MOREAU'S DECOMPOSITION THEOREM REVISITED

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ABSTRACT

Given two convex functions g and h on a Hilbert space, verifying $g + h = \frac{1}{2}\|.\|^2$, we show there necessarily exists a lower-semicontinuous convex function F such that $g = F \,\square\, \frac{1}{2}\|.\|^2$ and $h = F^* \,\square\, \frac{1}{2}\|.\|^2$. An explicit formulation of F is given as a deconvolution of a convex function by another one. The approach taken here as well as the way of factorizing g and h shed a new light on what is known as Moreau's theorem in the literature on Convex Analysis.

1 - INTRODUCTION

The starting point of our study was the following question, which takes root in the regularization processes studied in [9]: *Let $(H, <.,.>)$ be a Hilbert space, let f be a function on H and $\alpha > 0$ such that*

(1.1) *both* $\frac{\alpha}{2}\|.\|^2 - f$ *and* $\frac{\alpha}{2}\|.\|^2 + f$ *are convex functions on* H *(Here* $\|.\|$ *denotes the norm on* H *associated with the inner product* $<.,\cdot>$*).*
How to show that f is Gâteaux-differentiable on H with

(1.2) $\| f'(x) - f'(y)\| \leqslant \alpha \|x-y\|$ *for all* x, y *in* H *?*

The question of differentiability of f offers no difficulty since it readily comes from (1.1) that both $g := \frac{\alpha}{2}\|.\|^2 - f$ and $h := \frac{\alpha}{2}\|.\|^2 + f$ are finite convex functions on H, so that the directional derivative $f'(x,.)$ of f exists and satisfies:

(1.3) $f'(x,.) = \alpha < x, .> - g'(x,.) = h'(x,.) - \alpha < x, .>$
for all $x \in H$, whence $f'(x,.)$ is linear and continuous (since convex and concave) for all $x \in H$. The problem now is to prove that f' is Lipschitz on H, with Lipschitz constant α. It is clear, in view of (1.1), that α is the best Lipschitz constant one can expect on f'. Even if the problem can be reduced (by an argument of projection) to the same problem in a 2-dimensional context (cf.[6]), it is not simpler for all that. So, the question should be broached in a different way.
When reading (1.1), our first reaction is to observe that f is necessarily a d.c. function (i.e., a difference of convex functions) :

(1.4) $f = \frac{\alpha}{2}\|.\|^2 - g$ or $f = h - \frac{\alpha}{2}\|.\|^2$.

D.C. functions enjoy differentiability properties similar to those of convex functions, but to keep control of their derivatives is hopeless in general ([3, §II.2]). Things are however made easier since one of the functions involved in the decomposition of f is merely $\frac{\alpha}{2}\|.\|^2$. Referring back to (1.4), we see we are in the presence of two convex functions g and h such that

(1.5) $g + h = \alpha \|.\|^2$.

We thus reformulate the question posed at the beginning in the following way : *Let g and h be convex functions on H and $\alpha > 0$ such that*

(1.6) $g + h = \alpha \|.\|^2$

Show that both g and h are Gâteaux-differentiable on H with

(1.7) $< g'(x) - g'(y), h'(x) - h'(y) > \geqslant 0$ *for all x, y in H.*

Let us prove that the two formulations are equivalent.
Suppose we have answered the question in its second formulation and wish to answer it in its first one. Then, posing $g = \frac{\alpha}{2}\|.\|^2 - f$ and $h = \frac{\alpha}{2}\|.\|^2 + f$, we get that f is differentiable and

(1.8) $$< g'(x) - g'(y), h'(x) - h'(y) > = \alpha^2 \|x-y\|^2 - \|f'(x) - f'(y)\|^2 \geqslant 0 \text{ for all } x, y \in H,$$
which is (1.2) precisely.
Conversely, suppose we have answered the question in its original formulation and wish to answer it in its second one. Posing $f = \frac{\alpha}{2}\|.\|^2 - g = h - \frac{\alpha}{2}\|.\|^2$, we indeed have a function f such that both $\frac{\alpha}{2}\|.\|^2 + f$ and $\frac{\alpha}{2}\|.\|^2 + f$ are convex functions on H. Then, the differentiability of f induces that of g and h, and, in view of (1.8), the inequality (1.2) induces (1.7).

