DIRECTIONAL LIPSCHITZIAN OPTIMAL SOLUTIONS AND DIRECTIONAL DERIVATIVE FOR THE OPTIMAL VALUE FUNCTION IN NONLINEAR MATHEMATICAL PROGRAMMING

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This paper gives very accurate conditions to have Lipschitz behaviour for the optimal solutions of a mathematical programming problem with natural perturbations in some fixed direction. This result is then used to obtain the directional derivative for the optimal value function.

Key words: Mathematical programming, Lipschitzian solutions, optimal value function, directional derivative.

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1. INTRODUCTION

We consider in this paper a nonlinear mathematical programming problem with natural perturbations

$$P(tu):\ \text{s.t.}\quad \begin{array}{l} \min f_0(x)\ ,\ x \in R^n \\ f_i(x) \le t\, u_i\ ,\ i \in I \\ f_i(x) = t\, u_i\ ,\ i \in J \end{array}$$

where $t \ge 0$, I,J are finite sets of indices and $u \in R^{I \cup J}$ is a fixed direction for the perturbations. We are looking for the minimal assumptions to have the Lipschitz continuity of any local optimal solution $x(tu)$ of program $P(tu)$ near some optimal solution x^* of program $P(0)$.

The Lipschitz behaviour of the optimal solutions in parametric optimization have been studied by many authors. We cite the works of Aubin [1], Cornet-Vial [2] and Robinson [6] where this property has been obtained under regularity conditions related somehow with the Mangasarian-Fromovitz regularity condition. This regularity conditions restricts the program $P(0)$ to be defined in the interior of the domain of feasible perturbations; i.e., the perturbations v where the programs $P(v)$ have feasible solutions. Theorem 4.3 in Gauvin-Janin [3] is a tentative to have this Lipschitz property under more general regularity conditions. Our purpose in this paper is to give a more refined version of that result. An example is given which shows that we may have obtained the most accurate statement for a Lipschitz directional continuity property for the optimal solutions in mathematical programming. Finally the result is used to obtain a nice and simple formula for the directional derivative of the optimal value function of the mathematical programming problem. This last result comes to complete and to refine Theorem 3.6 in Gauvin-Tolle [4] where this formula was obtained with the assumption, among others, that a Lipschitz continuity property was satisfied for the optimal solutions.

It should be noticed that program $P(tu)$ as formulated contains the case where the feasible solutions are restrained to remain in some convex polyhedron because such set can always be defined by linear inequalities or equalities which can be included in the above formulation. Also a mathematical program with general nonlinear perturbations can be translated to the formulation with linear right-hand side perturbations only as it has been once shown by R.T. Rockafellar (see the Introduction in Gauvin-Janin [3]).

2. ASSUMPTIONS AND PRELIMINARIES

For an optimal solution x^* of $P(0)$, we denote by $\Omega(x^*)$ the set of Lagrange multipliers; i.e., the multipliers (λ_0,λ), $\lambda \in R^{I\cup J}$, such that

$$\lambda_0 \nabla f_0(x^*) + \sum_{i \varepsilon I \cup J} \lambda_i \nabla f_i(x^*) = 0$$

$$\lambda_0, \lambda_i \geq 0 \quad , \; i \varepsilon I$$

$$\lambda_i f_i(x^*) = 0 \; , \; i \varepsilon I.$$

The corresponding set of Kuhn-Tucker multipliers is denoted by

$$\Omega_1(x^*) = \{ \lambda \varepsilon R^{I \cup J} | \; (1,\lambda) \varepsilon \Omega(x^*) \}$$

with its recession cone

$$\Omega_0(x^*) = \{ \lambda \varepsilon R^{I \cup J} | \; (0,\lambda) \varepsilon \Omega(x^*) \}.$$

The first assumption is a very general regularity condition.

H_1: The set $\Omega_1(x^*)$ is nonempty.

The second assumption is a condition on the choice of the direction of perturbations.

