

ON THE SYMMETRY OF BLOWUP SOLUTIONS TO A MEAN FIELD EQUATION

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Received 15 October 1999; revised 15 May 2000

ABSTRACT. – In this article, we consider the mean field equation

$$\Delta u + \rho \left(\frac{e^u}{\int e^u} - \frac{1}{A} \right) = 0 \quad \text{in } \Sigma,$$

where Σ is a flat torus and A is the area of Σ . This paper is concerned with the symmetry induced by the phenomenon of concentration. By using the method of moving planes, we prove that blowup solutions often possess certain symmetry. In this paper, we consider cases when solutions blowup at one or two points. We also consider related problems for annulus domains of \mathbf{R}^2 .

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RÉSUMÉ. – Nous considérons l'équation de champ moyen

$$\Delta u + \rho \left(\frac{e^u}{\int e^u} - \frac{1}{A} \right) = 0 \quad \text{dans } \Sigma,$$

où Σ est un tore plat et A est l'aire de Σ . Cet article se rapporte à la symétrie induite par le phénomène de concentration. En utilisant la méthode de déplacement de plans, nous démontrons que les solutions avec singularités possèdent dans la plupart des cas des propriétés de symétrie. Dans cet article, nous considérons des cas où les solutions explosent en un ou deux points. Nous traitons également de problèmes reliés pour des domaines annulaires de \mathbf{R}^2 avec conditions de Dirichlet au bord.

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1. Introduction

Let Σ be a flat torus with a rectangle $\{(x_1, x_2) \mid |x_1| \leq a, |x_2| \leq b\}$ of \mathbf{R}^2 as its fundamental domain. For $\rho > 0$, we consider the equation

$$\Delta u + \rho \left(\frac{e^u}{\int e^u} - \frac{1}{A} \right) = 0 \quad \text{in } \Sigma, \quad (1.1)$$

where Δ is the Laplace operator and A is the area of Σ . Eq. (1.1) is the Euler–Lagrange equation of the following functional

$$J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 + \frac{\rho}{A} \int_\Sigma u - \rho \log \int_\Sigma e^u. \quad (1.2)$$

Clearly Eq. (1.1) and the functional J_ρ remain invariant under the adding a constant to u . Therefore we always seek solutions of (1.1) which are normalized by

$$\int_\Sigma u \, dx = 0. \quad (1.3)$$

Eq. (1.1) is generally known as a mean field equation because it is often derived from various Onsager’s vortex theories. Meanwhile, there are many recent works to relate (1.1) to some Chern–Simons–Higgs model. For more information concerning these developments, we refer the interested readers to [4–7,11,12,14,16,20–22,24] and references therein.

For $\rho < 8\pi$, the function J_ρ is coercive by the Moser–Trudinger inequality, and solutions of (1.1) can be obtained by minimization of J_ρ . In fact, even the compactness of solutions can be proved in this case. But, for $\rho \geq 8\pi$, it is quite a different story. Recently, there are several works to extend the existence of (1.1) for $\rho \in (8\pi, 16\pi)$. See [11,12,20,22]. In [14], Yan–Yan Li initiated a program to find solutions for $\rho \geq 8\pi$ by using the topological degree theory. He proved an uniform bound for solutions to Eq. (1.1) whenever ρ is contained in a compact set of $(8m\pi, 8(m+1)\pi)$ where $m \geq 0$ is an integer. Therefore, the Leray–Schauder degree for (1.1) remains the same when ρ is in the interval $(8m\pi, 8(m+1)\pi)$. In particular, if the degree is nonvanishing, then the existence of solutions can be guaranteed by the degree theory. However, the calculation of the topological degree seems not so easy. One of major difficulties is to prove whether $\rho_i > 8m\pi$ or $\rho_i < 8m\pi$ for a sequence of solutions of (1.1) with $\lim_{i \rightarrow +\infty} \rho_i = 8m\pi$ when blowup would actually happen. In [16], the second author has settled the question when Σ is the sphere S^2 , at least for the case of 8π and 16π . The major step in [16] is to show the symmetry induced from the concentration phenomenon. In this paper, we want to continue to study the symmetry of solutions due to the concentration phenomenon. We hope that this would be helpful when we come to computing the topological degree.

In the following, we always suppose that u_i is a solution of (1.1) with $\rho = \rho_i$ and u_i blows up at some points of Σ . Recall that by a result of [14], if $\rho_i \rightarrow 8m\pi$ and u_i blows up somewhere, then u_i has exactly m blowup points.

THEOREM 1.1. – *Let $\rho_i \rightarrow 8\pi$ as $i \rightarrow +\infty$ and the center $(0,0)$ of Σ is the maximum point of u_i . Then u_i is symmetric with respect to both x and y axis for i large. Furthermore, for any y fixed, $u_i(x, y)$ increases when x increases from the negative x up to 0 and decreases when x increases from 0 to the right hand side of the boundary. The same holds for x fixed.*

THEOREM 1.2. – *Let $\rho_i \rightarrow 16\pi$ as $i \rightarrow +\infty$ and one of local maximum points of u_i be located at the center. Then for i large, the other maximum point of u_i is located at the corner of the fundamental cell of Σ . Furthermore, $u_i(x, y)$ is symmetric with respect to both x and y axis.*

Naturally, we arise the question whether ρ_i is larger than 8π (or 16π) for solutions with one blowup point (or two blowup points). This question is important when we come to compute the Leray–Schauder degree for $\rho \in (8\pi, 16\pi)$. For S^2 , this question has been answered in [16]. For a compact Riemann surface Σ of positive genus, the problem remains unsolved. Nevertheless, we have the answer for a similar problem considered in an annulus of \mathbf{R}^2 . Let Ω be a smooth bounded domain of \mathbf{R}^2 . We consider the following equation.

$$\begin{cases} \Delta u + \frac{\rho e^u}{\int_{\Omega} e^u dx} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Eq. (1.4) is derived from Onsager’s vortex model for turbulent Euler flows. See [5,6] and [7] for details. For a non-simply-connected domain Ω , the existence of solutions was proved in [12] for $8\pi < \rho < 16\pi$. In fact, a minimax value α_{ρ} was defined and was proved to be a critical value. In [12], the authors raised the question whether solutions u_{ρ} remains bounded or not as $\rho \rightarrow 16\pi$, which equivalently is to ask whether solutions of (1.4) is uniformly bounded or not when ρ tends to 16π from below. In the following theorem, we answer the question for an annulus domain. For the rest of the section, Ω is always the annulus $\{x \mid a < |x| < b\}$ for some $a < b$.

THEOREM 1.3. – *Let u_i be a solution of (1.4) with $\rho_i \rightarrow 8\pi$ as $i \rightarrow +\infty$. Assume that u_i has one blowup point, then $\rho_i > 8\pi$ for large i and $u_i(x)$ is symmetric with respect to the line l_i , where l_i is the line connecting the maximum point of u_i and the origin.*

For solutions with two blowup points, we have

THEOREM 1.4. – *Let u_i be a solution of (1.4) with $\rho_i \rightarrow 16\pi$ such that u_i blows up at P and Q . Assume that P_i and Q_i are the two local maximum points near P and Q respectively. Then P_i, Q_i and the origin form a straight line l_i and u_i is symmetric with respect to l_i . Moreover, $\rho_i > 16\pi$.*

COROLLARY 1.5. – *Let Ω be an annulus in \mathbf{R}^2 . For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) > 0$ such that*

$$|u(x)| \leq C \quad \text{for } x \in \bar{\Omega}$$

holds for any solution u of (1.4) with $\rho \in (0, 8\pi] \cup [8\pi + \varepsilon, 16\pi]$.

The paper is organized as follows. In Section 2, both Theorem 1.1 and Theorem 1.2 are proved. Here we apply the well-known method of moving planes to prove our results. For Eq. (1.4) with an annulus Ω , we use the method of rotating planes to prove the axial symmetry of solutions. This is an important step for us to be able to use an isoperimetric inequality of C. Bandle. Together with the isoperimetric inequality of C. Bandle, Theorem 1.3 and Theorem 1.4 are established in Section 3 by this method. For Theorem 1.4, additional results concerning the regular part of Green’s function are required. See Lemma 3.3 and Lemma 3.4. We present their proofs here because the authors can not find them in the literature. In the final section, we will construct solutions with m blowup points in the annulus domain for ρ greater then $8m\pi$.

2. The method of moving planes

In this section, we begin with a proof of Theorem 1.1. We use the method of moving planes. For applications of this method, we refer the readers to [3,10,13,14] and [16].

Proof of Theorem 1.1. – After adding a constant, we may assume u_i is a solution of

$$\Delta u_i + \rho_i \left(e^{u_i} - \frac{1}{A} \right) = 0 \quad \text{on } \Sigma.$$

By translation, we may suppose that the position of the local maximum P_i of u_i is located at $(a/2, 0)$. For $\lambda > 0$, we let $\Sigma_\lambda = \{x \mid \lambda < x_1 < \lambda + a\}$, $T_\lambda = \{x \mid x_1 = \lambda\}$, and $x^\lambda = (2\lambda - x_1, x_2)$ is the reflection point of x with respect to T_λ . Set

$$w_\lambda(x) = u_i(x) - u_i(x^\lambda) \quad \text{for } x \in \Sigma_\lambda. \tag{2.1}$$

Then $w_\lambda(x)$ satisfies

$$\begin{cases} \Delta w_\lambda(x) + b_\lambda(x)w_\lambda(x) = 0 & \text{in } \Sigma_\lambda, \\ w_\lambda(x) = 0 & \text{on } \partial\Sigma_\lambda, \end{cases} \tag{2.2}$$

where

$$b_\lambda(x) = \rho_i \frac{e^{u_i(x)} - e^{u_i(x^\lambda)}}{u_i(x) - u_i(x^\lambda)}.$$

To start the method of moving planes, we need an estimate by Y.Y. Li [14]. Let $\{P_0^1, \dots, P_0^m\}$ be the blowup set of u_i . For any P_0^l , we set P_i^l to be the local maximum point of u_i near P_0^l , that is,

$$u_i(P_i^l) = \max_{|x - P_0^l| \leq \delta_0} u_i(x)$$

for some small $\delta_0 > 0$. Then the main theorem of [14] is stated as follows.