Starting from convex functions g and h such that $g + h = \alpha \|x\|^2$, we actually can prove more about g and h, namely that g and h can be *factorized* in the following form : $g = 2\alpha(F \,\square\, \frac{1}{2}\|.\|^2)$ and $h = 2\alpha\,(F^* \,\square\, \frac{1}{2}\|.\|^2)$ for some lower-semicontinuous convex function F. As a result, g and h will appear as Moreau-Yosida regularized versions of F and F^* respectively, so that all the announced properties on g and h follow.

2 - MOREAU'S DECOMPOSITION THEOREM REVISITED

2.1 - Let $\Gamma_0(H)$ denote the set of convex functions F from H into $(-\infty, +\infty]$ which are lower-semicontinuous and not identically equal to $+\infty$. What is known as Moreau's theorem in the context of Convex Analysis asserts the following : *for any* $F \in \Gamma_o(H)$

$$(2.1) \qquad F \,\square\, \frac{1}{2}\|.\|^2 + F^* \,\square\, \frac{1}{2}\|.\|^2 = \frac{1}{2}\|.\|^2. \qquad ([9])$$

By choosing F as the indicator function of a closed convex cone K of H, F^* is the indicator function of the polar cone K^o to K, $F \,\square\, \|.\|^2$ is the square of the distance function to K, so that (2.1) reads as a kind of Pythagore's theorem :

$$(2.2) \qquad d_K^2 + d_{K^o}^2 = \|.\|^2. \qquad ([7,9])$$

Such a decomposition has proved useful in all areas involving a Hilbertian structure (Euclidean spaces of matrices in Statistics, Sobolev spaces in Nonlinear Analysis [7,11], etc).

Our goal now is to prove a sort of *converse* to Moreau's theorem : starting with convex functions g and h such that $g + h = \frac{1}{2}\|.\|^2$, we want to factorize g and h in the form $F \,\square\, \frac{1}{2}\|.\|^2$ and $F^* \,\square\, \frac{1}{2}\|.\|^2$ respectively, by providing also an *explicit* formulation for F.

THEOREM (of factorization)

Let g and h be convex functions on H such that $g + h = \frac{1}{2} \|.\|^2$. There then exists $F \in \Gamma_o(H)$ such that

(2.3) $\quad g = F \,\square\, \frac{1}{2} \|.\|^2$ *and* $h = F^* \,\square\, \frac{1}{2} \|.\|^2$.

Moreover

(2.4) $\quad g'(x) \in \partial F\,(h'(x))$ *and* $h'(x) \in \partial F^*(g'(x))$ *for all* $x \in H$.

Before going into the details of the proof, we need to recall some facts about an operation on convex functions which has been recently introduced ([4]), and which bears the name of *deconvolution of a function by another one*.
Given φ and ψ in $\Gamma_0(H)$, the deconvolution of φ by ψ is the function denoted $\varphi \,\square\, \psi$ and defined as:

$$\forall\, x \in H,\ (\varphi \boxminus \psi)\,(x) = \sup_{\psi(u) < +\infty} \{\varphi(x+u) - \psi(u)\}.$$

The two main properties to be noticed are : $\varphi \boxminus \psi \in \Gamma_0(H)$ (or possibly identically equal to $+\infty$) and $(\varphi \boxminus \psi)^* = (\varphi^* - \psi^*)^{**}$ (see [5] and the references therein).

Proof of Theorem 1

We set $F = g \boxminus \frac{1}{2} \|.\|^2$, that is :

$$\forall\, x \in H,\ F(x) = \sup_{u \in H} \left\{ g(x+u) - \frac{1}{2} \|u\|^2 \right\}.$$

Since $g + h = \frac{1}{2} \|.\|^2$, we also have :

$$\forall\, x \in H,\ F(x) = \sup_{w \in H} \left\{ g(v) - \frac{1}{2} \|x-v\|^2 \right\}$$

$$= \sup_{v \in H} \left\{ \frac{1}{2} \|v\|^2 - h(v) - \frac{1}{2} \|x-v\|^2 \right\}$$

$$= \sup_{v \in H} \left\{ < x, v> -h(v) - \frac{1}{2} \|x\|^2 \right\}$$

$$= h^*(x) - \frac{1}{2} \|x\|^2 .$$

Whence

$$(2.5) \qquad F = g \boxminus \frac{1}{2} \|.\|^2 = h^* - \frac{1}{2} \|.\|^2 \qquad (\in \Gamma_0(H)).$$

