_i e^{q_j - q_i} \quad (x_i = e^{q_i},\ i = 1,\ldots m)$$

$$= \sup_{p\in P} \inf_{q\in\mathbb{R}^m} \sum_{i,j=1}^{m} a_{ij} p_i e^{q_j - q_i}$$

$$= \sup_{p\in P} \inf_{x>0} \sum_{i,j=1}^{m} \frac{p_i (Ax)_i}{x_i},$$

where we applied the minimax theorem to the linear-convex function

$$g(p,q) = \sum_{i,j=1}^{m} a_{ij} p_i e^{q_j - q_i}. \qquad \square$$

Remark. It is interesting to note that whereas (2.3) and (2.4) are "dual" under the interchange of inf and sup, the statement "dual" to (2.2) is false:

$$\lambda^0 \neq \inf_{p\in P} \sup_{x>0} \sum_{i=1}^{m} \frac{p_i (Ax)_i}{x_i} = +\infty. \qquad \square$$

Next we drop the requirement that the diagonal entries of $A$ be positive.

Theorem 2.2. Suppose $A$ is an $m\times m$ matrix with

$$a_{ij} > 0 \quad (1 \le i,j \le m,\ i \ne j).$$

(i) There exists a real number $\lambda^0 = \lambda^0(A)$ and a vector $x^0 > 0$ satisfying

$$Ax^0 = \lambda^0 x^0.$$

(ii) If $\lambda \in \mathbb{C}$ is any other eigenvalue of $A$, $\operatorname{Re}\lambda < \lambda^0$.

(iii) Furthermore,

$$\lambda^0 = \max_{x>0} \min_{1\le i\le m} \frac{(Ax)_i}{x_i} = \min_{x>0} \max_{1\le i\le m} \frac{(Ax)_i}{x_i} = \sup_{p\in P} \inf_{x>0} \sum_{i=1}^{m} \frac{p_i (Ax)_i}{x_i}.$$

(iv) For fixed entries $\{a_{ij} \mid 1 \le i,j \le m,\ i \ne j\}$, the function

$$f(a_{11}, a_{22}, \ldots, a_{mm}) \equiv \lambda^0(A)$$

is convex and nondecreasing.

*Proof*. Set

$$d \equiv \max\{|a_{11}|, \ldots, |a_{mm}|\} + 1,$$

and then apply Theorem 2.1 to

$$\hat{A} \equiv A + dI > 0$$

to establish (i) - (iii). Assertion (iv) follows at once from (iii), since

$$\begin{aligned} f(a_{11}, \ldots, a_{mm}) &= \sup_{p \in P} \inf_{x > 0} \sum_{i,j=1}^{m} \frac{a_{ij} p_i x_j}{x_i} \\ &= \sup_{p \in P} \left\{ \sum_{i=1}^{m} p_i a_{ii} + \inf_{x>0} \sum_{\substack{i,j=1 \\ i \ne j}}^{m} \frac{a_{ij} p_i x_j}{x_i} \right\}. \end{aligned}$$

This expression is convex and nondecreasing in the variables $a_{11}, a_{22}, \ldots, a_{mm}$. □

3. Statement of the PDE problem; estimates.

We assume now that $C = ((c_{ij}))$ is an $m\times m$ stochastic matrix; that is,

$$c_{k\ell} > 0 \quad (1 \le k,\ell \le m,\ k \ne \ell) \tag{3.1}$$

and

$$\sum_{\ell=1}^{m} c_{k\ell} = 0 \quad (1 \le \ell \le m). \tag{3.2}$$

Suppose also that the functions $a_{ij}^k, b_i^k, g_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(1 \le i,j \le n,$ $1 \le k \le m)$ are smooth, bounded, Lipschitz continuous and satisfy

$$\begin{cases} a_{ij}^k(x) = a_{ji}^k(x) & (1 \le i,j \le n) \\ a_{ij}^k(x)\xi_i\xi_j \ge \nu|\xi|^2 & (x,\xi \in \mathbb{R}^n) \end{cases} \tag{3.3}$$

for $k = 1,\dots,m$ and some constant $\nu > 0$. Assume further that

$$\begin{cases} G_0 = \operatorname{spt} g_k \text{ is bounded and } g_k \ge 0 & (k = 1,\dots,m). \end{cases} \tag{3.4}$$

We consider now the linear parabolic system

$$\begin{cases} u_{k,t}^{\varepsilon} = \frac{\varepsilon}{2} a_{ij}^k u_{k,x_ix_j}^{\varepsilon} + b_i^k u_{k,x_i}^{\varepsilon} + \frac{1}{\varepsilon} c_{k\ell} u_\ell^{\varepsilon} & \text{in } \mathbb{R}^n \times (0,\infty) \\ u_k^{\varepsilon} = g_k & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{3.5}$$

for $k = 1,\dots,m$. Here we employ a partial summation convention: the indices $i$ and $j$ are summed from 1 to $n$, the index $\ell$ is summed from 1 to $m$; the index $k$ is not summed.

According to the Perron-Frobenius theory, recalled in §2, there exists a unique vector $p > 0$ satisfying

$$\sum_{k=1}^{m} p_k = 1$$

and

$$\sum_{k=1}^{m} c_{k\ell} p_k = 0 \quad (i = 1,\ldots,m).$$

It is not particularly hard to prove (cf. [1, Section 4.2.11]) that as $\varepsilon \longrightarrow 0$ each of the function $u_k^\varepsilon$ converges on compact subsets of $\mathbb{R}^n \times (0,\infty)$ to the same Lipschitz function $u$, which satisfies the transport equation

$$\begin{cases} u_t = b_i u_{x_i} & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where

$$b_i \equiv p_k b_i^k \quad (i = 1,\ldots,n)$$

and

$$g \equiv p_k g_k.$$

Observe that whereas $u^\varepsilon = (u_1^\varepsilon,\ldots,u_m^\varepsilon)$ is everywhere positive on $\mathbb{R}^n \times (0,T]$ for each $T > 0$, the limit function $u$ has compact support. Following then Bensoussan-Lions-Papanicolaou [1,p.601] let us ask at what rate the functions $u^\varepsilon$ decay to zero off the support of $u$, and for this attempt a WKB-type representation of $u^\varepsilon$ of the form

$$u_k^\varepsilon = e^{-\frac{I+o(1)}{\varepsilon}} \quad \text{as } \varepsilon \longrightarrow 0 \ (k = 1,\ldots,m)$$

where $I = I(x,t)$ is to be determined. As in [6], [8], [9], we exploit W. Fleming's idea of writing

