

LOWER SEMICONTINUITY OF L^∞ FUNCTIONALS

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ABSTRACT. – We study the lower semicontinuity properties and existence of a minimizer of the functional

$$F(u) = \operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x))$$

on $W^{1,\infty}(\Omega; \mathbb{R}^m)$. We introduce the notions of Morrey quasiconvexity, polyquasiconvexity, and rank-one quasiconvexity, all stemming from the notion of quasiconvexity (= convex level sets) of f in the last variable. We also formally derive the Aronsson–Euler equation for such problems.

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RÉSUMÉ. – On examine les propriétés de semi-continuité inférieure et l'existence du minimiseur de la fonctionnelle

$$F(u) = \operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x))$$

sur $W^{1,\infty}(\Omega; \mathbb{R}^m)$. On introduit les idées du quasi-convexité de Morrey, du polyquasi-convexité, et du quasi-convexité du rang-un, qui suivent tous de l'idée de quasi-convexité (= les ensembles à niveau convexes) de f à la dernière variable. En plus, on en déduit dans les formes l'équation d'Aronsson–Euler pour de tels problèmes.

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1. Introduction

The major area of study in calculus of variations in the last thirty years has been the study of variational problems with vector valued functions and the associated necessary

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and sufficient conditions for lower semicontinuity of integral functionals. It is our goal to consider this problem for essential supremum functionals.

It is well known that for integral problems these necessary and sufficient conditions involve notions of convexity in some form. This has led to concepts of *quasiconvexity*, *polyconvexity*, and *rank-one convexity*. In this paper we change the terminology slightly to account for multiple uses of the word *quasiconvexity*. In place of *quasiconvex* when it relates to vector valued variational problems we use the term *Morrey convex* in honor of the founder of this condition. Here is the definition of Morrey convexity which is well known (see Dacorogna [9] and the references there) to be a necessary and sufficient for weak $W^{1,p}$ lower semicontinuity of the integral functional

$$F(u) = \int_{\Omega} f(Du) \, dx.$$

DEFINITION 1.1. – Let \mathbb{R}^{nm} denote the class of all $n \times m$ matrices with real entries. A function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is Morrey convex if for each $A \in \mathbb{R}^{nm}$

$$\frac{1}{|\Omega|} \int_{\Omega} f(A + D\varphi(x)) \, dx \geq f(A),$$

for each Ω , a bounded domain in \mathbb{R}^n , and $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$. Equivalently,

$$f(A) = \min \left\{ \frac{1}{|\Omega|} \int_{\Omega} f(A + D\varphi(x)) \, dx \mid \varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m) \right\}.$$

Observe the connection with Jensen’s inequality for convex functions. Indeed, this inequality has been fundamental to the existence theory in variational problems. In Barron, Jensen, Liu [7] we have derived an extended Jensen inequality which is just as fundamental for variational problems in L^∞ . It applies to *quasiconvex* functions, i.e., functions with convex level sets. In symbols, f is quasiconvex if $E_\gamma = \{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}$ is convex for any $\gamma \in \mathbb{R}$; equivalently,

$$f(\lambda x + (1 - \lambda)y) \leq f(x) \vee f(y), \quad \forall x, y \in \mathbb{R}^n, \lambda \in (0, 1).$$

Here is our extended Jensen inequality and its short proof.

THEOREM 1.2. – Let $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be lower semicontinuous and quasiconvex and let μ be a probability measure on \mathbb{R}^n supported on Ω . Let $\varphi \in L^1(\Omega; \mu)$ be a given function. Then

$$f\left(\int_{\Omega} \varphi \, d\mu\right) \leq \mu - \operatorname{ess\,sup}_{x \in \Omega} f(\varphi(x)).$$

The μ -essential supremum means we exclude sets of μ -measure zero.

Proof. – Define $\gamma = \mu - \operatorname{ess\,sup}_{x \in \Omega} f(\varphi(x))$ and $E_\gamma = \{q : f(q) \leq \gamma\}$. Then for μ -a.e. $x \in \Omega$, $\varphi(x) \in E_\gamma$. Since f is lsc and quasiconvex, E_γ is a closed convex set. Hence, since μ is a probability measure, $\int \varphi \, d\mu \in E_\gamma$ and the theorem is proved. \square

Notice that a quasiconvex function may be neither continuous nor even lower semicontinuous.

We now introduce a condition similar to Morrey convexity which we will see is (almost) necessary and sufficient for weak-* lower semicontinuity of functionals of the form

$$F(u) = F(u, \Omega) = \operatorname{ess\,sup}_{x \in \Omega} f(Du(x)).$$

DEFINITION 1.3. – $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is (weak) Morrey quasiconvex if for each $A \in \mathbb{R}^{nm}$,

$$f(A) \leq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi), \quad \forall \varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m).$$

Equivalently,

$$f(A) = \min_{\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m)} \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi),$$

where $Q = [0, 1]^n \subset \mathbb{R}^n$ throughout this paper denotes the standard unit cube.

The space of test functions in the classical definition of Morrey convexity may be replaced by $\varphi \in C_0^\infty(Q; \mathbb{R}^m)$. It is clear that one *cannot* replace the class of test functions in this definition by C_0^∞ since we do not have uniform approximation of the derivatives.

The following lemma shows that our definition of Morrey quasiconvexity satisfies the analogue of the quasiconvexity definition in convex analysis, i.e., that the level sets of a quasiconvex function must be convex.

LEMMA 1.4. – f is Morrey quasiconvex iff $E_c = \{A \in \mathbb{R}^{nm}: f(A) \leq c\}$ is Morrey convex for every $c \in \mathbb{R}$. That is,

$$\delta(A | E_c) = \begin{cases} 0, & \text{if } A \in E_c, \\ +\infty, & \text{if } A \notin E_c, \end{cases}$$

is a Morrey convex function for every $c \in \mathbb{R}$.

Proof. – Suppose that $\delta(A | E_c)$ is Morrey convex for every $c \in \mathbb{R}$. If f is not Morrey quasiconvex at $A \in \mathbb{R}^{nm}$, there is an $\varepsilon > 0$ and $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m)$ so that $f(A) - \varepsilon \geq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi)$. Choose $c = f(A) - \varepsilon$. Then $\delta(A | E_c) = +\infty$. Since $\delta(A | E_c)$ is Morrey convex, we conclude that $\delta(A + D\varphi | E_c) = +\infty$ on a subset of Q with positive measure. Hence, $c \geq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi) > c$, a contradiction and f must be Morrey quasiconvex.

Conversely, suppose f is Morrey quasiconvex at $A \in \mathbb{R}^{nm}$ and $c \in \mathbb{R}$ is arbitrary. Suppose that $f(A) \leq c$. Then $\delta(A | E_c) = 0$ and immediately, since $\delta \geq 0$, $\delta(A | E_c)$ is Morrey convex. If $f(A) > c$ then $\delta(A | E_c) = +\infty$. If $\delta(A | E_c)$ is not Morrey convex, there is a $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m)$ so that $\int_Q \delta(A + D\varphi | E_c) dx = 0$. But then, $f(A + D\varphi) \leq c$ a.e. on Q , i.e., $\operatorname{ess\,sup}_{x \in Q} f(A + D\varphi) \leq c < f(A)$, a contradiction to the assumption that f is Morrey quasiconvex. \square

For lower semicontinuity of functionals on L^∞ Morrey quasiconvexity is the analogue of Morrey convexity, but it turns out that for vector valued problems we need to modify

the notion to *weak* and *strong* Morrey quasiconvexity. We refer to the definition given above as the weak version. The strong version permits some play in the test functions φ on the boundary of the domain and this is what we need to get the lower semicontinuity result we are after. In the scalar case, i.e., $m = 1$ or $n = 1$, weak and strong Morrey quasiconvexity are equivalent. It is an important open problem to determine if they are also equivalent in the vector case. We conjecture that they are not equivalent.

