HIGH ORDER INVERSE FUNCTION THEOREMS

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Abstract We prove several first order and high order inverse mapping theorems for maps defined on a complete metric space and provide a number of applications.

Key words Inverse function theorem, set-valued map, controllability, reachable set, stability.

1 Introduction

Inverse function theorem is a natural tool to apply to many problems arising in control theory and optimization. The classical theorems, such as Ljusternik's theorem, are not always sufficient , and this because the data of the problems often happen to be "nonclassical" ones.

Such "unusual" situation does arise when one deals with

i) A map whose domain of definition is a metric space

ii) A map which is not single-valued

iii) A map for which the first order conditions are not sufficient to solve the problem.

This is why one has to look for different inverse function theorems adapted to new problems. During the last twenty years this task was undertaken in many papers (see for example [6], [5], [8], [9], [14], [21], [22] and bibliographies contained therein).

Let us recall first the classical result of functional analysis:

Theorem 1.1 *Let $f : U \to X$ be a continuously differentiable function from a Banach space U to a Banach space X and $\overline{u} \in U$. If the derivative $f'(\overline{u})$ is surjective, then for all $h > 0$, $f(\overline{u}) \in \text{Int } f(B_h(\overline{u}))$ and there exists $L > 0$ such that for all $x \in X$ near $f(\overline{u})$, $\text{dist }(\overline{u},\ f^{-1}(x)) \leq L\ \|f(\overline{u}) - x\|$.*

As it was observed in [9] the assumptions of theorem imply much stronger conclusions. In fact the very same proof allows to go beyond the above result and to prove the uniform open mapping principle and regularity of the inverse map f^{-1} on a neighborhood of the point $(f(\overline{u}), \overline{u})$.

The surjectivity assumption of the above theorem may be replaced by an equivalent assumption

$$(1) \qquad 0 \in \text{Int } \overline{f'(\overline{u})\ (B)}$$

Several extensions of Theorem 1.1 were derived in [9] via the same idea. Assuming that the space U is just a complete metric space and some "covering assumptions" on f one can obtain a result similar to Theorem 1.1. However verification of covering assumptions is not always simple.

In this paper we prove a **High Order Uniform Open Mapping Principle** for maps defined on a complete metric space. That is, we provide a sufficient condition for

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the existence of $\mathcal{L} > 0,\ k \geq 1$ such that

(2) $\qquad \forall\, u$ near $\overline{u}$ and $\forall$ small $h > 0,\quad f(u) + \mathcal{L}h^k B \in \operatorname{Int} f(B_h(u))$

To get regularity of the inverse map f^{-1} we use a high order analogue of Theorem 1.1 proved in [19]:

Theorem 1.2 (a general inverse function theorem). *Let G be a set-valued map from a complete metric space (U, d) to a metric space (X, d_X) having a closed graph and let $(\overline{u}, \overline{x}) \in$ Graph G. Assume that for some $k > 0,\ \rho > 0,\ \epsilon > 0,\ 0 \leq \alpha < 1$ we have*

(3) $\forall\, u \in B_\epsilon(\overline{u}),\ x \in G(u) \cap B_\epsilon(\overline{x}),\ h \in [0, \epsilon],\quad \sup_{b \in B_{\rho h^k}(x)} \operatorname{dist}(b, G(B_h(u))) \leq \alpha\rho h^k$

Then for every $h > 0$ satisfying $h/(1 - \alpha^{\frac{1}{k}}) + 2\rho h^k < \epsilon/2$ and all $u \in B_{\frac{\epsilon}{2}}(\overline{u})$, $x \in G(u) \cap B_{\frac{\epsilon}{2}}(\overline{x})$, $y \in B_{\rho h^k}(x)$ we have

$$\operatorname{dist}(u,\ G^{-1}(y)) \leq \frac{1}{1 - \alpha^{\frac{1}{k}}}\, h$$

In particular, for all $(u, x) \in$ Graph G near $(\overline{u}, \overline{x})$ and all y near $G(\overline{u})$

(4) $$\operatorname{dist}(u,\ G^{-1}(y)) \leq \frac{1}{\rho^{\frac{1}{k}}(1 - \alpha^{\frac{1}{k}})}\, d_X(x,\ y)^{\frac{1}{k}}$$

When X is a Banach space, assumption (3) can be formulated as

$$x + \rho h^k B \subset \overline{G(B_h(u)) + \alpha\rho h^k B}$$

It holds true for a map satisfying the Uniform Open Mapping Principle (2) and therefore the two theorems together bring a sufficient condition for the regularity of f^{-1}. The Uniform Open Mapping Theorem is proved via the Ekeland variational principle. The curious aspect of this approach lies in the use of an apparently first order result (Ekeland's principle) to derive high order sufficient conditions.

Let us explain briefly the main ideas. When the space U is just a metric space then one can neither differentiate the function f nor speak about the continuity of the derivative. In [18], [19] we proposed to replace the derivative by the variation of the map (which can be single-valued or set-valued). The first order variation $f^{(1)}(\overline{u})$ of a single-valued map is defined in such way that for a C^1 map f between two Banach spaces the set $\overline{f'(\overline{u})B}$ is equal to it. Condition (1) together with continuity of the derivative inherit then their natural extension

$$0 \in \operatorname{Int} \cup_{\epsilon > 0} \cap_{d(u,\overline{u}) \leq \epsilon} \overline{\text{co}}\, f^{(1)}(u)$$

High order variations were introduced in [13] (see also [19], [14], [16]), where several sufficient conditions for regularity of the inverse map f^{-1} were proved. When one restricts the attention to single-valued maps only, then results of [19] can be improved and proofs can be made simpler. In this paper on one hand we prove more precise results for single-valued maps on the other we overview the applications of the inverse function theorems given in [15], [17]-[19] and provide their several new consequences.

The plan of the paper is as follows: Variations are defined in Section 2, where also several examples are given. Sections 3 and 4 are devoted to first and high order inverse function theorems for a single valued map. In Section 5 we state several theorems for set-valued maps. Their proofs can be found in [19]. Examples of applications are provided in Section 6.

2 Variations of single-valued and set-valued maps

Consider a metric space (U,d) and a Banach space X. For all $u \in U$, $h > 0$ let $B_h(u)$ denote the closed ball in U of center u and radius h.

We recall first the notions of Kuratowski's *limsup* and *liminf*:
Let T be a metric space and $A_\tau \subset X$, $\tau \in T$ be a family of subsets of X. The Kuratowski *limsup* and *liminf* of A_τ at τ_0 are closed sets given by

$$\begin{array}{lcl} \limsup_{\tau\to\tau_0} A_\tau & = & \{\, v \in X \;:\; \liminf_{\tau\to\tau_0} \ \operatorname{dist}(v, A_\tau) \;=\; 0 \,\} \\ \liminf_{\tau\to\tau_0} A_\tau & = & \{\, v \in X \;:\; \lim_{\tau\to\tau_0} \ \operatorname{dist}(v, A_\tau) \;=\; 0 \,\} \end{array}$$

Definition 2.1 *Consider a function $G : U \to X$ and let $u \in U$, $k > 0$.*
i) The contingent variation of G at u is the closed subset of X given by

$$G^{(1)}(u) \;:=\; \limsup_{h\to 0+} \frac{G(B_h(u)) - G(u)}{h}$$

ii) The k-th order variation of G at u is the closed subset of X given by

$$G^k(u) \;:=\; \liminf_{h\to 0+,u'\to u} \frac{G(B_h(u')) - G(u')}{h^k}$$

In other words $v \in G^{(1)}(u)$ if and only if there exist sequences $h_i \to 0+$, $v_i \to v$ such that $G(u) + h_i v_i \in G(B_{h_i}(u))$. The word contingent is used because the definition reminds that of the contingent cone of Bouligand.

Similarly $v \in G^k(u)$ if and only if for all sequences $h_i \to 0+, u_i \to u$ there exists a sequence $v_i \to v$ such that $G(u_i) + h_i^k v_i \in G(B_{h_i}(u_i))$.

Clearly, $G^{(1)}(u)$ and $G^k(u)$ are closed sets starshaped at zero. When U is a Banach space and $G : U \to X$ is a Gâteaux differentiable at some $u \in U$ function, then $\overline{G'(u)(B)} \subset G^{(1)}(u)$. If moreover G is continuously Fréchet differentiable at u then $\overline{G'(u)(B)} = G^1(u) = G^{(1)}(u)$.

The notions of variation extend to set-valued maps in a natural way:

Let $G : U \to X$ be a set-valued map, that is for all $u \in U$, $G(u)$ is a (possibly empty) subset of X. The domain and the graph of G are given by

$$\operatorname{Dom} G = \{u \in U \mid G(u) \neq \emptyset\}, \;\; \operatorname{Graph} G = \{(u,x) \mid u \in \operatorname{Dom} G,\ x \in G(u)\}$$

Definition 2.2 *Let $G : U \to X$ be a set-valued map and $(u,x) \in$* Graph G, $k > 0$.
i) The contingent variation of G at (u,x) is the closed subset of X

$$G^{(1)}(u,x) \;:=\; \limsup_{h\to 0+} \frac{G(B_h(u)) - x}{h}$$

ii) The k-th order variation of G at (u,x) is the closed subset of X

$$G^k(u,x) \;:=\; \liminf_{\substack{(u',x') \to_G (u,x) \\ h \to 0+}} \frac{G(B_h(u')) - x'}{h^k}$$

where $\to_G$ denotes the convergence in Graph G.

When G is a single-valued map the point $(u, x) \in$ Graph G if and only if $G(u) = x$ and therefore in this case the variations in the sense of the first and the second definitions do coincide.