By inverting the role of g and h, we get in a same way :

$$(2.6) \qquad h \boxminus \frac{1}{2} \|.\|^2 = g^* - \frac{1}{2} \|.\|^2 \quad (\in \Gamma_0(H)).$$

But the formula giving the conjugate function of $g \boxminus \frac{1}{2} \|.\|^2$ (as aforesaid) yields that

$$\left(g \boxminus \frac{1}{2} \|.\|^2 \right)^* = \left(g^* - \frac{1}{2} \|.\|^2 \right)^{**} = g^* - \frac{1}{2} \|.\|^2 .$$

Thus, the function defined in (2.6) is nothing else than F^*. Consequently, the usual calculus rules on conjugate functions, applied to $h^* = F + \frac{1}{2} \|.\|^2$ and $g^* = F^* + \frac{1}{2} \|.\|^2$, induce that $g = F \square \frac{1}{2} \|.\|$ and $h = F^* \square \frac{1}{2} \|.\|^2$.

Now, calculus rules on subdifferentials, applied to $h^* = F + \frac{1}{2} \|.\|^2$ for example, yield that

$$\partial h^* (h'(x)) = \partial F(h'(x)) + \{h'(x)\} \text{ for all } x \in H.$$

But $x \in \partial h^*(h'(x))$ for all $x \in H$, whence

$g'(x) \in \partial F\,(h'(x))$ for all $x \in H$.

Remark 1 The factorization of g and h in the form $F \,\square\, \frac{1}{2}\|.\|^2$ and $F^* \,\square\, \frac{1}{2}\|.\|^2$ respectively, with $F \in \Gamma_0(H)$, is *unique* : indeed, if $\phi \in \Gamma_0(H)$ verifies $\phi \,\square\, \frac{1}{2}\|.\|^2 = g$ and $\phi^* \,\square\, \frac{1}{2}\|.\|^2 = h$, we get that

$$(2.7) \qquad \phi = h^* - \frac{1}{2}\|.\|^2 = \left(g^* - \frac{1}{2}\|.\|^2\right)^*,$$

that is $\phi = g \boxminus \frac{1}{2}\|.\|^2$.

Remark 2. The dual formulation of the theorem of factorization is as follows : *If k, $\ell \in \Gamma_o(H)$ satisfy $k \,\square\, \ell = \frac{1}{2}\|.\|^2$, there then exists an unique $K \in \Gamma_o(H)$ such that*

$$k = K + \frac{1}{2}\|.\|^2 \text{ and } \ell = K^* + \frac{1}{2}\|.\|^2.$$

Example. Let S be a nonempty closed convex set of H. We have that

$$\underbrace{\frac{1}{2}d_S^2}_{g} + \underbrace{\frac{1}{2}\left(\|.\|^2 - d_S^2\right)}_{h} = \frac{1}{2}\|.\|^2.$$

It is known that $h = \frac{1}{2}\left(\|.\|^2 - d_S^2\right)$ is convex ([1]) (*). Then the only solution F yielded by the factorization theorem is $F = \psi_S$ (the indicator function of S). Note incidentally the pairing result :

$$(2.8) \qquad \frac{1}{2}\left(\|.\|^2 - d_S^2\right) = \psi_S^* \,\square\, \frac{1}{2}\|.\|^2,$$

which also can be obtained from direct calculations or as an example of Moreau's theorem (cf. (2.1)).

2.2. *Applications*

2.2.1. As a first application of the factorization theorem, we look back at the question posed in the Introduction and which motivated our study.

Consider two convex functions g and h on H, $\alpha > 0$, such that $g + h = \alpha \|.\|^2$. According to the factorization theorem, there exists a unique $F \in \Gamma_0(H)$ such that :

$$g/2\alpha = F \,\square\, \frac{1}{2}\|.\|^2 \text{ and } h/2\alpha = F^* \,\square\, \frac{1}{2}\|.\|^2,$$
$$g'(x) \in \partial F(h'(x)) \text{ for all } x \in H.$$

Due to the monotonicity property of ∂F, the second relation above induces that

$$< g'(x) - g'(y), h'(x) - h'(y) > \geqslant 0 \text{ for all } x \in H,$$

which is the relation (1.7) required.