(3.6) $$v_k^\varepsilon = -\varepsilon \log u_k^\varepsilon \quad (x \in \mathbb{R}^n,\ t > 0);$$

so that

$$u_k^\varepsilon = e^{-\frac{v_k^\varepsilon}{\varepsilon}} \quad (k = 1,\dots,m),$$

and try to ascertain the limit of the functions $v_k^\varepsilon$ as $\varepsilon \to 0$.

Observe first that routine parabolic estimates using (3.1), (3.2) imply

$$0 < u_k^\varepsilon \le C_1 \equiv \max_{1\le k\le m} \|g^k\|_{L^\infty},$$

whence

(3.7)
$$v_k^\varepsilon > -\varepsilon \log C_1 \quad (k = 1,\dots,m).$$

We next employ (3.5) to compute that $v^\varepsilon = (v_1^\varepsilon,\dots,v_m^\varepsilon)$ solves this nonlinear system for $k = 1,\dots,m$:

(3.8)
$$\begin{cases} v^\varepsilon_{k,t} = \frac{\varepsilon}{2} a^k_{ij} v^\varepsilon_{k,x_i x_j} - \frac{1}{2} a^k_{ij} v^\varepsilon_{k,x_i} v^\varepsilon_{k,x_j} + b^k_i v^\varepsilon_{k,x_i} - c_{k\ell} e^{\frac{v_k^\varepsilon - v_\ell^\varepsilon}{\varepsilon}} \\ \qquad\qquad \text{in } \mathbb{R}^n \times (0,\infty), \\ v_k^\varepsilon = -\varepsilon \log g^k \quad \text{on } \operatorname{int} G_0 \times \{t = 0\}, \\ v_k^\varepsilon = +\infty \quad \text{on } (\mathbb{R}^n - G_0) \times \{t = 0\}. \end{cases}$$

Lemma 3.1. For each open set $Q \subset\subset \mathbb{R}^n \times (0,\infty)$ there exists a constant $C(Q)$, independent of $\varepsilon$, such that

$$\sup_Q |v_k^\varepsilon|, |Dv_k^\varepsilon| \le C(Q) \quad (k = 1,\dots,m).$$

Proof. 1. The proof is similar to the proof of Lemma 3.1 in [8], and we consequently emphasize only the important differences.

First of all we may assume that the ball $B(0,R)$ lies in $\operatorname{int} G_0$ for some fixed $R > 0$. Define then the scalar functions

$$z_k^1(x,t) \equiv \frac{1}{R^2-|x|^2} + \alpha t + \beta \quad (k = 1,\ldots,m)$$

for $\alpha,\beta > 0$ to be selected. Then

$$z^1_{k,t} - \frac{\varepsilon}{2} a^k_{ij} z^1_{k,x_i x_j} + \frac{1}{2} a^k_{ij} z^1_{k,x_i} z^1_{k,x_j} - b^k_i z_{k},_{x_i} + c_{k\ell} e^{\frac{z^1_k - z^1_\ell}{\varepsilon}}$$

$$= \alpha - \frac{\varepsilon}{2}\left[\frac{2a^k_{ij}\delta_{ij}}{(R^2-|x|^2)^2} + \frac{8a^k_{ij}x_i x_j}{(R^2-|x|^2)^3}\right]$$

$$+ \frac{2a^k_{ij}x_i x_j}{(R^2-|x|^2)^4} - \frac{2b^k_i x_i}{(R^2-|x|^2)^2} \geq 0 \quad \text{in} \quad B(0,R)\times(0,\infty),$$

provided $\alpha = \alpha(R)$ is large enough and $R > 0$ is selected so that

$$\beta \equiv \max_{1\leq k\leq n} (-\log(\inf_{B(0,R)} g^k)) < \infty.$$

Then

$$z^1_k \geq v^\varepsilon_k \quad \text{on} \quad [B(0,R)\times\{t = 0\}] \cup [\partial B(0,R)\times(0,\infty)].$$

Now the maximum principle applies to the nonlinear system (3.8) since (3.1) implies that the nonlinear term

$$c_{k\ell} e^{\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}}$$

is increasing with respect to $v^\varepsilon_k$ and decreasing with respect $v^\varepsilon_\ell$ $(\ell \neq k)$. Hence

$$z^1_k \geq v^\varepsilon_k \quad \text{in} \quad B(0,R)\times(0,\infty);$$

whence

$$|v^\varepsilon_k| \leq C \quad \text{in} \quad B(0,R/2)\times(0,T) \tag{3.9}$$

for each $T > 0$.

Now define

$$z_k^2(x,t) \equiv \rho\frac{|x|^2}{t} + \sigma t + \tau \quad (k = 1,\dots,m)$$

for $\rho,\sigma,\tau > 0$ to be selected. Then

$$z^2_{k,t} - \frac{\varepsilon}{2} a^k_{ij} z^2_{k,x_i x_j} + \frac{1}{2} a^k_{ij} z^2_{k,x_i} z^2_{k,x_j} - b^k_i z_{k,x_i} + c_{k\ell} e^{\frac{z_k^2 - z_\ell^2}{\varepsilon}} \ge 0$$

provided $\rho,\sigma > 0$ are large enough. Choosing now $\tau$ greater than the constant in (3.9) we apply the maximum principle to find

$$z_k^2 \ge v_k^\varepsilon \quad \text{in} \quad [\mathbb{R}^n - B(0,R/2)]\times(0,\infty) \quad (k = 1,\dots,m).$$

These inequalities and (3.9) yield the stated bounds on $|v_k^\varepsilon|$ on compact subsets of $\mathbb{R}^n\times(0,\infty)$, $(k = 1,\dots,m)$.