We also introduce the notions of *polyquasiconvex* and *rank-one quasiconvex* as generalizations of the classical notions polyconvex and rank-one convex. In the classical case we know that we have the implications

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ Morrey convex} \Rightarrow f \text{ rank-one convex.}$$

It is known that f Morrey convex does not imply f is polyconvex. The famous counterexample of Sverak [14] shows that

$$f \text{ rank-one convex} \not\Rightarrow f \text{ Morrey convex,}$$

at least when $m \geq 3, n \geq 2$. Obviously the question arises as to whether this can be extended to L^∞ but counterexamples, which are not easy to come by in the classical cases, are no easier in L^∞ . As is usual in a paper of this type, more questions are raised than are answered but we hope to return to these questions in a future paper. In particular we hope to resolve the relaxation question and the use of Young measures in such relaxations. Excellent references for these and many other considerations are Dacorogna [9] and Pedregal [13].

The main difficulty in dealing with L^∞ functionals is the fact that we do not have available to us the major tool used in L^p functionals, namely, the use of piecewise affine functions. Approximation of a $W^{1,\infty}$ function on a domain Ω by piecewise affine functions leaves a piece of small measure left over. In integral problems this portion is controlled since it is a small measure set, but in L^∞ small measure sets cannot be ignored, and functions on them cannot be controlled easily. Another idea which one might think of is the use of approximating L^∞ by L^p . In lower semicontinuity considerations this leads to then studying an iterated limit $\lim_{k \rightarrow \infty} \lim_{p \rightarrow \infty}$ – in which there is no reason to believe that the limits can be reversed.

Variational problems in L^∞ were first studied systematically by Aronsson [1–3]. Then Jensen in [12] considered the uniqueness question for the Aronsson equation arising in the minimization of $\text{ess sup}_{x \in \Omega} |Du|^2$ or $\text{ess sup}_{x \in \Omega} |Du|$. Barron and Ishii [6] initiated the study of optimal control via viscosity solutions in L^∞ and Barron and Liu [8] studied calculus of variations in L^∞ in the scalar case from the point of view of relaxation and duality. A survey of the foregoing and many other results is in Barron [5].

Finally, the motivation to consider L^∞ variational problems is provided by simple examples without simple solutions. A typical vector valued problem arising in elasticity (see for example [9]) developed by Ball [4] is to minimize the integral

$$I(u) = \int_{\Omega} W(x, Du(x)) + \psi(x, u(x)) \, dx,$$

where Ω is the reference configuration of a given elastic material and u is the deformation of the body. Usually it is assumed that $\det(Du(x)) > 0$, $x \in \Omega$. The stored energy of the configuration is measured in the function $W = W(x, Du(x))$ and the function ψ measures the body force per unit volume. It would seem to make sense that one should replace the integral by the essential supremum in certain bodies in which cracks are a primary consideration since in fact cracks occur due to pointwise excessive energy. Pointwise considerations arise in many practical applications including temperature distribution, chemotherapy, risk management, etc., and of course it is a fundamental problem in Chebychev approximation of functions.

2. Morrey quasiconvexity and lower semicontinuity in L^∞

We begin with a precise definition of what we mean by Morrey quasiconvexity. Then we show that this condition is necessary and sufficient for lower semicontinuity.

DEFINITION 2.1. – *A measurable function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is said to be (strong) Morrey quasiconvex, if for any $\varepsilon > 0$, for any $A \in \mathbb{R}^{nm}$, and any $K > 0$, there exists a $\delta = \delta(\varepsilon, K, A) > 0$ such that if $\varphi \in W^{1,\infty}(Q; \mathbb{R}^m)$ satisfies*

$$\|D\varphi\|_{L^\infty(Q)} \leq K, \quad \max_{x \in \partial Q} |\varphi(x)| \leq \delta,$$

then,

$$f(A) \leq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi(x)) + \varepsilon. \quad (2.1)$$

DEFINITION 2.2. – *A measurable function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is said to be weak Morrey quasiconvex, or $(0, 0)$ Morrey quasiconvex, if for any $A \in \mathbb{R}^{nm}$, and $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m)$, we have*

$$f(A) \leq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi(x)). \quad (2.2)$$

Remark 2.3. – One can easily check from the definitions that if $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is Morrey quasiconvex then it is automatically weak Morrey quasiconvex. In Section 3 we prove that for either $n = 1$ or $m = 1$ weak Morrey quasiconvexity also implies Morrey quasiconvexity. However, it is an open question whether Morrey quasiconvexity and weak Morrey quasiconvexity are equivalent for $m, n > 1$.

Throughout this paper *Morrey quasiconvexity* will mean strong Morrey quasiconvexity.

Recall from Definition 1.1 that if a measurable function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is *Morrey convex*, then for any $A \in \mathbb{R}^{nm}$,

$$f(A) \leq \frac{1}{|Q|} \int_Q f(A + D\varphi) \, dx, \quad \forall \varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m). \quad (2.3)$$

Just as a convex function is always quasiconvex, a Morrey convex function with appropriate growth conditions is always Morrey quasiconvex. In fact, we have

PROPOSITION 2.4. – Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be Morrey convex. For any $\varepsilon > 0$ there exists a $\delta > 0$, depending only on ε and the Lipschitz constant of f on Q , such that if $\varphi \in W^{1,\infty}(Q; \mathbb{R}^m)$ satisfies $\max_{x \in \partial Q} |\varphi(x)| \leq \delta$ then

$$f(A) \leq \frac{1}{|Q|} \int_Q f(A + D\varphi) \, dx + \varepsilon \leq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi(x)) + \varepsilon. \tag{2.4}$$

Proof. – Let $\delta > 0$ be chosen later. Let $\varphi \in W^{1,\infty}(Q; \mathbb{R}^m)$ with $\max_{x \in \partial Q} |\varphi(x)| \leq \delta$. Then there exists $\eta = \eta(\delta, \|D\varphi\|_{L^\infty(Q)}) > 0$ such that $|\varphi(y)| \leq 2\delta$ for any $y = (y_1, \dots, y_n) \in Q$ with $|y_i| \geq 1 - \eta$ for some $1 \leq i \leq n$. Let $\xi \in C_0^1(\mathbb{R}^n)$ be such that $\xi \equiv 1$ for $y \in (1 - \eta)Q$, $\xi \equiv 0$ for $y \notin Q$, and $|D\xi| \leq 4/\eta$. Then the Morrey convexity condition implies

$$f(A) \leq \frac{1}{|Q|} \int_Q f(A + D(\xi\varphi)) \, dx.$$

However,

$$\begin{aligned} & \frac{1}{|Q|} \int_Q f(A + D(\xi\varphi)) \, dx \\ &= \frac{1}{|Q|} \int_Q f(A + D\varphi) \, dx + \frac{1}{|Q|} \int_{Q \setminus (1-\eta)Q} (f(A + D(\xi\varphi)) - f(A + D\varphi)) \, dx \\ &= I + II. \end{aligned}$$

Since f is Morrey convex and hence convex in each component variable as well as locally Lipschitz, it follows that $\|Df\|_{L^\infty(Q)} \leq C$. Therefore,

$$|II| \leq C \int_{Q \setminus (1-\eta)Q} |D\varphi| + |D\xi||\varphi| \, dx \leq C(\|D\varphi\|_{L^\infty(Q)}\eta + \delta).$$

Choose now $\delta \leq \varepsilon/2C$ and $\eta = \eta(\varepsilon, \|D\varphi\|_{L^\infty(Q)})$ sufficiently small so that $|II| \leq \varepsilon$. Hence,

$$f(A) \leq I + II \leq \frac{1}{|Q|} \int_Q f(A + D\varphi) \, dx + \varepsilon.$$

Since

$$\frac{1}{|Q|} \int_Q f(A + D\varphi) \, dx \leq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi(x)),$$

we conclude that

$$f(A) \leq \frac{1}{|Q|} \int_Q f(A + D\varphi) \, dx + \varepsilon \leq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi(x)) + \varepsilon. \quad \square$$

One key property we need for Morrey quasiconvex functions is the following.