Variations of all orders can be used to prove sufficient conditions for the existence of a Hölder inverse for a single-valued and a set-valued map. They describe a local expansion of a map at a given point.

Let *co* ($\overline{co}$) denote the convex (closed convex) hull and B the closed unit ball in X. The following result was proved in [19]:

Theorem 2.3 *For every $(u, x) \in Graph\ (G)$, $k > 0$ we have*
i) For all $K \geq k$, $0 \in G^k(u,x) \subset G^K(u,x)$
ii) For all $s > 0$, $\mathbf{R}_+ G^k(u,x) \subset G^{k+s}(u,x)$
iii) For all $\lambda_i \geq 0, v_i \in G^k(u,x), i = 0, ..., m$ with $\sum_{i=0}^m \lambda_i = 1$, $\sum_{i=0}^m \lambda_i^k v_i \in G^k(u,x)$
iv) For all $v \in co\ G^k(u,x)$ *there exists* $\epsilon > 0$ *such that* $\epsilon v \in G^k(u,x)$
v) $\bigcup_{\lambda \geq 0} \lambda co\ G^k(u,x) = \bigcup_{\lambda \geq 0} \lambda G^k(u,x)$
vi) $\bigcup_{\lambda \geq 0} \lambda G^k(u,x) = X \iff 0 \in Int\ co\ G^k(u,x)$. *Moreover if $X = \mathbf{R}^n$ these conditions are equivalent to:* $\exists\, v_1, ..., v_p \in G^k(u,x)$ such that $0 \in Int\ co\ \{v_1, ..., v_p\}$.

Example 1. First order variation of a set-valued map

Consider Banach spaces P, X, Y, continuously differentiable functions $g : P \times X \to Y$, $h : P \times X \to \mathbf{R}^n$ and the set-valued map $G : P \times X \to P \times Y \times \mathbf{R}^n$ defined by

$$G(p,x) = \{ (p,\ g(p,x),\ h(p,x) + \rho) \mid \rho \in \mathbf{R}^n_+ \}$$

Then a direct calculation yields that for all $(p, x, q) \in P \times X \times \mathbf{R}^n_+$

$$G^1(p,x,p,g(p,x),h(p,x)+q) \supset$$
$$\left\{ (v,\ \tfrac{\partial g}{\partial p}(p,x)v + \tfrac{\partial g}{\partial x}(p,x)w,\ \tfrac{\partial h}{\partial p}(p,x)v + \tfrac{\partial h}{\partial x}(p,x)w + \rho) \mid \|(v,w)\| \leq 1,\ \rho \in \mathbf{R}^n_+ \right\}$$

Example 2. Contingent variation of end points of trajectories of a control system

Let U be a separable metric space, X be a Banach space and $f : X \times U \to X$ be a continuous, differentiable in the first variable function.

We assume that f is locally Lipschitz in the first variable uniformly on U, i.e. for all $x \in X$ there exist $L > 0$ and $\epsilon > 0$ such that for all $u \in U, f(\cdot, u)$ is L-Lipschitz on $B_\epsilon(x)$:

$$\|f(x',u) - f(x'',u)\| \leq L\ \|x' - x''\|, \text{ for all } x', x'' \in B_\epsilon(x)$$

Fix $T > 0$ and let $\mathcal{U}$ denote the set of all (Lebesgue) measurable functions $u : [0,T] \to U$. Define a metric d on $\mathcal{U}$ by setting $d(u,v) = \mu(\{t \in [0,T] \mid u(t) \neq v(t)\})$, where μ denotes the Lebesgue measure. The space $(\mathcal{U}, d)$ is complete (see Ekeland [10]).

Let $\{S(t)\}_{t \geq 0}$ be a strongly continuous semigroup of continuous linear operators from X to X and A be its infinitesimal generator, $x_0 \in X$. Consider the control system

$$\text{(5)} \qquad \begin{cases} x'(t) = Ax(t) + f(x(t), u(t)), & u \in \mathcal{U} \\ x(0) = x_0 \end{cases}$$

Recall that a continuous function $x : [0,T] \to X$ is called a mild trajectory of (5) if for some $u \in \mathcal{U}$ and all $0 \le t \le T$

$$x(t) \ = \ S(t)x_0 \ + \ \int_0^t S(t-s)f(x(s),u(s))ds$$

We denote by x_u the trajectory (when it is defined on the whole time interval $[0,T]$ and is unique) corresponding to the control u. Define the map $G : \mathcal{U} \to X$ by

$$G(u) \ = \ \{ \ x_u(T) \ \}$$

Let z be a mild trajectory of (5) on $[0,T]$ and $\overline{u}$ be the corresponding control. Consider the linear system

(6) $$Z' \ = \ AZ \ + \ \frac{\partial f}{\partial x}(z(t),\overline{u}(t))Z$$

and let $S_{\overline{u}}(t;s)$ denote its solution operator, where $S_{\overline{u}}(s;s) = Id, \ t \ge s$. Then for all u near $\overline{u}$, G is a well defined single-valued map. Moreover for almost all $t \in [0,T]$ we have

(7) $$\forall \, u \in U, \ S_{\overline{u}}(T;t)(f(z(t),u) - f(z(t),\overline{u}(t))) \ \in \ G^{(1)}(\overline{u})$$

We refer to [11] for the proof of this result.

Example 3. First order variation of end points of trajectories of a differential inclusion

Let X be a finite dimensional space, F be a set-valued map from X to X. We associate with it the differential inclusion

(8) $$x' \in F(x)$$

An absolutely continuous function $x \in W^{1,1}(0,T), \ T \ge 0$ (the Sobolev space) is called a trajectory of the differential inclusion (8) if for almost all $s \in [0,T]$, $x'(s) \in F(x(s))$. The set of all trajectories of (8) defined on the time interval $[0,T]$ and starting at ξ, $(x(0) = \xi)$ is denoted by $\mathcal{S}_{[0,T]}(\xi)$. The reachable map of (8) from ξ is the set-valued map $R : \mathbf{R}_+ \to X$ defined by

$$R(t) \ = \ \{ \ x(t) \mid \ x \in \mathcal{S}_{[0,t]}(\xi) \ \}$$

Assume that

$H_1)$ $\forall \, x \in X$ near ξ, $F(x)$ is a nonempty compact set and $0 \in F(\xi)$
$H_2)$ $\exists$ a neighborhood $\mathcal{N}$ of ξ, $L > 0$ such that $\forall \, x, y \in \mathcal{N}$, $F(x) \subset F(y) + L\|x-y\| B$

Hypothesis $H_2)$ means that F is Lipschitz in the Hausdorff metric on a neighborhood of ξ. Hypothesis $H_1)$ implies that $x \equiv \xi \in \mathcal{S}_{[0,\infty[}(\xi)$.

The derivative of F at $(\xi, 0)$ is the set-valued map $CF(\xi,0) : \ X \to X$ defined by

$$\forall \, u \in X, \ CF(\xi,0)u \ = \ \liminf_{\substack{(x,y) \to_F (\xi,0) \\ h \to 0+}} \frac{F(x+hu)-y}{h}$$

Fix $T > 0$ and consider the single-valued map $G : W^{1,1}(0,T) \supset \mathcal{S}_{[0,T]}(\xi) \to X$ defined by $G(x) = x(T)$. Let $K \subset \text{co } F(\xi)$ be a closed convex set having only finite number of extremal points. Then there exists $M > 0$ such that for every trajectory $w \in W^{1,1}(0,T)$ of the differential inclusion

$$(9) \qquad w' \in CF(\xi,0)w + K, \quad w(0) = 0$$

we have $\frac{w(T)}{M} \in G^1(\xi)$. The proof follows from the results of [15].

Example 4. High order variations of the reachable map

Let F be a set-valued map satisfying all the assumptions from the Example 3. Consider again the differential inclusion (8) and the reachable map $t \to R(t)$. It was shown in [17] that for all integer $k \geq 1$

$$(10) \qquad R^k(0,\xi) = \liminf_{h \to 0+} \frac{R(h) - \xi}{h^k}$$

A very same proof implies that the above holds true for all $k > 0$.

3 First order inverse of a single valued map

Consider a complete metric space (U, d), a Banach space X and a continuous map $G : U \to X$. Let $\overline{u} \in U$ be a given point. We study here a sufficient condition for the regularity of the inverse map $G^{-1} : X \to U$ defined by

$$G^{-1}(x) = \{ u \in U \mid G(u) = x \}$$

on a neighborhood of $(G(\overline{u}), \overline{u})$.