2. To estimate $|Dv_k^\varepsilon|$ we introduce the auxiliary functions

$$z_k^3 \equiv \zeta^2|Dv_k^\varepsilon|^2 - \lambda v_k^\varepsilon \quad (k = 1,\dots,m) \tag{3.10}$$

where $Q \subset\subset \mathbb{R}^n\times(0,\infty)$, $\zeta$ is a smooth cutoff function with compact support $Q' \supset\supset Q$, $\zeta \equiv 1$ on $Q$, and $\lambda > 0$ is to be selected below. Now choose an index $k \in \{1,\dots,m\}$ and a point $(x_0^k,t_0^k) \in \bar{Q}'$ such that

$$z_k^3(x_0^k,t_0^k) = \max_{1\le\ell\le m}\ \max_{\bar{Q}'} z_\ell^3. \tag{3.11}$$

If $(x_0^k,t_0^k) \in Q'$, then

$$0 = z^3_{k,x_j} = 2\zeta\zeta_{x_j}|Dv_k^\varepsilon|^2 + 2\zeta^2 v^\varepsilon_{k,x_r x_j} v^\varepsilon_{k,x_r} - \lambda v^\varepsilon_{k,x_j} \tag{3.12}$$

and

$$0 \le z^3_{k,t} - \frac{\varepsilon}{2} a^k_{ij} z^3_{k,x_i x_j} \tag{3.13}$$

at $(x_0^k,t_0^k)$. We utilize (3.8) and (3.10) to compute and then estimate the right hand side of (3.13):

$$0 \le 2\zeta\zeta_t|Dv_k^\varepsilon|^2 + 2\zeta^2 v_{k,x_r}^\varepsilon v_{k,x_r t}^\varepsilon - \lambda v_{k,t}^\varepsilon$$

$$- \frac{\varepsilon}{2} a_{ij}^k \Big[ (2\zeta\zeta_{x_i})_{x_j} |Dv_k^\varepsilon|^2 + 8\zeta\zeta_{x_i} v_{k,x_r}^\varepsilon v_{k,x_r x_j}^\varepsilon$$

$$+ 2\zeta^2 v_{k,x_r x_i}^\varepsilon v_{k,x_r x_j}^\varepsilon + 2\zeta^2 v_{k,x_r}^\varepsilon v_{k,x_r x_i x_j}^\varepsilon - \lambda v_{k,x_i x_j}^\varepsilon \Big]$$

$$\le -\lambda (v_{k,t}^\varepsilon - \frac{\varepsilon}{2} a_{ij}^k v_{k,x_i x_j}^\varepsilon)$$

$$+ 2\zeta^2 v_{k,x_r}^\varepsilon (v_{k,t x_r}^\varepsilon - \frac{\varepsilon}{2} a_{ij}^k v_{k,x_i x_j x_r}^\varepsilon)$$

$$- \varepsilon\nu\zeta^2 |D^2 v_k^\varepsilon|^2 + C|Dv_k^\varepsilon|^2 + \varepsilon C\zeta |Dv_k^\varepsilon||D^2 v_k^\varepsilon|$$

(3.14)

$$\le -\lambda(-\frac{1}{2} a_{ij}^k v_{k,x_i}^\varepsilon v_{k,x_j}^\varepsilon + b_i^k v_{k,x_i}^\varepsilon - c_{k\ell} e^{\frac{v_k^\varepsilon - v_\ell^\varepsilon}{\varepsilon}})$$

$$+ 2\zeta^2 v_{k,x_r}^\varepsilon (v_{k,t}^\varepsilon - \frac{\varepsilon}{2} a_{ij}^k v_{k,x_i x_j}^\varepsilon)_{x_r} + C|Dv_k^\varepsilon|^2$$

$$\le \lambda(\frac{1}{2} a_{ij}^k v_{k,x_i}^\varepsilon v_{k,x_j}^\varepsilon - b_i^k v_{k,x_i}^\varepsilon + c_{k\ell} e^{\frac{v_k^\varepsilon - v_\ell^\varepsilon}{\varepsilon}})$$

$$+ 2\zeta^2 v_{k,x_r}^\varepsilon (-\frac{1}{2} a_{ij}^k v_{k,x_i}^\varepsilon v_{k,x_j}^\varepsilon + b_i^k v_{k,x_i}^\varepsilon - c_{k\ell} e^{\frac{v_k^\varepsilon - v_\ell^\varepsilon}{\varepsilon}})_{x_r}$$

$$+ C|Dv_k^\varepsilon|^2$$

$$\le \lambda(\frac{1}{2} a_{ij}^k v_{k,x_i}^\varepsilon v_{k,x_j}^\varepsilon - b_i^k v_{k,x_i}^\varepsilon + c_{k\ell} e^{\frac{v_k^\varepsilon - v_\ell^\varepsilon}{\varepsilon}})$$

$$+ 2\zeta^2 v_{k,x_r}^\varepsilon (-a_{ij}^k v_{k,x_i}^\varepsilon v_{k,x_j x_r}^\varepsilon + b_i^k v_{k,x_i x_r}^\varepsilon)$$

$$- 2c_{k\ell}\zeta^2 v^\varepsilon_{k,x_r}\left(e^{\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}}\right)_{x_r} + C\zeta|Dv^\varepsilon_k|^3 + C|Dv^\varepsilon_k|^2.$$

We employ (3.12) now to compute

$$2\zeta^2 v^\varepsilon_{k,x_r}(-a^k_{ij}v^\varepsilon_{k,x_i}v^\varepsilon_{k,x_jx_r} + b^k_i v^\varepsilon_{k,x_ix_r})$$

$$= -a^k_{ij}v^\varepsilon_{k,x_i}(\lambda v^\varepsilon_{k,x_j} - 2\zeta\zeta_{x_j}|Dv^\varepsilon_k|^2) \tag{3.15}$$

$$+ b^k_i(\lambda v^\varepsilon_{k,x_i} - 2\zeta\zeta_{x_i}|Dv^\varepsilon_k|^2).$$