PROPOSITION 2.5. – Let $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be Morrey quasiconvex. Let $\{u_k\} \subset W^{1,\infty}(Q; \mathbb{R}^m)$ be a sequence converging to zero, weak* in $W^{1,\infty}(Q; \mathbb{R}^m)$. Then,

$$f(A) \leq \liminf_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in Q} f(A + Du_k(x)), \quad \forall A \in \mathbb{R}^{nm}. \tag{2.5}$$

Proof. – Since u_k converges to 0 weak-* in $W^{1,\infty}(Q; \mathbb{R}^m)$, we have the existence of a finite constant $K_0 = \sup_k \|Du_k\|_{L^\infty(Q)}$, and $u_k \rightarrow 0$ uniformly on Q . For any $\varepsilon > 0$ and $A \in \mathbb{R}^{nm}$, it then follows from the definition of Morrey quasiconvexity that there exists a $\delta = \delta(\varepsilon, K_0, A)$ such that

$$f(A) \leq \operatorname{ess\,sup}_{x \in Q} f(A + D\varphi(x)) + \varepsilon, \tag{2.6}$$

for any $\varphi \in W^{1,\infty}(Q; \mathbb{R}^m)$ satisfying $\|D\varphi\|_{L^\infty(Q)} \leq K_0$ and $\max_{x \in \partial Q} |\varphi(x)| \leq \delta$. On the other hand, we know that there exists a $k_0 = k_0(\delta) > 0$ so that $\max_{x \in Q} |u_k| \leq \delta$ for all $k \geq k_0$. Therefore, we have

$$f(A) \leq \operatorname{ess\,sup}_{x \in Q} f(A + Du_k(x)) + \varepsilon, \quad \forall k \geq k_0.$$

This implies

$$f(A) \leq \liminf_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in Q} f(A + Du_k(x)) + \varepsilon. \tag{2.7}$$

Since $\varepsilon > 0$ is arbitrary, (2.7) gives (2.5). \square

Now, we are ready to prove the first main result of our paper. It says that Morrey quasiconvexity gives a sufficient condition for lower semicontinuity.

THEOREM 2.6 (Sufficient condition). – Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy:

- (i) For any $(x, s) \in \mathbb{R}^n \times \mathbb{R}^m$, $f(x, s, \cdot) : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is Morrey quasiconvex;
- (ii) There exists a function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is continuous in its first variable and non-decreasing in its second variable, such that

$$|f(x_1, s_1, A) - f(x_2, s_2, A)| \leq \omega(|x_1 - x_2| + |s_1 - s_2|, |A|), \tag{2.8}$$

for any $(x_1, s_1), (x_2, s_2) \in \mathbb{R}^n \times \mathbb{R}^m$ and $A \in \mathbb{R}^{nm}$.

Then for any bounded domain $\Omega \subset \mathbb{R}^n$ the functional

$$F(u, \Omega) = \operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x))$$

is sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

Proof. – Let φ_k converge to 0 weak* in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. In particular, we may assume that $\varphi_k \rightarrow 0$ uniformly on Ω and $D\varphi_k \rightarrow 0$ weak* in $L^\infty(\Omega, \mathbb{R}^m)$. Set $u_k = u + \varphi_k$. We need to prove

$$\operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x)) \leq \liminf_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in \Omega} f(x, u_k(x), Du_k(x)). \tag{2.9}$$

Let $\{u_k\}$ denote a subsequence so that

$$\lim_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in \Omega} f(x, u_k(x), Du_k(x)) = \lim_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in \Omega} f(x, u_k(x), Du_k(x)). \tag{2.10}$$

Since u is Lipschitz continuous on Ω , it follows from Rademacher’s theorem and the Lebesgue density theorem that there exists an $\Omega_0 \subset \Omega$, with $|\Omega \setminus \Omega_0| = 0$, such that u is differentiable at any $x_0 \in \Omega_0$ and x_0 is a Lebesgue point of Du , namely

$$\lim_{x \rightarrow x_0} \frac{|u(x) - u(x_0) - Du(x_0)(x - x_0)|}{|x - x_0|} = 0, \tag{2.11}$$

and

$$\lim_{r \downarrow 0} \frac{1}{|Q_r(x_0)|} \int_{Q_r(x_0)} |Du - Du(x_0)| \, dx = 0. \tag{2.12}$$

Here $Q_r(x_0) = \{x_0 + rx : x \in Q\}$ denotes the cube with side length r and center at x_0 .

For $r > 0$ small, we define the rescaling maps:

$$v_{x_0}^r(x) = \frac{1}{r}(u(x_0 + rx) - u(x_0)) : Q \rightarrow \mathbb{R}^m,$$

and

$$\varphi_{k,x_0}^r(x) = \frac{1}{r}(\varphi_k(x_0 + rx) - \varphi_k(x_0)) : Q \rightarrow \mathbb{R}^m.$$

Then (2.11) and (2.12) imply

$$\lim_{r \downarrow 0} \max_{x \in Q} |v_{x_0}^r(x) - Du(x_0)x| = 0 \quad \text{and} \quad \lim_{r \downarrow 0} \|Dv_{x_0}^r - Du(x_0)\|_{L^1(Q)} = 0. \tag{2.13}$$

Notice also that $v_{x_0}^r$ is bounded in $W^{1,\infty}(Q; \mathbb{R}^m)$. Hence (2.13) implies that $v_{x_0}^r(x) \rightarrow Du(x_0)x$ weak* in $W^{1,\infty}(Q; \mathbb{R}^m)$.

For any fixed $r > 0$, we observe that

$$\sup_{k \geq 1} \max_{x \in Q} |\varphi_{k,x_0}^r(x)| \leq \sup_{k \geq 1} \|D\varphi_k\|_{L^\infty(\Omega)} < \infty,$$

and

$$\sup_{k \geq 1} \sup_{\{x,y \in Q, x \neq y\}} \frac{|\varphi_{k,x_0}^r(x) - \varphi_{k,x_0}^r(y)|}{|x - y|} \leq \sup_{k \geq 1} \|D\varphi_k\|_{L^\infty(\Omega)} < \infty.$$

Hence, by the Arzela–Ascoli theorem, we may assume that, for any $r > 0$ small, φ_{k,x_0}^r converges to φ uniformly on Q . Moreover, since $D\varphi_{k,x_0}^r(x) = D\varphi_k(x_0 + rx)$ for $x \in Q$ and $D\varphi_k \rightarrow 0$ weak* in $L^\infty(\Omega, \mathbb{R}^m)$, we have that, for any $r > 0$ small, $D\varphi_{k,x_0}^r \rightarrow 0$ weak* in $L^\infty(Q, \mathbb{R}^m)$. Therefore $D\varphi \equiv 0$ and $\varphi \equiv \text{constant}$ on Q . Since $\varphi_{k,x_0}^r(0) = 0$ we have $\varphi \equiv 0$ on Q .

In particular, we obtain that, for any $r > 0$ small, φ_{k,x_0}^r converges to 0 weak* in $W^{1,\infty}(Q; \mathbb{R}^m)$. By the Cauchy diagonal process, we have that for any $r_i \downarrow 0$ there exists

$k_i \uparrow \infty$ such that

$$v_{x_0}^{r_i}(x) \rightarrow Du(x_0)x \quad \text{and} \quad \varphi_{k_i, x_0}^{r_i}(x) \rightarrow 0, \tag{2.14}$$

weak* in $W^{1, \infty}(Q; \mathbb{R}^m)$. Applying Proposition 2.5, we can conclude

$$f(x_0, u(x_0), Du(x_0)) \leq \liminf_{k_i \uparrow \infty} \operatorname{ess\,sup}_{x \in Q} f(x_0, u(x_0), D(v_{x_0}^{r_i} + \varphi_{k_i, x_0}^{r_i})(x)) \tag{2.15}$$

On the other hand, by (2.8), we have

$$\begin{aligned} & \operatorname{ess\,sup}_{x \in Q_{r_i}(x_0)} f(x, u(x), Du_{k_i}(x)) \\ &= \operatorname{ess\,sup}_{y \in Q} f(x_0 + r_i y, u(x_0 + r_i y), Du_{k_i}(x_0 + r_i y)) \\ &\leq \operatorname{ess\,sup}_{y \in Q} f(x_0, u(x_0), Du_{k_i}(x_0 + r_i y)) \\ &\quad + \max_{y \in Q} \omega(|r_i y| + |u(x_0 + r_i y) - u(x_0)|, |Du_{k_i}(x_0 + r_i y)|) \\ &= I + II. \end{aligned}$$

It is easy to see that

$$I = \operatorname{ess\,sup}_{x \in Q} f(x_0, u(x_0), D(v_{x_0}^{r_i} + \varphi_{k_i, x_0}^{r_i})(x)).$$

