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MULTI-BUMP GROUND STATES OF THE GIERER–MEINHARDT SYSTEM IN $\mathbb{R}^2$

FONDAMENTALES MULTI-PICS DU SYSTÈME DE GIERER–MEINHARDT DANS $\mathbb{R}^2$

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ABSTRACT. – We consider the stationary Gierer–Meinhardt system in $\mathbb{R}^2$:

$$\begin{cases} \Delta A - A + \frac{A^2}{H} = 0 & \text{in } \mathbb{R}^2, \\ \Delta H - \sigma^2 H + A^2 = 0 & \text{in } \mathbb{R}^2, \\ A, H > 0;\ A, H \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

We construct multi-bump ground-state solutions for this system for all sufficiently small σ. The centers of these bumps are located at the vertices of a regular polygon, while the bumps resemble, after a suitable scaling in their A-coordinate, the unique radially symmetric solution of

$$\begin{cases} \Delta w - w + w^2 = 0 & \text{in } \mathbb{R}^2, \\ 0 < w(y) \to 0 & \text{as } |y| \to \infty. \end{cases}$$

A similar construction is made for vertices of two concentric polygons and a general procedure for detection of organized finite patterns is suggested.

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RÉSUMÉ. – On considère le système stationnaire de Gierer–Meinhardt dans $\mathbb{R}^2$:

$$\begin{cases} \Delta A - A + \frac{A^2}{H} = 0 & \text{dans } \mathbb{R}^2, \\ \Delta H - \sigma^2 H + A^2 = 0 & \text{dans } \mathbb{R}^2, \\ A, H > 0;\ A, H \to 0 & \text{quand } |x| \to +\infty. \end{cases}$$

On construit des solutions fondamentales multi-pics pour ce système, pour tout σ suffisamment petit. Les centres de ces pics sont localisés aux sommets d'un polygone régulier, tandis que les pics ressemblent, après un changement d'échelle approprié dans la coordonnée A, à l'unique solution radiale de

$$\begin{cases} \Delta w - w + w^2 = 0 & \text{dans } \mathbb{R}^2, \\ 0 < w(y) \to 0 & \text{quand } |y| \to \infty. \end{cases}$$

Une construction similaire est faite pour les sommets de deux polygones concentriques et une procédure générale de détection destructures finies organisées est suggérée.

1. Introduction

The following reaction–diffusion system was proposed in 1972 by Gierer and Meinhardt [5] as a model of biological pattern formation:

$$\begin{aligned} &a_t = d\Delta a - a + a^2/h && \text{in } \Omega, \\ &h_t = D\Delta h - h + a^2 && \text{in } \Omega, \\ &\frac{\partial a}{\partial \nu} = \frac{\partial h}{\partial \nu} = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The Gierer–Meinhardt system was used in [5] to model head formation of *Hydra*, an animal of a few millimeters in length, made up of approximately 100,000 cells of about fifteen different types. It consists of a "head" region located at one end along its length. Typical experiments with *hydra* involve removing part of the "head" region and transplanting it to other parts of the body column. Then, a new "head" will form if the transplanted area is sufficiently far from the (old) head. These observations led to the assumption of the existence of two chemical substances a *slowly* diffusing activator and a *rapidly* diffusing inhibitor, whose concentrations at the point $x \in \Omega$ and time $t > 0$ are represented, respectively, by the quantities $a(x,t)$ and $h(x,t)$. Their diffusion rates, given by the positive constants d and D are then assumed to be so that that $d \ll D$. The Gierer–Meinhardt system falls within the framework of a theory proposed by Turing [30] in 1952 as a mathematical model for the development of complex organisms from a single cell. He speculated that localized peaks in concentration of chemical substances, known as inducers or morphogens, could be responsible for a group of cells developing differently from the surrounding cells. Turing discovered through linear analysis that a large difference in relative size of diffusivities for activating and inhibiting substances carries instability of the homogeneous, constant steady state, thus leading to the presence of nontrivial, possibly stable stationary configurations. Activator-inhibitor systems have been used extensively in the mathematical theory of biological pattern formation, [17, 18]. Among them system (1.1) has been the object of extensive mathematical treatment in recent years. We refer the reader to the survey articles [19,34] for a description of progress made and references.

In particular, it has been a matter of high interest the study of nonconstant positive steady states, namely solutions of the elliptic system

$$\begin{aligned} &d\Delta a - a + a^2/h = 0 \quad \text{in } \Omega,\\ &D\Delta h - h + a^2 = 0 \quad \text{in } \Omega,\\ &\frac{\partial a}{\partial \nu} = 0 = \frac{\partial h}{\partial \nu} \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Problem (1.2) is quite difficult to solve, in general, since it does not have *variational structure*. A first step in studying (1.2) is to study its *shadow system*, an idea due to Keener [15] and Nishiura [28]. Let us observe that dividing the second equation by D, letting formally $D \to \infty$, and making use of the boundary conditions we obtain that $h = \xi =$ constant and the system becomes equivalent to

$$\begin{gathered} d\Delta a - a + a^2/\xi = 0 \quad \text{in } \Omega,\\ \xi = \frac{1}{|\Omega|}\int_\Omega a^2,\\ a > 0 \quad \text{in } \Omega, \qquad \frac{\partial a}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

(This is the so-called *shadow system* associated to (1.2).) Setting $w(y) = \xi^{-1}a(d^{1/2}y)$ transforms the system to the scalar equation

$$\begin{aligned} &\Delta w - w + w^2 = 0 \quad \text{in } \Omega_d,\\ &w > 0 \quad \text{in } \Omega_d, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega_d. \end{aligned}$$

Here Ω_d denotes the expanding domain $d^{-1/2}\Omega$. Conversely, a solution w of this equation determines one of the system. The study of nonconstant solutions of this and related equations as d approaches zero has been an object of extensive study in recent years. Since the domain is expanding as $d \to 0$, it is natural to search for solutions w which resemble, after a convenient translation of the origin, a solution of the limiting problem

$$\begin{aligned} &\Delta w - w + w^2 = 0 \quad \text{in } \mathbb{R}^n,\\ &0 < w(y) \to 0 \quad \text{as } |y| \to \infty. \end{aligned} \tag{1.4}$$

It is well known that this problem has a solution for $n \leqslant 5$. This solution is unique up to translations, and radially symmetric. Solutions of this type, when regarded in the original coordinates, exhibit *point concentration* in the activator a (spike shape) around one or several distinguished points of the closure of the domain Ω as $d \to 0$. A number of interesting results concerning this scalar problem have been derived in recent years. For the subcritical case, we refer the reader to the articles [8,10–12,14, 23,31], and the references therein, starting with the pioneering works [16,20–22]. For the critical exponent case, we refer to the papers [6,7,9,24], and the references therein. A good review of the subject is to be found in [19].