Furthermore, by (3.11),

$$-2\zeta^2 v^\varepsilon_{k,x_r}\left(e^{\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}}\right)_{x_r} = -2\zeta^2 \frac{e^{\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}}}{\varepsilon}(|Dv^\varepsilon_k|^2 - Dv^\varepsilon_k \cdot Dv^\varepsilon_\ell)$$

$$\le \frac{e^{\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}}}{\varepsilon}(\zeta^2|Dv^\varepsilon_\ell|^2 - \zeta^2|Dv^\varepsilon_k|^2)$$

$$\le \lambda e^{\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}}\left(\frac{v^\varepsilon_\ell - v^\varepsilon_k}{\varepsilon}\right).$$

Insert this estimate and (3.15) into (3.14), to find

$$\frac{\lambda\nu}{2}|Dv^\varepsilon_k|^2 \le \frac{\lambda}{2}a^k_{ij}v^\varepsilon_{k,x_i}v^\varepsilon_{k,x_j} \le \lambda c_{k\ell}e^{\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}}\left(1 - \left(\frac{v^\varepsilon_k - v^\varepsilon_\ell}{\varepsilon}\right)\right)$$

$$+ C\zeta|Dv^\varepsilon_k|^3 + C|Dv^\varepsilon_k|^2 + C\lambda|Dv^\varepsilon_k|.$$

Thus

$$\lambda|Dv^\varepsilon_k|^2 \le C\lambda + C\zeta|Dv^\varepsilon_k|^3 + C|Dv^\varepsilon_k|^2 \tag{3.16}$$

at $(x^k_0, t^k_0)$. Now choose

$$\lambda \equiv \mu(\max_{Q'}(\zeta|Dv^\varepsilon_k|) + 1),$$

and select the constant $\mu$ so large that (3.16) forces the inequality

$$|Dv_k^{\varepsilon}|^2 \le C \tag{3.17}$$

at $(x_0^k, t_0^k)$. But then (3.11) and (3.17) imply

$$\max_{1\le \ell\le m} \max_{Q'} \zeta |Dv_\ell^{\varepsilon}|^2 \le \max_{1\le \ell\le m} \max_{Q'} z_\ell^3 + C\lambda \le C + C\lambda$$

$$\le C + C \max_{Q'} (\zeta |Dv_k^{\varepsilon}|);$$

and so

$$\max_{1\le \ell\le m} \max_{Q'} \zeta |Dv_\ell^{\varepsilon}| \le C. \tag{3.18}$$

If $(x_0^k, t_0^k) \in \partial Q'$, $\zeta(x_0^k, t_0^k) = 0$; and easy estimates lead also to (3.18). Since $\zeta \equiv 1$ on $Q$, this gives the desired estimate. □

Remark. This proof is related to one in Koike [13], and follows unpublished work of Evans-Ishii.

Lemma 3.2. For each open set $Q \subset\subset \text{int } G_0 \times [0,\infty)$ there exists a constant $C(Q)$, independent of $\varepsilon$, such that

$$\sup_Q |v_k^{\varepsilon}|, |Dv_k^{\varepsilon}| \le C(Q) \qquad (k = 1,\dots,m).$$

Proof. The proof is similar, except that we must consider also the case that the point $(x_0^k, t_0^k)$ from (3.11) lies in $\text{int } G_0 \times \{t=0\}$. But since $g_k > 0$ in $\text{int } G_0$ for $k = 1,\dots,m$, the requisite estimates are easy. □

## 4. Convergence.

We next demonstrate that the functions $\{v_k^\varepsilon\}_{\varepsilon>0}$ converge uniformly on compact subsets to limit functions $v_k$ $(k = 1,\ldots,m)$, and furthermore that $v_1 = v_2 = \ldots = v_m \equiv v$. A major difficulty is that we do not have any obvious uniform control over the t-dependence of the functions $v_k^\varepsilon$. Indeed, Lemma 3.1 provides us with uniform estimates on only one of the four terms in the PDE (3.8). Nevertheless we are able as follows to argue directly that the differences $\{v_k^\varepsilon - v_\ell^\varepsilon\}_{\varepsilon>0}$ $(k,\ell = 1,\ldots,m)$ converge locally uniformly to zero, and to control the rate of convergence, we adapt here some ideas from [14].

Lemma 4.1. There exists a function $v \in C(\mathbb{R}^n\times(0,\infty))$ and a sequence $\varepsilon_j \longrightarrow 0$ such that

$$v_k^{\varepsilon_j} \longrightarrow v \quad \text{as} \quad \varepsilon_j \longrightarrow 0 \quad (k = 1,\ldots,m), \tag{4.1}$$

uniformly on compact subsets of $\mathbb{R}^n\times(0,\infty)$.

Proof. We first claim that for each open set $Q \subset\subset \mathbb{R}^n\times(0,\infty)$

$$\max_{i,j} \|v_i^\varepsilon - v_j^\varepsilon\|_{L^\infty(Q)} \le O(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{4.2}$$

To prove this choose a cutoff function $\zeta$ with compact support in $\mathbb{R}^n\times(0,\infty)$ and set

$$\alpha_\varepsilon \equiv \max_{i,j} \|\zeta^2(v_i^\varepsilon - v_j^\varepsilon)\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}. \tag{4.3}$$