H_2: The direction u satisfies

$$\lambda^T u > 0 \; , \; \forall \lambda \varepsilon \Omega_0(x^*) \; , \; \lambda \neq 0.$$

This assumption implies that the family $\{ \nabla f_i(x^*) \; | \; i \varepsilon J \}$ related with the <u>equality</u> constraints is linearly independent (see (3) of Remark 3.1 in Gauvin-Janin [3]).

Both assumptions implies that the set $\Omega_1(x^*,u)$ of optimal solutions for the linear program

$$\max \{ -\lambda^T u \; | \; \lambda \varepsilon \Omega_1(x^*) \}$$

is nonempty and bounded even if $\Omega_1(x^*)$ is unbounded (see

(4) of Remark 3.1 in Gauvin-Janin [3]). By duality, this implies that the linear program $LP(x^*,u)$:

$$\min \nabla f_0(x^*)y \ , \ y \in R^n$$

subject to

$$\nabla f_i(x^*)y \leq u_i \ , \ i \in I(x^*) = \{ \ i \in I \mid f_i(x^*) = 0 \ \}$$
$$\nabla f_i(x^*)y = u_i \ , \ i \in J$$

is feasible and bounded. We denote by $Y(x^*,u)$ the set of optimal solutions for that program, by

$$I(x^*,u) = \{ \ i \in I(x^*) \mid \exists \ y \in Y(x^*,u) \ \text{ such that } \nabla f_i(x^*)y = u_i \ \}$$

the set of indices corresponding to possible binding inequality constraints for some optimal solution and by

$$I^*(x^*,u) = \{ \ i \in I(x^*,u) \mid \sup \{ \ \lambda_i \mid \lambda \in \Omega_1(x^*,u) \ \} > 0 \ \}$$

the subset of indices corresponding to nonnul optimal multipliers. Finally we denote by

$$E = \{ \ y \in R^n \mid \nabla f_i(x^*)y = 0 \ , \ i \in I(x^*,u) \cup J \ \}$$

the tangent subspace at x^* to the inequality constraints related with this last set of indices together with the equality constraints.

If we let $\nabla^2 L(x,\lambda)$ be the Hessian of the Lagrangian

$$L(x,\lambda) = f_0(x) + \sum_{i \in I \cup J} \lambda_i \ f_i(x),$$

the third and last assumption is a weak second-order sufficient optimality condition related with the above tangent subspace.

H_3: For any $y \in E$, $y \neq 0$, there exists $\lambda^* \in \Omega_1(x^*,u)$ such that

$$y^T \nabla^2 L(x^*,\lambda^*) y > 0.$$

This condition is weak in the sense that it does not need to hold for all $\lambda^* \in \Omega_1\ (x^*,u)$ or all $\lambda \in \Omega_1(x^*)$.

By arguments similar of those in Theorem 2.2 in Gauvin-Janin [3], assumption H_3 implies that x^* is a strict local optimum for the enlarged program

$$\min f_0(x)\ ,\ x \in R^n$$
$$\text{s.t.}$$
$$f_i(x) \le 0\ ,\ i \in I^*(x^*,u)$$
$$f_i(x) = 0\ ,\ i \in J.$$

Since x^* is assumed to be an optimum of the original program $P(0)$, we must have that x^* is also a strict optimum of $P(0)$. Therefore we have a neighborhood $X(x^*)$ of x^* where $f_0(x^*) < f_0(x)$ for any feasible point x of $P(0)$.

By a local optimal solution of $P(tu)$ near x^* we mean any optimal solution $x(tu)$ of the restricted program

$$P(tu|x^*) : \min \{ f_0(x) \mid x \in R(tu) \cap X(x^*) \}$$

where $R(tu)$ is the set of feasible solutions of $P(tu)$. Under assumptions H_1 and H_2, the proof of Theorem 3.2 in Gauvin-Janin [3] shows that it is possible to construct a feasible are $x(t) \in R(tu)$, $t \in [0,t_0[$, for some $t_0 > 0$, such that $\lim_{t \downarrow 0} x(t) = x^*$. Since $f_0(x(tu)) \leq f_0(x(t))$, we have, for any cluster point x^{**} of $x(tu)$ as $t \downarrow 0$,

$$f_0(x^{**}) \leq \limsup_{t \downarrow 0} f_0(x(tu)) \leq \lim_{t \downarrow 0} f_0(x(t)) = f_0(x^*).$$