In the case of finite D and bounded domain case, the construction of multiple peak solutions began with the work of I. Takagi [29]. There he constructed multiple symmetric peaks in the one-dimensional case. In high-dimensional case, Ni and Takagi [23] constructed multiple boundary spikes in the case of axially symmetric domains, assuming that D is large. Multiple interior spikes for finite D case in a bounded two-dimensional domain are constructed in [35,36] and [37]. The stability of multiple spikes as well as the dynamics of spikes are considered in [1,2,4,13,25–27,33,36,37] and references therein.

It is of course natural to ask whether these solutions, single or multiple spikes, will actually correspond to limiting configurations solutions of the full system when D becomes finite and d very small. In fact, though tiny, variations of the inhibitor may lead to localized organized patterns which are lost in the limit. This has been recently established for the ground-state problem in the real line in [3]. (Similar results have been obtained independently in [4].) The presence of such steady configurations appears driven by smallness of the *relative size* $\sigma^2 = d/D$ of the diffusion rates of the activating and inhibiting substances. In the shadow system, geometry of the domain is to be held responsible for the presence of multi-peak patterns (see for example [12]). Let us make in (1.2) the scaling

$$u(x) = \sigma^{-1} a(d^{1/2}x), \qquad v(x) = \sigma^{-1} h(d^{1/2}x).$$

Then similarly as one gets formally the ground state problem (1.4) from the shadow system (1.3) we obtain, letting $d \to 0$ in (1.2) with σ stabilized and $n = 2$, the *limiting system*

$$\begin{aligned} &\Delta u - u + u^2/v = 0 \quad \text{in } \mathbb{R}^2, \\ &\Delta v - \sigma^2 v + u^2 = 0 \quad \text{in } \mathbb{R}^2, \\ &u, v > 0, \quad u, v \to 0 \quad \text{as } |x| \to +\infty. \end{aligned} \tag{1.5}$$

This setting is rather natural, since it may correspond to a very large domain with the pattern formation process taking place only very far away from the boundary. On the other hand solutions to this problem would play the role of “basic cells” after scaling, for solutions of the system in a bounded domain. As we will see, a notable feature of this ground-state problem in the plane is the presence of solutions with any prescribed number of bumps in the activator as the parameter σ gets smaller. These bumps are separated from each other at a distance $O(|\log \log \sigma|)$ and approach a single universal profile given by the unique radial solution of (1.4). These solutions are lost in the limiting shadow-system, since, up to translations, only one ground state of Eq. (1.4) exists. This unveils a new side of the rich and complex structure of the solution set of the Gierer–Meinhardt system in the plane and gives rise to a number of questions. The multi-bump solutions we predict in the results to follow correspond, respectively, to bumps arranged at the vertices of a k-regular polygon and at those of two concentric regular polygons. These arrangements with one extra bump at the origin are also considered.

In the sequel by $U(x)$ we denote the unique radially symmetric solution of

$$\begin{aligned} &\Delta U - U + U^2 = 0 \quad \text{in } \mathbb{R}^2, \\ &0 < U(x) \to 0 \qquad\quad \text{as } |x| \to \infty. \end{aligned} \tag{1.6}$$

Let us set

$$\tau_\sigma = \left(\frac{k}{2\pi} \log \frac{1}{\sigma} \int_{\mathbb{R}^2} U^2(y)\, dy \right)^{-1}. \tag{1.7}$$

Our first result is the following:

THEOREM 1.1. – *Let $k \geqslant 1$ be a fixed positive integer. There exists $\sigma_k > 0$ such that, for each $0 < \sigma < \sigma_k$, problem* (1.5) *admits a solution (u, v) with the following property*:

$$\lim_{\sigma \to 0} \left| \tau_\sigma u_\sigma(x) - \sum_{i=1}^{k} U(x - \xi_i) \right| = 0, \tag{1.8}$$

uniformly in $x \in \mathbb{R}^2$. Here the points ξ_i correspond to the vertices of a regular polygon centered at the origin, with sides of equal length l_σ satisfying

$$l_\sigma = \log \log \frac{1}{\sigma} + \mathrm{O}\left(\log \log \log \frac{1}{\sigma} \right). \tag{1.9}$$

Finally, for each $1 \leqslant j \leqslant k$ we have

$$\lim_{\sigma \to 0} \left| \tau_\sigma v_\sigma(\xi_j + y) - 1 \right| = 0,$$

uniformly on compacts in y.

Our second result gives existence of a solution with bumps at vertices of two concentric polygons.

THEOREM 1.2. – *Let $k \geqslant 1$ be a fixed positive integer. There exists $\sigma_k > 0$ such that, for each $0 < \sigma < \sigma_k$, problem* (1.5) *admits a solution (u, v) with the following property*:

$$\lim_{\sigma \to 0} \left| \tau_\sigma u_\sigma(x) - \sum_{i=1}^{k} \left[U(x - \xi_i) + U\left(x - \xi_i^*\right) \right] \right| = 0, \tag{1.10}$$

uniformly in $x \in \mathbb{R}^2$. *Here the points* ξ_i *and* ξ_i^* *are the vertices of two concentric regular polygons. They satisfy*

$$\xi_j = \rho_\sigma e^{\frac{2j\pi}{k}i}, \quad \xi_j^* = \rho_\sigma^* e^{\frac{2\pi j}{k}i}, \quad j = 1, \ldots, k,$$

where

$$\rho_\sigma = \frac{1}{|1 - e^{\frac{2\pi i}{k}}|} \log\log\frac{1}{\sigma} + \mathrm{O}\left(\log\log\log\frac{1}{\sigma}\right),$$

and

$$\rho_\sigma^* = \left(1 + \frac{1}{|1 - e^{\frac{2\pi i}{k}}|}\right) \log\log\frac{1}{\sigma} + \mathrm{O}\left(\log\log\log\frac{1}{\sigma}\right).$$

A similar assertion to (1.9) *holds for* v_σ, *around each of the* ξ_i *and the* ξ_i^**'s.*

THEOREM 1.3. – *Let* $k \geqslant 1$ *be given. Then there exists solutions which are exactly as those in Theorems* 1.1 *and* 1.2 *but with an additional bump at the origin. More precisely, with* $U(x)$ *added to* $\sum_{i=1}^k U(x - \xi_i)$ *in* (1.8) *and added to* $\sum_{i=1}^k [U(x - \xi_i) + U(x - \xi_i^*)]$ *in* (1.10).