2.2.2. A second application of the factorization theorem is the following result.

COROLLARY 2. *Let $f : H \longrightarrow \mathbb{R}$ be a Gâteaux-differentiable function and $\alpha > 0$. Then the next statements are equivalent*:

(2.9) $|< f'(x) - f'(y), x-y >| \leqslant \alpha \|x-y\|^2$ for all $x, y \in H$;

(2.10) $\| f'(x) - f'(y) \| \leqslant \alpha \|x-y\|$ for all $x, y \in H$.

Although it was known for C^2 - functions, this equivalence is rather surprising ; clearly, (2.9) which involves f on line segments is easier to check.

(*) Actually, h is convex whatever S be. But to ensure the convexity of g also, we need the convexity of S.

To prove that (2.9) implies (2.10), it suffices to observe that both $\frac{\alpha}{2}\|.\|^2 - f$ and $\frac{\alpha}{2}\|.\|^2 + f$ are convex functions on H ; (2.10) then follows from the equivalence properties stated in the Introduction.

Corollary 2 answers a question the first author alluded to in [3, p. 48 bottom] concerning the comparison between (globally) $C^{1,1}$ functions f and those satisfying an inequality like (2.9).

2.2.3. A third application of the factorization theorem is a characterization of the so-called α-strongly convex functions. We recall that, given $\alpha > 0$, $f \in \Gamma_0(H)$ is said to be *α-strongly convex* (or strongly convex with modulus α) if

$$f(tx + (1-t)\,x') \leqslant t\,f(x) + (1-t)\,f(x') - \frac{\alpha}{2}\,t(1-t)\,\|x-x'\|^2$$

for all x, x' in H and $t \in]0,1[$. In other words, that means that $f - \frac{\alpha}{2}\|.\|^2$ is still a convex function ($\in \Gamma_0(H)$). The next characterization of α-strongly convex functions has also been observed by Volle ([10]) who, furthermore, introduced a new conjugacy mapping for such functions by substituting the "coupling functional"

$(x, y) \longmapsto \frac{\alpha}{2}\|x-y\|^2$ for the usual bilinear functional

$(x, y) \longmapsto \langle x, y \rangle$.

<u>COROLLARY 3</u>. *Let $f \in \Gamma_0(H)$. The following are equivalent :*

(2.11) *f is α-strongly convex ;*

(2.12) $\frac{1}{2\alpha}\|.\|^2 - f^* \in \Gamma_0(H)$;

(2.13) *There exists $\varphi \in \Gamma_0(H)$ such that* $f \,\square\, \varphi = \frac{\alpha}{2}\|.\|^2$.

Condition (2.12) actually says more than what is stated : since f^* is itself in $\Gamma_0(H)$, condition (2.12) implies that

f^* is finite on H ; in fact we will see in the course of the proof that f^* is a $C^{1,1}$ function (*).

Likewise, a consequence of (2.13) is that $\varphi^* = \frac{1}{2\alpha}\|.\|^2 - f^*$, whence the exhibited function φ is α-strongly convex ; indeed,

$$(2.14) \qquad \varphi = \left(\frac{1}{2\alpha}\|.\|^2 - f^*\right)^* = \frac{\alpha}{2}\|.\|^2 \boxminus f,$$

$$(2.15) \qquad f = \left(\frac{1}{2\alpha}\|.\|^2 - \varphi^*\right)^* = \frac{\alpha}{2}\|.\|^2 \boxminus \varphi.$$

Proof. (2.12) $\Longrightarrow$ (2.11). Let g denote the convex function $\frac{1}{2\alpha}\|.\|^2 - f^*$. Since $\alpha\, g + \alpha\, f^* = \frac{1}{2}\|.\|^2$, the theorem of factorization yields that there exists $F \in \Gamma_0(H)$ such that $\alpha\, f^* = F \,\square\, \frac{1}{2}\|.\|^2$. Consequently, f assigns $\frac{1}{\alpha} F^*(\alpha x) + \frac{\alpha}{2}\|x\|^2$ to $x \in H$, so that $f - \frac{\alpha}{2}\|.\|^2$ is still a convex function. We thus have proved f is α-strongly convex.