But since x^* is the unique optimum for the restricted program $P(0|x^*)$, we must necessarily have $x^{**} = x^*$ and consequently

$$\lim_{t \downarrow 0} x(tu) = x^*.$$

It should be noticed from above that the assumptions H_1 and H_2 implies that the program $P(tu)$, $t \in [0, t_0[$, is feasible even if $v = 0$ is at the boundary of the domain of feasible perturbations

$$\text{dom } R = \{ v \mid R(v) \neq \phi \}.$$

In that case the condition on the direction u in H_2 implies that u must be pointing toward the interior of dom R. Example (3.1) in Gauvin-Janin [3] illustrates that situation.

3. LIPSCHITZ DIRECTIONAL CONTINUITY FOR THE OPTIMAL SOLUTIONS

The next result on the Lipschitz directional continuity for the optimal solution is a generalization and a refinement of Theorem 4.3 in Gauvin-Janin [3].

Theorem 1

Let x^* be an optimal solution of program $P(0)$ with assumptions H_1, H_2 and H_3 satisfied. Then, for any local optimal solution $x(tu)$ of $P(tu)$ near x^*, we have

$$\limsup_{t \downarrow 0} |x(tu)-x^*|/t < +\infty.$$

Proof.

As previously noticed, assumption H_3 implies the existence of a neighborhood $X(x^*)$ where x^* is the unique optimum of the restricted program $P(0|x^*)$; and by assumption H_1 and H_2, we also have

$$\lim_{t \downarrow 0} x(tu) = x^*$$

for any optimal solution $x(tu)$ of the restricted program $P(tu|x^*)$ which is then feasible for some nontrivial interval $[0,t_0[$.

Now let suppose the conclusion is false and let take any sequence $\{t_n\}$, $t_n \downarrow 0$, such that

$$\lim_{n\to\infty} |x(t_n u)-x^*|/t_n = +\infty,$$

$$\lim_{n\to\infty} (x(t_n u)-x^*)/|x(t_n u)-x^*| = y$$

for some cluster point y of the bounded set

$$\{(x(tu)-x^*)/|x(tu)-x^*|\}.$$

From Lemma 3.1 and Theorem 3.2 in Gauvin-Janin [3], we have, for n large enough, for some δ and for any $\lambda \in \Omega_1(x^*,u)$,

$$-t_n \lambda^T u + \frac{1}{2} (x(t_n u)-x^*)^T \nabla^2 L(x^*,\lambda)(x(t_n u)-x^*)$$
$$+ |x(t_n u)-x^*|^2 \theta_\lambda (x(t_n u)$$
$$\le f_0(x(t_n u)) - f_0(x^*)$$
$$\le -t_n \lambda^T u + \delta t_n^2 \qquad (1)$$

where $\lim_{x \to x^*} \theta_\lambda (x) = 0$. We also have

$$f_i(x(t_n u)) - f_i(x^*) \begin{cases} \le t_n u_i , & i \varepsilon I^*(x^*,u) \\ = t_n u_i , & i \varepsilon J. \end{cases}$$

Therefore we have, for some β and some $s_i^n \varepsilon [0,1]$,

$$\nabla f_0(x(s_0^n t_n u))(x(t_n u)-x^*) \le \beta t_n$$

$$\nabla f_0(x(s_i^n t_n u))(x(t_n u)-x^*) \begin{cases} \le t_n u_i , & i \varepsilon I^*(x^*,u) \\ = t_n u_i , & i \varepsilon J. \end{cases}$$

Since we have assumed that

$$\lim_{n\to\infty} t_n / |x(t_n u)-x^*| = 0,$$

we can divided all above inequalities and equalities by $|x(t_n u)-x^*|$ and take the limits to obtain

$$\nabla f_0(x^*) y \le 0$$

$$\nabla f_i(x^*) y \begin{cases} \le 0 , & i \varepsilon I^*(x^*,u) \\ = 0 , & i \varepsilon J. \end{cases}$$