THEOREM 2.1. – *There is a constant $C > 0$ such that*

$$|u_i(x) - U_i(x)| \leq C \tag{2.3}$$

holds for $|x - P_i| \leq \delta_0$ and i large, where $\delta_0 > 0$ is any fixed small number and $U_i(x)$ is defined by

$$U_i(x) = \log \frac{e^{u_i(P_i)}}{\left(1 + \frac{\rho_i}{8} e^{u_i(P_i)} |x - P_i|^2\right)^2}$$

for $|x - P_i| \leq \delta_0$. Furthermore, let $\bar{u}_i = -\int_{\Sigma} u_i(x) dx$. Then

$$u_i(x) - \bar{u}_i \rightarrow 8\pi \sum_{l=1}^m G(\cdot, P_0^l) - \frac{8\pi m}{A} \int_{\Sigma} G(\cdot, y) d\mu(y), \tag{2.4}$$

in $C^2(\Sigma \setminus \{P_0^1, \dots, P_0^m\})$, where $G(\cdot, P_0^l)$ is the Green function with a singularity at P_0^l .

From (2.3) and (2.4), we see that

$$|u_i(x) + u_i(P_i^l)| \leq C$$

holds when $|x - P_i^l| \geq \delta_0$ for each l , and

$$|u_i(P_i^{l'}) - u_i(P_i^l)| \leq C \tag{2.5}$$

for $1 \leq l', l \leq m$. Inequality (2.5) is important for the case of two blowup points in this article.

Now we go back to the proof of Theorem 1.1. We claim

Step 1.

$$w_0(x) > 0 \quad \text{for } x \in \Sigma_0. \tag{2.6}$$

To prove (2.6), we note that by Theorem 2.1, $e^{u_i(x)} \rightarrow 0$ as $i \rightarrow +\infty$ for $x \in \Sigma \setminus \Sigma_0$. Suppose (2.6) is false, that is, the set $\Omega = \{x \in \Sigma_0 \mid w_0(x) < 0\}$ is a non-empty set. Then by (2.2)

$$\Delta w_0(x) + \rho_i e^{u_i(x^-)} w_0(x) \leq 0$$

for $x \in \Omega$ where $x^- = (-x_1, x_2)$. Therefore the first eigenvalue $\lambda_1(\Omega)$ of the linear equation $\Delta + \rho_i e^{u_i(x^-)}$ on Ω is nonpositive. On the other hand, since $e^{u_i(x^-)} \rightarrow 0$ as $i \rightarrow +\infty$ and the first eigenvalue of Δ on Σ_0 is π^2/a^2 , the first eigenvalue of $\Delta + \rho_i e^{u_i(x^-)}$ tends to π^2/a^2 as $i \rightarrow +\infty$, which yields a contradiction. This proves (2.6).

Set

$$\lambda_0 = \sup\{\lambda \in [0, a/2] \mid w_{\tilde{\lambda}}(x) \geq 0 \text{ for } x \in \Sigma_{\tilde{\lambda}} \text{ and } 0 \leq \tilde{\lambda} \leq \lambda\}.$$

Claim

Step 2. $\lambda_0 = a/2$ and $w_{a/2}(x) \equiv 0$ for $x \in \Sigma_{a/2}$.

Suppose $\lambda_0 < a/2$. Then by the continuity, $w_{\lambda_0}(x) \geq 0$ for $x \in \Sigma_{\lambda_0}$. By the strong maximum principle and the Hopf boundary Lemma, we have

$$w_{\lambda_0}(x) > 0 \quad \text{and} \quad \frac{\partial w_{\lambda_0}(x)}{\partial \nu} < 0 \quad \text{for } x \in \partial \Sigma_{\lambda_0}, \tag{2.7}$$

where ν is the outnormal. By the definition of λ_0 , there are a sequence of $\lambda_j > \lambda_0$ such that $\lim_{j \rightarrow +\infty} \lambda_j = \lambda_0$ and

$$w_{\lambda_j}(x_j) = \inf_{\Sigma_{\lambda_j}} w_{\lambda_j}(x) < 0,$$

for some $x_j \in \Sigma_{\lambda_j}$. After passing to a subsequence, $x_0 = \lim_{j \rightarrow +\infty} x_j$. Then either $x_0 \in \Sigma_{\lambda_0}$ and $w_{\lambda_0}(x_0) = 0$ or $x_0 \in \partial \Sigma_{\lambda_0}$ and $\nabla w_{\lambda_0}(x_0) = 0$. Obviously, either case yields a contradiction to (2.7). Hence $\lambda_0 = a/2$, and $w_{a/2}(x) \geq 0$ in $\Sigma_{a/2}$. Of course, we can apply the same procedure from $\lambda = a$ to $\lambda = a/2$ and obtain $u(x^\lambda) \geq u(x)$ also for $x \in \Sigma_{a/2}$. Thus, we have $u(x^\lambda) = u(x)$ for $\lambda = a/2$. Clearly, the conclusion of Theorem 1.1 follows immediately. \square

Now we come to the case of two blowup points. Let P_i and Q_i are two local maximum points of u_i such that both $u_i(P_i)$ and $u_i(Q_i) \rightarrow +\infty$ as $i \rightarrow +\infty$. Let $P = \lim_{i \rightarrow +\infty} P_i$ and $Q = \lim_{i \rightarrow +\infty} Q_i$. If there exists an open half Σ' of the fundamental domain of Σ such that Σ' contains both P and Q , then we can apply the method above to yield a contradiction. The reason is as follows. Without loss of generality, we may suppose that P and Q are contained in Σ_0 . First, we assume that P_i and Q_i are not contained in the same vertical line. By step 1 of the proof above, we still have $w_0(x) > 0$ in Σ_0 . And as before, we can move the vertical line until it contains one of the two local maximum points, say P_i . Since Q_i is not on the same vertical line, solutions can not be symmetric with respect to this vertical line. Thus, we can continue our procedure by moving the line to cross P_i and u_i in the right-hand half is greater than the left-hand half. Clearly, it yields a contradiction to the fact that P_i is a local maximum point. If P_i and Q_i are contained on the same vertical line, then we can move horizontal lines and obtain the same conclusion. Therefore, we conclude that if P is at the center of the fundamental domain, then Q must be at the corner. In the following, we want to prove that for large i , if P_i is at the center, then Q_i must be at the corner.

Proof of Theorem 1.2. – Suppose that P_i and Q_i are contained in an open half of the fundamental domain of Σ . Without loss of generality, one may assume that P_i and Q_i are contained in $(0, a) \times (0, b)$ such that $\lim_{i \rightarrow +\infty} P_i = (0, 0)$ and $\lim_{i \rightarrow +\infty} Q_i = (a, b)$. By translation, we may assume $P_i = (t_i, 0)$, $Q_i = (a_i, b_i)$ with $0 < a_i < a, 0 < b_i < b$ and $|P_i| = |Q_i - (a, b)|$. Note that under the assumption of the position of Q_i , $u_i(x)$ could not be symmetric with respect to x_1 and x_2 . Let δ_0 be a small positive number and $u_i(P_i) = \max_{|x - P_i| \leq \delta_0} u_i(x)$ and $u_i(Q_i) = \max_{|x - Q_i| \leq \delta_0} u_i(x)$. Then $|u_i(P_i) - u_i(Q_i)| \leq c$ for some constant c independent of i by (2.5). We want to prove

$$w_0(x) = u_i(x_1, x_2) - u_i(-x_1, x_2) > 0 \quad \text{for } 0 < x_1 < a. \tag{2.8}$$

Once (2.8) is established, we could follow step 2 of Theorem 1.1 to show that $w_\lambda(x) \geq 0$ for $x \in \Sigma_\lambda = \{x \mid \lambda < x_1 < \lambda + a\}$ when $0 < \lambda \leq 2t_i$. Thus, it yields a contradiction to the fact that u_i has a local maximum at P_i . We prove (2.8) by two cases.

Case 1. $e^{u_i(P_i)} |t_i|^2 \rightarrow +\infty$ as $i \rightarrow +\infty$.

Note that by (2.5), we also have

$$e^{u_i(Q_i)}|Q_i - (a, b)|^2 \rightarrow +\infty \quad \text{as } i \rightarrow +\infty.$$

For any $x \in \Sigma_0$, we have

$$e^{u_i(P_i)}|P_i|^2 \leq e^{u_i(P_i)}|x^- - P_i|^2. \tag{2.9}$$

Thus, by Theorem 2.1, we have for $x \in B_{\delta_0} \cap \{x \mid x_1 > 0\}$,

$$\begin{aligned} u_i(x^-) &\leq c + \log \frac{e^{u_i(P_i)}}{\left(1 + \frac{\rho_i}{8} e^{u_i(P_i)}|x^- - P_i|^2\right)^2} \\ &\leq c_1 + \log \left(\frac{1}{e^{u_i(P_i)}|x^- - P_i|^4}\right), \end{aligned}$$

and by (2.9),

$$e^{u_i(x^-)} \leq e^{c_1} (e^{-u_i(P_i)}|x^- - P_i|^{-4}) = o(1)|x^- - P_i|^{-2} \leq o(1)|x|^{-2}. \tag{2.10}$$

If $|x| \geq \delta_0$, then (2.10) holds obviously. Similar inequality holds near Q_i also.