Since

$$\max_{x \in Q} |Du_{k_i}(x_0 + r_i y)| \leq \|Du_{k_i}\|_{L^\infty(\Omega)} \leq C < \infty, \forall k_i,$$

where $C = \sup_{k_i} \|Du_{k_i}\|_{L^\infty(\Omega)}$, we have

$$|II| \leq \max_{y \in Q} \omega(|r_i y| + |u(x_0 + r_i y) - u(x_0)|, C).$$

Combined with the continuity of u and ω in its first variable, yields

$$\lim_{r_i \downarrow 0} II = 0.$$

Therefore, we obtain

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in \Omega} f(x, u_k(x), Du_k(x)) \\ &= \lim_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in \Omega} f(x, u_k(x), Du_k(x)) \\ &\geq \liminf_{k_i \uparrow \infty} \operatorname{ess\,sup}_{x \in \Omega} f(x, u_{k_i}(x), Du_{k_i}(x)) \\ &\geq \liminf_{r_i \downarrow 0} \operatorname{ess\,sup}_{x \in Q_{r_i}(x_0)} f(x, u_{k_i}(x), Du_{k_i}(x)) \\ &= \liminf_{r_i \downarrow 0} \operatorname{ess\,sup}_{y \in Q} f(x_0 + r_i y, u(x_0 + r_i y), Du_{k_i}(x_0 + r_i y)) \end{aligned}$$

$$\begin{aligned}
 &= \liminf_{r_i \downarrow 0} \operatorname{ess\,sup}_{y \in Q} f(x_0, u(x_0), D(v_{x_0}^{r_i} + \varphi_{k_i, x_0}^{r_i})(y)) \\
 &\geq f(x_0, u(x_0), Du(x_0)) \quad \text{by (2.15).}
 \end{aligned}$$

Since this holds for any $x_0 \in \Omega_0$ and $|\Omega \setminus \Omega_0| = 0$, we have proved (2.9) and the proof of Theorem 2.6 is complete. \square

Now we prove that Morrey quasiconvexity is also a necessary condition for the sequential weak* lower semicontinuity of the functional $F(u, \Omega)$ on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

THEOREM 2.7 (Necessary condition). – *Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy the condition (2.8). For any bounded $\Omega \subset \mathbb{R}^n$, let $F(u, \Omega) = \operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x))$. Assume that $F(\cdot, \Omega)$ is sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Then $p \mapsto f(x, u, p)$ is Morrey quasiconvex for any $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$.*

To prove Theorem 2.7 we first need the following lemma. The second part of this lemma exhibits the importance of the extended Jensen inequality.

LEMMA 2.8. – *Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy (2.8). If $F(\cdot, \Omega)$ is sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$, then*

- (i) *For any $(x_0, s_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $f(x_0, s_0, \cdot) : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is lower semicontinuous.*
- (ii) *For any $(x_0, s_0) \in \mathbb{R}^n \times \mathbb{R}^m$, we have*

$$f\left(x_0, s_0, \frac{1}{|Q|} \int_Q Du(x) \, dx\right) \leq \operatorname{ess\,sup}_{x \in Q} f(x_0, s_0, Du(x)) \quad (2.16)$$

for any $u \in W^{1,\infty}(Q; \mathbb{R}^m)$ with Du Q -periodic.

- (iii) *For any $(x_0, s_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $f(x_0, s_0, \cdot) : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfies the weak Morrey quasiconvexity property:*

$$f(x_0, s_0, A) = \inf_{x \in Q} \{ \operatorname{ess\,sup} f(x_0, s_0, A + D\varphi(x)) : \varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m) \}. \quad (2.17)$$

Proof. – (i) For $(x_0, s_0) \in \mathbb{R}^n \times \mathbb{R}^m$. Let $\{A_j\} \subset \mathbb{R}^{nm}$ satisfy $A_j \rightarrow A$. Then $u_j(x) = s_0 + A_j(x - x_0)$ converges to $u(x) = s_0 + A(x - x_0)$ in $W^{1,\infty}(B_r(x_0); \mathbb{R}^m)$ for any $r > 0$. Hence we have

$$\begin{aligned}
 \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, u(x), Du(x)) &= \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, s_0 + A(x - x_0), A) \\
 &\leq \liminf_{j \rightarrow \infty} \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, u_j(x), Du_j(x)) \\
 &= \liminf_{j \rightarrow \infty} \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, s_0 + A_j(x - x_0), A_j). \quad (2.18)
 \end{aligned}$$

On the other hand, since f satisfies (2.8) we have

$$\lim_{r \rightarrow 0} \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, s_0 + A(x - x_0), A) = f(x_0, s_0, A),$$

and

$$\lim_{r \rightarrow 0} \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, s_0 + A_j(x - x_0), A_j) = f(x_0, s_0, A_j),$$

uniformly in j . Hence, sending r to zero, (2.18) implies

$$f(x_0, s_0, A) \leq \liminf_{j \rightarrow \infty} f(x_0, s_0, A_j).$$

(ii) For $(x_0, s_0) \in \mathbb{R}^n \times \mathbb{R}^m$. We first extend $u \in W^{1,\infty}(Q; \mathbb{R}^m)$ to \mathbb{R}^n with Du as a Q -periodic function. Define $u_j(x) = u(j(x - x_0))/j + s_0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then we have

$$u_j(x) \rightarrow s_0 + \left(\frac{1}{|Q|} \int_Q Du \, dx \right) (x - x_0),$$

weak* in $W^{1,\infty}(B_r(x_0); \mathbb{R}^m)$, for any $r > 0$. In particular, $Du_j \rightarrow \frac{1}{|Q|} \int_Q Du \, dx$ weak* in $L^\infty(B_r(x_0), \mathbb{R}^m)$. Hence we have

$$\begin{aligned} & \operatorname{ess\,sup}_{x \in B_r(x_0)} f \left(x, s_0 + \left(\frac{1}{|Q|} \int_Q Du \, dx \right) (x - x_0), \frac{1}{|Q|} \int_Q Du \, dx \right) \\ & \leq \liminf_{j \rightarrow \infty} \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, u_j(x), Du_j(x)) \\ & = \liminf_{j \rightarrow \infty} \operatorname{ess\,sup}_{x \in B_r(x_0)} f \left(x, s_0 + \frac{u(j(x - x_0))}{j}, Du(j(x - x_0)) \right). \end{aligned} \tag{2.19}$$

Since f satisfies (2.8) we know, setting $A = \frac{1}{|Q|} \int_Q Du \, dx$

$$\lim_{r \rightarrow 0} \operatorname{ess\,sup}_{x \in B_r(x_0)} f(x, s_0 + A(x - x_0), A) = f(x_0, s_0, A),$$

and

$$\begin{aligned} & \lim_{r \rightarrow 0} \operatorname{ess\,sup}_{x \in B_r(x_0)} f \left(x, s_0 + \frac{u(j(x - x_0))}{j}, Du(j(x - x_0)) \right) \\ & = \lim_{r \rightarrow 0} \operatorname{ess\,sup}_{x \in B_r(x_0)} f \left(x_0, s_0 + \frac{u(0)}{j}, Du(j(x - x_0)) \right), \end{aligned}$$

uniformly in j . Here we have used the fact that

$$\lim_{x \rightarrow x_0} \sup_j \frac{|u(j(x - x_0)) - u(0)|}{j} = 0.$$

On the other hand, since Du is Q -periodic we have

$$\lim_{j \rightarrow \infty} \operatorname{ess\,sup}_{x \in B_r(x_0)} f \left(x_0, s_0 + \frac{u(0)}{j}, Du(j(x - x_0)) \right) = \operatorname{ess\,sup}_{x \in Q} f(x_0, s_0, Du(x)).$$

Putting these together and sending r to zero in (2.19), we obtain

$$f \left(x_0, s_0, \frac{1}{|Q|} \int_Q Du \, dx \right) \leq \operatorname{ess\,sup}_{x \in Q} f(x_0, s_0, Du(x)).$$

Observe that if we were to assume the condition $f(x_0, s_0, p) = f(x_0, s_0, -p)$ we could drop the assumption that Du is Q -periodic.