But it exists a $\lambda \varepsilon \Omega_1(x^*,u)$ with $\lambda_i > 0$, $i \varepsilon I^*(x^*,u)$, such that

$$-\nabla f_0(x^*) = \sum_{i\varepsilon K} \lambda_i \nabla f_i(x^*)$$

where $K = I^*(x^*,u) \cup J$. From above, we then have

$$0 \leq -\nabla f_0(x^*)y = \sum_{i\varepsilon I^*(x^*,u)} \lambda_i \nabla f_i(x^*)y \leq 0$$

which implies that

$$\nabla f_i(x^*)y = 0 \ , \ i \ \varepsilon \ I^*(x^*,u);$$

therefore $y \ \varepsilon \ E$.

From H_3, it exists $\lambda^* \ \varepsilon \ \Omega_1(x^*,u)$ and $\alpha > 0$ such that

$$y^T \nabla^2 L(x^*,\lambda^*)y = 2\alpha.$$

For n large enough, we can write

$$(x(t_n u)-x^*)^T \nabla^2 L(x^*,\lambda^*)(x(t_n u)-x^*) > \alpha|x(t_n u)-x^*|^2.$$

For n large enough we also have $\theta_\lambda(x(t_n u)) \geq -\alpha/4$ in the left-hand side of inequality (1). The two previous inequalities put together in the left-hand side of (1) reduce that inequality to

$$|x(t_n u)-x^*|^2/t_n^2 \leq 4\delta/\alpha < +\infty$$

which is in contradiction with what we have assumed at the beginning.

□

The following example shows that Theorem 1 is perhaps the most accurate statement for a Lipschitz directional continuity property for the optimal solutions in mathematical programming.

Example 1

$$\min f_0 = -x_2$$

s.t.

$$f_1 = -x_1^2 + x_2 \le tu_1$$

$$f_2 = x_1^2 + x_2 \le tu_2$$

For $t = 0$, the optimum is $x_1^* = x_2^* = 0$ where $I(x^*) = \{1,2\}$. The multipliers are

$$\Omega_1(x^*) = \{\lambda \ge 0 \mid \lambda_1+\lambda_2 = 1\}$$

with in this case $\Omega_0(x^*) = \{ 0 \}$; therefore assumption H_2 is satisfied for any direction $u = (u_1,u_2)$. The set of optimal multipliers is

$$\Omega_1(x^*,u) = \begin{cases} \Omega_1(x^*) & \text{if } u_1-u_2 = 0 \\ \{(0,1)\} & \text{if } u_1-u_2 > 0 \\ \{(1,0)\} & \text{if } u_1-u_2 < 0. \end{cases}$$

The linear program $LP(x^*,u)$ becomes

$$\min - y_2$$

s.t.

$$y_2 \le u_1$$

$$y_2 \le u_2$$

for which we have

$$I^*(x^*,u) = I(x^*,u) = \begin{cases} \{1,2\} & \text{if } u_1-u_2 = 0 \\ \{2\} & \text{if } u_1-u_2 > 0 \\ \{1\} & \text{if } u_1-u_2 < 0 \end{cases}$$

The corresponding tangent subspace is

$$E = \{ (y_1,y_2) \mid y_2 = 0 \}.$$

On this subspace, the Hessian of the Lagrangian has value

$$y^T \nabla^2 L(x^*,\lambda)y = -2(\lambda_1-\lambda_2)y_1^2.$$

For the case where $u_1-u_2 \geq 0$, we have $\lambda^* = (0,1) \in \Omega_1(x^*,u)$, therefore H_3 is satisfied with

$$y^T \nabla^2 L(x^*,\lambda^*)y = 2y_1^2 > 0 \ , \ \forall y_1 \neq 0.$$

In that case, the optimal solution of $P(tu)$ is

$$x(tu) = \{ (0,tu_2) \}$$

for which the Lipschitz continuity holds.