Suppose that $w_0(x) < 0$ for some $x \in \Sigma_0$. Without loss of generality, we may assume that the maximum of $-w_0$ occurs at some point $x^0 \in \Sigma_0$ with $x_1^0 \leq a/2$. Let $h(x) = (x_1)^\alpha$ for $0 < \alpha < 1$. Clearly, $h(x)$ satisfies

$$\Delta h(x) = \alpha(\alpha - 1)(x_1)^{-2}h(x). \tag{2.11}$$

Set $\bar{w}(x) = -w_0(x)/h(x)$. Since w_0 is C^1 on $x_1 = 0$, we have $\bar{w}(x) = 0$ on $x_1 = 0$. Then by a straightforward computation, $\bar{w}(x)$ satisfies

$$\begin{cases} \Delta \bar{w}(x) + 2\nabla(\log h) \cdot \nabla \bar{w} + (b_0(x) + \alpha(\alpha - 1)|x_1|^{-2})\bar{w}(x) \geq 0, \\ \bar{w}(x) = 0 \quad \text{on } \partial\Omega_0, \end{cases} \tag{2.12}$$

where

$$b_0(x) = \rho_i \frac{e^{u_i(x)} - e^{u_i(x^-)}}{u_i(x) - u_i(x^-)} \quad \text{and} \quad \Omega_0 = \{x \in \Sigma_0 \mid w_0(x) < 0\}.$$

Let y be a maximum point of \bar{w} . Clearly, $y_1 \leq x_1^0 \leq a/2$, and $b_0(y) \leq \rho_i e^{u_i(y^-)} \leq o(1)|y|^{-2}$ by (2.10). Let i be large enough so that $o(1) + \alpha(\alpha - 1) < 0$. Then, applying the maximum principle to (2.12) at y , it yields a contradiction. Hence (2.8) is proved in case 1.

Case 2. Assume $e^{u_i(P_i)}|P_i|^2 \leq c$ for some constant $c > 0$.

To prove the positivity of $w_0(x)$ in Σ_0 , we want to prove

$$\lim_{i \rightarrow +\infty} (e^{u_i(P_i)}|x|^2) = +\infty \quad \text{and} \quad \lim_{i \rightarrow +\infty} e^{u_i(Q_i)}|x - (a, b)|^2 = +\infty \tag{2.13}$$

for all x whenever $w_0(x) < 0$. Once (2.13) is proved, we could use $h(x)$ of (2.11) and follow the argument of case 1 to show $w_0(x) > 0$ for $x \in \Sigma_0$.

Obviously, it suffices to prove (2.13) in a neighborhood of P_i and Q_i . First, suppose $\lim_{i \rightarrow +\infty} e^{u_i(P_i)} |P_i|^2 > 0$. Then we also have $\lim_{i \rightarrow +\infty} e^{u_i(Q_i)} |Q_i - (a, b)|^2 > 0$. In the following, we prove (2.13) for x near P_i . For the case when x is near Q_i , the proof is similar. Let

$$v_i(y) = u_i(e^{-\frac{u_i(P_i)}{2}}y) - u_i(P_i). \tag{2.14}$$

Then by Theorem 2.1, after passing to a subsequence, $v_i(y)$ converges to $v(y) \equiv -2\log(1 + \frac{\rho}{8}|y - \xi_0|^2)$ uniformly in any compact set of \mathbf{R}^2 , where

$$0 \neq \xi_0 = \lim_{i \rightarrow +\infty} e^{\frac{u_i(P_i)}{2}} P_i \in \mathbf{R}_+^2.$$

It is easy to see $w_0(e^{-u_i(P_i)/2}y) = v_i(y) - v_i(y^-)$ where $y^- = (-y_1, y_2)$. Thus, by the explicit expression of v ,

$$\lim_{i \rightarrow +\infty} w_0(e^{-\frac{u_i(P_i)}{2}}y) = v(y) - v(y^-) > 0$$

for any $y \in \mathbf{R}^2$ and $y_1 > 0$. Now suppose there are some y_i such that y_i is bounded and $w_0(e^{-u_i(P_i)/2}y_i) < 0$. Without loss of generality, we may assume $\lim_{i \rightarrow +\infty} y_i = y_0 = (y_{0,1}, y_{0,2})$. Since $w_0(e^{-u_i(P_i)/2}y)$ converges to a positive function for $y_1 > 0$, we have $y_{0,1} = 0$. From it, y_i can be chosen to be a local minimum point of w_0 for $\{(y_1, y_{i,2}) \mid 0 \leq y_1 \leq 1\}$. Hence $\frac{\partial}{\partial y_1}(v(y_0) - v(y_0^-)) = 0$ at $y = y_0$. By a straightforward computation, we have

$$\frac{\partial}{\partial y_1}(v(y_0) - v(y_0^-)) = \frac{-\rho\xi_{0,1}}{1 + \frac{\rho}{8}|y_0 - \xi_0|^2} < 0,$$

a contradiction. Thus, we have $|y_i| \rightarrow +\infty$ whenever $w_0(e^{-u_i(P_i)/2}y_i) \leq 0$. Clearly, (2.13) follows readily. By the similar method, we can prove (2.13) near Q_i .

Now suppose $\lim_{i \rightarrow +\infty} e^{u_i(P_i)} |P_i|^2 = 0$. Let

$$N_i = \max_{\Sigma_0} |w_0(x)| = |w_0(x_i)|$$

for some $x_i \in \Sigma_0$. We claim that x_i tends to either $(0,0)$ or $(a, \pm b)$ and for any $\delta_0 > 0$,

$$\sup_{\Sigma_0(\delta_0)} |w_0(x)| = o(1)N_i, \tag{2.15}$$

where $\Sigma_0(\delta_0) = \{x \in \Sigma_0 \mid |x - P_i| \geq \delta_0 \text{ and } |x - Q_i| \geq \delta_0\}$.

We prove (2.15) by contradiction. Assume that for a small $\delta_0 > 0$, there are subsequence of u_i (still denoted by u_i) such that

$$\sup_{\Sigma_0(\delta_0)} |w_0(x)| \geq c_0 N_i \tag{2.16}$$

holds for some constant c_0 . Set $\hat{w}_i(x) = N_i^{-1}w_0(x)$. Then $|\hat{w}_i(x)| \leq 1$ for $x \in \bar{\Sigma}_0$. Since $b_0(x) \rightarrow 0$ uniformly in any compact set of $\bar{\Sigma} \cap \bar{\Sigma}_0 \setminus \{0, (a, \pm b)\}$, by the elliptic

estimates, there are a subsequence of \hat{w}_i (still denoted by \hat{w}_i) such that \hat{w}_i converges to a harmonic function \hat{w} , where \hat{w} is smooth in $\bar{\Sigma} \cap \bar{\Sigma}_0$ except for the points $(0,0)$, $(a, \pm b)$. Since $|\hat{w}(x)| \leq 1$ in $\bar{\Sigma}_0$ and $\hat{w}(x) = 0$ for $x \in (\partial \Sigma_0 \setminus \{(0, 0), (a, \pm b)\}) \cap \bar{\Sigma}$, by the reflection and the elliptic regularity for bounded harmonic functions, we conclude that $\hat{w}(x)$ is smooth for $x \in \bar{\Sigma}_0$ and then, $\hat{w}(x) \equiv 0$ for $x \in \Sigma_0$. However, it obviously contradicts to the assumption (2.16). Hence the claim (2.15) is proved.

Now suppose that the maximum of w_0 occurs in a neighborhood of P_i . If the maximum of w_0 occurs near Q_i , we can prove (2.13) by the same argument as follows. Define v_i as (2.14), and \tilde{w}_0 by

$$\tilde{w}_0(y) = w_0(e^{-\frac{u_i(P_i)}{2}} y) \tag{2.17}$$

for $|y| \leq e^{u_i(P_i)/2} \delta_0$ for some fixed number $\delta_0 > 0$. Then \tilde{w}_0 satisfies

$$\Delta \tilde{w}_0 + \tilde{b}(y) \tilde{w}_0(y) = 0 \quad \text{for } |y| \leq e^{u_i(P_i)/2} \delta_0,$$

where

$$\tilde{b}(y) = \frac{e^{v_i(y)} - e^{v_i(y^-)}}{v_i(y) - v_i(y^-)}.$$

By Theorem 2.1, $\tilde{b}(y)$ is bounded by

$$|\tilde{b}(y)| \leq A(1 + |y|^2)^{-2} \tag{2.18}$$

for $|y| \leq e^{u_i(P_i)/2} \delta_0$ and for some constant A . Applying the Green representation formula, we have

$$|\tilde{w}_0(x)| \leq \frac{1}{2\pi} \int_{B_i} \log \frac{|x^- - y|}{|x - y|} |\tilde{b}_0(y)| |\tilde{w}_0(y)| + o(1)N_i, \tag{2.19}$$

where $B_i = \{y \mid y_1 \geq 0 \text{ and } |y| \leq e^{u_i(P_i)/2} \delta_0\}$,

$$G_0(x, y) = \frac{1}{2\pi} \log \frac{|x^- - y|}{|x - y|}$$

is the Green function for the half plane $\{y \mid y_1 > 0\}$, and the term $o(1)N_i$ comes from the boundary value of \tilde{w}_0 and (2.15). Let y_i be the maximum point of \tilde{w}_0 . Then by using (2.18), we estimate the integral of the right hand side of (2.19) by $(1 + |y_i|)^{-1}N_i$. Thus, (2.19) implies

$$N_i \leq c(1 + |y_i|)^{-1}N_i,$$

which implies y_i is bounded. Let $\lim_{i \rightarrow +\infty} y_i = y_0$. Then by elliptic estimates, there are a subsequence of $N_i^{-1} \tilde{w}_0$ such that $N_i^{-1} \tilde{w}_0$ converges to $w \neq 0$ in C_{loc}^2 which satisfies

$$\begin{cases} \Delta w + \rho e^{v(y)} w = 0 & \text{for } y_1 \geq 0, \\ w(y) = 0 & \text{for } y_1 = 0. \end{cases}$$

By (2.19), we have

$$|\tilde{w}_0(y)| \leq c N_i (1 + |y|)^{-1} + o(1)N_i$$

for $|y| \leq e^{u_i(P_i)/2} \delta_0$. Hence $|w(y)| \leq c(1 + |y|)^{-1}$, and then by Lemma 2.2 below, $w(y) = c \frac{\partial v}{\partial y_1}(y)$ for some constant $c \neq 0$. Let $\xi_i = e^{u_i(P_i)/2} P_i$. Since $\tilde{w}_0(\xi_i) > 0$ and $\frac{\partial v}{\partial y_1} < 0$ for $y_1 > 0$, we have $c < 0$. Hence, on any compact set, $\tilde{w}_0(y) > 0$ for i large. This proves (2.13) near P_i .