(2.11) $\Longrightarrow$ (2.13). Let χ denote the convex function $\frac{f}{\alpha} - \frac{1}{2}\|.\|^2$; we set $\varphi = \alpha\, \chi^* + \frac{\alpha}{2}\|.\|^2$. Starting from the relation $\frac{f}{\alpha} = \chi + \frac{1}{2}\|.\|^2$,
we get successively

(*) The equivalence of (2.11) and (2.12) appears also as a by-product of more general results on the duality relations between uniformly convex functions and uniformly smooth convex functions ([2]).

$$\left(\frac{f}{\alpha}\right)^* = \chi^* \,\square\, \frac{1}{2}\|.\|^2$$

(2.16)

$$= \frac{1}{2}\|.\|^2 - \left(\chi \,\square\, \frac{1}{2}\|.\|^2\right)$$ by Moreau's theorem.

Let us calculate $g = \left(\frac{f}{\alpha}\right) \square \left(\frac{\varphi}{\alpha}\right)$. Since $g^* = \left(\frac{f}{\alpha}\right)^* + \left(\frac{\varphi}{\alpha}\right)^*$, we infer from the definition of φ and (2.16) :

$$g^* = \frac{1}{2}\|.\|^2 - \left(\chi \,\square\, \frac{1}{2}\|.\|^2\right) + \chi \,\square\, \frac{1}{2}\|.\|^2 = \frac{1}{2}\|.\|^2.$$

Whence $g = \frac{1}{2}\|.\|^2$ and (2.13) is secured.

(2.13) $\Longrightarrow$ *(2.12)* From $f \,\square\, \varphi = \frac{\alpha}{2}\|.\|^2$ we derive $f^* + \varphi^* = \frac{1}{2\alpha}\|.\|^2$, so that $\frac{1}{2\alpha}\|.\|^2 - f^* = \varphi^* \in \Gamma_0(H)$. ∎

3 - COMPARISON WITH MOREAU'S APPROACH

In his seminal 1965 paper ([8]), Moreau extensively studied the functions of the form $F \,\square\, \frac{1}{2}\|.\|^2$, $F \in \Gamma_0(H)$, and defined the so-called *proximal mapping* prox_F which assigns to $x \in H$ the unique point where the infimum of $u \longmapsto F(u) + \frac{1}{2}\|x - u\|^2$ is achieved. Among other properties, he proved that prox_F is a Lipschitz mapping (with Lipschitz constant 1) and that prox_F is actually a gradient mapping (i.e., there is a differentiable function Φ, called primitive function of prox_F, such that $\Phi'(x) = \text{prox}_F(x)$ for all $x \in H$).

In a much less read section ([8, §9]), Moreau introduced a binary relation between convex functions by defining what he meant by "a convex function g less convex than a convex function f". More interesting is the characterization of such a relationship when f is $\frac{1}{2}\|.\|^2$ precisely, which now allows us to make connections with our approach.

According to Moreau ([8, définition 9.b]), a convex function g is less convex than a convex function f (or f is more convex than g) if there exists a convex function h such that f = g + h. He then proved the equivalence of the following properties ([8, Proposition 9.b and Proposition 10.b] :

(3.1) *$g \in \Gamma_o(H)$ is less convex than $\frac{1}{2}\|.\|^2$;*

(3.2) *The conjugate function of $g \in \Gamma_o(H)$ is more convex than $\frac{1}{2}\|.\|^2$;*

(3.3) *g is the primitive function of a proximal mapping ;*

(3.4) *$g \in \Gamma_o(H)$ is differentiable and g' is Lipschitz on H with a Lipschitz constant 1.*

(3.1) expresses the existence of a convex function h such that $g + h = \frac{1}{2}\|.\|^2$, which is precisely the situation we have considered here. According to (3.4), such a g is differentiable and $\|g'(x) - g'(y)\| \leqslant \|x-y\|$ for all x, y $\in$ H ; the property we were looking for from the beginning is stronger, namely : $\|g'(x) - g'(y) - \frac{x-y}{2}\| \leqslant \frac{1}{2}\|x-y\|$ (cf. Introduction).

Moreover, the factorization of g (and h) does not appear explicitly and a characterization like (3.3) uses heavily the properties of the proximal mapping.

Our approach, based on the deconvolution operation, allowed us to get at an explicit formulation of F in the factorization theorem (Theorem 1), thereby shedding a new light on Moreau's theorem.

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