The method employed in the proof of the above results consists of a Lyapunov–Schmidt type reduction. Fixing m points which satisfy the constraints

$$\frac{2}{3}\log\log\frac{1}{\sigma} \leqslant |\xi_j - \xi_i| \leqslant b\log\log\frac{1}{\sigma},$$

for all $i \neq j$ and with some $b > 1$ to be determined later, an auxiliary problem is solved uniquely, and solutions satisfying the required conditions will be precisely those satisfying a nonlinear system of equations of the form

$$c_{i\alpha}(\xi_1, \xi_2, \ldots, \xi_m) = 0, \quad i = 1, \ldots, m, \ \alpha = 1, 2,$$

where for such a class of points the functions $c_{i\alpha}$ satisfy

$$c_{i\alpha}(\xi_1, \ldots, \xi_k) = \frac{\partial}{\partial \xi_{i\alpha}}\left[\sum_{i \neq j} F\big(|\xi_j - \xi_i|\big)\right] + \varepsilon_{i\alpha}, \tag{1.11}$$

function $F : \mathbb{R}_+ \to \mathbb{R}$ is of the form

$$F(r) = \frac{c_7 \log r}{\log 1/\sigma} + c_8 U(r),$$

c_7 and c_8 are universal constants and

$$\varepsilon_{i\alpha} = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right),$$

for some $\gamma > 0$. Although (1.11) does not have a variational structure, solutions of the problem $c_{i\alpha} = 0$ are close to critical points of the functional $\sum_{i \neq j} F(|\xi_j - \xi_i|)$. In spite of the simple form of this functional, its critical points are highly degenerate because of the invariance under rotations and translations of the problem. Thus, to get solutions using degree theoretical arguments, we need to restrict ourselves to classes of points enjoying symmetry constraints. This is how Theorems 1.1–1.3 are established. On the other hand, we believe strongly that finer analysis may yield existence of more complex patterns. This, as well as the stability of these patterns, are among issues we intend to study in the future. In this regard, we can mention that the construction of such arrangements of multi-bumps can be obtained in patterns such as a symmetric six-arm snowflake and a bounded (hexagonal) honeycomb.

The rest of the paper will be devoted to the proofs of these results. In Section 2 we set up the scheme of proof, in particular we explain why the constant τ_σ is the right scaling factor to get the desired multi-bump expansion. The program there outlined is carried over the following sections.

2. The scheme of the proof

Our strategy of the proof of the main results is based on the idea of solving the second equation in (1.5) for v and then working with a nonlocal elliptic PDE rather than directly with the system. It is however convenient to do this by replacing first u by $\tau_\sigma u$ and v by $\tau_\sigma v$, which transforms (1.5) into the problem

$$\begin{aligned} &\Delta u - u + u^2/v = 0 \quad \text{in } \mathbb{R}^2,\\ &\Delta v - \sigma^2 v + \tau_\sigma u^2 = 0 \quad \text{in } \mathbb{R}^2,\\ &u, v > 0, \quad u, v \to 0 \quad \text{as } |x| \to +\infty. \end{aligned} \tag{2.1}$$

With the choice of the parameter τ_σ as in (1.7),

$$\tau_\sigma = \left(\frac{k}{2\pi} \log \frac{1}{\sigma} \int_{\mathbb{R}^2} U^2(y)\, dy \right)^{-1}, \tag{2.2}$$

we obtain

$$u \sim \sum_{i=1}^{k} U(x - \xi_i), \qquad v \sim 1,$$

i.e., the height of the bumps near the ξ_i's remains bounded as $\sigma \to 0$. We should point out here that the situation is similar in dimension $N = 1$, see [3], with the scaling factor τ_σ of order σ^{-1}. On the other hand, in dimension $N \geqslant 3$, it is not at all clear what the right scaling factor should be. This difference is due to the behavior of the fundamental solution of $-\Delta + \sigma^2$.

In the sequel, by $T(h)$ we denote the unique solution of the equation

$$\begin{aligned} &-\Delta v+\sigma^2 v=\tau_\sigma h \quad \text{in } \mathbb{R}^2,\\ &v(x)\to 0 \qquad\qquad \text{as } |x|\to+\infty, \end{aligned} \tag{2.3}$$

for $h\in L^2(\mathbb{R}^2)$, namely $T=\tau_\sigma(-\Delta+\sigma^2)^{-1}$. Solving the second equation for v in (2.1) we get $v=T(u^2)$, which leads to the nonlocal PDE for u

$$-\Delta u+u-\frac{u^2}{T(v^2)}=0. \tag{2.4}$$

We consider points $\xi_1,\xi_2,\ldots,\xi_k$ in $\mathbb{R}^2$ which are the candidates for the location of spikes. We will assume that for some $b>1$,

$$\frac{2}{3}\log\log\frac{1}{\sigma}<|\xi_j-\xi_i|<b\log\log\frac{1}{\sigma} \quad \forall i\neq j. \tag{2.5}$$

Let us write

$$W(x)=\sum_{i=1}^k U(x-\xi_i).$$

We look for a solution to (2.4) in the form $u=W+\phi$, where ϕ is a lower order term. Then, formally, we have

$$T\big(u^2\big)=T\big(W^2\big)+2T(W\phi)+\text{l.o.t.}$$

where l.o.t. correspond to lower order terms. We denote $V=T(W^2)$. By $K(|y|)$ we denote the fundamental solution to $-\Delta+1$ in the plane. We can write

$$T\big(W^2\big)=\tau_\sigma\int W^2(y)K\big(\sigma|x-y|\big)\,dy\sim\tau_\sigma\sum_{i=1}^k\int U^2(x-\xi_i)K\big(\sigma|x-y|\big)\,dx$$

where the integration extends over all $\mathbb{R}^2$. For $|x|=\mathrm{o}(\sigma^{-1})$ we have $K(\sigma|x|)=-\frac{1}{2\pi}\log(\sigma|x|)+\mathrm{O}(1)$. Using this, and the definition of τ_σ we get that near the ξ_i's,

$$V(x)=1+\text{l.o.t.}$$

Arguing similarly we get

$$T(W\phi)=\omega\int W\phi\,dx+\text{l.o.t.},\qquad \omega=\frac{\tau_\sigma}{2\pi}\log\frac{1}{\sigma}=\frac{1}{k\int U^2}.$$