Theorem 3.1 (Inverse Mapping Theorem I) *Let $\overline{u} \in U$ and assume that for some $\epsilon > 0$, $\rho > 0$*

$$(11) \qquad \rho B \subset \bigcap_{d(u,\overline{u}) \leq \epsilon} G^{(1)}(u)$$

Then for every $u \in B_{\frac{\epsilon}{2}}(\overline{u})$ and $h \in [0, \frac{\epsilon}{2}]$, $G(u) + h\rho \overset{o}{B} \subset G(B_h(u))$, (where $\overset{o}{B}$ denotes the open unit ball in X). Furthermore for every $u \in B_{\frac{\epsilon}{4}}(\overline{u})$, $x \in X$ satisfying $\|x - G(u)\| < min\{ \frac{\epsilon}{8}, \frac{\epsilon\rho}{4} \}$

$$(12) \qquad dist\,(u, G^{-1}(x)) \leq \frac{1}{\rho} \|G(u) - x\|$$

Remark Inequality (12) means that G is pseudo-Lipschitz at $(G(\overline{u}), \overline{u})$ with the Lipschitz constant ρ^{-1} (see Aubin [1]). □

Theorem 3.2 (Inverse Mapping Theorem II) *Assume the norm of X is Gâteaux differentiable away from zero. If for some $\epsilon > 0$, $\rho > 0$*

$$\rho B \subset \bigcap_{d(u,\overline{u}) \leq \epsilon} \overline{co}\, G^{(1)}(u)$$

Then for every $u \in B_{\frac{\epsilon}{2}}(\overline{u})$ and $h \in [0, \frac{\epsilon}{2}]$, $G(u) + h\rho \overset{o}{B} \subset G(B_h(u))$. Moreover for every $u \in B_{\frac{\epsilon}{4}}(\overline{u})$, $x \in X$ satisfying $\|x - G(u)\| < min\{\frac{\epsilon}{8}, \frac{\epsilon\rho}{4}\}$

$$(13) \qquad \text{dist}\,\left(u, G^{-1}(x)\right) \leq \frac{1}{\rho} \|G(u) - x\|$$

Corollary 3.3 *Assume that X is a finite dimensional space and that*

$$0 \in \text{Int} \liminf_{u \to \overline{u}} \; \overline{co}\, G^{(1)}(u)$$

Then there exist $\epsilon > 0$, $\rho > 0$ such that all conclusions of Theorem 3.2 are valid.

Proof (of Theorem 3.1) Fix $u \in B_{\frac{\epsilon}{2}}(\overline{u})$, $0 < h \le \frac{\epsilon}{2}$ and assume for a moment that there exists $x \in X$ satisfying

$$\|x - G(u)\| < h\rho, \quad x \notin G(B_h(u)) \tag{14}$$

Set $\Theta^2 = \|x - G(u)\| / h\rho$. Then $0 < \Theta < 1$. Applying the Ekeland variational principle [10] to the complete metric space $B_h(u)$ and the continuous function $y \to \|G(y) - x\|$ we prove the existence of $\overline{y} \in B_{\Theta h}(u)$ such that for all $y \in B_h(u)$

$$\|G(\overline{y}) - x\| \le \|G(y) - x\| + \Theta\, \rho\, d(y, \overline{y}) \tag{15}$$

Observe that $\overline{y} \in \text{Int}\, B_h(u)$ and, by (14), $x \ne G(\overline{y})$. Set $w = -\rho(G(\overline{y}) - x)/\|G(\overline{y}) - x\|$. By our assumption there exist $h_i \to 0+$, $w_i \to w$ such that $G(\overline{y}) + h_i w_i \in G(B_{h_i}(\overline{y}))$. Hence, from (15) we deduce that for all large i

$$\begin{array}{c}\|G(\overline{y}) - x\| \le \|G(\overline{y}) + h_i w - x\| + h_i\|w_i - w\| + \Theta\rho h_i = \\ \left(1 - h_i \frac{\rho}{\|G(\overline{y}) - x\|}\right) \|G(\overline{y}) - x\| + h_i\|w_i - w\| + \Theta\rho h_i\end{array}$$

and therefore $h_i\rho \le h_i\|w_i - w\| + \Theta\rho h_i$. Dividing by ρh_i and taking the limit yields $1 \le \Theta$. The obtained contradiction ends the proof of the first statement. The second one results from Theorem 1.2. □.

Proof of Theorem 3.2 Fix $u \in B_{\frac{\epsilon}{2}}(\overline{u})$, $0 < h \le \frac{\epsilon}{2}$ and assume for a moment that there exists $x \in X$ satisfying (14). Let Θ, $\overline{y}$ be as in the proof of Theorem 3.1. By differentiability of the norm, there exists $p \in X^\star$ of $\|p\| = 1$ such that for all $h_j \to 0+$, $v_j \to v$ we have

$$\|G(\overline{y}) + h_j v_j - x\| = \|G(\overline{y}) - x\| + < p, h_j v > + o_v(h_j) \tag{16}$$

where $\liminf_{j\to\infty} o_v(h_j)/h_j = 0$. Fix $v \in G^{(1)}(\overline{y})$. Then from (15), (16) and Definition 2.1 we obtain $0 \le < p, h_j v > + \Theta\rho h_j + o_v(h_j)$. Dividing by h_j and taking the limit yields $< p, v > \ge -\Theta\rho$. Hence

$$\forall\, v \in \overline{co}\, G^{(1)}(\overline{y}), \quad < p, v > \ge -\Theta\rho \tag{17}$$

Since $d(\overline{y}, \overline{u}) \le d(\overline{y}, u) + d(u, \overline{u}) \le \Theta h + \frac{\epsilon}{2} < \epsilon$, by the assumption of theorem, $\rho B \subset \overline{co}\, G^{(1)}(\overline{y})$. From (17) we deduce that $-\rho \ge \inf_{v \in \overline{co}\, G^{(1)}(\overline{y})} < p, v > \ge -\Theta\rho$. But $0 < \Theta < 1$ and $\rho > 0$ and we obtained a contradiction. The second statement follows from the first one and Theorem 1.2. □

The following theorem provides a stronger sufficient condition for local invertibility but does not allow to estimate the Lipschitz constant.

Theorem 3.4 *Assume that X is either separable or reflexive and that its norm is Gâteaux differentiable away from zero. Let $\overline{u} \in U$. Further assume that there exist $\epsilon > 0$ and a compact $Q \subset X$ such that*

$$Int \bigcap_{u \in B_\epsilon(\overline{u})} \left(\overline{co}\, G^{(1)}(u) + Q\right) \ne \emptyset \tag{18}$$

Then the following statements are equivalent

$$i) \quad \left(\liminf_{u\to\overline{u}} \overline{co}\, G^{(1)}(u) \;+\; G^1(\overline{u})\right)^+ \;=\; \{\,0\,\}$$

ii) for some $\delta > 0$, $L > 0$ and for all $(u,x) \in B_\delta(\overline{u}) \times B_\delta(G(\overline{u}))$

$$dist\;\left(u,\; G^{-1}(x)\right) \;\leq\; L\,\|G(u) \;-\; x\|$$

In particular if for some $\delta > 0$, $G(\overline{u})$ is a boundary point of $G(B_\delta(\overline{u}))$, then there exists a non zero $p \in X^\star$ such that

$$(19) \qquad \forall\, w \in \liminf_{u\to\overline{u}} \overline{co}\, G^{(1)}(u) \;+\; G^1(\overline{u}),\; < p,\; w > \geq 0$$

Proof Clearly $ii)$ implies that for all $u \in U$ near $\overline{u}$ and all small $h > 0$, $G(u) + \frac{1}{L}h\;\overset{o}{B} \subset G(B_h(u))$. Thus $\frac{1}{L}B \subset G^1(\overline{u})$ and $i)$ follows. To show that $i) \implies ii)$, by Theorem 1.2, it is enough to prove that for some $\rho > 0$ and all $u \in U$ near $\overline{u}$ and all small $h > 0$

$$G(u) \;+\; \rho h\;\overset{o}{B} \subset\; G(B_h(u))$$

Assume for a moment that for some $t_i \to \overline{u}$, $\overline{h}_i \to 0+$ there exist

$$(20) \qquad \overline{y}_i \notin G(B_{\overline{h}_i}(t_i)), \quad \|\overline{y}_i - G(t_i)\| \;<\; \frac{\overline{h}_i}{i^2},\;\; i = 1,2,...$$

We shall derive a contradiction. Set $K_i = B_{\overline{h}_i}(t_i)$. We apply the Ekeland variational principle ([10]) to the continuous functions $K_i \ni u \to \|G(u) - \overline{y}_i\|$, $i = 1,2,...$ to prove the existence of $u_i \in B_{\frac{\overline{h}_i}{i}}(t_i)$ such that for all $u \in K_i$

$$(21) \qquad \|G(u_i) - \overline{y}_i\| \;\leq\; \|G(u) - \overline{y}_i\| \;+\; \frac{1}{i}\,d(u,u_i)$$

By differentiability of the norm and by (20), there exist $p_i \in X^\star$, $\|p_i\|_{X^\star} = 1$ such that for all $h_j \to 0+, v_j \to v$, we have

$$\|G(u_i) \;+\; h_jv_j \;-\; \overline{y}_i\| \;=\; \|G(u_i) \;-\; \overline{y}_i\| + \;< p_i, h_jv > \;+\; o_{i,v}(h_j)$$

where $\lim_{h\to 0+} o_{i,v}(h)/h = 0$. Fix $v \in G^{(1)}(u_i))$ and let $h_j \to 0+, v_j \to v$ be such that $G(u_i) + h_jv_j \in G(B_{h_j}(u_i))$. Setting $G(u) = G(u_i) + h_jv_j$ in (21) we obtain

$$(22) \qquad 0 \;\leq\; < p_i, h_jv > \;+\; o_{i,v}(h_j) \;+\frac{1}{i}\,h_j$$

Dividing by h_j and taking the limit when $j \to \infty$ yields that for all $v \in G^{(1)}(u_i)$, $< p_i, v > \geq -\frac{1}{i}$ and therefore

$$(23) \qquad \forall\, v \in \overline{co}\, G^{(1)}(u_i),\;\; < p_i, v > \geq\; -\frac{1}{i}$$

Let $p \in X^\star$ be a weak-$\star$ cluster point of $\{p_i\}$. Then

$$(24) \qquad \forall\, w \in \liminf_{u\to\overline{u}}\; \overline{co}\, G^{(1)}(u),\;\; < p, w > \geq\; 0$$

Fix next $v \in G^1(\overline{u})$ and choose $h_i \to 0+$ in such way that

$$\lim_{i\to\infty} \frac{o_{i,v}(h_i)}{h_i} = 0 \text{ and } B_{h_i}(u_i) \subset B_{\overline{h}_i}(t_i)$$