(iii) Notice we can rewrite (2.16) as

$$\begin{aligned}
 f(x_0, s_0, A) &= \inf \left\{ \operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, D\varphi(y)) : \varphi \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m), \right. \\
 &\quad \left. D\varphi \text{ is } Q\text{-periodic, } \frac{1}{|Q|} \int_Q D\varphi \, dx = A \right\} \\
 &= \inf \left\{ \operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, A + D\psi(y)) : \psi \in W_0^{1,\infty}(Q; \mathbb{R}^m) \right\}. \tag{2.20}
 \end{aligned}$$

In fact, given $\psi \in W_0^{1,\infty}(Q; \mathbb{R}^m)$ set $\varphi(y) = Ay + \psi(y)$. This shows the first infimum in (2.20) is not greater than the second. For the reverse, given any $\varphi \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$ with $D\varphi$ Q -periodic and $1/|Q| \int_Q D\varphi \, dx = A$, take $\psi \equiv 0$. Then, by (ii), we have

$$\begin{aligned}
 &\inf \left\{ \operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, A + D\psi(y)) : \psi \in W_0^{1,\infty}(Q; \mathbb{R}^m) \right\} \\
 &\leq f(x_0, s_0, A) \leq \operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, D\varphi(y)).
 \end{aligned}$$

This shows that the second infimum in (2.20) is not greater than the first and hence they are equal. This gives the weak Morrey quasiconvexity of f in its last variable. \square

Now we can return to the proof of Theorem 2.7.

Proof of Theorem 2.7. – We prove it by contradiction. Suppose that f is not Morrey quasiconvex in its last variable. Then there exist $(x_0, s_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $\varepsilon_0 > 0$, $K_0 > 0$, $A_0 \in \mathbb{R}^{nm}$, and sequences $\{\varphi_k\} \subset W^{1,\infty}(Q; \mathbb{R}^m)$ such that

$$\sup_k \|D\varphi_k\|_{L^\infty(Q)} \leq K_0, \quad \max_{x \in \partial Q} |\varphi_k(x)| = \delta_k \rightarrow 0 \quad \text{as } k \rightarrow \infty, \tag{2.21}$$

but

$$f(x_0, s_0, A_0) > \operatorname{ess\,sup}_{x \in Q} f(x_0, s_0, A_0 + D\varphi_k(x)) + \varepsilon_0, \tag{2.22}$$

for all $k \geq 1$. It follows from (2.21) that we may assume that there exists a $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m)$ such that $\varphi_k \rightarrow \varphi$ weak* in $W^{1,\infty}(Q; \mathbb{R}^m)$. For any $r > 0$, we see that

$$s_0 + A_0(x - x_0) + r\varphi_k\left(\frac{x - x_0}{r}\right) \rightarrow s_0 + A_0(x - x_0) + r\varphi\left(\frac{x - x_0}{r}\right),$$

weak* in $W^{1,\infty}(Q_r(x_0); \mathbb{R}^m)$, as $k \rightarrow \infty$. Since $F(\cdot, Q_r(x_0))$ is sequentially weak* lower semicontinuous on $W^{1,\infty}$, we have

$$\begin{aligned}
 &\operatorname{ess\,sup}_{y \in Q} f(x_0 + ry, s_0 + rA_0y + r\varphi(y), A_0 + D\varphi(y)) \\
 &= \operatorname{ess\,sup}_{x \in Q_r(x_0)} f\left(x, s_0 + A_0(x - x_0) + r\varphi\left(\frac{x - x_0}{r}\right), A_0 + D\varphi\left(\frac{x - x_0}{r}\right)\right)
 \end{aligned}$$

$$\begin{aligned} &\leq \liminf_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in Q_r(x_0)} f\left(x, s_0 + A_0(x - x_0) + r\varphi_k\left(\frac{x - x_0}{r}\right), A_0 + D\varphi_k\left(\frac{x - x_0}{r}\right)\right) \\ &= \liminf_{k \rightarrow \infty} \operatorname{ess\,sup}_{y \in Q} f(x_0 + ry, s_0 + rA_0y + r\varphi_k(y), A_0 + D\varphi_k(y)). \end{aligned}$$

Notice that

$$\lim_{r \downarrow 0} \max_{y \in Q} |x_0 + ry - x_0| = 0,$$

$$\lim_{r \downarrow 0} \max_{y \in Q} |s_0 + rA_0y + r\varphi(y) - s_0| = 0,$$

and

$$\lim_{r \downarrow 0} \sup_k \max_{y \in Q} |s_0 + rA_0y + r\varphi_k(y) - s_0| = 0.$$

Since f satisfies (ii) of Theorem 2.6, we have, by sending r to zero,

$$\operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, A_0 + D\varphi(y)) \leq \liminf_{k \rightarrow 1} \operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, A_0 + D\varphi_k(y)). \tag{2.23}$$

This, combined with (2.22), implies

$$\operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, A_0 + D\varphi(y)) + \varepsilon_0 < f(x_0, s_0, A_0). \tag{2.24}$$

On the other hand, since $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m)$, it follows from (iii) of Lemma 2.8 that

$$f(x_0, s_0, A_0) \leq \operatorname{ess\,sup}_{y \in Q} f(x_0, s_0, A_0 + D\varphi(y)).$$

This contradicts (2.24). The proof is complete. \square

We finish this section with an existence theorem for minimizing problems in L^∞ .

THEOREM 2.9. – *If, in addition to the conditions in Theorem 2.6, let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy the following coercivity condition:*

$$f(x, s, A) \geq C_1|A|^p - C_2, \quad \forall (x, s, A) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \tag{2.25}$$

for some $C_1 > 0$, $C_2 > 0$, and $p > 0$. Then for any bounded domain $\Omega \subset \mathbb{R}^n$ and $\psi \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ there exists at least one function $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$, with $u|_{\partial\Omega} = \psi|_{\partial\Omega}$, so that

$$\begin{aligned} &\operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x)) \\ &= \inf_{x \in \Omega} \{ \operatorname{ess\,sup}_{x \in \Omega} f(x, v(x), Dv(x)) : v \in W^{1,\infty}(\Omega; \mathbb{R}^m), v|_{\partial\Omega} = \psi|_{\partial\Omega} \}. \end{aligned}$$

Proof. – First notice that

$$\begin{aligned} c &\equiv \inf_{x \in \Omega} \{ \operatorname{ess\,sup}_{x \in \Omega} f(x, v(x), Dv(x)) : v \in W^{1,\infty}(\Omega; \mathbb{R}^m), v|_{\partial\Omega} = \psi|_{\partial\Omega} \} \\ &\leq \operatorname{ess\,sup}_{x \in \Omega} f(x, \psi(x), D\psi(x)) < \infty. \end{aligned}$$

Moreover, $c \geq -C_2 > -\infty$.

Let $\{u_k\} \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ be a minimizing sequence with $u_k|_{\partial\Omega} = \psi|_{\partial\Omega}$, and

$$\lim_{k \rightarrow \infty} \operatorname{ess\,sup}_{x \in \Omega} f(x, u_k(x), Du_k(x)) = c.$$

It follows from the coercivity condition (2.25) that $\|Du_k\|_{L^\infty(\Omega)}$ is bounded. By the Poincaré inequality we see that $\|u_k\| \leq \|\psi\| + \|Du_k\|$ in the $L^\infty(\Omega)$ norm and so $\{u_k\}$ is bounded in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Therefore, we may assume that there exists $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $u|_{\partial\Omega} = \psi|_{\partial\Omega}$ so that $u_k \rightarrow u$ weak* in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. It then follows from Theorem 2.6 that u is a minimizer and $\operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x)) = c$. \square

Remark 2.10. – The coercivity condition of the theorem is essential for the existence of a minimizer. Indeed, consider $F(u) = \operatorname{ess\,sup}_{x \in [0,1]} |x u'(x)|$, with $u(0) = 1, u(1) = 0$. Clearly this is not coercive. Consider the sequence

$$u_n(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1/n, \\ -\log x / \log n, & \text{if } 1/n \leq x \leq 1. \end{cases}$$

Then $u_n \in W^{1,\infty}([0, 1])$ and

$$x u'_n(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/n, \\ -1/\log n, & \text{if } 1/n \leq x \leq 1. \end{cases}$$

Hence $F(u_n) = 1/\log n \rightarrow 0$. The infimum of F is therefore zero, but clearly no Lipschitz function assuming the boundary data can give $F(u) = 0$.