We may as well assume

$$\alpha_\varepsilon > 0.$$

Fix $T > 0$ so large that

$$\mathrm{spt}(\zeta) \subset \mathbb{R}^n\times(0,T),$$

and define

$$\Phi^{\varepsilon}_{ij}(x,y,t) \equiv \zeta(x,t)\zeta(y,t)(v^{\varepsilon}_i(x,t) - v^{\varepsilon}_j(y,t)) - \frac{|x-y|^2}{\varepsilon} - \frac{t}{2T}\alpha_{\varepsilon} \tag{4.4}$$

for $1 \le i,j \le m$, $x,y \in \mathbb{R}^n$, $0 \le t \le T$. Choose now a subsequence of the $\varepsilon$'s (which for simplicity of notation we continue to write as "$\varepsilon$") and integers $k,\ell \in \{1,\ldots,m\}$ such that

$$\sup_{(x,y,t)} \Phi^{\varepsilon}_{k\ell}(x,y,t) = \max_{i,j} \sup_{(x,y,t)} \Phi^{\varepsilon}_{ij}(x,y,t) \tag{4.5}$$

for all small $\varepsilon > 0$. Next, passing to a further subsequence of the $\varepsilon$'s if necessary, select indices $r,s \in \{1,\ldots,m\}$ and points $(x^{\varepsilon},t^{\varepsilon}) \in \operatorname{spt}\zeta$ such that

$$\zeta^2(x^{\varepsilon},t^{\varepsilon})(v^{\varepsilon}_r - v^{\varepsilon}_s)(x^{\varepsilon},t^{\varepsilon}) = \alpha_{\varepsilon} \tag{4.6}$$

for all small $\varepsilon > 0$. Then (4.5) implies

$$\begin{aligned} \sup_{(x,y,t)} \Phi^{\varepsilon}_{k\ell}(x,y,t) &\ge \sup_{(x,y,t)} \Phi^{\varepsilon}_{rs}(x,y,t) \\ &\ge \Phi^{\varepsilon}_{rs}(x^{\varepsilon},x^{\varepsilon},t^{\varepsilon}) \\ &= \zeta^2(x^{\varepsilon},t^{\varepsilon})(v^{\varepsilon}_r - v^{\varepsilon}_s)(x^{\varepsilon},t^{\varepsilon}) - \frac{t^{\varepsilon}}{2T}\alpha_{\varepsilon} \\ &\ge \frac{\alpha_{\varepsilon}}{2}. \end{aligned} \tag{4.7}$$

Since $\Phi^{\varepsilon}_{k\ell}(x,y,t) \le 0$ if either $(x,t) \notin \operatorname{spt}(\zeta)$ or $(y,t) \notin \operatorname{spt}(\zeta)$, there exist points $(x_{\varepsilon},t_{\varepsilon}),(y_{\varepsilon},t_{\varepsilon}) \in \operatorname{spt}(\zeta)$ such that

$$\Phi^{\varepsilon}_{k\ell}(x_{\varepsilon},y_{\varepsilon},t_{\varepsilon}) = \sup_{(x,y,t)} \Phi^{\varepsilon}_{k\ell}(x,y,t). \tag{4.8}$$

Then since

$$\Phi^{\varepsilon}_{k\ell}(x_{\varepsilon},y_{\varepsilon},t_{\varepsilon}) \ge \Phi^{\varepsilon}_{k\ell}(x_{\varepsilon},x_{\varepsilon},t_{\varepsilon}),$$

we deduce from (4.4) and Lemma 3.1 that

$$(4.9) \qquad |x_\varepsilon - y_\varepsilon| \le C\varepsilon$$

for some appropriate constant $C$.

Now inasmuch as $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ is a maximum point for $\Phi^\varepsilon_{k\ell}$ we have

$$\Phi^\varepsilon_{k\ell,t}(x_\varepsilon, y_\varepsilon, t_\varepsilon) = 0,\ D^2_x\Phi^\varepsilon_{k\ell}(x_\varepsilon, y_\varepsilon, t_\varepsilon) \le 0,\ D^2_y\Phi^\varepsilon_{k\ell}(x_\varepsilon, y_\varepsilon, t_\varepsilon) \le 0.$$

These facts imply:

$$(4.10) \quad \begin{cases} [\zeta_t(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon) + \zeta(x_\varepsilon, t_\varepsilon)\zeta_t(y_\varepsilon, t_\varepsilon)](v^\varepsilon_k(x_\varepsilon, t_\varepsilon) \\ \quad - v^\varepsilon_\ell(y_\varepsilon, t_\varepsilon)) \\ + \zeta(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon)(v^\varepsilon_{k,t}(x_\varepsilon, t_\varepsilon) - v^\varepsilon_{\ell,t}(y_\varepsilon, t_\varepsilon)) = \frac{\alpha_\varepsilon}{2T}, \end{cases}$$

$$(4.11) \quad \begin{cases} -\zeta(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon)a^k_{ij}(x_\varepsilon)v^\varepsilon_{k,x_ix_j}(x_\varepsilon, t_\varepsilon) \\ \ge -\frac{2}{\varepsilon}\, a_{ii}(x^k_\varepsilon) \\ + \zeta(y_\varepsilon, t_\varepsilon)(v^\varepsilon_k(x_\varepsilon, t_\varepsilon) - v^\varepsilon_\ell(y_\varepsilon, t_\varepsilon))a^k_{ij}(x_\varepsilon)\zeta_{x_ix_j}(x_\varepsilon, t_\varepsilon) \\ + 2\zeta(y_\varepsilon, t_\varepsilon)a^k_{ij}(x_\varepsilon)v^\varepsilon_{k,x_i}(x_\varepsilon, t_\varepsilon)\zeta_{x_j}(x_\varepsilon, t_\varepsilon), \end{cases}$$

and

$$(4.12) \quad \begin{cases} -\zeta(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon)a^\ell_{ij}(y_\varepsilon)v^\varepsilon_{\ell,x_ix_j}(y_\varepsilon, t_\varepsilon) \\ \le \frac{2}{\varepsilon}\, a^\ell_{ii}(y_\varepsilon) \\ - \zeta(x_\varepsilon, t_\varepsilon)(v^\varepsilon_k(x_\varepsilon, t_\varepsilon) - v^\varepsilon_\ell(y_\varepsilon, t_\varepsilon))a^\ell_{ij}(y_\varepsilon)\zeta_{x_ix_j}(y_\varepsilon, t_\varepsilon) \\ + 2\zeta(x_\varepsilon, t_\varepsilon)a^\ell_{ij}(y_\varepsilon)v^\varepsilon_{\ell,x_i}(y_\varepsilon, t_\varepsilon)\zeta_{x_j}(y_\varepsilon, t_\varepsilon). \end{cases}$$

Recalling Lemma 3.1 we deduce from these complicated expressions the useful facts that

$$
(4.13)\quad \begin{cases} \zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon,t_\varepsilon)v^\varepsilon_{k,t}(x_\varepsilon,t_\varepsilon) - v^\varepsilon_{\ell,t}(y_\varepsilon,t_\varepsilon) = O(1) \\ -\frac{\varepsilon}{2}\,\zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon,t_\varepsilon)a^k_{ij}(x_\varepsilon)v^\varepsilon_{k,x_ix_j}(x_\varepsilon,t_\varepsilon) \geq O(1) \\ -\frac{\varepsilon}{2}\,\zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon,t_\varepsilon)a^\ell_{ij}(y_\varepsilon)v^\varepsilon_{\ell,x_ix_j}(y_\varepsilon,t_\varepsilon) \leq O(1), \end{cases}
$$

as $\varepsilon \longrightarrow 0$.