Let $v_i \to v$ be such that $G(u_i) + h_i v_i \in G(B_{h_i}(u_i))$. Then from (22) there exist $\epsilon_i \to 0+$ such that

$$< p_i, v > \geq -\epsilon_i - \frac{1}{i}$$

Since p is a weak$-\star$ cluster point of $\{p_i\}$, we infer from the last inequality that for all $v \in G^1(\overline{u})$, $< p, v > \geq 0$. This, (24) and i) together yield that $p = 0$. To end the proof it remains to show that $p \neq 0$. Indeed by (18) there exist $z \in X$, $\rho > 0$ and $a_i \in \overline{\text{co}}\, G^{(1)}(u_i)$, $q_i \in Q$, $w_i \in B$ such that $< p_i, w_i > \geq 1 - \frac{1}{i}$, $z - \rho w_i = a_i + q_i$. From (23), $< p_i, z - \rho w_i - q_i > \geq -\frac{1}{i}$. Hence

$$< p_i, z - q_i > \geq \rho \left(1 - \frac{1}{i}\right) - \frac{1}{i}$$

Let $\{p_{i_j}\}, \{q_{i_j}\}$ be such subsequences that $p_{i_j} \to p$ weakly-$\star$ and $q_{i_j} \to q \in Q$. Then taking the limit in the last inequality yields $< p, z - q > \geq \rho > 0$ and therefore $p \neq 0$. The obtained contradiction proves *ii*). □

Corollary **3.5** *Let X be a Hilbert space, H be a closed subspace of X of finite co-dimension. Assume that there exist $\rho > 0$, $z \in X$ such that*

$$\forall\, u \text{ near } \overline{u}, \quad z + \rho B_H \subset \overline{\text{co}}\, G^{(1)}(u)$$

where B_H denote the closed unit ball in H. If $0 \in \text{Int} \liminf_{u\to\overline{u}} \overline{\text{co}}\, G^{(1)}(u)$, then for all $h > 0$, $G(\overline{u}) \in \text{Int}\, G(B_h(\overline{u}))$.

Proof Observe that $\rho B \subset \rho B_H + \rho B_{H^\perp}$. Thus for all u near $\overline{u}$, $z + \rho B \subset z + \rho B_H + \rho B_{H^\perp} \subset \overline{\text{co}}\, G^{(1)}(u) + \rho B_{H^\perp}$. Since $B_{H^\perp}$ is a compact set, Theorem 3.4 ends the proof. □

4 High order inverse of a single valued map

Let U be a complete metric space, X be a uniformly smooth Banach space (see [7]) and $G : U \to X$ be a continuous function. In this section we prove higher order sufficient conditions for regularity of the inverse map G^{-1}.

Theorem **4.1** **(High Order Inverse Mapping Theorem I)** *Let $\overline{u} \in U$ and assume that for some $k \geq 1$, $M > 0$, $\rho > 0$ and for all $u \in U$ near $\overline{u}$ and all small $h > 0$*

$$\rho B \subset \overline{\text{co}} \left(\frac{G(B_h(u)) - G(u)}{h^k} \cap MB + G^{(1)}(u) \right) \tag{25}$$

Then there exists $\mathcal{L} > 0$ such that for all $u \in U$ near $\overline{u}$, for all $x \in X$ near $G(\overline{u})$ and all $h > 0$ small enough

$$G(u) + \mathcal{L} h^k B \subset G(B_h(u)); \quad \text{dist}\,(u, G^{-1}(x)) \leq \frac{1}{\mathcal{L}^{\frac{1}{k}}} \|G(u) - x\|^{\frac{1}{k}}$$

Corollary 4.2 *Assume that X is a finite dimensional space and for some $k \geq 1$, the convex cone spanned by $G^k(\overline{u}, \overline{x})$ is equal to X. Then the conclusions of Theorem 4.1 hold true.*

Proof From Theorem 2.3 we deduce that for some $v_i \in G^k(\overline{u})$, $i = 1, ..., p$ we have $0 \in \text{Int } co\{v_1, ..., v_p\}$. Then for some $\epsilon > 0$ and for all $v_i' \in B_\epsilon(v_i)$ we have $0 \in \text{Int } co\{v_1', ..., v_p'\}$. On the other hand, by the definition of variation for every $1 \leq i \leq p$ there exists $\delta_i > 0$ such that

$$d(\overline{u}, u) + h \leq \delta_i \implies \text{dist}\left(v_i, \frac{G(B_h(u)) - G(u)}{h^k}\right) \leq \epsilon$$

Hence the assumption of Theorem 4.1 is satisfied. □

Proof (of Theorem 4.1) Assume for a moment that there exist $\overline{u}_i \to \overline{u}$, $\overline{h}_i \to 0+$, $x_i \in G(\overline{u}_i) + 2^{-k} i^{-k} \overline{h}_i^k B$ satisfying

$$x_i \notin G(B_{\overline{h}_i}(\overline{u}_i)) \tag{26}$$

Applying the Ekeland variational principle to the complete metric space $B_{\overline{h}_i}(\overline{u}_i)$ and the continuous function $u \to \|G(u) - x_i\|^{\frac{1}{k}}$ we prove the existence of $u_i \in B_{\frac{\overline{h}_i}{2}}(\overline{u}_i)$ such that $\|G(u_i) - x_i\|^{\frac{1}{k}} \leq \|G(\overline{u}_i) - x_i\|^{\frac{1}{k}}$ and for all $u \in B_{\overline{h}_i}(\overline{u}_i)$

$$\|G(u_i) - x_i\|^{\frac{1}{k}} \leq \|G(u) - x_i\|^{\frac{1}{k}} + \frac{1}{i} d(u, u_i) \tag{27}$$

By (26) and the smooth differentiability of the norm, there exist $p_i \in X^*$ of $\|p_i\| = 1$ and a function $o : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying $\lim_{h \to 0+} o(h)/h = 0$ such that for all $u \in B_{\overline{h}_i}(\overline{u}_i)$ we have

$$\|G(u) - x_i\| \leq \|G(u_i) - x_i\| + < p_i, G(u) - G(u_i) > + o(\|G(u) - G(u_i)\|)$$

Hence

$$\begin{cases} \|G(u) - x_i\|^{\frac{1}{k}} \leq (\|G(u_i) - x_i\| + < p_i, G(u) - G(u_i) > + o(\|G(u) - G(u_i)\|))^{\frac{1}{k}} = \\ \|G(u_i) - x_i\|^{\frac{1}{k}} \left(1 + < p_i, \frac{G(u) - G(u_i)}{\|G(u_i) - x_i\|} > + \frac{o(\|G(u) - G(u_i)\|)}{\|G(u_i) - x_i\|}\right)^{\frac{1}{k}} \leq \\ \|G(u_i) - x_i\|^{\frac{1}{k}} \left(1 + \frac{1}{k} < p_i, \frac{G(u) - G(u_i)}{\|G(u_i) - x_i\|} > + \overline{o}\left(\frac{\|G(u) - G(u_i)\|}{\|G(u_i) - x_i\|}\right)\right) \end{cases}$$

where $\overline{o} : \mathbf{R}_+ \to \mathbf{R}_+$ satisfies $\lim_{h \to 0+} \overline{o}(h)/h = 0$. This and (27) yield that for all $u \in B_{\overline{h}_i}(\overline{u}_i)$

$$\begin{cases} 0 \leq < p_i, G(u) - G(u_i) > + k \|G(u_i) - x_i\| \, \overline{o}\left(\frac{\|G(u) - G(u_i)\|}{\|G(u_i) - x_i\|}\right) \\ \quad + \frac{k}{i} \|G(u_i) - x_i\|^{\frac{k-1}{k}} d(u, u_i) \end{cases} \tag{28}$$

Fix $v \in G^{(1)}(u_i)$ and let $h_j \to 0+, v_j \to v$ be such that $G(u_i) + h_j v_j \in G(B_{h_j}(u_i))$. Then from (28) we obtain

$$0 \leq < p_i, h_j v_j > + o(h_j) + \frac{k}{i} \|G(u_i) - x_i\|^{\frac{k-1}{k}} h_j$$

Dividing by h_j and taking the limit yields

$$\forall\, v \in G^{(1)}(u_i), \quad < p_i,\, v > \; \geq \; -\frac{k}{i}\, \|G(u_i) - x_i\|^{\frac{k-1}{k}} \tag{29}$$

On the other hand, by (28), for $h_i = \|G(u_i) - x_i\|^{\frac{1}{k}} / i^{\frac{1}{2k}}$ and for all $v \in \frac{G(B_{h_i}(u_i)) - G(u_i)}{h_i^k}$

$$0 \;\leq\; < p_i,\, v > \; + k\sqrt{i}\, \overline{o}\left(\frac{\|v\|}{\sqrt{i}}\right) \; + \; k\, i^{-\frac{k+1}{2k}} \tag{30}$$

Adding (29) and (30) yields

$$\begin{cases} \forall\, v \in \frac{G(B_{h_i}(u_i)) - G(u_i)}{h_i^k} \cap MB + G^{(1)}(u_i) \\ < p_i, v > \; \geq -\frac{k}{i}\; \|G(u_i) - x_i\|^{\frac{k-1}{k}} - \sqrt{i}\,\overline{o}\left(\frac{1}{\sqrt{i}}\right) - k i^{-\frac{k+1}{2k}} \end{cases} \tag{31}$$