On the other hand if we consider instead $F(u) = \operatorname{ess\,sup}_{x \in [0,1]} |u'(x)|$ with $u(0) = 1, u(1) = 0$ there is a unique minimizer and it is $u^*(x) = 1 - x$. Indeed, $u(1) - u(0) = \int_0^1 u'(x) dx$ implies that $F(u) \geq 1$ for any Lipschitz u with $u(0) = 1, u(1) = 0$, and $F(u^*) = 1$. The uniqueness of the minimizer follows from the fact that

$$\operatorname{ess\,sup}_{x \in [x_0, x_1]} |(u^*)'(x)| = 1 \quad \text{for any } 0 \leq x_0 < x_1 \leq 1.$$

This is Theorem 1 of [1].

Remark 2.11. – This theorem establishes the existence of a minimizer but not an absolute minimizer, i.e., a function which minimizes F on any subdomain. This is what we need to use the Aronsson–Euler equation derived in the last section. In a companion paper *The Euler equation and absolute minimizers of L^∞ functionals* (to appear in Archives Rat. Mech. Anal.), we do establish the existence of an absolute minimizer for problems with $u : \mathbb{R}^n \rightarrow \mathbb{R}$.

3. Various classes of quasiconvex functions

In this section, we introduce various classes of functions, which are natural extensions to L^∞ of the well-known concepts of convexity, polyconvexity, Morrey convexity, and rank one convexity. For convenience we rewrite the definition of quasiconvexity.

DEFINITION 3.1 (Quasiconvexity). – A measurable function $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is called quasiconvex, if

$$f(tA + (1-t)B) \leq \max\{f(A), f(B)\}, \quad \forall A, B \in \mathbb{R}^{nm}, 0 \leq t \leq 1. \quad (3.1)$$

Remark 3.2. – Is every quasiconvex function Morrey convex? Define $f(M) = 0$ for $|M| < 1$ and $f(M) = 2$ otherwise. Then f is quasiconvex, but not continuous. Therefore f is not Morrey convex (since every Morrey convex function must be locally Lipschitz). Quasiconvex or even Morrey quasiconvex functions need not be continuous.

We first prove that the stronger and easier to check condition of quasiconvexity is enough for weak* lower semicontinuity in $W^{1,\infty}$. We restrict ourselves to the simpler case $f = f(Du)$, with the extension to $f(x, u, Du)$ causing only minor technical difficulties.

THEOREM 3.3. – Let $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be quasiconvex and lower semicontinuous. Then, for any bounded domain $\Omega \subset \mathbb{R}^n$, $F(u, \Omega) = \text{ess sup}_{x \in \Omega} f(Du(x))$ is sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

Proof. – For any $r \in \mathbb{R}$, let $E_r = \{A \in \mathbb{R}^{nm}: f(A) \leq r\}$. Then since f is lower semicontinuous and quasiconvex E_r is a closed convex set. Let $d(\cdot, E_r): \mathbb{R}^{nm} \rightarrow \mathbb{R}$ denote the distance function to E_r , i.e.,

$$d(A, E_r) = \inf_{B \in E_r} |A - B| = \inf_{B \in \mathbb{R}^{nm}} \delta(B | E_r) + |A - B|,$$

where $|\cdot|$ is a norm on \mathbb{R}^{nm} . Then since E_r is closed and convex, $d(\cdot, E_r)$ is Lipschitz continuous and convex. Hence it is well-known (see, [9] for example) that

$$G(u, \Omega) = \int_{\Omega} d(Du(x), E_r) dx$$

is sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$. Thus, if u_k converges to $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ weak* in $W^{1,\infty}(\Omega; \mathbb{R}^m)$, we have

$$\int_{\Omega} d(Du(x), E_r) dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} d(Du_k(x), E_r) dx. \quad (3.2)$$

Let

$$r_0 = \liminf_k \text{ess sup}_{x \in \Omega} f(Du_k(x)) = \lim_{i \rightarrow \infty} \text{ess sup}_{x \in \Omega} f(Du_{k_i}(x)).$$

Then for any $\varepsilon > 0$, there exists $i_0 = i_0(\varepsilon) > 0$ such that for $i \geq i_0$

$$f(Du_{k_i}(x)) \in E_{r_0+\varepsilon}, \quad \text{for a.e. } x \in \Omega,$$

so that

$$d(Du_{k_i}(x), E_{r_0+\varepsilon}) = 0, \quad \text{a.e. } x \in \Omega.$$

Hence (3.2) implies

$$\int_{\Omega} d(Du(x), E_{r_0+\varepsilon}) \, dx = 0.$$

This implies

$$f(Du(x)) \leq r_0 + \varepsilon \quad \text{a.e. } x \in \Omega.$$

Since $\varepsilon > 0$ is arbitrary, this gives

$$\operatorname{ess\,sup}_{x \in \Omega} f(Du(x)) \leq \liminf_k \operatorname{ess\,sup}_{x \in \Omega} f(Du_k(x)),$$

and completes the proof. \square

The proof of this theorem can be used to extend a result of Ioffe [10] to L^∞ in the scalar case. In particular, we will consider

$$F(u, v) = \mu - \operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), v(x)), \tag{3.3}$$

where $(\Omega, \mathcal{A}, \mu)$ is a measure space with μ nonnegative and finite and $f: \Omega \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow [0, \infty]$ is $\mathcal{A} \times \mathcal{B}_m \times \mathcal{B}_n$ measurable, where \mathcal{B}_m denotes the Borel subsets of \mathbb{R}^m .

THEOREM 3.4. – *Assume that for μ -a.e. $x \in \Omega$ $f(x, \cdot, \cdot)$ is lower semicontinuous on $\mathbb{R}^m \times \mathbb{R}^n$ and for every $u \in \mathbb{R}^m$ $f(x, u, \cdot)$ is quasiconvex on \mathbb{R}^n . Then the functional in (3.3) is sequentially lower semicontinuous on $L^\infty_\mu(\Omega; \mathbb{R}^m) \times L^\infty_\mu(\Omega; \mathbb{R}^n)$ using the strong topology on $L^\infty_\mu(\Omega; \mathbb{R}^m)$ and the weak* topology on $L^\infty_\mu(\Omega; \mathbb{R}^n)$.*

Proof. – The proof is similar to that of the previous theorem but here we can use the indicator function

$$\delta(x, u(x), v(x) \mid E_r) \quad \text{with } E_r = \{(y, \xi, \eta) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^n \mid f(y, \xi, \eta) \leq r\}.$$

With our hypotheses on f the indicator function satisfies all of the hypotheses of Ioffe’s theorem [10] and so we can complete the proof as before using the integral functional

$$G(x, u(x), v(x)) = \int_{\Omega} \delta(x, u(x), v(x) \mid E_r) \, dx.$$

The proof is simplified by use of the indicator function which can be used due to the fact that Ioffe’s theorem permits extended real valued integrands. \square

Naturally, this theorem includes as a special case the variational problem $F(u) = \mu - \operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), u'(x))$.

Now we turn to an extension of the idea of a polyconvex function.

DEFINITION 3.5 (Polyquasiconvexity). – *A measurable function $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is called polyquasiconvex if there exists a quasiconvex function $g: \mathbb{R}^{c(n,m)} \rightarrow \mathbb{R}$ such that*

$f(A) = g(T(A))$, where $c(n, m)$ is given by

$$c(n, m) = \sum_{s=1}^{\min(n,m)} \frac{m!n!}{s!^2(m-s)!(n-s)!}$$

and $T(A) : \mathbb{R}^{nm} \rightarrow \mathbb{R}^{c(n,m)}$ is the map consisting of A and all of its $s \times s$ minors for $s \leq \min(n, m)$.

A *polyconvex* function satisfies the same definition except that the function g must be convex, and not just quasiconvex. These types of functions are important in variational problems arising in elasticity.

It is clear from the definition that every quasiconvex function is polyquasiconvex and any polyconvex function is also polyquasiconvex.