To prove (2.13) near Q_i , we note that

$$N_i^{-1} \frac{\partial \tilde{w}_0}{\partial y_1}(\xi_i) = N_i^{-1} \frac{\partial v_i}{\partial y_1}(\xi_i^-) = N_i^{-1} \frac{\partial^2 v_i}{\partial y_1^2}(\eta_i)(-2\xi_{i,1}), \tag{2.20}$$

where $\frac{\partial v_i}{\partial y_1}(\xi_i) = 0$ is used and $|\eta_i| \leq |\xi_i|$. Since $\frac{\partial^2 v}{\partial y_1^2}(0) < 0$, we have

$$\lim_{i \rightarrow +\infty} N_i^{-1} \cdot \xi_{i,1} = \lim_{i \rightarrow +\infty} N_i^{-1} e^{\frac{u_i(P_i)}{2}} |P_i| > 0. \tag{2.21}$$

By (2.21), we have

$$\lim_{i \rightarrow +\infty} N_i^{-1} e^{\frac{u_i(Q_i)}{2}} |Q_i - (a, b)| > 0$$

also. Thus, by using calculation similar to (2.20) near Q_i , we have

$$\lim_{i \rightarrow +\infty} N_i^{-1} |\nabla w_0^*(\xi_i^*)| = \lim_{i \rightarrow +\infty} N_i^{-1} e^{\frac{u_i(Q_i)}{2}} |Q_i - (a, b)| > 0,$$

where

$$w_0^*(y) = w_0((a, b) + e^{-\frac{u_i(Q_i)}{2}} y) \quad \text{and} \quad \xi_i^* = e^{\frac{u_i(Q_i)}{2}} (Q_i - (a, b)).$$

Therefore, $N_i^{-1} w_0^*(y)$ converges to a nonzero limit. Since Q_i is a local maximum, the limiting function is positive on \mathbf{R}_+^2 . Thus, for any compact set of $\bar{\mathbf{R}}_+^2$, $N_i^{-1} w_0^*(y)$ is positive on any compact set of \mathbf{R}_+^2 and for i large. This proves (2.13) near Q_i .

By (2.13), $w_0(x) > 0$ for $x \in \Sigma_0$. Thus, by moving the line $x_1 = \mu$ cross t_i , we prove $w_\mu(x) > 0$ in Σ_μ for $0 \leq \mu \leq 2t_i$. Clearly, it yields a contradiction. Therefore, the positions of P_i and Q_i are proved as claimed.

To prove the symmetry with respect to x , we consider $P_i = (0, 0)$, $Q_i = (a, b)$ and set $w_0(x) = u_i(x_1, x_2) - u_i(-x_1, x_2)$ for $x \in \Sigma_0$. Suppose $w_0(x) \not\equiv 0$ for $x \in \Sigma_0$. Then as the proof in case 2, (2.15) holds for any $\delta_0 > 0$. Thus, the maximum of $|w_0(x)|$ can occur only near either P_i or Q_i (not like the situation in case 2, $w_0(x)$ may not simultaneously be positive near P_i and near Q_i). Suppose the maximum of $|w_0(x)|$ occurs near P_i . Then by the Green formula (2.19), $\tilde{w}_i(y) = N_i^{-1} w(e^{-u_i(P_i)/2} y)$ converges to $w(y)$ in $C_{loc}^2(\mathbf{R}_+^2)$, where $w(y)$ satisfies

$$\begin{cases} \Delta w + \rho e^{v(y)} w = 0 & \text{in } \mathbf{R}_+^2 = \{(y_1, y_2) \mid y_1 \geq 0\}, \\ w(y_1, 0) = 0, & y_1 \in \mathbf{R}, \end{cases}$$

and $|w(y)| \leq c(1 + |y|)^{-1}$. By the reflection and Lemma 2.2, $w(y) = c \frac{\partial v}{\partial y_1}$ for some constant $c \neq 0$. But, since $\nabla \tilde{w}_i(0) = 2\nabla v_i(0) = 0$, we have $0 = \frac{\partial w}{\partial y_1}(0) = c \frac{\partial^2 v}{\partial y_1^2}(0) = \frac{\rho c}{2}$, which implies $c = 0$, a contradiction. Thus, the maximum $|w_0(x)|$ must occur near Q_i . But the same argument is applied to yield a contradiction when the maximum point is near Q_i . Therefore, $w_0(x)$ must vanish completely in Σ_0 . The symmetry of u_i follows readily. \square

LEMMA 2.2. – *Let $\varphi(x)$ be a solution of*

$$\Delta \varphi + e^{v(x)} \varphi = 0 \quad \text{in } \mathbf{R}^2, \tag{2.22}$$

where $v(x) = -2 \log(1 + |x|^2/8)$. Suppose that $\varphi(x)$ is bounded in \mathbf{R}^2 . Then

$$\varphi(x) = \sum_{j=0}^2 a_j \psi_j(x)$$

for some constants $a_j \in \mathbf{R}$, where $\psi_j(x) = (1 + |x|^2/8)^{-1} x_j$ for $j = 1, 2$ and $\psi_0(x) = (1 + |x|^2/8)^{-1} (1 - |x|^2/8)$.

Proof. – Set $\varphi_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(x) \cos k\theta \, d\theta$ for $k \geq 1$. Then φ_k satisfies

$$\begin{cases} \Delta \varphi_k(r) - \frac{k^2}{r^2} \varphi_k(r) + e^{v(r)} \varphi_k(r) = 0, \\ \varphi_k(0) = 0. \end{cases} \tag{2.23}$$

Let $\tilde{\varphi}_1 = \frac{1}{2\pi} \int_0^{2\pi} \psi_1(x) \cos \theta \, d\theta$. Then $\tilde{\varphi}_1$ satisfies (2.23) for $k = 1$. By the uniqueness of solutions of ODE, $\varphi_1(r) = c\tilde{\varphi}_1(r)$ for some constant $c \in \mathbf{R}$.

Suppose $\varphi_k(r) \not\equiv 0$ for some $k \geq 2$. Since $\tilde{\varphi}_1(r) > 0$, then, by the comparison with $\tilde{\varphi}_1(r)$, $\varphi_k(r)$ never vanishes for $r > 0$ and $k \geq 2$. By the assumption, $\varphi_k(r)$ is bounded. By an elementary argument and (2.22), $r\varphi'_k(r) \rightarrow 0$ as $r \rightarrow +\infty$. Comparing with $\tilde{\varphi}_1$, we have,

$$0 = \int_0^\infty (\tilde{\varphi}_1 \Delta \varphi_k - \varphi_k \Delta \tilde{\varphi}_1) r \, dr = (k^2 - 1) \int_0^\infty \frac{\varphi_k(r) \tilde{\varphi}_1(r)}{r} \, dr,$$

a contradiction since $\varphi_k(r)\tilde{\varphi}_1(r)$ has only one sign. Then, $\varphi_k(r) \equiv 0$ for $k \geq 2$. The conclusion of Lemma 2.2 follows. \square

3. Mean field equation on annulus domains

In this section, we consider a sequence of solutions u_i of

$$\begin{cases} \Delta u_i + \frac{\rho_i e^{u_i}}{\int_\Omega e^{u_i} \, dx} = 0 & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \{x \mid a < |x| < b\}$ for some $a < b$. Suppose that u_i has one single blow up point at P and $\max_{\bar{\Omega}} u_i = u_i(P_i) \rightarrow +\infty$ as $i \rightarrow +\infty$. Without loss of generality, we

may assume $P_i = (r_i, \pi/2)$ in the polar coordinate. Following the proof of Theorem 1.1, we can show that

$$w_0(x) = u_i(x) - u_i(x^-) > 0 \tag{3.1}$$

for $x_2 > 0$, where $x^- = (x_1, -x_2)$. Instead of the method of moving planes which was used in the second step of the proof of Theorem 1.1, we use the method of “rotating planes”. For any $\theta \in [0, \pi/2]$, let l_θ denote the line $\{(t \cos \theta, t \sin \theta) \mid t \in \mathbf{R}\}$ and Σ_θ be one of the components of $\Omega \setminus l_\theta$ such that $P_i \in \Sigma_\theta$. Set

$$w_\theta = u(x) - u(x^\theta), \tag{3.2}$$

where x^θ is the reflection point of x with respect to l_θ . For $\theta = 0$, we have $w_0(x) > 0$ for $x \in \Sigma_0$ by (3.1). Set

$$\theta_0 = \sup\{\theta \in [0, \pi/2] \mid w_{\tilde{\theta}}(x) > 0 \text{ for } x \in \Sigma_{\tilde{\theta}} \text{ and } 0 \leq \tilde{\theta} < \theta\}.$$

By the argument similar to step 2 of the proof of Theorem 1.1, we can prove $\theta_0 = \pi/2$ and $w_{\pi/2}(x) \equiv 0$, that is, $u_i(x)$ is symmetric with respect to x_1 , and

$$w_\theta(x) > 0 \quad \text{for } x \in \Sigma_\theta \text{ and } 0 \leq \theta < \frac{\pi}{2}. \tag{3.3}$$

By the Hopf lemma and (3.3), we have

$$\frac{\partial w_\theta}{\partial \nu_\theta}(x) > 0 \quad \text{for } x \in \partial \Sigma_\theta \cap l_\theta, \tag{3.4}$$

where $\nu_\theta(x)$ is the outnormal at $x \in \partial \Sigma_\theta \cap l_\theta$. Obviously, by (3.4), we have

$$\frac{\partial u_i}{\partial \theta}(x) = x_1 \frac{\partial u_i}{\partial x_2}(x) - x_2 \frac{\partial u_i}{\partial x_1}(x) > 0 \quad \text{for } x_1 > 0. \tag{3.5}$$

Note that $\partial u_i / \partial \theta$ is odd in x_1 . Hence $\partial u_i(x) / \partial \theta = 0$ on $x_1 = 0$, which implies we have

LEMMA 3.1. – *Let $\phi = \partial u_i / \partial \theta$. Then ϕ satisfies*

$$\begin{cases} \Delta \phi + \rho_i e^{v_i} \phi = 0 & \text{in } \Omega^+, \\ \phi(x) > 0 & \text{in } \Omega_+ \text{ and } \phi = 0 \text{ on } \partial \Omega^+, \end{cases} \tag{3.6}$$

where $\Omega^+ = \{(x_1, x_2) \in \Omega \mid x_1 \geq 0\}$ and $v_i = u_i - \log(\int_\Omega e^{u_i})$.