Observe that for all large i, $d(u_i, \overline{u}) \leq d(u_i, \overline{u}_i) + d(\overline{u}_i, \overline{u}) \leq \overline{h}_i/2 + d(\overline{u}_i, \overline{u}) < \epsilon$. Hence assumption (25) of theorem contradicts (31). This proves that (26) can not hold. The second statement of theorem follows from the first one and Theorem 1.2. □

Theorem 4.3 (High Order Inverse Function Theorem II) *Let $\overline{u} \in U$ and assume that (18) holds true for some $\epsilon > 0$ and a compact set $Q \subset X$. If for some $k \geq 1$, $0 \in Int\; \overline{co}\; G^k(\overline{u})$, then there exists $L > 0$ such that for all $u \in U$ near $\overline{u}$ and for all $x \in X$ near $G(\overline{u})$*

$$dist\, \left(u,\; G^{-1}(x)\right) \;\leq\; L\; \|G(u) - x\|^{\frac{1}{k}}$$

Proof — By Theorem 1.2 it is enough to show that for some $\rho > 0$ and all $u \in U$ near $\overline{u}$ and all small $h > 0$

$$G(u) \; + \; \rho h^k \; \mathring{B} \subset \; G(B_h(u))$$

If we assume the contrary, then, by the proof of Theorem 4.1 there exist $\overline{u}_i \to \overline{u}$, $u_i \to \overline{u}$, $x_i \to G(\overline{u})$, $h_i \to 0+$, $\overline{h}_i \to 0+$, $p_i \in X^\star$ of $\|p_i\|_{X^\star} = 1$ and a function $\overline{o} : \mathbf{R}_+ \to \mathbf{R}_+$ such that $B_{h_i}(u_i) \subset B_{\overline{h}_i}(\overline{u}_i)$, $x_i \neq G(u_i)$, $\lim_{h \to 0+} \overline{o}(h)/h = 0$ and

$$\begin{cases} \forall\, u \in B_{\overline{h}_i}(u_i), \;\; 0 \;\leq < p_i, G(u) - G(u_i) > \; + \\ \\ k\; \|G(u_i) - x_i\|\; \overline{o}\left(\frac{\|G(u) - G(u_i)\|}{\|G(u_i) - x_i\|}\right) \; + \; \frac{k}{i}\; \|G(u_i) - x_i\|^{\frac{k-1}{k}}\; d(u, u_i) \end{cases} \tag{32}$$

$$\forall\, w \in \frac{G(B_{h_i}(u_i)) - G(u_i)}{h_i^k}, \quad 0 \leq \frac{1}{k} < p_i, w > + \sqrt{i}\; \overline{o}\left(\frac{\|w\|}{\sqrt{i}}\right) + i^{-\frac{k+1}{2k}}$$

Fix $v \in G^k(\overline{u})$ and let $v_i \to v$ be such that $G(u_i) + h_i^k v_i \in G(B_{h_i}(u_i))$. Thus we deduce from the last inequality

$$0 \;\leq\; \frac{1}{k} \;\; < p_i, v_i > \; + \sqrt{i}\; \overline{o}\left(\frac{\|v_i\|}{\sqrt{i}}\right) \; + \; i^{-\frac{k+1}{2k}}$$

Let p be a weak-$\star$ cluster point of $\{p_i\}$ (it exists because X is reflexive). Then taking the limit in the above inequality we obtain $< p, v > \geq 0$ and since v is arbitrary

$$p \;\in\; \left(G^k(x_0, y_0)\right)^+ \;=\; \left(\overline{co}\; G^k(x_0, y_0)\right)^+ \;=\; \{\, 0 \,\}$$

We show next that p can not be equal to zero. Fix $v \in G^{(1)}(u_i)$ and let $h_j \to 0+$, $v_j \to v$ be such that $G(u_i)+h_j v_j \in G(B_{h_j}(u_i))$. Setting $G(u) = G(u_i)+h_j v_j$ in (32), dividing by h_j and taking the limit yield $0 \le \frac{1}{k} < p_i, v > +\frac{1}{i}\|G(u_i) - x_i\|^{\frac{k-1}{k}}$. Hence for a sequence $\epsilon_i \to 0+$ we have

$$\forall\, v \in \overline{co}\, G^{(1)}(u_i),\ \ < p_i, v > \ \ge\ -\epsilon_i$$

The end of the proof is similiar to that of Theorem 3.4. Let $z,\ \rho,\ w_i,\ a_i,\ q_i$ be as in the proof of Theorem 3.4. Then $< p_i,\ z - q_i > \ \ge\ < p_i,\ \rho w_i > \ + \ < p_i,\ a_i > \ \ge \rho\left(1-\frac{1}{i}\right) - \epsilon_i$. Consider subsequences $\{p_{i_j}\}$, $\{q_{i_j}\}$ such that $\{p_{i_j}\}$ converges weakly-* to p and $q_{i_j} \to q \in Q$. Then the last inequality implies that $< p, z - q > \ge \rho$ which yelds that p can not be equal to zero and completes the proof. □

5 Inverse of a set-valued map

Consider again a complete metric space (U,d) and a Banach space X. Let G be a set-valued map from U to X, whose graph is closed in $U \times X$. Consider a point $(\overline{u},\overline{x}) \in$ Graph G. We proved in [19] sufficient conditions for regularity of the inverse map

$$G^{-1}(x) \ := \ \{\, u \in U :\ x \in G(u)\, \}$$

on a neighborhood of $(\overline{x},\overline{u})$. In this case the results are more restrictive, we only state them (the corresponding proofs can be found in [19]).

Theorem 5.1 *Let $\overline{x} \in G(\overline{u})$ and assume that for some $\epsilon > 0$, $\rho > 0$*

$$(33)\qquad \rho B \ \subset \bigcap_{\substack{(u,x) \in Graph(G)\\ d(u,\overline{u}) \le \epsilon,\ \|x-\overline{x}\| \le \epsilon}} G^{(1)}(u,x)$$

Then for every $(u_1,x_1) \in Graph\ G \cap B_{\frac{\epsilon}{4}}(\overline{u}) \times B_{\frac{\epsilon}{4}}(\overline{x})$, $x_2 \in X$ satisfying $\|x_2 - x_1\| < min\{\ \frac{\epsilon}{8},\ \frac{\epsilon\rho}{4}\ \}$

$$(34)\qquad dist\ \left(u_1,\ G^{-1}(x_2)\right)\ \le\ \frac{1}{\rho}\ \|x_1 - x_2\|$$

Theorem 5.2 (A characterization of the image) *Assume that X is either separable or reflexive and let $\overline{x} \in G(\overline{u})$. Assume further that there exist closed convex subsets $K(u,x) \subset G^{(1)}(u,x)$, $\epsilon > 0$ and a compact set $Q \subset X$ such that*

$$(35)\qquad Int \bigcap_{\substack{(u,x) \in Graph\ G\\ (u,x) \in B_\epsilon(\overline{u}) \times B_\epsilon(\overline{x})}} (K(u,x)\ +\ Q)\ \ne\ \emptyset$$

Then at least one of the following two statements holds true:

i) There exist $L > 0$, $\delta > 0$ such that for all $(u_1,x_1,x_2) \in (Graph\ G \cap B_\delta(\overline{u}) \times B_\delta(\overline{x})) \times B_\delta(\overline{x})$

$$dist\ \left(u_1,\ G^{-1}(x_2)\right)\ \le\ L\,\|x_1\ -\ x_2\|$$

ii) There exists a non zero $p \in X^\star$ such that

$$(36)\qquad \forall\, w\ \in \liminf_{(u,x)\to_G(\overline{u},\overline{x})} K(u,x),\ \ < p, w > \ \ge 0$$

Consequently if for some $\delta > 0$, $\overline{x}$ is a boundary point of $G\left(B_\delta(\overline{u})\right)$, then there exists a non zero $p \in X^\star$ such that (36) is satisfied.

When the norm of X is differentiable, then a stronger result may be proved:

Theorem 5.3 *Assume that the norm of X is Gâteaux differentiable away from zero and let $\overline{x} \in G(\overline{u})$. If for some $\epsilon > 0$, $\rho > 0$, $M > 0$*

(37) $$\rho B \subset \bigcap_{\substack{(u,x) \in Graph\,(G) \\ (u,x) \in B_\epsilon(\overline{u}) \times B_\epsilon(\overline{x})}} \overline{co}\,(G^{(1)}(u,x) \cap MB)$$

then for every $(u_1, x_1) \in Graph\ G \cap B_{\frac{\epsilon}{4}}(\overline{u}) \times B_{\frac{\epsilon}{4}}(\overline{x})$, $x_2 \in X$ satisfying $\|x_2 - x_1\| < min\left\{\frac{\epsilon}{8}, \frac{\epsilon\rho}{4}\right\}$

$$\text{dist}\,(u_1,\ G^{-1}(x_2)) \ \le\ \frac{1}{\rho}\ \|x_1 - x_2\|$$

Theorem 5.4 *Assume that X is either separable or reflexive and that its norm is Gâteaux differentiable away from zero. Let $\overline{x} \in G(\overline{u})$. Further assume that there exist $\epsilon > 0$, $M > 0$ and a compact $Q \subset X$ such that*

(38) $$Int \bigcap_{\substack{(u,x) \in Graph(G) \\ (u,x) \in B_\epsilon(\overline{u}) \times B_\epsilon(\overline{x})}} \left(\overline{co}\,(G^{(1)}(u,x) \cap MB) + Q\right) \ne \emptyset$$