PROPOSITION 3.6. – *Let $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be polyquasiconvex and lower semicontinuous. Then, for any bounded domain $\Omega \subset \mathbb{R}^n$, $F(u, \Omega) = \text{ess sup}_{x \in \Omega} f(Du(x))$ is sequentially weak* lower semi-continuous on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.*

Proof. – Let $g : \mathbb{R}^{c(n,m)} \rightarrow \mathbb{R}$ be quasiconvex so that $f(A) = g(T(A))$. Assume that u_k converges to u weak* in $W^{1,\infty}(\Omega; \mathbb{R}^m)$. It is well-known (see [9] for example) that

$$T(Du_k) \rightarrow T(Du), \quad \text{weak* in } W^{1,\infty}(\Omega; \mathbb{R}^m).$$

Now we can apply the same argument of Proposition 3.3 to show that

$$\text{ess sup}_{x \in \Omega} g(T(Du(x))) \leq \liminf_{k \rightarrow \infty} \text{ess sup}_{x \in \Omega} g(T(Du_k(x))). \tag{3.4}$$

This finishes the proof. \square

The last notion of convexity we extend is rank one convexity:

DEFINITION 3.7 (Rank one quasiconvexity). – *A measurable function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is rank one quasiconvex if (3.1) holds for any $A, B \in \mathbb{R}^{nm}$ with $\text{rank}(A - B) \leq 1$.*

Rank one convexity means $f(tA + (1 - t)B) \leq tf(A) + (1 - t)f(B)$, for $\text{rank}(A - B) \leq 1$.

It is clear that any quasiconvex function is rank one quasiconvex and any rank one convex function is also rank one quasiconvex.

PROPOSITION 3.8. – *Let $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy the weak Morrey quasiconvexity property:*

$$f(A) \leq \text{ess sup}_{x \in Q} f(A + D\varphi(x)), \quad \text{for any } A \in \mathbb{R}^{nm} \text{ and } \varphi \in W_0^{1,\infty}(Q; \mathbb{R}^m).$$

Then f is rank one quasiconvex.

Proof. – A result in Dacorogna [9] asserts that for any $A, B \in \mathbb{R}^{nm}$ with $\text{rank}(A - B) \leq 1$, any $\varepsilon > 0$, and $t \in [0, 1]$, there exist two subdomains $Q_1^\varepsilon, Q_2^\varepsilon$, with $Q_1^\varepsilon \cap Q_2^\varepsilon = \emptyset$, and

$\varphi_\varepsilon \in W_0^{1,\infty}(Q; \mathbb{R}^m)$ such that

$$||Q_1^\varepsilon| - t|Q|| \leq \varepsilon, \quad ||Q_2^\varepsilon| - (1 - t)|Q|| \leq \varepsilon,$$

$$D\varphi_\varepsilon = \begin{cases} (1 - t)(A - B) & \text{in } Q_1^\varepsilon, \\ -t(A - B) & \text{in } Q_2^\varepsilon, \end{cases}$$

and

$$\|D\varphi_\varepsilon\|_{L^\infty(Q)} \leq K(A, B) < \infty.$$

Moreover, it follows from the construction in [9] that Q_1^ε and Q_2^ε are increasing as ε decreases. Letting $\varepsilon \downarrow 0$, we may assume that $Q_i = \lim_{\varepsilon \downarrow 0} Q_i^\varepsilon$, $i = 1, 2$, exists, and there exists a $\varphi_0 \in W_0^{1,\infty}(Q; \mathbb{R}^m)$ such that $D\varphi_\varepsilon \rightarrow D\varphi_0$ weak* in $L^\infty(Q, \mathbb{R}^m)$.

Furthermore, we know that the interiors of Q_1 and Q_2 have empty intersection, $|Q_1| = t|Q|$, $|Q_2| = (1 - t)|Q|$, $D\varphi_0 = (1 - t)(A - B)$ in Q_1 , and $D\varphi_0 = -t(A - B)$ in Q_2 . Notice also that $|Q_1 \cup Q_2| = |Q|$.

Now we apply the weak Morrey quasiconvexity of f to obtain

$$\begin{aligned} f(tA + (1 - t)B) &\leq \operatorname{ess\,sup}_{x \in Q} f(tA + (1 - t)B + D\varphi_0(x)) \\ &= \operatorname{ess\,sup}_{x \in Q_1 \cup Q_2} f(tA + (1 - t)B + D\varphi_0(x)) \\ &= \max\left\{ \operatorname{ess\,sup}_{x \in Q_1} f(tA + (1 - t)B + D\varphi_0(x)), \right. \\ &\quad \left. \operatorname{ess\,sup}_{x \in Q_2} f(tA + (1 - t)B + D\varphi_0(x)) \right\} \\ &= \max\{f(A), f(B)\}. \end{aligned}$$

This shows that f is rank one quasiconvex. \square

Summarizing our results we have proved the following corollary.

COROLLARY 3.9. – *Let $f: \mathbb{R}^m \rightarrow \mathbb{R}$ lower semicontinuous be given. Then the following holds:*

- (1) *If f is quasiconvex, then f is polyquasiconvex.*
- (2) *If f is polyquasiconvex, then f is Morrey quasiconvex.*
- (3) *If f is Morrey quasiconvex, then f is weak Morrey quasiconvex.*
- (4) *If f is weak Morrey quasiconvex, then f is rank one quasiconvex.*
- (5) *If either $n = 1$ or $m = 1$, then all these notions are equivalent.*

Thus,

$$f \text{ quasiconvex} \Rightarrow f \text{ polyquasiconvex} \Rightarrow$$

$$f \text{ Morrey quasiconvex} \Rightarrow f \text{ weak Morrey quasiconvex} \Rightarrow f \text{ rank one quasiconvex}.$$

Given an arbitrary function $f: \mathbb{R}^m \rightarrow \mathbb{R}$ we could define the greatest Morrey quasiconvex minorant, greatest polyquasiconvex minorant, etc., as the relaxation of f . These considerations will be addressed later.

4. Convex properties do not carry over

Lemma 1.4 of the introduction is useful primarily in that it gives us a tool to prove theorems about Morrey quasiconvex functions and functionals in L^∞ by reducing them to Morrey convex functions and associated integral functionals. Unfortunately, this generally does not work in the vector valued case because most of the results in that case need a growth condition on the integrand, which, of course, the indicator function does not satisfy.

One might think that the way to get around the extended real valued problem is to take the inf convolution of the indicator function. This procedure then will satisfy the growth conditions, but then a new problem arises in the vector valued case. While the inf convolution of a convex function is convex, it is not true, as we will verify in this section, that the inf convolution of a Morrey convex function is Morrey convex. Hence, for vector valued problems we would convert the Morrey convex function $\delta(A | E_c)$ into a function which is not Morrey convex and so lower semicontinuity theorems would not apply.

Recall that for a given function $f : \mathbb{R}^{nm} \rightarrow \mathbb{R}$, say lower semicontinuous and bounded from below, the ε -inf convolution of f is defined by

$$f_\varepsilon(A) = \inf_{B \in \mathbb{R}^{nm}} \left\{ f(B) + \frac{1}{2\varepsilon} |A - B|^2 \right\}.$$

It is well known that if f is convex, then f_ε is convex for any $\varepsilon > 0$. Furthermore, if f is quasiconvex, i.e., has convex level sets, then f_ε is also quasiconvex. In the following example we will show that if f is Morrey convex, or even polyconvex, then f_ε may fail to be even rank one convex.

Define $f : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ by $f(A) = \det(A)$. Set $A = (a_{ij})$, $B = (b_{ij})$, $i, j = 1, 2$.

We compute f_ε from

$$f_\varepsilon(A) = \inf_B \left(b_{11}b_{22} - b_{12}b_{21} + \frac{1}{2\varepsilon} \left[\sum_{i,j=1}^2 (a_{ij} - b_{ij})^2 \right] \right).$$

The necessary conditions for a minimum point become the system of equations

$$\begin{aligned} b_{22} - \frac{a_{11} - b_{11}}{\varepsilon} &= 0, & b_{11} - \frac{a_{22} - b_{22}}{\varepsilon} &= 0, \\ -b_{21} - \frac{a_{12} - b_{12}}{\varepsilon} &= 0, & -b_{12} - \frac{a_{21} - b_{21}}{\varepsilon} &= 0. \end{aligned}$$

Solving these equations we obtain

$$\begin{aligned} b_{11} &= \frac{1}{\alpha} (a_{11} - \varepsilon a_{22}), & b_{22} &= \frac{1}{\alpha} (a_{22} - \varepsilon a_{11}), \\ b_{12} &= \frac{1}{\alpha} (a_{12} + \varepsilon a_{21}), & b_{21} &= \frac{1}{\alpha} (a_{21} + \varepsilon a_{12}), \end{aligned}$$

where $\alpha = 1 - \varepsilon^2$. Hence,

$$f_\varepsilon(A) = \frac{1}{\varepsilon^2} \left[\det(A) - \frac{\varepsilon}{2} (a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2) \right].$$

We claim that f_ε is not Morrey convex. Indeed f_ε is not rank-one convex. In fact, setting

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

we have

$$f_\varepsilon(A + tI) = f_\varepsilon(A) + \frac{1}{\varepsilon^2} \left(-\frac{\varepsilon}{2} t^2 + (a_{22} - \varepsilon a_{11})t \right).$$

This is clearly not convex with respect to t .