For the case where $u_1-u_2 < 0$, we only have $\lambda^* = (1,0)$ in $\Omega_1(x^*,u)$; therefore H_3 is not satisfied since

$$y^T \nabla^2 L(x^*,\lambda^*)y = -2y_1^2 < 0 \ , \ \forall y_1 \neq 0.$$

The optimal solutions of $P(tu)$ are in that case

$$x(tu) = \{ (\pm\sqrt{(t(u_2-u_1)/2} \ , \ t(u_1+u_2)/2) \}$$

and the Lipschitz continuity does not hold.

4. DIRECTIONAL DERIVATIVE FOR THE OPTIMAL VALUE FUNCTION

We now consider the optimal value function of program $P(tu)$:

$$p(tu) = \inf \{ f_0(x) \mid x \in R(tu) \}$$

where, as previously, $R(tu)$ is the set of feasible solutions. We need also to consider the local optimal value function for the restricted program $P(tu|x^*)$:

$$p(tu|x^*) = \inf \{ f_0(x) \mid x \in R(tu) \cap X(x^*) \}$$

where $X(x^*)$ is some neighborhood of an optimal solution x^* of $P(0)$.

The next result is a more refined and a more accurate statement for the result of Theorem 3.6 in Gauvin-Tolle [4].

<u>Theorem 2</u>

Let x^* and $x(tu)$ be optimal solutions respectively of program $P(0)$ and $P(tu)$ with the assumptions

(1) $\Omega_1(x^*) \neq \phi$ and u satisfied

$$\lambda^T u > 0 , \forall \lambda \in \Omega_0(x^*) , \lambda \neq 0$$

(2) $\limsup_{t \downarrow 0} |x(tu) - x^*|/t < +\infty$.

Then the optimal value function has a directional derivative given by

$$p'(0;u) = \lim_{t \downarrow 0} (p(tu) - p(0))/t = \max_{\lambda \in \Omega_1(x^*)} \{ -\lambda^T u \}$$

Proof.

When t is small enough, we have

$$|x(tu)-x^*|/t \le k.$$

for some k. For any $\lambda \in \Omega_1(x^*)$, we have

$$\begin{aligned} p(tu)-p(0) &= f_0(x(tu)) - f_0(x^*) \\ &\ge L(x(tu),\lambda) - L(x^*,\lambda) - t\lambda^T u \\ &= -t\lambda^T u + \nabla L(x^*,\lambda)(x(tu)-x^*) \\ &+ \frac{1}{2}(x(tu)-x^*)^T \nabla^2 L(x_u(t),\lambda)(x(tu)-x^*) \end{aligned}$$

for some $x_u(t) \in [x^*, x(tu)]$. Since $\nabla L(x^*,\lambda) = 0$ and

$$\begin{aligned} \lim_{t\downarrow 0} |x(tu)-x^*|/\sqrt{t} &= \lim_{t\downarrow 0} (|x(tu)-x^*|/t)(\sqrt{t}) \\ &\le k \lim_{t\downarrow 0} (\sqrt{t}) = 0, \end{aligned}$$

we necessarily have

$$\liminf_{t\downarrow 0} (p(tu)-p(0))/t \ge \max_{\lambda\in\Omega_1(x^*)} \{ -\lambda^T u \}.$$

This maximum is attained under assumption (1) as previously noticed.

On the other hand, we have, by Theorem 3.2 in Gollan [5] or Theorem 3.2 in Gauvin-Janin [3],

$$\limsup_{t\downarrow 0}(p(tu)-p(0))/t \le \max_{\lambda\in\Omega_1(x^*)} \{ -\lambda^T u\}.$$

The result follows from both inequalities.

□

To complete finally the result of Theorem 3.6 in Gauvin-Tolle [4], we can state the following theorem. The family $R(tu)$, $t \in [0,t_0[$, is said to be uniformly compact if the closure of $\bigcup_{t\in[0,t_0[} R(tu)$ is compact (see also the inf-boundedness condition in Rockafellar [7]). Let $S(0)$ be the set of optimal solution for $P(0)$.