To estimate ρ_i , we need an isoperimetric inequality which was due to C. Bandle. See [1].

LEMMA 3.2. – *Suppose u is a solution of $\Delta u + \rho e^u = 0$ in ω , where ω is a simply connected domain. If $\rho \int_\omega e^u dx < 4\pi$, then the first eigenvalue of $\Delta + \rho e^u$ for the Dirichlet problem is positive.*

Clearly, $\rho_i \geq 8\pi$ by Lemma 3.1 and Lemma 3.2. However, we require a stronger version of Lemma 3.2 to ensure $\rho_i > 8\pi$. In order to see whether Lemma 3.2 holds or

not when $\rho \int_{\omega} e^u dx = 4\pi$, we have to go back to the proof of Lemma 3.2. Since the proof is not very long, we present here for the completeness.

Proof of Theorem 1.3. – Let $v_i = u_i + c_i$ for some constant c_i such that v_i satisfies $\Delta v_i + e^{v_i} = 0$ in Ω and $\phi = \partial v_i / \partial \theta$. Then $\rho_i = \int_{\Omega} e^{v_i} dx \rightarrow 8\pi$ as $i \rightarrow +\infty$. Set $\Omega^+ = \{(x_1, x_2) \mid x_1 > 0\}$. So, $\phi(x) > 0$ in Ω^+ , and $\phi(x) = 0$ on $\partial\Omega^+$. Let $v(x) = v(|x|) = -2 \log(1 + \frac{1}{8}|x|^2)$ be the solution of

$$\begin{cases} \Delta v + e^v = 0 & \text{in } \mathbf{R}^2, \\ \int_{B_1} e^v dx = 4\pi. \end{cases} \tag{3.7}$$

Now assume $\int_{\Omega^+} e^{v_i} dx = 4\pi$. Let $\Omega_t^+ = \{x \in \Omega^+ \mid \phi(x) > t\}$ for $t > 0$. Set B_t^* be the open ball with center 0 such that

$$\int_{B_t^*} e^v dx = \int_{\Omega_t^+} e^{v_i} dx = A(t). \tag{3.8}$$

Define the spherically decreasing rearrangement ϕ^* of ϕ to be a non-negative function in B_1 such that

$$\phi^*(x) = \phi^*(|x|) = \sup\{t \mid x \in B_t^*\}. \tag{3.9}$$

Obviously, $\phi^*(x) > t_0$ if and only if $x \in B_{t_0}^*$ (here we use the fact that B_t^* is open). Thus, $\phi^*(x)$ has the same distribution function (with respect to the measure $e^v dx$) as ϕ (with respect to the measure $e^{v_i} dx$). So, the identity

$$\int_{B_1} \phi^{*2} e^v dx = \int_{\Omega^+} \phi^2 e^{v_i} dx \tag{3.10}$$

holds. To prove

$$\int_{\Omega^+} |\nabla \phi|^2 dx \geq \int_{B_1} |\nabla \phi^*|^2 dx. \tag{3.11}$$

We need Bol's inequality which states that for any $\omega \subset\subset \Omega^+$,

$$\frac{1}{2}(8\pi - A(\omega))A(\omega) \leq l^2(\partial\omega), \tag{3.12}$$

provided that $A(\Omega^+) \leq 8\pi$, where

$$A(\omega) = \int_{\omega} e^{v_i} dx \quad \text{and} \quad l(\partial\omega) = \int_{\partial\omega} e^{\frac{1}{2}v_i} d\sigma.$$

We note that the equality of (3.12) holds when the measure $e^v dx$ is used and ω is a ball with center 0.

Since ϕ satisfies an elliptic equation, it is easy to prove the Lebesgue measure of $\partial\Omega_t$ is equal to zero for any $t > 0$. Thus, $A(t)$ is continuous and strictly decreasing in t . By

the co-area formulas,

$$-A'(t) = \int_{\{\phi=t\}} \frac{e^{v_i}}{|\nabla\phi|} ds \quad \text{a.e. } t > 0, \tag{3.13}$$

and

$$-\frac{d}{dt} \left(\int_{\Omega_t^+} |\nabla\phi|^2 dx \right) = \int_{\{\phi=t\}} |\nabla\phi| ds \quad \text{a.e. } t > 0 \tag{3.14}$$

hold. By the Schwarz inequality and Bol's inequality, we have

$$\begin{aligned} -\frac{d}{dt} \left(\int_{\Omega_t^+} |\nabla\phi|^2 dx \right) &\geq \left(\int_{\{\phi=t\}} e^{v_i/2} dx \right)^2 \left(\int_{\{\phi=t\}} \frac{e^{v_i}}{|\nabla\phi|} dx \right)^{-1} \\ &= l^2(\partial\Omega_t^+) (-A'(t))^{-1} \\ &\geq \frac{1}{2} (8\pi - A(t)) A(t) (-A'(t))^{-1} \\ &= -\frac{d}{dt} \left(\int_{B_t^*} |\nabla\phi^*|^2 dx \right) \end{aligned} \tag{3.15}$$

for a.e. $t > 0$, where the equality of (3.12) is used for the metric $e^v |dx|^2$. Integrating (3.15), one has

$$\int_{\Omega^+} |\nabla\phi|^2 dx \geq \int_{B_1} |\nabla\phi^*|^2 dx \tag{3.16}$$

as claimed by (3.11).

Together (3.10) and (3.11), we have

$$\int_{B_1} |\nabla\phi^*|^2 - \int_{B_1} \phi^{*2} e^v dx \leq \int_{\Omega^+} |\nabla\phi|^2 - \int_{\Omega^+} \phi^2 e^{v_i} dx = 0. \tag{3.17}$$

Since the first eigenvalue of $\Delta + e^v$ for the Dirichlet problem is equal to zero, (3.17) implies

$$\int_{B_1} |\nabla\phi^*|^2 - \int_{B_1} \phi^{*2} e^v dx = 0,$$

that is, each inequality in (3.15) must be an equality. In particular,

$$e^{v_i} = \Phi_i(\phi) |\nabla\phi|^2 \tag{3.18}$$

holds for some function Φ_i . Clearly, Φ_i is continuous at $\phi = t$ as long as t is not a critical value of ϕ . Since $\phi = 0$ is not a critical value of ϕ , one has $|\nabla\phi|(x) \equiv \text{constant}$ whenever $|x| = b$ or $|x| = a$. Since $\partial\phi/\partial\theta \equiv 0$ for $x \in \partial\Omega$, $\phi_r(b, \theta)$ is a constant independent of θ .

However,

$$\int_0^{2\pi} \phi_r(b, \theta) \, d\theta = \int_0^{2\pi} (u_r)_\theta(b, \theta) \, d\theta = 0,$$

which implies $|\nabla\phi(x)| \equiv 0$ on $\partial\Omega$. Clearly, it yields a contradiction to the Hopf boundary lemma. \square

Let $G(x, y)$ denote the Green function of Δ with vanishing boundary value. Set

$$\psi(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}, \quad \text{and} \tag{3.19}$$

$$\varphi(x) = \psi(x, x) \tag{3.20}$$

is the regular part of the Green function. Recall that $\Omega = \{x \mid a < |x| < b\}$.

LEMMA 3.3. – *Let Ω be the radially symmetric annulus $\{x \mid a < |x| < b\}$ and $\varphi(x)$ be the regular part of the Green function. Then $\varphi(x) = \varphi(|x|)$ and $\varphi(r)$ has a unique critical point r_0 . Furthermore, r_0 is the maximum point and $\varphi''(r_0) < 0$.*

Proof. – By the uniqueness of the Green function, it is easy to see that $\varphi(x)$ is radially symmetric. Since $\lim_{x \rightarrow \partial\Omega} \varphi(x) = -\infty$, $\varphi(r)$ has a critical point r_0 at least. We want to prove $\varphi''(r_0) < 0$ for any critical point r_0 . Then Lemma 3.3 follows readily.