Then

$$\frac{u^2}{v}=\frac{W^2+2W\phi+\text{l.o.t.}}{V+T(W\phi)+\text{l.o.t.}}=\frac{W^2}{V}+2W\phi-W^2\omega\int W\phi+\text{l.o.t.}$$

Substituting all this in (2.4) we obtain the equation for ϕ

$$-\Delta\phi + (1-2W)\phi + 2\omega W^2 \int W\phi = \mathcal{S} + N(\phi), \tag{2.6}$$

where $\mathcal{S} = \Delta W - W + \frac{W^2}{V}$ and $N(\phi)$, defined by

$$N(\phi) = \left[\frac{(W+\phi)^2}{T((W+\phi)^2)} - \frac{W^2}{V} - 2W\phi + 2W^2\omega \int W\phi\right],$$

represents higher order terms in ϕ.

Thus we have reduced the problem of finding solutions to (2.1) to the problem of solving (2.6) for ϕ. We set $\frac{\partial W}{\partial \xi_{j\alpha}} = Z_{j\alpha}$. Rather than solving directly problem (2.6), we consider first the following auxiliary problem: given points ξ_i, find a function ϕ such that for certain constants $c_{i\alpha}$ the following equation is satisfied:

$$L\phi = \mathcal{S} + N(\phi) + \sum_{j,\alpha} c_{j\alpha} Z_{j\alpha}, \tag{2.7}$$

$$\langle \phi, Z_{j\alpha}\rangle = 0, \quad j = 1,\ldots,k, \tag{2.8}$$

where

$$L\phi = -\Delta\phi + (1-2W)\phi + 2\omega W^2 \int W\phi \tag{2.9}$$

and $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product.

We will prove in Section 4 that this problem is uniquely solvable within a class of small functions ϕ for all points $(\xi_1,\ldots,\xi_k)$ satisfying constraints (2.5). Besides, the resulting constants $c_{i\alpha}(\xi_1,\ldots,\xi_m)$ admit the expansion (1.11). We will of course get a solution of the full problem whenever the points ξ_i are adjusted in such a way that all of $c_{i\alpha}$'s vanish. We show the existence of such points in Section 5, where Theorems 1.1–1.3 are finally established. In remainder of the paper we rigorously carry out the program outlined above. In particular, we will need to understand invertibility properties of the linear operator L first. We will do this in the next section.

3. The linear operator

PROPOSITION 3.1. – *Let U be the unique, positive, radially symmetric solution to* (1.6).

(a) *There exists a positive constant μ_0 such that, as $r \to \infty$, the following formula holds*

$$U(r) = \mu_0 r^{-1/2} e^{-r}\left[1 + \mathrm{O}(r^{-1})\right].$$

Moreover $U'(r) < 0$, $r > 0$ and $U'(r) = -U(r)[1+\mathrm{O}(r^{-1})]$, as $r \to \infty$; a similar formula holds for $U''(r)$.

(b) *Let $L_0 = -\Delta + (1-2U)\mathrm{id}$. Then we have*

$$\mathrm{Ker}(L_0) = \mathrm{span}\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\right\}.$$

(c) *Let* L *be the operator defined in* (2.9) *and let*

$$L^*\phi = -\Delta\phi + (1-2W)\phi + 2\omega W \int W^2\phi$$

be its formal adjoint. If we denote

$$Z_{j\alpha} = \frac{\partial W}{\partial \xi_{j\alpha}},$$

then, for all $j = 1,\dots,k,\ \alpha = 1,2$, *we have*

$$LZ_{j\alpha} = \mathrm{o}\Big(\exp\Big(-\frac{1}{2}\min_{l\neq m}|\xi_l - \xi_m|\Big)\Big),$$
$$L^*Z_{j\alpha} = \mathrm{o}\Big(\exp\Big(-\frac{1}{2}\min_{l\neq m}|\xi_l - \xi_m|\Big)\Big).$$

Similar estimates hold for Sobolev norms of $LZ_{j\alpha}$ *and* $L^*Z_{j\alpha}$.

We shall carry out the analysis of the linear operator L in a framework of weighted L^∞ spaces. For this purpose we consider the following norms for a function defined on $\mathbb{R}^2$: given points $\xi_1,\dots,\xi_k$ we define

$$\|\phi\|_* = \sup_{x\in\mathbb{R}^2} e^{2\mu \min_{i\leqslant k}|x-\xi_i|}\big|\phi(x)\big|, \tag{3.1}$$

where $0<\mu<1/8$ is a fixed number. We also consider

$$\|h\|_{**} = \sup_{x\in\mathbb{R}^2} e^{3\mu \min_{i\leqslant k}|x-\xi_i|}\big|h(x)\big|. \tag{3.2}$$

This choice of norms will become clear later. In the sequel we will not emphasize the dependence of the norms of a particular value of μ. We first consider a problem that will later give rise to the finite-dimensional reduction. Given a function h, $\|h\|_{**}<\infty$ find a ϕ and constants $c_{j\alpha},\ j=1,\dots,k,\ \alpha=1,2$, such that one has

$$\begin{aligned} &L\phi = h + \sum_{j,\alpha} c_{j\alpha}Z_{j\alpha} \quad \text{in } \mathbb{R}^2,\\ &\phi(x)\to 0 \qquad \text{as } |x|\to\infty,\\ &\langle\phi, Z_{j\alpha}\rangle = 0 \quad \text{for } j=1,\dots,k,\ \alpha=1,2. \end{aligned} \tag{3.3}$$

By $\mathbf{c}$ we will denote a vector with components $c_{j\alpha}$.

We refer to a pair $(\phi,\mathbf{c})$ as a solution to (3.3). We have the following existence result for (3.3).