<u>Theorem 3</u>

Let $S(0)$ be nonempty and $R(tu)$ be uniformly compact on the nontrivial interval $[0,t_0[$. If, for any optimum $x^* \varepsilon S(0)$, the assumptions H_1, H_2 and H_3 are satisfied, then the directional derivative of the optimal value function exists and is given by

$$p'(0;u) = \lim (p(tu)-p(0))/t = \min_{x^* \varepsilon S(0)} \max_{\lambda \varepsilon \Omega_1(x^*)} \{ -\lambda^T u \}.$$

Proof.

Because $R(tu)$ is uniformly compact and $S(0)$ nonempty, $p(tu)$ is lower semi-continuous at $t = 0^+$ (see Lemma 2.1 in Gauvin-Tolle [4]). By H_3, each $x^* \varepsilon S(0)$ is a strict optimum of $P(0)$; therefore it exists a neighborhood $X(x^*)$ where x^* is the unique optimum of the restricted program $P(0|x^*)$. By finite covering, the number of optimum points in $S(0)$ must then be finite. By H_1 and H_2, as previously noticed, the restricted programs $P(tu|x^*)$ are all feasible for $t \varepsilon [0,t_0[$, for some $t_0 > 0$. Therefore the local optimal value function are also upper semi-continuous at $t = 0^+$; therefore continuous at $t = 0^+$.

For t in some nontrivial interval, we then have

$$p(tu) = \min_{x^* \varepsilon S(0)} p(tu|x^*).$$

For each $x^* \varepsilon S(0)$, we have by Theorem 1 that any optimal solution $x(tu)$ of the restricted program $P(tu|x^*)$

is Lipschitz continuous at $t = 0^+$. We then applied Theorem 2 for each local optimal value function $p(tu|x^*)$ to obtain finally

$$\begin{aligned} p'(0;u) &= \lim_{t\downarrow 0} (p(tu)-p(0))/t \\ &= \min_{x^*\varepsilon S(0)} \lim_{t\downarrow 0} (p(tu|x^*)-p(0|x^*))/t \\ &= \min_{x^*\varepsilon S(0)} p'(0|x^*;u) \\ &= \min_{x^*\varepsilon S(0)} \max_{\lambda\varepsilon\Omega_1(x^*)} \{ -\lambda^T u \} \end{aligned}$$

□

When the program $P(0)$ is convex, the set $\Omega_1(x^*)$ is identical for all $x^*\varepsilon S(0)$; therefore the above formula reduces to

$$p'(0;u) = -\min_{\lambda\varepsilon\Omega_1(x^*)} \{ \lambda^T u \}$$

which is the classical result of convex programming under the Slater regularity condition ($\Omega_1(x^*)$ nonempty and bounded or equivalently $\Omega_0(x^*) = \{0\}$).

In the case where the Lipschitz property does not hold for the optimal solution, we can still have a result on the directional derivative for the optimal value function if the optimal solutions satisfy the Holderian continuity; i.e., for any local optimal solution $x(tu)$ of $P(tu)$ near x^*

$$\limsup_{t\downarrow 0} |x(tu)-x^*| \,/\, \sqrt{t} < +\infty$$

(see Corollary 4.1 in Gauvin-Janin [3]). In that case, the formula for the directional derivative for the local optimal value function is given by

$$p'(0|x^*;u) = \inf_{y\varepsilon D(x^*)} \sup_{\lambda\varepsilon\Omega_1(x^*:y)} \{ -\lambda^T u \} \qquad (2)$$

where

$$D(x^*) = \{y \mid \nabla f_i(x^*)y \leq 0,\ i\varepsilon I(x^*)\cup\{0\};\ \nabla f_i(x^*)y = 0,\ i \varepsilon J\}$$

is the set of critical directions at x^* and where

$$\Omega_1(x^*:y) = \{ \lambda \varepsilon \Omega_1(x^*) \mid y^T \nabla^2 L(x^*,\lambda)y \geq 0 \}$$

is the subset of multipliers satisfying the second-order necessary optimality condition for that critical direction y.