Let $y_0 = r_0 e_2 = (0, r_2)$. Since $\psi(x, y) = \psi(y, x)$, we have

$$\nabla_x \psi(y_0, y_0) = \nabla_y \psi(y_0, y_0) = 0, \tag{3.21}$$

where

$$\nabla_x \psi(x, y) = \left(\frac{\partial \psi}{\partial x_1}(x, y), \frac{\partial \psi}{\partial x_2}(x, y) \right)$$

is the first derivative of ψ with respect to x -variable. Also, we have

$$\psi(x, y_0) = \psi(x^-, y_0), \tag{3.22}$$

where $x^- = (-x_1, x_2)$. Set

$$\frac{\partial \psi}{\partial \theta}(x, y_0) = x_1 \frac{\partial \psi}{\partial x_2}(x, y_0) - x_2 \frac{\partial \psi}{\partial x_1}(x, y_0).$$

Note that $\psi(x, y_0) = \frac{1}{2\pi} \log |x - y_0|$ for $x \in \partial\Omega$. Together with (3.22), $\frac{\partial \psi}{\partial \theta}(x, y_0)$ satisfies

$$\frac{\partial \psi}{\partial \theta}(x, y_0) \leq 0 \quad \text{for } x \in \partial\Omega^+,$$

where $\Omega^+ = \{x \in \Omega \mid x_1 > 0\}$. By the maximum principle, $\frac{\partial \psi}{\partial \theta}(x, y_0) < 0$ for $x \in \Omega^+$. Thus,

$$\psi(|x|e_2, y_0) \leq \psi(x, y_0)$$

for $x \in \Omega$. By (3.21) and the maximum principle, we have

$$\frac{\partial^2 \psi}{\partial x_1^2}(y_0, y_0) \geq 0,$$

which implies

$$\frac{\partial^2 \psi}{\partial x_2^2}(y_0, y_0) \leq 0. \tag{3.23}$$

On the other hand, $\frac{\partial \psi}{\partial y_2}(x, y_0)$ is a harmonic function which by a straightforward computation, satisfies

$$\frac{\partial \psi}{\partial y_2}(x, y_0) \begin{cases} > 0 & \text{if } |x| = a \text{ or } x = (b \cos \theta, b \sin \theta) \text{ with } -\pi/2 \leq \theta < \theta_0, \\ < 0 & \text{if } x = (b \cos \theta, b \sin \theta) \text{ with } \theta_0 < \theta \leq \pi/2, \end{cases} \tag{3.24}$$

where $\sin \theta_0 = r_0/b$. Note that $\frac{\partial \psi}{\partial y_2}(x, y_0) = \frac{\partial \psi}{\partial y_2}(x^-, y_0)$. Hence the nodal line $\{x \mid \frac{\partial \psi}{\partial y_2}(x, y_0) = 0\}$ must intersect with the positive x_2 -axis only at $x = r_0 e_2$. Otherwise, by the symmetry of $\frac{\partial \psi}{\partial y_2}(x, y_0)$ in x_1 , the nodal line would enclose a region in Ω , which violates the maximum principle. Therefore, by (3.24) and (3.21),

$$\begin{aligned} \frac{\partial \psi}{\partial y_2}(r_0 e_2, y_0) &= 0 \quad \text{and,} \\ \frac{\partial \psi}{\partial y_2}(r e_2, y_0) &< 0 \quad \text{for } r_0 < r \leq b. \end{aligned}$$

By the Hopf lemma, one has

$$\frac{\partial^2 \psi}{\partial x_2 \partial y_2}(y_0, y_0) < 0. \tag{3.25}$$

Clearly, by (3.23) and (3.25), we have

$$\psi''(r_0) = \frac{\partial^2 \psi}{\partial x_2^2}(y_0, y_0) + \frac{\partial^2 \psi}{\partial y_2^2}(y_0, y_0) + 2 \frac{\partial^2 \psi}{\partial x_2 \partial y_2}(y_0, y_0) < 0. \quad \square$$

LEMMA 3.4. – *Let $G(x, y)$ be the Green function and $y_1 = r_1 e_2$ for some $a < r_1 < b$. Then $G(r e_2, y_1)$ increases in r for $r \in [a, r_1]$ and decreases in r for $r \in (r_1, b]$.*

Proof. – Let

$$\frac{\partial G}{\partial \theta}(x, y_1) = x_1 \frac{\partial G}{\partial x_2}(x, y_1) - x_2 \frac{\partial G}{\partial x_1}(x, y_1).$$

In a neighborhood of y_1 , $\frac{\partial G}{\partial \theta}(x, y_1)$ satisfies

$$\frac{\partial G}{\partial \theta}(x, y_1) = \frac{x_1 r_1}{|x - y_1|^2} + \text{smooth function}. \tag{3.26}$$

Since $\frac{\partial G}{\partial \theta}(x, y_0) = 0$ for $x \in \Omega^+ \setminus \{y_1\}$ where $\Omega^+ = \{x \in \Omega \mid x_1 > 0\}$, by the maximum principle and (3.26), we have

$$\frac{\partial G}{\partial \theta}(x, y_1) > 0 \quad \text{for } x \in \Omega_1.$$

There,

$$G(x, y_1) < G(|x|e_2, y_1) \quad \text{for } x \in \Omega. \quad (3.27)$$

Now suppose $G(re_2, y_1)$ has a critical point in $[a, r_1)$. Since $G(ae_2, y_1) = 0$ and $\lim_{r \rightarrow r_1} G(re_2, y_1) = +\infty$, we may assume $G(re_2, y_1)$ has a local maximum at $r = r_0$. Together with (3.27), we have $G(x, y_1)$ has a local maximum at $x = r_0e_2$, which violate the strong maximum principle. \square

Now we are in the position to finish the proof of Theorem 1.4.

Proof of Theorem 1.4. – Suppose that u_i blows up at the two points P and Q , and P_i and Q_i are local maximum points of u_i near P and Q respectively. As in Theorem 1.2, we first claim that P_i , Q_i and the origin O is on a straight line. Suppose the claim does not hold. By composing with a rotation, we assume that $P_i = (t_i, s_i)$ and $Q_i = (t_i, \tilde{s}_i)$ for some $t_i > 0$, and $|P_i - P| = |Q_i - Q|$. By following the argument of (2.8), we can show that

$$w_0(x) = u_i(x) - u_i(x^-) > 0 \quad \text{for } x \in \Omega^+,$$

where $\Omega^+ = \{x \mid x_1 > 0\}$. Then, as in the beginning of this section, we could use the method of rotating planes to yield a contradiction. Hence we conclude that P_i , Q_i and O are on the same straight line l_i . The symmetry of u_i with respect to l_i can be proved similarly.

For simplicity of notations, we may assume both P_i and Q_i are located on the x_2 -axis. Without loss of generality, we may assume that P_i is on the positive x_2 -axis. We claim

$$Q_i \text{ is on the negative } x_2\text{-axis.} \quad (3.28)$$

We prove (3.28) by contradiction. Suppose Q_i is on the positive x_2 -axis. Let $P = (0, r_2)$ and $Q = (0, r_1)$. By Theorem 2.1, $\frac{\rho_i e^{u_i}}{\int_{\Omega} e^{u_i} dx}$ converges to $8\pi(\delta(P) + \delta(Q))$, where $\delta(P)$ is the Dirac measure at P . Thus, $u_i(x)$ converges to $8\pi[G(x, P) + G(x, Q)]$ in $C_{\text{loc}}^2(\Omega \setminus \{P, Q\})$. By the Pohozaev identity, we have for any unit vector e in \mathbf{R}^2 ,

$$0 = \int_{|x-P|=r} \langle e, \nabla u_i \rangle \frac{\partial u_i}{\partial \nu} - \frac{\langle e, \nu \rangle}{2} |\nabla u_i|^2 d\sigma.$$

By passing to the limit, we have

$$0 = \int_{|x-P|=r} \langle e, \nabla G \rangle \frac{\partial G}{\partial \nu} - \frac{\langle e, \nu \rangle}{2} |\nabla G|^2 d\sigma, \quad (3.29)$$

where

$$\begin{aligned} G(x) &= 8\pi[G(x, P) + G(x, Q)] \\ &= 4 \log\left(\frac{1}{|x - P|}\right) + 8\pi[\psi(x, P) + G(x, Q)] \end{aligned}$$

by (3.19). By a straightforward computation, (3.29) implies

$$\langle e, \nabla_x(\psi(x, P) + G(x, Q)) \rangle = 0 \quad \text{at } x = P \tag{3.30}$$

as $r \rightarrow 0$. Since $\psi(x, P)$ and $G(x, Q)$ is an even function in x_1 , (3.30) is equivalent to

$$\frac{\partial G(x, Q)}{\partial x_2} + \frac{\partial \psi(x, P)}{\partial x_2} = 0 \quad \text{at } x = P,$$

that is,

$$2 \frac{\partial G(P, Q)}{\partial x_2} + \frac{\partial \varphi}{\partial x_2}(P) = 0. \tag{3.31}$$

Similarly, when the Pohozaev identity is applied at Q , we have

$$2 \frac{\partial G}{\partial x_2}(Q, P) + \frac{\partial \varphi}{\partial x_2}(Q) = 0. \tag{3.32}$$

Set $P = (0, r_1)$ and $Q = (0, r_2)$. Suppose

$$a < r_2 < r_1 < b. \tag{3.33}$$

Then, by Lemma 3.4, we have

$$\frac{\partial G}{\partial x_2}(Q, P) > 0 > \frac{\partial G}{\partial x_2}(P, Q).$$

Thus, by (3.31) and (3.32), we have

$$\frac{d}{dr}\varphi(Q) < 0 < \frac{d}{dr}\varphi(P). \tag{3.34}$$

Let r_0 be the maximum point $\varphi(r)$. Then by Lemma 3.3, (3.34) implies

$$r_1 < r_0 < r_2,$$

which yields a contradiction to (3.33). The other assumption $a < r_1 < r_2 < b$ yields a similar contradiction by Lemma 3.3, Lemma 3.4 and (3.32). Hence the claim (3.28) is proved.

Set $\phi_i(x) = \frac{\partial u_i}{\partial \theta}(x)$. Then $\phi_i(x)$ satisfies $\Delta \phi_i + e^{v_i} \phi_i = 0$ in Ω^+ and $\phi_i = 0$ on $\partial \Omega^+$, where $v_i = u_i + c_i$ for some suitable constant c_i and $\Omega^+ = \{(x_1, x_2) \in \Omega \mid x_1 > 0\}$. Note that ϕ_i cannot be one sign in Ω^+ because u_i has two local maximum point P_i and Q_i which are located in the opposite direction of the x_2 -axis. Furthermore, the nodal line $\{x \in \Omega^+ \mid \phi_i(x) = 0\}$ must intersect with the boundary of $\partial \Omega^+$ at two points at least. This can be seen by the scaling in the following way.