5. Euler Lagrange equation

In this section we derive the necessary conditions for an *absolute minimizer* of the functional

$$F(u, \Omega) = \operatorname{ess\,sup}_{x \in \Omega} f(x, u(x), Du(x))$$

with condition $u = \psi$ on $\partial\Omega$. Here ψ is a given Lipschitz function on $\partial\Omega$ and an absolute minimizer is defined precisely in:

DEFINITION 5.1. – *A function $u^* \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ is an absolute minimizer of $F(u, \Omega)$ if, for any open $\Omega' \subset \overline{\Omega'} \subset \Omega$, u^* is a minimizer of $F(u, \Omega')$, where $u^*|_{\partial\Omega'} = u|_{\partial\Omega'}$. That is, u^* minimizes F on every subdomain.*

The concept of absolute minimizer localizes the problem and this is necessary since the essential supremum function is a global operator. In integral problems the concept of absolute minimizer is not needed. To see this, suppose that u is a minimizer of $\int_\Omega f(x, u, Du) \, dx$ with given boundary data g . Let $\Omega' \subset \overline{\Omega'} \subset \Omega$ be any given subdomain and let v be an appropriate function on Ω' with $v = u$ on $\partial\Omega'$. Define $v = u$ on $\overline{\Omega} \setminus \Omega'$. Then,

$$\begin{aligned} \int_{\Omega'} f(x, u, Du) \, dx &= \int_{\Omega} f(x, u, Du) \, dx - \int_{\Omega \setminus \Omega'} f(x, u, Du) \, dx \\ &\leq \int_{\Omega} f(x, v, Dv) \, dx - \int_{\Omega \setminus \Omega'} f(x, v, Dv) \, dx \\ &= \int_{\Omega'} f(x, v, Dv) \, dx. \end{aligned}$$

Hence any minimizer for an integral problem is immediately an absolute minimizer. It is easy to see that the preceding argument fails for L^∞ problems.

Aronsson [1] derived the Euler equation for L^∞ by using L^p approximations. He noticed that the property of absolutely minimizing is critical in stating that the candidate function satisfies the Euler equation and in all subsequent results depending on the Euler equation. In this section we will not use L^p approximations to derive the Aronsson–Euler equation. Instead we will directly calculate the directional derivative of the functional and then directly use the absolute minimizing property to get the equation we are after. This formal proof, which was carried out in [5] and [8] for the scalar case, distills the critical property of absolute minimizing.

We use the notation

$$f_u(x, u, p) = \begin{pmatrix} \frac{\partial f}{\partial u_1} \\ \vdots \\ \frac{\partial f}{\partial u_m} \end{pmatrix}, \quad f_p(x, u, p) = \begin{pmatrix} \frac{\partial f}{\partial p_{1,1}} & \cdots & \frac{\partial f}{\partial p_{1,n}} \\ \vdots & \cdots & \vdots \\ \frac{\partial f}{\partial p_{m,1}} & \cdots & \frac{\partial f}{\partial p_{m,n}} \end{pmatrix}.$$

THEOREM 5.2. – *Let $\Omega \subset \mathbb{R}^n$ be open with compact closure and let $\psi \in C^2(\Omega, \mathbb{R}^m)$ be given. Suppose $f(x, u, p)$ and f_x, f_u, f_p are continuous. If $u^* \in C^2(\Omega, \mathbb{R}^m)$ is an absolute minimizer with $u = \psi$ on $\partial\Omega$, then u^* must satisfy the system*

$$f_p(x, u^*(x), Du^*(x)) \cdot D_x(f(x, u^*(x), Du^*(x))) = 0, \quad x \in \Omega. \tag{5.1}$$

Formal proof. – First, observe that we may replace the essential supremum by maximum since we assume u^* is smooth.

Define

$$S = \{x \in \overline{\Omega} \mid F(u^*, \Omega) = f(x, u^*(x), Du^*(x))\}.$$

If $x_0 \in S$ is an interior point of $\overline{\Omega}$ then we immediately have the condition

$$Df(x, u^*(x), u^*(x))|_{x=x_0} = 0.$$

Under our assumption on f , F is directionally differentiable for every direction $\gamma \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ and, by Danskin’s theorem [11], for example,

$$\begin{aligned} 0 &= DF(u^* + \delta\gamma, \Omega) |_{\delta=0} \\ &= \max\{f_u(x, u^*(x), Du^*(x))\gamma(x) + f_p(x, u^*(x), Du^*(x))D\gamma(x) \mid x \in S\}. \end{aligned}$$

For any $x_0 \in \Omega \setminus S$, we choose $\varepsilon > 0$ sufficiently small so $\Omega' = B_\varepsilon(x_0) \subset \Omega$. Now we need the absolute minimizing property of u^* . We know that u^* minimizes $F(u, B_\varepsilon(x_0))$ and hence for any $\gamma \in W_0^{1,\infty}(B_\varepsilon(x_0); \mathbb{R}^m)$

$$\begin{aligned} 0 &= DF(u^* + \delta\gamma, \Omega') |_{\delta=0} \\ &= \max\{f_u(x, u^*(x), u^*(x))\gamma(x) + f_p(x, u^*(x), Du^*(x))D\gamma(x) \mid x \in S_0\}, \tag{5.2} \end{aligned}$$

where

$$S_0 = \{x \in \overline{B_\varepsilon(x_0)} \mid F(u^*, B_\varepsilon(x_0)) = f(x, u^*(x), Du^*(x))\}.$$

Let us choose the direction $\gamma \in W_0^{1,\infty}(B_\varepsilon(x_0); \mathbb{R}^m)$

$$\gamma_i(x) = \frac{\varepsilon^2}{2} - \frac{|x - x_0|^2}{2}, \quad i = 1, 2, \dots, m.$$

Then, from (5.2) we get for some $y \in B_\varepsilon(x_0) \cap S_0$, assuming y is an interior max,

$$0 = f_u(y, u^*(y), Du^*(y)\gamma(y) - f_p(y, u^*(y), Du^*(y))) \cdot (y - x_0).$$

Notice that $y \rightarrow x_0$ as $\varepsilon \rightarrow 0$ and

$$\lim_{\varepsilon \rightarrow 0} \frac{\gamma(y)}{\varepsilon} = 0.$$

Therefore dividing by ε and then sending $\varepsilon \rightarrow 0$ we get

$$f_p(x_0, u^*(x_0), Du^*(x_0)) = 0.$$

We conclude that u^* must be a solution of the problem

$$\begin{cases} f_p(x, u^*(x), Du^*(x))D_x f(x, u^*(x), Du^*(x)) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega. \end{cases} \quad \square \quad (5.3)$$

The equation reduces to $\Delta_\infty u = 0$ in the case $f(x, u, p) = |p|^2$.

Remark 5.3. – In the paper *The Euler equation and absolute minimizers of L^∞ functionals* we prove the existence of an absolute minimizer for the case $u : \mathbb{R}^n \rightarrow \mathbb{R}$, i.e., $m = 1$. We also prove that an absolute minimizer is a viscosity solution of (5.3). The existence and uniqueness of an absolute minimizer in the case $m > 1$ is open.

Remark 5.4. – The special case $f(x, u, Du) = |Du|$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ was studied extensively in Jensen [12] extending the C^2 approach of Aronsson [1–3] to the $W^{1,\infty}$ case. In this case, the necessary conditions for an absolute minimizer results in the Euler equation

$$\Delta_\infty u = Du^t \cdot D^2 u \cdot Du = 0, \quad x \in \Omega, \quad u(x) = \psi(x), \quad x \in \partial\Omega,$$

where Du^t is the transpose of the vector Du . This equation is referred to as the Aronsson equation or the ∞ -Laplace equation and plays the same role in L^∞ as the Laplacian plays in the integral minimum problem for $\int |Du|^2 dx$.

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