This nice formula is not so simple and sometime may be quite difficult to evaluate! Nevertheless the formula can be useful as illustrated by Example 1 where we have noticed that for direction u, with $u_1-u_2 < 0$, the optimal solutions

$$x(tu) = \{ (\pm\sqrt{t(u_2-u_1)/2}\ ,\ t(u_1+u_2)/2 \}$$

are not Lipschitzian but are Holderian. For that example, the optimal value function

$$p(tu) = -t(u_1+u_2)/2$$

has a directional derivative with value

$$p'(0;u) = -(u_1+u_2)/2.$$

Since $u_1-u_2 < 0$, the value given by the formula of Theorem 2 is

$$\max_{\lambda \varepsilon \Omega_2(x^*)} \{ -\lambda^T u \} = \max_{0 \le \lambda_1 \le 1} \{ -\lambda_1(u_1-u_2)-u_2 \}$$
$$= -u_1$$

which disagree with the real value since Theorem 2 does not apply for that case. If we refer to the above valid formula (2), we have the set of critical directions at $x^* = 0$ given by

$$D(x^*) = \{ y \varepsilon R^2 \mid y_2 = 0 \}$$

for which we have

$$y^T \nabla^2 L(x^*,\lambda)y = -2(\lambda_1-\lambda_2)y_1^2 \ge 0$$

if and only if $\lambda_1 \le \lambda_2$. Therefore

$$\Omega_1(x^*:y) = \{ (\lambda_1,\lambda_2) \mid \lambda_1+\lambda_2 = 1 , 0 \le \lambda_1 \le \frac{1}{2} \}$$

and, since $u_1-u_2 < 0$, the formula gives the value

$$\max_{\lambda \varepsilon \Omega_1(x^*:y)} \{ -\lambda^T u \} = \max_{0 \le \lambda_1 \le 1/2} \{ -\lambda_1(u_1-u_2)-u_2 \}$$
$$= - \frac{1}{2} (u_1+u_2)$$

which is in agreement with the real value for the directional derivative.

Assumption H_3 can be equivalently formulated by

$$H_3: \sup_{\lambda \varepsilon \Omega_1(x^*,u)} y^T \nabla^2 L(x^*,\lambda) y > 0 , \forall y \varepsilon E, y \ne 0.$$

This assumption needed for the Lipschitzian property in Theorem 1 is contained in the following less restrictive condition used to obtain the Holderian property in Theorem 4.1 in Gauvin-Janin [3]

$$H_4:\ \sup_{\lambda \in \Omega_1(x^*)} y^T \nabla^2 L(x^*,\lambda) y > 0\ ,\ \forall\ y \in D(x^*),\ y \neq 0.$$

We can put together Theorem 3 above and Corollary 4.1 in Gauvin-Janin [3] to obtain the following nice and clear result for the existence and value for the directional derivative of the optimal value function in mathematical programming. This last theorem is related with a similar result in Rockafellar [7].

Theorem 4

Let $S(0)$ be nonempty and $R(tu)$ be uniformly compact in some nontrivial interval $[0,t_0[$. At any optimum $x^* \in S(0)$, we assume that H_1 and H_2 are satisfied with at least one of the following weak second-order sufficient optimality conditions:

$$H_3:\ \sup_{\lambda \in \Omega_1(x^*,u)} y^T \nabla^2 L(x^*,\lambda)\ y > 0\ ,\ \forall y \in E,\ y \neq 0,$$

$$H_4:\ \sup_{\lambda \in \Omega_1(x^*)} y^T \nabla^2 L(x^*,\lambda)\ y > 0\ ,\ \forall y \in D(x^*),\ y \neq 0.$$

Then the directional derivative $p'(0;u) = \lim_{t \downarrow 0} (p(tu)-p(0))/t$ of the optimal value function exists and is given by

$$p'(0;u) = \min_{x^* \in S(0)} \begin{cases} \max_{\lambda \in \Omega_1(x^*)} \{-\lambda^T u\} & \text{if } H_3 \text{ is satisfied} \\ \inf_{y \in D(x^*)} \sup_{\lambda \in \Omega_1(x^*:y)} \{-\lambda^T u\} & \text{if not and if } H_4 \text{ is satisfied} \end{cases}$$

□

So far, we don't know any example where the optimal value function has directional derivative without at least one of the condition H_3 or H_4 satisfied.

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