THEOREM 3.1. – *There exist positive numbers* R *and* C *such that, for any points* $\xi_1, \dots, \xi_k$ *satisfying* $|\xi_i - \xi_j| > R$ *for all* $i \neq j$*, and* h *locally Hölder continuous with* $\|h\|_{**} < \infty$*, problem* (3.3) *has a unique solution* $\phi = \mathcal{T}(h)$ *and* $\mathbf{c} = \mathbf{c}(h)$*. Moreover,*

$$\left\|\mathcal{T}(h)\right\|_* \leqslant C\|h\|_{**}. \tag{3.4}$$

The main ingredient in the proof of this result is the following lemma.

LEMMA 3.1. – *Assume that* ξ_j^n*,* $j = 1, \dots, k$*, are such that* $\min_{i \neq j} |\xi_i^n - \xi_j^n| \to \infty$*,* $\|h_n\|_{**} \to 0$*, and that* ϕ_n *solves*

$$L\phi_n = h_n + \sum_{j,\alpha} c_{j\alpha}^n Z_{j\alpha} \quad \text{in } \mathbb{R}^2,$$

$$\phi_n(x) \to 0 \qquad \text{as } |x| \to \infty,$$

$$\langle \phi_n, Z_{j\alpha} \rangle = 0 \quad \text{for } j = 1, \dots, k,\ \alpha = 1, 2.$$

Then $\|\phi_n\|_* \to 0$.

Proof. – We will argue by contradiction. Without loss of generality we can assume that $\|\phi_n\|_* = 1$. Our first observation is that $c_{j\alpha}^n \to 0$. Indeed, multiplying the equation by $Z_{j\alpha}$ and integrating by parts we get

$$\left\langle \phi_n, L^* Z_{j\alpha} \right\rangle = c_{j\alpha}^n \int Z_{j\alpha}^2 + \sum_{(m,\beta) \neq (j,\alpha)} c_{m\beta}^n \langle Z_{m\beta}, Z_{j\alpha} \rangle + \langle h_n, Z_{j\alpha} \rangle.$$

Using Proposition 3.1, by rather standard calculations, it follows that $c_{j\alpha}^n \to 0$ as $n \to \infty$.

Our next goal is to prove that

$$\int W\phi_n \to 0 \quad \text{as } n \to \infty.$$

To this end consider test function

$$Z = x \cdot \nabla W + 2W$$

and let

$$L_0 = -\Delta + (1 - 2U)\mathrm{id}.$$

We first claim that

$$L_0 Z = -2W + \mathrm{o}(1).$$

Indeed if we set $u_\lambda(x) = \lambda^2 W(\lambda x)$, then

$$\Delta u_\lambda(x) = \lambda^2 u_\lambda - u_\lambda^2 + \sum_{i \neq j} U(\lambda x - \xi_i) U(\lambda x - \xi_j).$$

Since $Z = \frac{\partial u_\lambda}{\partial \lambda}|_{\lambda=1}$ the claim follows now from Proposition 3.1 and the above.

Decompose $\phi_n = a_n W + \psi_n$ where $\langle W, \psi_n\rangle = 0$. Then $L\psi_n = L_0\psi_n$ and we have

$$\mathrm{o}(1) = \langle L\phi_n, Z\rangle = a_n\langle LW, Z\rangle + \langle L_0\psi_n, Z\rangle.$$

But

$$\langle L_0\psi_n, Z\rangle = \langle \psi_n, L_0 Z\rangle = -2\langle W, \psi_n\rangle + \mathrm{o}(1) = \mathrm{o}(1)$$

and

$$\langle LW, Z\rangle = \langle W^2, Z\rangle + \mathrm{o}(1) = \frac{1}{3}\int x\cdot\nabla W^3 + 2\int W^3 + \mathrm{o}(1) = \frac{4}{3}\int W^3 + \mathrm{o}(1).$$

It follows that $a_n \to 0$, or $\langle W, \phi_n\rangle = \mathrm{o}(1)$. Going back to the equation satisfied by ϕ_n, we see then that

$$-\Delta\phi_n + (1-2W)\phi_n = \mathrm{o}(1)\left(W^2 + \sum_{j,\alpha} Z_{j\alpha}\right) + h_n \equiv g_n + h_n,$$

with $\|g_n\|_{**} \to 0$. We can rewrite this relation as

$$\phi_n(x) = \int K\big(|x-y|\big)(2W\phi_n + g_n + h_n)\,dy \equiv \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where K is the fundamental solution of $-\Delta + 1$ in $\mathbb{R}^2$.

Using the definition of the norm $\|\cdot\|_*$ and normalization $\|\phi_n\|_* = 1$ we get

$$\begin{aligned}\mathrm{I} &\leqslant 2\int K\big(|x-y|\big)W(y)\big|\phi_n(y)\big|\,dy\\ &\leqslant 2\int K\big(|x-y|\big)W(y)e^{-2\mu\min_{j\leqslant k}|y-\xi_j|}\,dy\\ &\leqslant Ce^{-(1+2\mu)\min_{j\leqslant k}|x-\xi_j|}.\end{aligned}$$

Furthermore we have

$$\mathrm{II} \leqslant \mathrm{o}(1)\int K\big(|x-y|\big)\left[W^2(y) + \sum_{j,\alpha}|Z_{j\alpha}|\right]dy \leqslant \mathrm{o}(1)e^{-\min_{j\leqslant k}|x-\xi_j|}.$$

Finally we get

$$\begin{aligned}\mathrm{III} &\leqslant C\|h_n\|_{**}\int K\big(|x-y|\big)e^{-3\mu\min_{j\leqslant k}|y-\xi_j|}\,dy\\ &\leqslant \mathrm{o}(1)\int K\big(|x-y|\big)e^{-3\mu\min_{j\leqslant k}|y-\xi_j|}\,dy\\ &\leqslant \mathrm{o}(1)e^{-3\mu\min_{j\leqslant k}|x-\xi_j|}.\end{aligned}$$

Combining the above inequalities we obtain

$$\big|\phi_n(x)\big| \leqslant \nu_0 e^{-3\mu\min_{j\leqslant k}|x-\xi_j|}, \tag{3.5}$$

with some $\nu_0 > 1$ independent on n. Consequently,

$$e^{2\mu \min_{j\leqslant k} |x-\xi_j|}\left|\phi(x)\right| \leqslant \nu_0 e^{-\mu \min_{j\leqslant k} |x-\xi_j|}.$$