Then the following statements are equivalent

$$i) \quad \left(\liminf_{(u,x) \to_G (\overline{u},\overline{x})} \overline{co}\,(G^{(1)}(u,x) \cap MB) \ +\ G^1(\overline{u},\overline{x})\right)^+ \ =\ \{\,0\,\}$$

ii) for some $\delta > 0$, $L > 0$ and for all $(u_1, x_1, x_2) \in (Graph\ G \cap B_\delta(\overline{u}) \times B_\delta(\overline{x})) \times B_\delta(\overline{x})$

$$dist\ \left(u_1,\ G^{-1}(x_2)\right) \ \le\ L\,\|x_1\ -\ x_2\|$$

In particular, if for some $\delta > 0$, $\overline{x}$ is a boundary point of $G(B_\delta(\overline{u}))$, then there exists a non zero $p \in X^\star$ such that

(39) $$\forall\, w \in \liminf_{(u,x) \to_G (\overline{u},\overline{x})} \overline{co}\left(G^{(1)}(u,x) \cap MB\right) \ +\ G^1(\overline{u},\overline{x}), \quad < p,\ w > \ \ge 0$$

Observe that when X is a finite dimensional space, then the condition (38) is always satisfied with Q equal to the unit ball and $M = 1$. Hence

Corollary 5.5 *Assume that X is a finite dimensional space and for some $M > 0$*

$$0 \in \text{Int} \liminf_{(u,x) \to_G (\overline{u},\overline{x})} \overline{\text{co}}\ \left(G^{(1)}(u,x) \cap MB\right)$$

Then for all $h > 0$, $\overline{x} \in \text{Int}\ G(B_h(\overline{u}))$.

The high order results also have their analogs in the set-valued case.

Theorem 5.6 *Let $\overline{x} \in G(\overline{u})$ and assume that for some $k \geq 1$, $\rho > 0$, $M > 0$ and all $(u,x) \in$ Graph G near $(\overline{u}, \overline{x})$ and all small $h > 0$*

$$\rho B \subset \overline{co}\left(\frac{G(B_h(u)) - x}{h^k} \cap MB\right)$$

or equivalently

$$\inf_{\|p\|_{X^*}=1} \left\{< p, v > \mid v \in \frac{G(B_h(u)) - x}{h^k} \cap MB\right\} \geq \rho$$

If the space X is uniformly smooth, then there exists $L > 0$ such that for all $(u_1, x_1) \in$ Graph G near $(\overline{u}, \overline{x})$ and all $x_2 \in X$ near $\overline{x}$

$$dist\ \left(u_1,\ G^{-1}(x_2)\right) \leq L\,\|x_1 - x_2\|^{\frac{1}{k}}$$

Theorem 5.7 (High Order Inverse Function Theorem) *Let $\overline{x} \in G(\overline{u})$ and assume that (38) holds true for some $\epsilon > 0$, $M \geq 0$ and a compact set $Q \subset X$. If the space X is uniformly smooth and for some $k \geq 1$, $0 \in Int\ \overline{co}\ G^k(\overline{u}, \overline{x})$, then there exists $L > 0$ such that for all $(u_1, x_1) \in$ Graph G near $(\overline{u}, \overline{x})$ and for all $x_2 \in X$ near $\overline{x}$*

$$dist\ \left(u_1,\ G^{-1}(x_2)\right) \leq L\ \|x_1 - x_2\|^{\frac{1}{k}}$$

6 Applications

6.1 Taylor coefficients and inverse of a vector-valued function

Consider a function f from a Banach space X to a Hilbert space Y and a point $\overline{x} \in X$. We assume that $f \in C^k$ at $\overline{x}$ for some $k \geq 1$. Then for a neighborhood $\mathcal{N}$ of $\overline{x}$ and for all $x \in \mathcal{N}$ there exist i-linear forms $A_i(x)$, $i = 1, ..., k$ such that $A_i(\cdot)$ is continuous at $\overline{x}$ and

$$\lim_{h \to 0+} \sup_{\|w\| \leq 1} \left\| f(x + hw) - f(x) - \sum_{i=1}^{k} \frac{h^i}{i!} A_i(x) w ... w \right\| / h^k = 0$$

uniformly in $x \in \mathcal{N}$.

Theorem 6.1 (Second order invertibility condition) *Assume that $f \in C^2$ at $\overline{x}$, that $\mathrm{Im} f'(\overline{x})$ is a closed subspace of Y and for some $\alpha > 0$, $\epsilon > 0$ and all $x \in B_\epsilon(\overline{x})$ the following holds true*

$$(40) \qquad \left\{ \begin{array}{l} \forall\, y \in \mathrm{Im}\ f'(\overline{x})^{\perp} \text{ of } \|y\| = 1\ \exists\, w \in X \text{ of } \|w\| \leq 1 \text{ such that} \\ < y, f''(\overline{x})ww > \geq \alpha, \quad < y, f'(x)w > \geq \alpha \|f'(x)w\| \end{array} \right.$$

Then there exists $L > 0$ such that for all x near $\overline{x}$ and for all y near $f(\overline{x})$

$$\mathrm{dist}\ (x,\ f^{-1}(y)) \leq L\sqrt{\|f(x) - y\|}$$

Proof It is not restrictive to assume that $\alpha \leq 1$. Since $f \in C^2$ at $\overline{x}$, there exists a function $o : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{h \to 0+} o(h^2)/h^2 = 0$ and for all x near $\overline{x}$ and all $w \in X$ of $\|w\| \leq 1$ we have

$$\frac{f'(x)w}{h} + \frac{1}{2} f''(\overline{x})ww \subset \frac{f(B_h(x)) - f(x)}{h^2} + \left(\frac{o(h^2)}{h^2} + \|f''(x) - f''(\overline{x})\|\right) B$$

Hence, by the separation theorem and Theorem 4.1 it is enough to show that for some $M > 0,\ \rho > 0,\ \bar\epsilon > 0$ and all $y \in Y$ of $\|y\| = 1,\ x \in B_{\bar\epsilon}(\bar x),\ h > 0$

$$\sup\left\{ < y, e > \mid e \in \left\{ \frac{f'(x)w}{h} + \frac{1}{2} f''(\bar x)ww \mid \|w\| \le 1 \right\} \cap MB + f'(x)B \right\} \ge \rho$$

Set $H = \operatorname{Im} f'(\bar x)$, $M = 1 + \|f''(\bar x)\|$ and let $\gamma > 0$ be such that $\gamma B_H \subset f'(\bar x)B$. Then for some $0 < \bar\epsilon < \epsilon$ and all $x \in B_{\bar\epsilon}(\bar x)$

$$(41) \qquad \begin{cases} \forall\, y \in H \quad \text{of} \quad \|y\| = 1, & \sup_{\|w\| \le 1} < y, f'(x)w > \ \ge \frac{\gamma}{2} \\ \forall\, y \in H^\perp \quad \text{of} \quad \|y\| = 1, & \sup_{\|w\| \le 1} < y, f'(x)w > \ \le \frac{\alpha\gamma}{16M} \end{cases}$$

Set $\rho = \min\{\frac{\alpha\gamma}{16M}, \frac{\alpha}{8}\}$. Fix $x \in B_{\bar\epsilon}(\bar x),\ h > 0,\ y \in Y$ of $\|y\| = 1$ and let $y_1 \in H, y_2 \in H^\perp$ be such that $y = y_1 + y_2$. If $\|y_1\| \ge \frac{\alpha}{4M}$ then from (41) we obtain

$$\begin{aligned} \sup_{\|w\| \le 1} < y, f'(x)w > \ &\ge \sup_{\|w\| \le 1} < y_1, f'(x)w > - \sup_{\|w\| \le 1} < y_2, f'(x)w > \\ &\ge \tfrac{\gamma}{2} \|y_1\| - \tfrac{\alpha\gamma}{16M} \ \ge \ \rho \end{aligned}$$

If $\|y_1\| \le \frac{\alpha}{4M}$ then $\|y_2\| \ge 1 - \frac{\alpha}{4M}$ and, by (40), for some $w \in X$ of $\|w\| \le 1$

$$< y_2, f'(x)w > \ \ge \ \alpha \|y_2\| \ \|f'(x)w\|, \qquad < y_2, f''(\bar x)ww > \ \ge \ \alpha \|y_2\|$$

If $\left\|f'(x)\frac{w}{h}\right\| \le 1$ then $\left\|f'(x)\frac{w}{h} + \frac{1}{2}f''(\bar x)ww\right\| \le M$ and

$$< y, f'(x)\frac{w}{h} + \frac{1}{2}f''(\bar x)ww > \ge < y_2, \frac{1}{2}f''(\bar x)ww > - \|y_1\| M \ge \frac{\alpha}{2}\|y_2\| - \frac{\alpha}{4} \ \ge \ \frac{\alpha}{4} - \frac{\alpha^2}{8M} \ \ge \rho$$

If $\left\|f'(x)\frac{w}{h}\right\| \ge 1$ then setting $\overline{w} = \frac{hw}{\|f'(x)w\|}$ we obtain $\|\overline{w}\| \le 1$, $\left\|f'(x)\frac{\overline{w}}{h} + \frac{1}{2}f''(\bar x)\overline{w}\,\overline{w}\right\| \le M$ and

$$< y, f'(x)\frac{\overline{w}}{h} + \frac{1}{2}f''(\bar x)\overline{w}\,\overline{w} > \ge < y_2, f'(x)\frac{\overline{w}}{h} > - \|y_1\| M \ \ge \alpha \|y_2\| - \frac{\alpha}{4} \ge \alpha\left(\frac{3}{4} - \frac{\alpha}{4M}\right) \ge \rho$$

The proof is complete. □

Theorem 6.2 (A high order condition) *Assume that for some $\lambda > 0$ and for all x near $\bar x$ and $y \in Y$ of $\|y\| \le 1$ there exists $\lambda' \ge \lambda$ such that*

$$\lambda' y \in \overline{\text{co}}\,\{A_i(x)w...w \mid \|w\| \le 1,\ A_j(x)w...w = 0 \ \text{for} \ 1 \le j < i \le k\}$$

Then f^{-1} is pseudohölderian on a neighborhood of $(f(\bar x), \bar x)$ with the Holder exponent $\frac{1}{k}$.