Set

$$\tilde{\phi}_i(y) = \phi_i(e^{-\frac{u_i(P_i)}{2}}y)$$

as in (2.17). Then by multiplying a constant, $\tilde{\phi}_i(y)$ converges to $c \frac{\partial v}{\partial y_1}$ in $C^2_{loc}(\mathbf{R}^2)$, where $c \neq 0$ and $\frac{\partial v}{\partial y_1}(y) = -y_1/(1 + \frac{1}{8}|y|^2)$. Note that u_i has a local maximum at P_i . Thus, ϕ_i is positive in a neighborhood of P_i in Ω^+ . The same holds for Q_i also, that is, $\phi_i(x)$ is negative in a neighborhood of Q_i . This proves that the nodal line of ϕ_i intersects at least twice with $\partial\Omega^+$.

Since Ω^+ is simply connected, the fact about the nodal line implies that Ω^+ contains two disjoint, simply-connected subdomain Ω_1^+ and Ω_2^+ where $\phi_i(x)$ vanishes on the boundary of Ω_i^+ , $i = 1, 2$. By Lemma 3.2,

$$\rho_i \int_{\Omega_k^+} e^{u_i} dx \geq 4\pi \tag{3.35}$$

for $k = 1, 2$. If there exists $\Omega_{k_0}^+$ such that

$$\rho_i \int_{\Omega_{k_0}^+} e^{u_i} dx > 4\pi.$$

Then $\rho_i > 16\pi$ as claim. If (3.35) turns out to be an equality for each k , then by using the spherical symmetrization as (3.9), we can prove that $|\nabla\phi_i|(x) \equiv \text{constant}$ for $|x| = b$. Since $\phi_{i\theta} \equiv 0$ for $|x| = b$, we have $\phi_{ir}(b, \theta)$ is a constant independent of θ . By integrating $\phi_{ir}(b, \theta)$ along θ , we obtain a contradiction as the last step of the proof of Theorem 1.3. This ends the proof of Theorem 1.4. \square

4. Existence of blowing-up solutions

In this section, we want to prove the existence of blowup solutions of the following equation:

$$\begin{cases} \Delta u_i + \lambda_i e^{u_i} = 0 & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\lambda_i \rightarrow 0$. In particular, we want to seek solutions such that

$$\lambda_i \int_{\Omega} e^{u_i} dx \rightarrow 8m\pi$$

as $\lambda_i \rightarrow 0$ when Ω is a radially symmetric annulus. Write $\lambda_i = \rho_i(\int_{\Omega} e^{u_i} dx)^{-1}$. Then u_i is a solution of (1.4) with ρ_i such that $\rho_i \rightarrow 8m\pi$. Our main result in this section is Theorem 4.2 below.

When Ω is a simply-connected domain under some nondegenerate condition, solutions with one blowup point was constructed in [17,18] and [25]. In general, there

seems no results about the existence of solutions with one point blowup. For Eq. (1.4) with $\rho \in (8\pi, 16\pi)$, solutions were obtained by Ding et al. [12]. However, it is not clear whether their solution would blowup or not as $\rho \downarrow 8\pi$. (By Theorem 1.4, their solutions remains bounded as $\rho \uparrow 16\pi$.) On the other hand, since the nonlinear term $e^u > 0$ at $u = 0$, there always exists a minimal solutions $u_\lambda \rightarrow 0$ as $\lambda \rightarrow 0$. It is easy by the well-known Mountain Pass lemma to prove that for any fixed small $\lambda > 0$, (4.1) possesses a second solution u_λ at least, and the solution u_λ must blow up at some point as $\lambda \rightarrow 0$. It is plausible to guess that u_λ has one single blowup point only. This result might be known to experts. But, there seems no proofs in the literature as far as the authors know. For the sake of the completeness, we present the proof here. When Ω is an annulus, we also construct solutions with m blowup points which satisfies

$$\lambda_i \int_{\Omega} e^{u_i(x)} dx > 8m\pi. \tag{4.2}$$

In the following, we consider a more general nonlinear term $f(t)$ which satisfies $f(0) > 0$ and

$$\lim_{t \rightarrow +\infty} f(t)e^{-t} = 1. \tag{4.3}$$

Denote $F(t) = \int_0^t f(s) ds$. Let v_λ be the minimal solution of

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Set

$$J_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} \{F(v_\lambda + u) - F(v_\lambda) - f(v_\lambda)u\} \tag{4.5}$$

for $u \in H^1(\Omega)$. Then $J_\lambda \in C^1(H^1(\Omega))$ and satisfies the Palais–Smale condition. Suppose that there is $e_\lambda \in H^1(\Omega)$ such that $J_\lambda(e_\lambda) \leq 0$. Let

$$c_\lambda \equiv \inf_{\gamma} \max_{0 \leq t \leq 1} J_\lambda(\gamma(t)) \tag{4.6}$$

where γ is any path connecting 0 and e_λ . The well-known Mountain Pass Lemma says that c_λ is a critical value of J_λ . Let w_λ be a critical point of J_λ with $J_\lambda(w_\lambda) = c_\lambda$. Then $u_\lambda \equiv v_\lambda + w_\lambda$ is a solution of (4.1). It is easy to see that solutions obtained in this manner must blow up as $\lambda \rightarrow 0$. We will prove that, by choosing a particular e_λ , u_λ blows up at a single point as $\lambda \rightarrow 0$.

THEOREM 4.1. – *Let u_λ be the solution described above. Then u_λ blows up at a single point as $\lambda \rightarrow 0$.*

Proof. – Let B be a ball contained in Ω , and U_λ^ε be the large-norm solution of

$$\begin{cases} \Delta u + \lambda e^{(1-\varepsilon)u} = 0, \\ u(x) > 0 & \text{in } B, \\ u(x) = 0 & \text{on } \partial B, \end{cases} \tag{4.7}$$

where ε is a fixed small positive number. Then we have

$$\lambda \int_B e^{(1-\varepsilon)U_\lambda^\varepsilon} = \frac{8\pi}{1-\varepsilon} (1 + o(1)) \quad \text{as } \lambda \downarrow 0, \tag{4.8}$$

$$\lambda \int_B U_\lambda^\varepsilon e^{(1-\varepsilon)U_\lambda^\varepsilon} = \int_B |\nabla U_\lambda^\varepsilon|^2 = \frac{16\pi}{(1-\varepsilon)^2} (1 + o(1)) \log \frac{1}{\lambda}, \tag{4.9}$$

and

$$\int_B U_\lambda^\varepsilon \, dx \leq c \tag{4.10}$$

where c is a constant independent of λ and ε .

Extend U_λ^ε to be zero outside B , then $U_\lambda^\varepsilon \in H^1(\Omega)$. We have

$$\begin{aligned} J_\lambda((1+\varepsilon)U_\lambda^\varepsilon) &= \frac{(1+\varepsilon)^2}{2} \int_\omega |\nabla U_\lambda^\varepsilon|^2 \, dx - \lambda \int_\Omega [F((1+\varepsilon)U_\lambda^\varepsilon + v_\lambda) \\ &\quad - F(v_\lambda) - f(v_\lambda)(1+\varepsilon)U_\lambda^\varepsilon] \, dx. \end{aligned}$$

When the value of $U_\lambda^\varepsilon(x)$ is large, by (4.3),

$$\begin{aligned} F((1+\varepsilon)U_\lambda^\varepsilon + v_\lambda) &\geq e^{(1-\varepsilon)[(1+\varepsilon)U_\lambda^\varepsilon + v_\lambda]} - c_1(\varepsilon) \\ &\geq c_2 \varepsilon^2 (U_\lambda^\varepsilon)^2 e^{(1-\varepsilon)U_\lambda^\varepsilon} - c_1(\varepsilon). \end{aligned}$$

When ε is fixed, we have

$$\int_\Omega (U_\lambda^\varepsilon)^2 e^{(1-\varepsilon)U_\lambda^\varepsilon} / \int_\Omega U_\lambda^\varepsilon e^{(1-\varepsilon)U_\lambda^\varepsilon} \rightarrow +\infty$$

as $\lambda \rightarrow 0$. By observing that v_λ is uniformly bounded in λ , it is easy to see that

$$J_\lambda((1+\varepsilon)U_\lambda^\varepsilon) \rightarrow -\infty \quad \text{as } \lambda \downarrow 0.$$

Let $e_\lambda = (1+\varepsilon)U_\lambda^\varepsilon$, then for small λ ,

$$J_\lambda(e_\lambda) \leq 0,$$

and let

$$c_\lambda \equiv \inf_\gamma \max_{0 \leq t \leq 1} J_\lambda(\gamma(t)),$$

where γ is any continuous path connecting 0 and e_λ . By the Mountain Pass Lemma, we know that c_λ is a critical value. Let w_λ be one of the critical points of J_λ with $J_\lambda(w_\lambda) = c_\lambda$. Then $u_\lambda = w_\lambda + v_\lambda$ is a solution of (4.4).