Since $\|\phi_n\|_* = 1$, the above inequality implies that at least for one index m we have

$$\sup_{\left\{|x-\xi_m|<\frac{\log \nu_0}{\mu}\right\}} \left|\phi_n(x)\right| > \nu_0^{-2}.$$

We set $\tilde{\phi}_n(y) = \phi_n(y+\xi_m)$. A standard compactness argument then yields the existence of a subsequence of $\tilde{\phi}_n$ which converges uniformly over compacts to a nontrivial solution ϕ of the equation

$$-\Delta\phi + (1-2U)\phi = 0,$$

which decays exponentially to zero at infinity. Moreover, $\langle \phi_n, Z_{j\alpha}\rangle = 0$, estimate (3.5) and Dominated Convergence Theorem yield

$$\left\langle \phi, \frac{\partial U}{\partial y_\alpha}\right\rangle = 0, \quad \alpha = 1, 2,$$

hence, from Proposition 3.1, we obtain $\phi \equiv 0$. We have reached a contradiction which concludes the proof of the lemma. □

Proof of Theorem 3.1. – Let us set

$$\mathcal{H} = \left\{\phi \in H^1\left(\mathbb{R}^N\right) \mid \langle \phi, Z_{j\alpha}\rangle = 0,\ j = 1, \ldots, k,\ \alpha = 1, 2\right\}.$$

Observe that ϕ solves problem (3.3) if and only if $\phi \in \mathcal{H}$ satisfies

$$\int (\nabla\phi\nabla\psi + \phi\psi) - \langle 2W\phi, \psi\rangle + 2\omega\langle W, \phi\rangle\left\langle W^2, \psi\right\rangle = \langle h, \psi\rangle, \quad \forall\psi \in \mathcal{H}.$$

This equation that can be rewritten in $\mathcal{H}$ in the form

$$\phi + S(\phi) = \bar{h}, \tag{3.6}$$

where S is a linear compact operator in $\mathcal{H}$ and $\bar{h} \in \mathcal{H}$.

Using Fredholm's alternative to show that this equation is uniquely solvable it suffices to check that Eq. (3.6) for $\bar{h} \equiv 0$ has only the zero solution. To this end, we can just assume the opposite, namely the existence of points ξ_i^n such that $|\xi_i^n - \xi_j^n| \to \infty$, so that ϕ_n solves

$$L(\phi_n) = \sum_{j,\alpha} c_{j\alpha}^n Z_{j\alpha} \quad \text{in } \mathbb{R}^2,$$
$$\phi_n(x) \to 0 \qquad \text{as } |x| \to \infty,$$
$$\langle \phi_n, Z_{j\alpha}\rangle = 0 \quad j = 1, \ldots, k,\ \alpha = 1, 2,$$

and $\|\phi_n\|_* = 1$. But this contradicts the previous lemma. Once we know ϕ we can determine a unique $\mathbf{c}$ from the system of equations

$$c_{j\alpha}\int Z_{j\alpha}^2 + \sum_{(m,\beta)\neq(j,\alpha)} c_{m\beta}\langle Z_{m\beta}, Z_{j\alpha}\rangle = \langle \phi, L^* Z_{j\alpha}\rangle - \langle h, Z_{j\alpha}\rangle.$$

Estimate (3.4) follows now immediately from Lemma 3.1. □

4. Basic estimates

In this section and those to follow, we make the following assumptions on the points $\xi_1,\dots,\xi_k$:

$$\frac{2}{3}\log\log\frac{1}{\sigma} \leqslant |\xi_i - \xi_j| \quad \forall i \neq j, \tag{4.1}$$

and for a certain number $b > 1$

$$|\xi_i| \leqslant \frac{b}{2}\log\log\frac{1}{\sigma} \quad \forall i. \tag{4.2}$$

The estimates obtained below will be uniform on points ξ_i satisfying these constraints, and valid for all sufficiently small $\sigma > 0$. Observe that from (4.2) it follows

$$|\xi_i - \xi_j| \leqslant b\log\log\frac{1}{\sigma}, \quad i \neq j.$$

We also notice that from our argument in the following sections one can show that it suffices to take $b > 20$.

For the rest of this section as well as in the remainder of this paper the same symbol γ will designate different positive numbers taken in each step smaller if necessary.

Our immediate purpose is to work out estimates for the solution V of the problem

$$-\Delta V + \sigma^2 V = \tau_\sigma\left[\sum_{i=1}^{k} U\big(|x-\xi_i|\big)\right]^2,$$
$$V(x) \to 0 \quad \text{as } |x| \to \infty,$$

where τ_σ is given in (2.2) Denote by Z_0 the solution of

$$-\Delta Z_0 + \sigma^2 Z_0 = U\big(|x|\big)^2,$$
$$Z_0(x) \to 0 \quad \text{as } |x| \to \infty,$$

and by $\theta_{ij}(x)$, $i \neq j$, that of

$$\begin{aligned} -\Delta\theta + \sigma^2\theta &= U\big(|x-\xi_i|\big)U\big(|x-\xi_j|\big), \\ \theta(x) &\to 0 \quad \text{as } |x| \to \infty. \end{aligned} \tag{4.3}$$

Then we have

$$V(x) = \tau_\sigma \sum_{i=1}^{k} Z_0\big(|x - \xi_i|\big) + \tau_\sigma \sum_{i \neq j} \theta_{ij}(x).$$

We will now study $Z_0(|x|)$. Let $K(|x|)$ be the fundamental solution of

$$-\Delta K + K = \delta_0,$$

where δ_0 is the Dirac mass at the origin.

LEMMA 4.1. – *The following expansion of K holds*:

$$K(r) = -\frac{1}{2\pi} \log r + c_1 + c_2 r^2 \log r + \psi(r),$$

for $0 < r < 1$, where ψ is a smooth function, with $\psi(0) = 0$, $\psi'(0) = 0$ and c_1, c_2 are universal constants.