Proof Observe that for all $x \in \mathcal{N},\ w \in X$ of $\|w\| \le 1$

$$\frac{\sum_{i=1}^k h^i A_i(x)w...w}{i!\, h^k} \subset \frac{f(B_h(x)) - f(x)}{h^k} + \frac{o(h^k)}{h^k}B$$

where $\lim_{h \to 0+} \frac{o(h^k)}{h^k} = 0$. Hence, by Theorem 4.1, it is enough to show that for some $M > 0,\ \rho > 0,\ \epsilon > 0$ and all $x \in B_\epsilon(\bar x)$, all small $h > 0$ and every $y \in Y$ of $\|y\| = 1$

$$\sup\left\{ < y, e > \ \mid \ e \in \left\{ \sum_{i=1}^k \frac{h^i A_i(x)w...w}{i!\, h^k} \mid \ \|w\| \le 1 \right\} \cap MB \right\} \ \ge \ \rho$$

Let $\epsilon > 0$ be such that $\max_{1\leq i\leq k} \sup_{x\in B_\epsilon(\overline{x})} \|A_i(x)w...w\| < \infty$ and such that the assumption of theorem is satisfied on $B_\epsilon(\overline{x})$. Fix $y \in Y$ of $\|y\| = 1$ and $x \in B_\epsilon(\overline{x})$, $0 < h \leq 1$ and let $w_x \in B$, $1 \leq s \leq k$ be such that $< y, A_s(x)w_x...w_x > \geq \lambda$ and for all $1 \leq j < s$, $A_j(x)w_x...w_x = 0$. Setting $w = h^{\frac{k}{s}-1}w_x$ we obtain $\|w_x\| \leq 1$ and

$$\sum_{i=1}^{k} \frac{h^i A_i(x)w...w}{i!\, h^k} = \frac{1}{s!}A_s(x)w_x...w_x + \sum_{i=s+1}^{k} \frac{h^{\frac{i-s}{s}k}}{i!}A_i(x)w_x...w_x$$

Then

$$< y, \sum_{i=1}^{k} \frac{h^i A_i(x)w...w}{i!\, h^k} > \geq \frac{\lambda}{k!} - \left\| \sum_{i=s+1}^{k} \frac{h^{\frac{i-s}{s}k}}{i!}A_i(x)w_x...w_x \right\|$$

Since the right-hand side of the above inequality converges to $\frac{\lambda}{k!}$ when $h \to 0+$ uniformly in $x \in B_\epsilon(\overline{x})$ we end the proof. □

6.2 Stability

Consider a Banach space X, finite dimensional spaces P, Y and continuously differentiable functions $g : P \times X \to Y$, $h : P \times X \to \mathbf{R}^n$. For all $p \in P$ define the set

(42) $$D_p = \{ x \in X \mid g(p,x) = 0, \;\; h(p,x) \leq 0 \}$$

Let $(\overline{p}, \overline{x})$ be such that

$$g(\overline{p},\overline{x}) = 0, \;\; h(\overline{p},\overline{x}) \leq 0$$

We study here the map $p \to D_p$ on a neighborhood of $(\overline{p}, \overline{x})$.

Theorem 6.3 (first order condition) *Assume that for some $\overline{w} \in X$*

$$\frac{\partial g}{\partial x}(\overline{p},\overline{x})\, \overline{w} = 0, \quad \frac{\partial h}{\partial x}(\overline{p},\overline{x})\, \overline{w} < 0$$

Then there exist $\epsilon > 0$, $L > 0$ such that for all p, $p' \in B_\epsilon(\overline{p})$, $x \in D_p \cap B_\epsilon(\overline{x})$

$$\text{dist}\,(x, D_{p'}) \leq L\,\|p - p'\|$$

Remark The above result is the well known Mangasarian and Fromowitz condition for stability (see [24]). It was also proved in [25] via an inverse mapping theorem involving the inverse of a closed convex process. The proof given below uses the variational inverse function theorem (Corollary 4.2).

Proof Define the set-valued map $G : P \times X \to P \times Y \times \mathbf{R}^n$ by

$$G(p,x) = \{ (\, p, \; g(p,x), \; h(p,x) + \rho) \mid \rho \in \mathbf{R}^n_+ \}$$

and set $\xi = (\overline{p},\overline{x},\overline{p},g(\overline{p},\overline{x}),h(\overline{p},\overline{x}))$. Then, by Example 1,

$$G^1(\xi) \supset \left\{ (v, \frac{\partial g}{\partial p}(\overline{p},\overline{x})v + \frac{\partial g}{\partial x}(\overline{p},\overline{x})w, \frac{\partial h}{\partial p}(\overline{p},\overline{x})v + \frac{\partial h}{\partial x}(\overline{p},\overline{x})w + \rho) \mid \; \|(v,w)\| \leq 1, \rho \in \mathbf{R}^n_+ \right\}$$

Let $\overline{w}$ be as in the assumptions of theorem. Without any loss of generality we may assume that $\|\overline{w}\| \leq 1$. Then $(0,0,\frac{\partial h}{\partial x}(\overline{p},\overline{x})\overline{w}) \in G^1(\xi)$. Hence for all $\mu \geq 0$ and all $(v,w) \in P \times X$

$$\{(v, \frac{\partial g}{\partial p}(\overline{p},\overline{x})v+\frac{\partial g}{\partial x}(\overline{p},\overline{x})w, \frac{\partial h}{\partial p}(\overline{p},\overline{x})v+\frac{\partial h}{\partial x}(\overline{p},\overline{x})(w+\mu\overline{w})+\rho) \mid \rho \in R^n_+\} \subset \bigcup_{\lambda\geq 0} \lambda \mathrm{co}\ G^1(\xi)$$

Fix $(v,y,z) \in P \times Y \times \mathbf{R}^n$. Since $\frac{\partial g}{\partial x}(\overline{p},\overline{x})$ is surjective, there exists $w \in X$ such that

$$y - \frac{\partial g}{\partial p}(\overline{p},\overline{x})v \;=\; \frac{\partial g}{\partial x}(\overline{p},\overline{x})w$$

On the other hand, since $\frac{\partial h}{\partial x}(\overline{p},\overline{x})\overline{w} < 0$ there exists $\mu > 0$ such that

$$z - \frac{\partial h}{\partial p}(\overline{p},\overline{x})v - \frac{\partial h}{\partial x}(\overline{p},\overline{x})w \;\in\; \mu\frac{\partial h}{\partial x}(\overline{p},\overline{x})\overline{w} \;+\; \mathbf{R}^n_+$$

Therefore $\bigcup_{\lambda\geq 0} \lambda co\ G^1(\xi) = P \times Y \times \mathbf{R}^n$ and we may apply Corollary 4.2. Thus there exists $L > 0$ such that for all (p,x) near $(\overline{p},\overline{x})$ and all z near $(p, g(p,x), h(p,x))$

$$\text{dist }\left((p,x), G^{-1}(z)\right) \leq L\ \|(p, g(p,x), h(p,x)) - z\|$$

Fix (p,x) sufficiently close to $(\overline{p},\overline{x})$ with $x \in D_p$, p' sufficiently close to $\overline{p}$ and $z = (p', 0, h(p,x))$ and let $(p'',x') \in G^{-1}(z)$ be such that $\|(p,x) - (p'',x')\| \leq L\|p - p'\|$ Then $z \in G(p'',x') = (p'', g(p'',x'), h(p'',x')) + \mathbf{R}^n_+$. This yields that $p'' = p'$, $g(p',x') = 0$ and $h(p,x) \in h(p',x') + \mathbf{R}^n_+$. Therefore $h(p',x') \leq h(p,x) \leq 0$. Consequently $x' \in D_{p'}$ and

$$\|x - x'\| \leq \|(p,x) - (p',x')\| \leq L\|p - p'\|$$

Second order sufficient conditions require a more fine analysis. Theorem 6.1 can be applied to the case when equality constraints only are present. We state next a result for inequality constraints.