We evaluate the $k^{th}$ equation in the PDE (3.8) at $(x_\varepsilon,t_\varepsilon)$ and the $\ell^{th}$ equation at $(y_\varepsilon,t_\varepsilon)$. Subtracting and employing the estimates from (4.13) and Lemma 3.1 we discover

(4.14)

$$
\zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon,t_\varepsilon)\left[c_{kp}e^{\frac{v^\varepsilon_k(x_\varepsilon,t_\varepsilon)-v^\varepsilon_p(x_\varepsilon,t_\varepsilon)}{\varepsilon}} - c_{\ell q}e^{\frac{v^\varepsilon_\ell(y_\varepsilon,t_\varepsilon)-v^\varepsilon_q(y_\varepsilon,t_\varepsilon)}{\varepsilon}}\right] \leq O(1),
$$

where the indices $p$ and $q$ are summed from 1 to m. Since $c_{kp} > 0$ if $p \neq k$, we deduce

$$
(4.15)\quad \begin{aligned} &\zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon t_\varepsilon)c_{k\ell}e^{\frac{v^\varepsilon_k(x_\varepsilon,t_\varepsilon)-v^\varepsilon_\ell(x_\varepsilon,t_\varepsilon)}{\varepsilon}} \\ &\quad\leq \zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon,t_\varepsilon)c_{\ell q}e^{\frac{v^\varepsilon_\ell(y_\varepsilon,t_\varepsilon)-v^\varepsilon_q(y_\varepsilon,t_\varepsilon)}{\varepsilon}} + O(1), \end{aligned}
$$

summed for $q = 1$ to $m$.

We must estimate the right hand side of this expression. But (4.5) and (4.8) imply for each $q \in \{1,\dots,m\}$ that

$$
\Phi^\varepsilon_{k\ell}(x_\varepsilon,y_\varepsilon,t_\varepsilon) \geq \Phi_{kq}(x_\varepsilon,y_\varepsilon,t_\varepsilon).
$$

Thus

$$
\begin{aligned} &\zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon,t_\varepsilon)[v^\varepsilon_k(x_\varepsilon,t_\varepsilon) - v^\varepsilon_\ell(y_\varepsilon,t_\varepsilon)] \\ &\qquad\geq \zeta(x_\varepsilon,t_\varepsilon)\zeta(y_\varepsilon,t_\varepsilon)[v^\varepsilon_k(x_\varepsilon,t_\varepsilon) - v^\varepsilon_q(y_\varepsilon,t_\varepsilon)] \end{aligned}
$$

whence, since $\zeta(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon) > 0$, we have

$$(4.16) \qquad v_\ell^\varepsilon(y_\varepsilon, t_\varepsilon) - v_q^\varepsilon(y_\varepsilon, t_\varepsilon) \leq 0 \quad (q = 1, \ldots, m).$$

This inequality inserted into (4.15) implies

$$\zeta(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon)[v_k^\varepsilon(x_\varepsilon, t_\varepsilon) - v_\ell^\varepsilon(x_\varepsilon, t_\varepsilon)] \leq O(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

But then (4.9) and Lemma 3.1 in turn yield

$$\zeta(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon)[v_k^\varepsilon(x_\varepsilon, t_\varepsilon) - v_\ell^\varepsilon(y_\varepsilon, t_\varepsilon)] \leq O(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Finally we observe from (4.7) and (4.8) that

$$\frac{\alpha_\varepsilon}{2} \leq \Phi_{k\ell}^\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon) \leq \zeta(x_\varepsilon, t_\varepsilon)\zeta(y_\varepsilon, t_\varepsilon)(v_k^\varepsilon(x_\varepsilon, t_\varepsilon) - v_\ell^\varepsilon(y_\varepsilon, t_\varepsilon))$$
$$\leq O(\varepsilon) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

This proves (4.2).

Returning now to the system of PDE (3.8) we observe that Lemma 3.1 and estimate (4.2) yield

$$v_{k,t}^\varepsilon - \frac{\varepsilon}{2} v_{ij}^k v_{k,x_i x_j}^\varepsilon = O(1) \quad \text{as} \quad \varepsilon \longrightarrow 0$$

on compact sets for $k = 1, \ldots, m$. Consequently standard parabolic estimates (see, for example, Evans-Ishii [6]) provide a uniform Hölder modulus of continuity on compact sets for the functions $v_k^\varepsilon$ in the $t$-variable. Consequently there exists a sequence $\varepsilon_j \longrightarrow 0$ and functions $v_k \in C(\mathbb{R}^n \times (0,\infty))$ such that

$$v_k^{\varepsilon_j} \longrightarrow v_k \qquad\qquad (k = 1, \ldots, m)$$

uniformly on compact sets. But then finally observe from estimate (4.2) that in fact

$$v_1 = v_2 = \ldots = v_m \equiv v.$$

We next demonstrate that $v$ is a viscosity solution of a certain Hamilton-Jacobi PDE. For fixed $x, p \in \mathbb{R}^n$, set

$$B(x,p) \equiv \operatorname{diag}(\ldots, \tfrac{1}{2} a^k_{ij}(x) p_i p_j - b^k_i(x) p_i, \ldots)$$

and then define the Hamiltonian

$$(4.17) \quad \begin{cases} H(x,p) = \lambda^0(B(x,p) + C) \\ \qquad = \text{the eigenvalue of the matrix } A(x,p) \equiv \\ \qquad\ B(x,p) + C \text{ with the largest real part.} \end{cases}$$