Proof. – Let $h(r) = K(r) + \frac{1}{2\pi} \log r$. Then h satisfies

$$-\Delta h + h = \frac{1}{2\pi} \log r,$$
$$h(1) = K(1).$$

Consequently, h is a radially symmetric function which is of class C^1 in $B(0,1)$. More precisely, it can be written in the form:

$$h(r) = c_1 + c_2 r^2 \log r + \psi(r),$$

with ψ as in the statement of the lemma, and the desired expansion follows. □

Our next purpose is to estimate $Z_0(x)$ in the range $|x| < 10b \log\log \frac{1}{\sigma}$. Our starting point is that Z_0 can be represented in the following form:

$$Z_0\big(|x|\big) = \int K\big(\sigma|x - y|\big) U^2\big(|y|\big)\, dy.$$

We can expand Z_0 as

$$Z_0\big(|x|\big) = -\frac{1}{2\pi} \int \log\big(\sigma|x - y|\big) U^2\big(|y|\big)\, dy + c_1 \int U^2 + \mathrm{O}\big(\sigma^2 \log \sigma\big),$$

whenever $|x| < 10b \log\log \frac{1}{\sigma}$. The quantity $\mathrm{O}(\sigma^2 \log \sigma)$, as well as its derivative, is small, uniformly in x in this range.

Let us consider now the quantity

$$H\big(|x|\big) = -\frac{1}{2\pi} \int \log|x - y| U^2\big(|y|\big)\, dy. \tag{4.4}$$

$H(r)$ can be written explicitly, for $r > 1$, as

$$H(r) = c_4 - \int_1^r \frac{ds}{s} \int_0^s U^2(\rho)\rho\, d\rho,$$

for a certain constant c_4. Hence, for $|x| > 1$,

$$H\big(|x|\big) = -\big(\log|x|\big) \int U^2 + f\big(|x|\big)$$

where f and its derivative are uniformly bounded.

Let us now consider the functions $\theta_{ij}(x)$ given in (4.3). Since θ_{ij} can be represented as

$$\theta_{ij}(x) = \int K\big(\sigma|x-y|\big)U(y-\xi_i)U(y-\xi_j)\, dy.$$

Then, for $|x| < 10b \log\log\frac{1}{\sigma}$, the following uniform expansion holds

$$\theta_{ij}(x) = -\frac{1}{2\pi} \int \log\big(\sigma|x-y|\big)U(y-\xi_i)U(y-\xi_j)\, dy + c_1 \int U(y-\xi_i)U(y-\xi_j)\, dy + \mathrm{o}(\sigma).$$

Using Proposition 3.1 one can show that there is a $\gamma > 0$ such that

$$\int \left(c_1 - \frac{1}{2\pi}\log|x-y|\right)U(y-\xi_i)U(y-\xi_j)\, dy = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^\gamma}\right),$$

uniformly on $|x| < 10b \log\log\frac{1}{\sigma}$; a similar estimate holds for the derivative of the above expression with respect to x. Let us set

$$\delta_*\big(|z|\big) = \int U(y)U(y-z)\, dy. \tag{4.5}$$

We have that $|U(y)| \leqslant Ce^{-|y|}$, hence, for $\rho = |z|^{-1}$, $\hat{z} = \rho z$,

$$\delta_*\big(|z|\big) \leqslant Ce^{-(1-\rho)|z|} \int e^{-\rho(|x|+|x-z|)}\, dx \leqslant Ce^{-|z|}|z|^2 \int e^{-(|y|+|y-\hat{z}|)}\, dy.$$

This implies

$$\delta_*\big(|z|\big) \leqslant Ce^{-|z|}|z|^2. \tag{4.6}$$

A similar estimate is valid for the derivative of δ_*. Hence

$$\delta_*\big(|\xi_i - \xi_j|\big) \leqslant Ce^{-|\xi_i-\xi_j|}|\xi_i - \xi_j|^2.$$

Thus, combining the above estimates we obtain:

LEMMA 4.2. –
(a) *The following estimate holds uniformly for* $|x| < 10b \log\log\sigma$

$$V(x) = 1 + \tau_\sigma \sum_{i=1}^{k} H\big(|x-\xi_i|\big) + c_5 \sum_{i\neq j} \delta_*\big(|\xi_j-\xi_i|\big) + \mathrm{O}\bigg(\frac{1}{\log 1/\sigma}\bigg).$$

A similar estimate holds for the derivatives of V *with respect to* x. *The function* $H(|x|)$ *is given by* (4.4) *and, for* $|x| > 1$, *has the expansion,*

$$H\big(|x|\big) = -\big(\log|x|\big)\int U^2 + f\big(|x|\big)$$

with a smooth, bounded f. *The function* δ_* *is given by* (4.5) *and satisfies estimate* (4.6).
(b) *If* $|x| > 10b\log\log\sigma$ *then the following lower estimate is true*

$$V(x) \geqslant \frac{c_6}{\log 1/\sigma} e^{-2\sigma|x|}. \tag{4.7}$$

Estimate (4.7) can be proven by using a suitable barrier function. We omit the details.

5. Further estimates

For brevity we shall denote $U_i(x) = U(|x-\xi_i|)$ and $W = \sum_{i=1}^{k} U_i$. Our purpose in this section is to derive estimates for the quantity

$$\mathcal{S} \equiv \Delta W - W + \frac{W^2}{V}, \tag{5.1}$$

which can be rewritten as

$$\mathcal{S} = \frac{W^2}{V} - \sum_{i=1}^{k} U_i^2.$$

Our first result is the following:

LEMMA 5.1. – *Let the number* $\mu > 0$ *in the definition of the norm* $\|\cdot\|_{**}$ *be such that* $\mu < 1/6$. *For all points* ξ_i *satisfying constraints* (4.1), (4.2), *and all sufficiently small* σ *we have*

$$\|\mathcal{S}\|_{**} \leqslant \frac{1}{(\log 1/\sigma)^{1/2+\gamma}},$$

where $\gamma > 0$.

Proof. – Let us assume first $|x| < 10b\log\log\frac{1}{\sigma}$. We write

$$\mathcal{S} = \frac{1-V}{V}\sum_{i=1}^{k} U_i^2 + V^{-1}\sum_{i\neq j} U_i U_j = I_1 + I_2. \tag{5.2}$$

To begin with, observe that, in the region under consideration,

$$V = 1 + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1-\delta}}\right),$$

for any $\delta > 0$. Hence

$$I_1 = \left(\sum_{i=1}^{k} U_i^2\right)\mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1-\delta}}\right).$$

On the other hand,

$$V^{-1}U_iU_j \leqslant 2U_iU_j \leqslant Ce^{-3\mu(|x-\xi_i|+|x-\xi_j|)/2}e^{-(1-3\mu/2)|\xi_i-\xi_j|}.$$

Hence

$$I_2 \leqslant Ce^{-3\mu \min_i |x-\xi_i|}\frac{1}{(\log 1/\sigma)^{\frac{2}{3}(1-3\mu/2)}}$$

in this region. Choosing $\mu < 1/6$, we then get

$$|\mathcal{S}| \leqslant Ce^{-3\mu \min_i |x-\xi_i|}\frac{1}{(\log 1/\sigma)^{\frac{1}{2}+\gamma}}, \quad 0 < \gamma < 1/6 - \mu, \tag{5.3}$$

for all small σ, provided that $|x| < 10b \log\log \frac{1}{\sigma}$.