Theorem 6.4 (Second order condition) *Assume that* $g \equiv 0$, $h \in C^2$ *at* $(\overline{p},\overline{x})$ *and let* H *be the largest subspace contained in* $\mathrm{Im}\ \frac{\partial h}{\partial x}(\overline{p},\overline{x}) + \mathbf{R}^n_+$. *If there exists* $\alpha > 0$ *such that for all* (p,x) *near* $(\overline{p},\overline{x})$

$$\begin{cases} \forall\, y \in H^\perp \text{ of } \|y\| = 1,\ \exists\, w \in X,\ \|w\| \leq 1,\ \rho_i \in \mathbf{R}^n_+,\ i = 1,2,\ \|\rho_i\| \leq 1, \text{ such that} \\ < y, \frac{\partial^2 h}{\partial x^2}(\overline{p},\overline{x})ww + \rho_1 > \geq \alpha, \quad < y, \frac{\partial h}{\partial x}(p,x)w + \rho_2 > \geq \alpha\left\|\frac{\partial h}{\partial x}(p,x)w\right\| \end{cases}$$

then there exist $\epsilon > 0$, $L > 0$ *such that for all* $p, p' \in B_\epsilon(\overline{p})$, $x \in D_p \cap B_\epsilon(\overline{x})$

$$\text{dist }(x,\ D_{p'}) \;\leq\; L\sqrt{\|p - p'\|}$$

Proof Consider the set-valued map $G : P \times X \to P \times \mathbf{R}^n$ defined by

$$G(p,x) \;=\; \{\ (p,\ h(p,x) + \rho) \mid\ \rho \in \mathbf{R}^n_+\ \}$$

Then for all $e \geq h(p,x)$ and for all (p,x) near $(\overline{p},\overline{x})$, $t > 0$

$$\begin{array}{l}\left\{(v, \frac{\partial h}{\partial p}(p,x)v + \frac{\partial h}{\partial x}(p,x)\frac{w}{t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(\overline{p},\overline{x})ww + \rho) \mid \|(v,w)\| \leq 1, \rho \in \mathbf{R}^n_+\right\} \\ \quad\subset\ \frac{G((p,x)+tB)-(p,e)}{t^2} + \left(\frac{o(t^2)}{t^2} + \left\|\frac{\partial^2 h}{\partial x^2}(\overline{p},\overline{x}) - \frac{\partial^2 h}{\partial x^2}(p,x)\right\|\right) B\end{array}$$

where $o(t^2)/t^2 \to 0$ when $t \to 0+$ and

$$\left(0,\ \frac{\partial h}{\partial x}(p,x)B \ + \ \mathbf{R}^n_+\right) \ \subset \ G^{(1)}(p,x,p,e)$$

Let $\delta > 0$ be such that

$$M_1 \ := \ \sup_{(p,x)\in B_\delta(\overline{p},\overline{x})} \left\|\frac{\partial h}{\partial p}(p,x)\right\| \ < \ \infty$$

Fix $(a,q) \in P \times Y$ of $\|(a,q)\| = 1$. If $\|q\| \le \frac{1}{2(M_1+1)}$ then $\|a\| \ge 1 - \frac{1}{2(M_1+1)}$ and

$$\sup_{\|v\|\le 1} \left\langle (a,q), (v, \frac{\partial h}{\partial p}(p,x)v)\right\rangle \ \ge \ \|a\| \ - \ \|q\|\, M_1 \ \ge \ 1 \ - \ \frac{1+M_1}{2(M_1+1)} \ = \ \frac{1}{2}$$

By Theorem 5.7 it remains to show the existence of $\gamma > 0$, $M > 0$ such that for all $y \in Y$ of $\|y\| = 1$ and all (p,x) near $(\overline{p},\overline{x})$ and all small $t > 0$

$$\sup\Big\{< y, e > \mid e \in \Big\{\tfrac{\partial h}{\partial x}(p,x)\tfrac{w}{t} + \tfrac{1}{2}\tfrac{\partial^2 h}{\partial x^2}(\overline{p},\overline{x})ww + \rho \mid \|w\| \le 1, \rho \in \mathbf{R}^n_+\Big\} \cap MB$$
$$+ \left(\tfrac{\partial h}{\partial x}(p,x)B \ + \ \mathbf{R}^n_+\right) \cap MB\Big\} \ge \ \gamma$$

But this follows from the assumptions by the arguments similar to the proof of Theorem 6.1. □

6.3 Interior points of reachable sets of a control system

We consider the control system described in Example 2 and we impose the same assumptions on f, X, S. For all $T > 0$ denote by $R(T)$ the reachable set of (5) at time T, i.e.,

$$R(T) \ = \ \{x(T) \mid x \text{ is a mild trajectory of } (5)\}$$

Let z be a mild trajectory of (5) on $[0,T]$ and $\overline{u}$ be the corresponding control. We provide here a sufficient condition for $z(T) \in \text{Int } R(T)$ and study how much we have to change controls in order to get in neighboring points of $z(T)$.

Consider the linear control system

$$\text{(43)} \qquad \begin{cases} w'(t) \ = \ Aw(t) \ + \ \frac{\partial f}{\partial x}(z(t),\overline{u}(t))w(t) \ + \ v(t) \\ w(0) = 0; \ \ v(t) \in \overline{\text{co}}\, f(z(t),U) - f(z(t),\overline{u}(t)) \end{cases}$$

and let $R^L_{\overline{u}}(T)$ denote its reachable set by the mild trajectories at time T.

Theorem 6.5 *Under the above assumptions assume that for all $x \in X$ the set $\overline{f(x,U)}$ is bounded and for all $u \in U$, $t \in [0,T]$, $\frac{\partial f}{\partial x}(\cdot,u)$ is continuous at $z(t)$. If $0 \in \text{Int } \overline{R^L_{\overline{u}}(T)}$ then $z(T) \in \text{Int } R(T)$ and there exist $\epsilon > 0$, $L > 0$ such that for every control $u \in \mathcal{U}$ satisfying $d(u,\overline{u}) \le \epsilon$ and all $b \in B_\epsilon(z(T))$ there exists a trajectory-control pair (x_v, v) which verifies*

$$x_v(T) \ = \ b; \quad \mu(\{t \in [0,T] \mid u(t) \ne v(t)\}) \ \le \ L\ \|b - x_u(T)\|$$

In particular for every $b \in B_\epsilon(z(T))$ there exists a control $u \in \mathcal{U}$ such that

$$x_u(T) \ = \ b; \quad \mu(\{t \in [0,T] \mid u(t) \ne \overline{u}(t)\}) \ \le \ L\,\|b - z(T)\|$$

The above result was proved in [18] therefore we only sketch the idea of the proof. From (7) we deduce that for almost all $t \in [0,T]$

$$S_{\overline{u}}(T;t)\ \left(\overline{co}\ f(z(t),U) - f(z(t),\overline{u}(t))\right)\ \subset\ \overline{co}\ G^{(1)}(\overline{u})$$

and for the same reasons for all u near $\overline{u}$ and for almost all $t \in [0,T]$

$$S_u(T;t)\left(\overline{co}f(x_u(t),U) - f(x_u(t),u(t))\right)\ \subset\ \overline{co}\ G^{(1)}(u) \tag{44}$$

On the other hand

$$R_u^L(T) = \{\int_0^T S_u(T;t)v(t)dt : v(t) \in \overline{co}f(x_u(t),U) - f(x_u(t),u(t)) \text{ is measurable}\}$$

and therefore integrating (44) we obtain $R_u^L(T) \subset T\ \overline{co}\ G^{(1)}(u)$. Hence

$$\frac{1}{T}\ \overline{R_u^L(T)}\ \subset\ \overline{co}\ G^{(1)}(u)$$

Since $0 \in \text{Int}\ \overline{R_{\overline{u}}^L(T)}$ and $\overline{R_u^L(T)}$ is a closed convex set, the separation theorem and regularity of the data imply that for some $\delta > 0$

$$0\ \in\ \text{Int} \bigcap_{d(u,\overline{u})\leq\delta} \overline{R_u^L(T)}$$

This allow to apply Theorem 3.2 (see [18] for details of the proof). □

6.4 Local controllability of a differential inclusion

We consider the dynamical system described by a differential inclusion from the Example 3 and we assume H_1) and H_2). Let $T > 0$ be a given time. We study here sufficient conditions for $\xi \in \text{Int}\ R(T)$ and the regularity of the "inverse". Consider the following linearized inclusion

$$w' \in CF(\xi,0)w\ +\ co\ F(\xi);\ \ w(0)\ =\ 0 \tag{45}$$

and let $R^L(T)$ denote its reachable set at time T.

Theorem 6.6 *If $0 \in \text{Int}\ R^L(T)$ then $\xi \in \text{Int}\ R(T)$ and there exists a constant $L > 0$ such that for every $x \in S_{[0,T]}(\xi)$ sufficiently close to the constant trajectory ξ (in $W^{1,1}(0,T)$) and for all $b \in X$ near ξ there exists $y \in S_{[0,T]}(\xi)$, satisfying*

$$y(T)\ =\ b;\ \ \ \|y - x\|_{W^{1,1}(0,T)}\ \leq\ L\,\|b - x(T)\|$$

In particular this implies that for all b near ξ there exists $x \in S_{[0,T]}(\xi)$ with

$$x(T)\ =\ b;\ \ \ \|x - \xi\|_{W^{1,1}(0,T)}\ \leq L\ \|b - \xi\|$$

Proof The map G defined in Example 3 is continuous on its domain of definition. It was proved in [15] that if $0 \in \text{Int}\ R^L(T)$ then there exists a compact convex set $K \subset coF(\xi)$ having only finite number of extremal points such that the reachable set $R_K^L(T)$ at time T of the differential inclusion (9) satisfies $0 \in \text{Int}\ R_K^L(T)$. From the Example 3 we also know that for some $M > 0$

$$\frac{1}{M}\ R_K^L(T)\ \subset\ G^1(\xi)$$

Applying Corollary 4.2 we end the proof.

6.5 Small time local controllability of differential inclusions

Consider again dynamical system described in Example 3 and satisfying H_1), H_2). We study here sufficient conditions for

$$\forall\, T > 0, \quad \xi \in \text{Int } R(T)$$

Set

$$V = \{ v \in X \mid \exists\, k > 0 \text{ such that } \forall\, t \geq 0,\ \xi + t^k v \in R(t) + o(t^k) \}$$

Theorem 6.7 *Under the above assumptions assume that for every $x \in X$, $F(x)$ is a convex set. If the convex cone spanned by V is equal to X, then for all $T > 0$, $\xi \in \text{Int } R(T)$.*

The above result was proved in [17]. It follows from Theorem 5.7, (10) and from the existence of $\epsilon > 0$ such that Graph$R \cap B_\epsilon(0, \xi)$ is a closed set.

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