According to Theorem 2.2, $H(x,p)$ is real, the mapping $p \mapsto H(x,p)$ is convex and increasing, and

$$(4.18) \quad H(x,p) = \max_{z>0} \min_{1 \le k \le m} \frac{(A(x,p)z)_k}{z_k} = \min_{z>0} \max_{1 \le k \le m} \frac{(A(x,p)z)_k}{z_k}.$$

Lemma 4.2. The function $v$ is a viscosity solution of the Hamilton-Jacobi equation

$$v_t + H(x, Dv) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Proof. Let $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and suppose that $v - \phi$ has a strict local maximum at some point $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$. We must prove

$$(4.19) \quad \phi_t(x_0,t_0) + H(x_0, D\phi(x_0,t_0)) \le 0.$$

To simplify notation slightly, let us suppose $v^\varepsilon_k \longrightarrow v$ uniformly on compact sets. Then there exist points $(x^\varepsilon_k, t^\varepsilon_k) \in \mathbb{R}^n \times (0,\infty)$ such that

$$(4.20) \quad (x^\varepsilon_k, t^\varepsilon_k) \longrightarrow (x_0, t_0) \quad \text{as } \varepsilon \longrightarrow 0 \ (k = 1, \ldots, m),$$

and

(4.21) $v_k^\varepsilon - \phi$ has a local maximum at $(x_k^\varepsilon, t_k^\varepsilon)$ $(k = 1,\dots,m)$.

Since $v_k^\varepsilon$ is smooth, the maximum principle and (3.8) imply that

$$(4.22)\quad \left\{\phi_t - \frac{\varepsilon}{2}a_{ij}^k\phi_{x_ix_j} + \frac{1}{2}a_{ij}^k\phi_{x_i}\phi_{x_j} - b_i^k\phi_{x_i} + c_{k\ell}e^{\frac{v_k^\varepsilon - v_\ell^\varepsilon}{\varepsilon}} \le 0\right.$$

at the point $(x_k^\varepsilon, t_k^\varepsilon)$, $k = 1,\dots,m$. Fix $1 \le k \le m$. Then (4.21) implies for $\ell = 1,\dots,m$ that

$$v_\ell^\varepsilon(x_\ell^\varepsilon, t_\ell^\varepsilon) - \phi(x_\ell^\varepsilon, t_\ell^\varepsilon) \ge v_\ell^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) - \phi(x_k^\varepsilon, t_k^\varepsilon).$$

Hence

$$\begin{aligned} v_k^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) - v_\ell^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) &\ge v_k^\varepsilon(x_k^\varepsilon, t_k^\varepsilon) - v_\ell^\varepsilon(x_\ell^\varepsilon, t_\ell^\varepsilon) \\ &\quad + \phi(x_\ell^\varepsilon, t_\ell^\varepsilon) - \phi(x_k^\varepsilon, t_k^\varepsilon) \quad (\ell = 1,\dots m). \end{aligned}$$

We insert this inequality into (4.22) to find

$$(4.23)\quad \left\{\phi_t - \frac{\varepsilon}{2}a_{ij}^k\phi_{x_ix_j} + \frac{1}{2}a_{ij}^k\phi_{x_i}\phi_{x_j} - b_i^k\phi_{x_i} + c_{k\ell}\frac{z_\ell^\varepsilon}{z_k^\varepsilon} < 0,\right.$$

where $z_k^\varepsilon \equiv \exp\left(\frac{\phi(x_k^\varepsilon, t_k^\varepsilon) - v_k^\varepsilon(x_k^\varepsilon, t_k^\varepsilon)}{\varepsilon}\right) > 0$ $(k = 1,\dots,m)$. Recall now (4.17). Then, since $\phi$ is smooth, (4.23) implies

$$\phi_t(x_0, t_0) + \frac{(Az^\varepsilon)_k}{z_k^\varepsilon} \le o(1) \quad \text{as} \quad \varepsilon \longrightarrow 0,\ k = 1,\dots,m,$$

where $A = A(x_0, D\phi(x_0, t_0))$. But then

$$\phi_t(x_0, t_0) + \max_{1 \le k \le m} \frac{(Az^\varepsilon)_k}{z_k^\varepsilon} \le o(1) \quad \text{as} \quad \varepsilon \longrightarrow 0;$$

whence the "min-max" characterization of $H$ afforded by (4.18) implies

$$\phi_t(x_0,t_0) + H(x_0,D\phi(x_0,t_0)) = \phi_t(x_0,t_0) + \min_{z>0} \max_{1\le k\le m} \frac{(Az)_k}{z_k}$$

$$\le \phi_t(x_0,t_0) + \max_{1\le k\le m} \frac{(Az^{\varepsilon})_k}{z_k^{\varepsilon}} \le o(1) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

This proves (4.19). The proof of the opposite inequality in case $v - \phi$ has a strict local minimum follows similarly, using the max-min characterization of $H$ provided by (4.18). □

Remarks. (i) In view of this Lemma, Lemma 3.1 and Crandall-Lions [3], we see that $v$ is locally Lipschitz in the variable $t$.

(ii) From Lemma 3.2 we see also that $v$ is continuous on compact subsets of $\text{int } G_0 \times [0,\infty)$, and that

(4.24) $$v = 0 \quad \text{on} \quad \text{int } G_0.$$

## 5. Indentification of the action function.

In view of the definition of the $v_k^\varepsilon = -\varepsilon \log u_k^\varepsilon$, we obtain

Theorem 5.1. The functions $u_k^\varepsilon$ converge to zero uniformly on compact subsets of $\{v > 0\}$.