To compute $\int_\Omega |\nabla u_\lambda|^2$, one have

$$\lambda \int_\Omega u_\lambda f(u_\lambda) dx = \int_\Omega |\nabla u_\lambda|^2 = 2J_\lambda(w_\lambda) + 2\lambda \int_\Omega F(u_\lambda) - F(v_\lambda) - f(v_\lambda)w_\lambda + O(1).$$

Fixed $\varepsilon > 0$, one have

$$\lambda \int_\Omega F(u_\lambda) < \varepsilon \lambda \int_\Omega u_\lambda F(u_\lambda) < c\varepsilon \lambda \int_\Omega u_\lambda f(u_\lambda)$$

for all small λ . Hence

$$\begin{aligned} O(1) + \int_\Omega |\nabla u_\lambda|^2 &\leq \frac{2}{1 - c\varepsilon} J_\lambda(w_\lambda) \\ &\leq \frac{2}{1 - c\varepsilon} \max_{0 \leq t \leq 1 + \varepsilon} J_\lambda(tU_\lambda^\varepsilon) \\ &\leq \frac{(1 + \varepsilon)^2}{1 - c\varepsilon} \int_{\bar{\Omega}} |\nabla U_\lambda^\varepsilon(x)|^2 dx + O(1) \\ &\leq \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^2(1 - c\varepsilon)} 16\pi \log 1/\lambda(1 + o(1)), \end{aligned}$$

by (4.5). Let ε satisfy

$$\frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^2(1 - c\varepsilon)} < \frac{3}{2}.$$

By Lemma 4.3 (below), we know that the blow-up point of u_λ consist of one single-point only and $\lambda \int_\Omega f(u) dx = 8\pi(1 + o(1))$. Hence the proof of Theorem 4.1 is complete. \square

THEOREM 4.2. – *Let Ω be a radially symmetric annulus and $f(t) = e^t$. Let m be any positive integer. Then for small $\lambda > 0$, there exists a solutions u_λ of (4.1) such that*

$$\lambda \int_\Omega e^{u_\lambda} dx > 8m\pi, \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \lambda \int_\Omega e^{u_\lambda} dx = 8m\pi. \tag{4.11}$$

Proof. – We will prove Theorem 4.2 in a similar manner. Let T_m be the rotation with an angle $2\pi/m$. Consider J_λ of (4.5) in the pace $\mathring{H}^1_m(\Omega)$, where $\mathring{H}^1_m(\Omega) = \{u \in \mathring{H}^1(\Omega) \mid u(T_m(x)) = u(x)\}$. As in the proof of Theorem 4.1, we construct a $U_\lambda^\varepsilon \in \mathring{H}^1_m(\Omega)$ such that

$$\lambda \int_\Omega U_\lambda^\varepsilon e^{(1-\varepsilon)U_\lambda^\varepsilon} = \frac{16m\pi}{(1 - e)^2} \log \frac{1}{\lambda}(1 + o(1)). \tag{4.12}$$

And, we have

$$J_\lambda((1 + \varepsilon)U_\lambda^\varepsilon) \rightarrow -\infty \quad \text{as } \lambda \rightarrow 0.$$

Let $e_\lambda = (1 + \varepsilon)U_\lambda^\varepsilon$, and

$$c_\lambda = \inf_\gamma \max_{0 \leq t \leq 1} J_\lambda(\gamma(t)),$$

where γ is any continuous path in $\overset{\circ}{H}^1_m(\Omega)$ connecting 0 and e_λ . Hence there exists a critical point $w_\lambda \in \overset{\circ}{H}^1_m(\Omega)$ with $J_\lambda(w_\lambda) = c_\lambda$. Then $u_\lambda = v_\lambda + w_\lambda$ is a solution of (4.4), and invariant under T_m . In a similar manner as shown above, we can choose ε small so that

$$\int_\Omega |\nabla u_\lambda|^2 \leq 24m\pi \log 1/\lambda(1 + o(1)) \quad \text{as } \lambda \downarrow 0.$$

Since the number of blow-up points of u_λ is a multiple of m , by (4.15) of Lemma 4.3 (below), the above inequality implies that the blow-up set of u_λ consists of m points. Hence, $\lambda \int_\Omega f(u_\lambda) dx = 8\pi m(1 + o(1))$ as $\lambda \downarrow 0$.

We give a sketch of the proof for (4.11). For simplicity, we assume $m = 3$. The proof for the general case is similar. Now suppose u_i has the local maximum at P_i, Q_i, \tilde{Q}_i , where P_i is on the positive x_1 -axis and Q_i, \tilde{Q}_i are located at the rays having the angle $2\pi/3$ and $4\pi/3$ to the positive x_1 -axis respectively. we want to prove $u_i(x)$ is symmetric with respect to the x_1 -axis. Note that Q_i is the reflection point of \tilde{Q}_i with respect to the x_1 -axis.

Set $w(x_1, x_2) = u_i(x_1, x_2) - u_i(x_1, -x_2)$ for $x_2 \geq 0$. Then w satisfies

$$\begin{cases} \Delta w + b(x)w(x) = 0 & \text{in } \Omega^+ = \{x \in \Omega \mid x_2 \geq 0\}, \\ w|_{\partial\Omega^+} = 0. \end{cases} \tag{4.13}$$

Suppose $w(x) \not\equiv 0$ in Ω^+ . Then it can be proved as (2.15) such that for any $\delta > 0$,

$$\sup_{\Omega^+(\delta)} |w(x)| = o(1)N_i \tag{4.14}$$

holds, where $o(1) \rightarrow 0$ as $i \rightarrow +\infty$,

$$N_i = \max_{\Omega^+} |w(x)|$$

and

$$\Omega^+(\delta) = \{x \in \Omega^+ \mid |x - Q_i| \geq \delta \text{ and } |x - P_i| \geq \delta\}.$$

Now suppose the maximum of w occurs near Q_i . Note that $w(x) = 0$ for $x \in \{(t \cos \frac{2\pi}{3}, t \sin \frac{2\pi}{3}), t \geq 0\}$ because u_i is invariant under the group action of T_3 . By rotation and translation, we assume $Q_i = (0, 0)$, $w(x)$ satisfies

$$\begin{cases} \Delta w(x) + b(x)w(x) = 0 & \text{for } |x| \leq \delta \text{ and } x_2 \geq 0, \\ w(x_1, 0) = 0, \end{cases}$$

and the maximum of $w(x)$ occurs near Q_i , say,

$$|w(x_i)| = N_i.$$

By scaling as (2.17), we consider $\tilde{w}(y) = w(e^{-u_i(Q_i)/2}y)$ for $|y| \leq e^{u_i(Q_i)/2}\delta$. Then $\tilde{w}(y)$ satisfies

$$\Delta \tilde{w} + \tilde{b}(y)\tilde{w}(y) = 0 \quad \text{for } |y| \leq e^{\frac{u_i(Q_i)}{2}}\delta$$

with

$$|\tilde{b}(y)| \leq c(1 + |y|^2)^{-2}.$$

We apply the Green representation formula (2.19) to prove $e^{u_i(Q_i)/2}|x_i|$ is bounded. Furthermore, $N_i^{-1}\tilde{w}(x)$ tends to some function $\tilde{w}(x)$ in $C_{loc}^2(\mathbf{R}^2)$, where \tilde{w} satisfies

$$\begin{cases} \Delta \tilde{w} + e^{v(x)}\tilde{w} = 0 & \text{in } \mathbf{R}_+^2, \\ \tilde{w}(x_1, 0) = 0. \end{cases}$$

By the reflection, we have

$$\tilde{w}(x) = \frac{cx_1}{1 + \frac{1}{8}|x|^2} \quad \text{in } \mathbf{R}_+^2$$

for some constant $c \neq 0$. On the other hand,

$$\tilde{w}(0) = |\nabla \tilde{w}(0)| = 0$$

because u_i has local maximum at Q_i and \tilde{Q}_i . Then

$$0 = \frac{\partial \tilde{w}}{\partial x_1}(0, 0) = c \neq 0$$

yields a contradiction. Hence the maximum of w must occur near P_i . By using the same argument, it also yields a contradiction. Therefore the symmetry of u_i is proved.

Let $\varphi = \partial u_i / \partial \theta$. Then φ vanishes at the rays $\{(t \cos \theta, t \sin \theta) \mid t \geq 0\}$ with $\theta = k\pi/3$, $k = 0, 1, \dots, 5$. By the isoperimetric inequality (see the proof of Theorem 1.3), we have

$$\lambda_i \int_{0 \leq \theta \leq \frac{\pi}{3}} e^{u_i(x)} dx > 4\pi.$$

Therefore

$$\lambda_i \int_{\Omega} e^{u_i}(x) dx > 24\pi$$

as claimed. \square

LEMMA 4.3. – *Suppose that u_i is a sequence of solutions of (4.4) with $\lambda_i \rightarrow 0$. Then after passing to a subsequence (still denoted by u_i), one of the following statements hold.*

- (i) $\lambda_i \int_{\Omega} f(u_i) dx \rightarrow +\infty$ as $i \rightarrow +\infty$. In this case, $u_i(x) \rightarrow +\infty$ uniformly in any compact set of Ω ,
- (ii) $\lim_{i \rightarrow +\infty} \lambda_i \int_{\Omega} f(u_i) dx = 8m\pi$ for some integer $m \in \{0, 1, 2, \dots\}$. Furthermore, we have (a) if $m = 0$, then $u_i(x) \rightarrow 0$ uniformly for $x \in \bar{\Omega}$, (b) if $m > 0$, then u_i has exactly m blowup points $\{P_1, \dots, P_m\}$ and $\lambda_i f(u_i(x)) \rightarrow 8\pi \sum_{j=1}^m \delta_{P_j}$ with δ_{P_j} being the Dirac measure at P_j . Moreover, as $i \rightarrow \infty$

$$\lambda_i \int_{|P_j-x| \leq \delta} f(u_i(x)) u_i(x) dx = 16m\pi \log \frac{1}{\lambda_i} (1 + o(1)). \quad (4.15)$$

Proof. – The claims of (i) and (ii) are well-known now except (4.15). (4.15) is a consequence of the estimate of local maximum of u_i near each blowup point P_j :

$$\max_{|x-P_j| \leq \delta} u_i(x) = 2 \log \frac{1}{\lambda_i} (1 + o(1)) \quad (4.16)$$

for each j and small $\delta > 0$. The upper bound of $\max_{|x-P_j| \leq \delta} u_i(x)$ can be obtained by the so-called “sup + inf” inequality which was proved in [3] and [8]. The lower bound can be obtained by the Green representation formulas and the scaling method. For the details of proof, we refer to [9]. \square

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