We next identify $v$, using ideas set forth in [8]. For this let us first note that $H$ satisfies

$$(5.1) \qquad |H(x,p) - H(x,\hat{p})| \le C|p - \hat{p}|(|p| + |\hat{p}| + 1)$$

and

$$(5.2) \qquad |H(x,p) - H(\hat{x},p)| \le C|x - \hat{x}|(|p|^2 + 1)$$

for appropriate constants $C$ and all $x,\hat{x},p,\hat{p} \in \mathbb{R}^n$. In addition,

$$(5.3) \qquad a|p|^2 - b \le |H(x,p)| \le A|p|^2 + B$$

for all $x,p \in \mathbb{R}^n$ and certain constants $a,b,A,B$.

We recall next that the Lagrangian associated with $H$ is

$$L(x,q) \equiv \sup_{p\in\mathbb{R}^n} (q\cdot p - H(x,p)) \quad (x,q \in \mathbb{R}^n).$$

$L$ satisfies continuity and growth estimates similar to (5.1) - (5.3).

Theorem 5.2. We have

$$v(x,t) = I(x,t) \equiv \inf_{x(\cdot)\in X}\left\{\int_0^t L(x(s), -\dot{x}(s))ds \,|\, x(0) = x,\ x(t) \in G_0\right\},$$

where

$$X \equiv H^1_{loc}([0,\infty);\mathbb{R}^N) = \{x : [0,\infty)\longrightarrow\mathbb{R}^n \,|\, x(\cdot) \text{ is absolutely continuous, } \dot{x} \in L^2(0,T) \text{ for each } T > 0\}.$$

Proof. Choose a smooth function $\eta : \mathbb{R}^n\longrightarrow\mathbb{R}$ satisfying

$$0 \le \eta \le 1, \ \eta = 0 \ \text{ on } \ G_0, \ \eta > 0 \ \text{ on } \ \mathbb{R}^n - G_0.$$

Fix $r \in \{1,2,\dots\}$ and write

$$\hat{g}_k^{\varepsilon}(x) \equiv g_k(x) + e^{\frac{-r\eta(x)}{\varepsilon}} \quad (x \in \mathbb{R}^n).$$

Let $\hat{u}^{\varepsilon} = (\hat{u}_1^{\varepsilon},\dots,\hat{u}_m^{\varepsilon})$ solve (3.5), but with $\hat{g}_k^{\varepsilon}$ replacing $g_k$. Since $\hat{g}_k^{\varepsilon} \ge g_k$, $\hat{u}_k^{\varepsilon} \ge u_k^{\varepsilon}$ and so $\hat{v}_k^{\varepsilon} \le v_k^{\varepsilon}$, where

$$\hat{v}_k^{\varepsilon} = -\varepsilon \log \hat{u}_k^{\varepsilon} \quad (k = 1,\dots,m).$$

Now

$$\hat{v}_k^{\varepsilon}(x,0) = -\varepsilon \log \hat{g}_k^{\varepsilon}(x) = \begin{cases} -\varepsilon \log(g_k(x)+1) & (x \in G_0) \\ r\eta(x) \ \ (x \in \mathbb{R}^n - G_0). \end{cases}$$

The estimates and convergence arguments developed above apply also to the $\hat{v}^{\varepsilon}$. Thus, passing if necessary to another subsequence, we have

$$\hat{v}^{\varepsilon_j} \longrightarrow \hat{v}$$

uniformly on compact subsets of $\mathbb{R}^n \times (0,\infty)$, where $\hat{v}$ is a viscosity solution of

$$\hat{v}_t + H(x,D\hat{v}) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty).$$

Furthermore, since $\hat{v}^{\varepsilon}$ is well behaved at $t = 0$, the technique for estimating the gradient works even on compact subsets of $\mathbb{R}^n \times [0,\infty)$, and thus $\hat{v}$ is Lipschitz on compact subsets of $\mathbb{R}^n \times [0,\infty)$. Finally, a simple barrier argument shows that

$$\hat{v}(x,0) = r\eta(x) \quad (x \in \mathbb{R}^n).$$

Then we have, according to [8], that

$$v(x,t) \geq \hat{v}(x,t) = \inf_{x(\cdot)\in X} \left\{\int_0^t L(x(s),-\dot{x}(s))ds + r\eta(x(t)) \mid x(0) = x\right\}.$$

Let $r \longrightarrow \infty$ to establish

$$(5.4) \quad v(x,t) \geq \inf_{x(\cdot)\in X} \left\{\int_0^t L(x(s),-\dot{x}(s))ds \mid x(0) = x,\ x(t) \in G_0\right\}.$$

On the other hand, fix $T,\delta,\rho > 0$ and choose $R > 0$ so large that

$$G_0 \subset B(0,R).$$

Consider the cylinder $C \equiv B(0,R)\times[\delta,T]$ and suppose $(x,t) \in$ int $C$. Then, again using [8], we see that

$$\begin{aligned} v(x,t) = \inf_{x(\cdot)\in X} \Bigg\{&\int_0^{(t-\delta)\wedge\tau} L(x(s),-\dot{x}(s))ds \\ &+ \chi_{[\tau<t-\delta]} v(x(\tau),t-\tau) \\ &+ \chi_{[\tau\geq t-\delta]} v(x(t-\delta),\delta) \mid x(0) = x\Bigg\} \\ \leq \inf_{x(\cdot)\in X} \Bigg\{&\int_0^t L(x(s),-\dot{x}(s))ds \\ &+ v(x(t-\delta),\delta) \mid x(0) = x, |x(s)| \leq R \text{ for } \\ &0 \leq s \leq t-\delta,\ x(t-\delta) \in G_0^\rho\Bigg\}, \end{aligned}$$

where

$$\tau \equiv \inf\{s \geq 0 \mid |x(s)| = R\}$$

and

$$G_0^\rho \equiv \{y \in G_0 \mid \text{dist}\,(y,\partial G_0) \geq \rho\}.$$

Let $R \longrightarrow \infty$, $\delta \longrightarrow 0$ and $\rho \longrightarrow 0$, in that order, and recall (4.24) to find

$$v(x,t) \le \inf_{x(\cdot)\in X} \left\{\int_0^t L(x(s),-\dot{x}(s))ds \mid x(0) = x,\ x(t) \in G_0\right).$$

This inequality and (5.4) complete the proof. □

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