Assume now $|x| > 10b \log\log \frac{1}{\sigma}$. Then, recalling estimate (4.7), we get, assuming also that $b > 1$,

$$|\mathcal{S}| \leqslant C\log\frac{1}{\sigma}\left(\sum_{i=1}^{k} U_i^2\right)e^{\sigma|x|} \leqslant C\log\frac{1}{\sigma}\left(\sum_{i=1}^{k} U_i^2\right)^{\frac{1}{2}} e^{-\frac{7}{2}\log\log\frac{1}{\sigma}}$$
$$\leqslant \left(C\log\frac{1}{\sigma}\right)^{-1} e^{-\min_i |x-\xi_i|}. \tag{5.4}$$

Combining relations (5.3) and (5.4), the assertion of the lemma immediately follows. □

Another quantity whose estimates will be crucial for the remaining arguments is

$$\mathcal{I} = \int \mathcal{S} Z_{i\alpha}. \tag{5.5}$$

We shall consider $i = 1 = \alpha$ only, since the other cases are similar. Observe that $\frac{\partial U(x-\xi_1)}{\partial \xi_{11}} = -\frac{\partial U(x-\xi_1)}{\partial x_1}$ and thus we have

$$-\mathcal{I} = \int (1-V)V^{-1}\sum_{i=1}^{k} U_i^2 \frac{\partial U}{\partial x_1}(x-\xi_1)\,dx$$
$$+ \int V^{-1}\sum_{i\neq j} U_iU_j \frac{\partial U}{\partial x_1}(x-\xi_1)\,dx = I_1 + I_2.$$

We will estimate separately I_1 and I_2. In fact we will find the following expansions:

$$I_2 = -c_7 \frac{\partial}{\partial \xi_{11}} \sum_{j \neq 1} U(\xi_j - \xi_1) + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right) \tag{5.6}$$

and

$$I_1 = -c_8 \frac{\partial}{\partial \xi_{11}} \sum_{i \neq 1} \frac{\log|\xi_i - \xi_1|}{\log \sigma} + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right). \tag{5.7}$$

Here c_7 and c_8 are absolute constants and γ some positive number.

We will establish first (5.6). Using Lemma 4.2 we obtain

$$\int V^{-1} \sum_{i \neq j} U_i U_j \frac{\partial U_1}{\partial x_1} = \int \sum_{i \neq j} U_i U_j \frac{\partial U_1}{\partial x_1} + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right).$$

Let us estimate $\int U_i U_j \frac{\partial U_1}{\partial x_1}$ for $i \neq j$. We observe that if $i, j \neq 1$, then

$$\int U_i U_j \frac{\partial U_1}{\partial x_1} = \mathrm{O}\left(e^{-|\xi_i - \xi_1| - |\xi_i - \xi_1|}\right) = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{4/3}}\right).$$

On the other hand, if $i = 1$, $j \neq 1$ we get

$$\int U_1 U_j \frac{\partial U_1}{\partial x_1} = -\frac{1}{2} \frac{\partial}{\xi_{11}} \int U^2(x - \xi_1) U(x - \xi_j)\, dx.$$

To analyze this last quantity, we will use the following intermediate result.

LEMMA 5.2. – *Let $h(|x|)$ be a nonnegative, radially symmetric function such that $h(|x|) \leqslant Ce^{-\alpha|x|}$ for some $1 < \alpha \leqslant 2$ and let*

$$F(z) = \int h\big(|x|\big) U(x - z)\, dx.$$

There is a number $c_5 > 0$ such that

$$F(z) = c_5 U(z) + \psi(z),$$

where $\psi(z)$, as well as its derivative, satisfy $\psi(z) = \mathrm{O}(e^{-\alpha r})$ as $r \to +\infty$.

Proof. – Let us observe that F is radially symmetric and satisfies

$$-\Delta F(z) + F(z) = \int h\big(|x|\big) U(x - z)^2\, dx \equiv h_0(z).$$

Thus h_0 is radially symmetric, and $|h_0(z)| \leqslant Ce^{-\alpha|z|}$. Using the ODE satisfied by F we see that it can be represented, thanks to the variation of parameters formula, as

$$F(r) = \lambda_0 K(r) - K(r) \int_r^\infty \frac{ds}{sK(s)^2} \int_s^\infty h_0(t) K(t) t\, dt,$$

where

$$\lambda_0 = \int_0^\infty \frac{ds}{sK(s)^2} \int_s^\infty h_0(t)K(t)t\,dt > 0.$$

Similarly, since U satisfies

$$-\Delta U(z) + U(z) = U(z)^2,$$

we get, with

$$\lambda_1 = \int_r^\infty \frac{ds}{sK(s)^2} \int_s^\infty U^2(t)K(t)t\,dt > 0,$$

that

$$U(r) = \lambda_1 K(r) - K(r) \int_r^\infty \frac{ds}{sK(s)^2} \int_s^\infty U^2(t)K(t)t\,dt.$$

Then, choosing $c_5 = \frac{\lambda_0}{\lambda_1}$, the result of the lemma follows with

$$\psi(r) = K(r) \int_r^\infty \frac{ds}{sK(s)^2} \int_r^\infty \left[h_0(t) - c_5 U^2(t)\right] K(t)t\,dt.$$

This concludes the proof. □

Using Lemma 5.2, we thus get that for a certain universal constant $c_7 > 0$,

$$\int U^2(x-\xi_1)U(x-\xi_j)\,dx = 2c_7 U(\xi_j - \xi_1) + \mathrm{O}\left(e^{-2|\xi_j-\xi_1|}\right),$$

with a similar estimate for its derivative. Hence

$$\int V^{-1} \sum_{i\neq j} U_i U_j \frac{\partial U_1}{\partial x_1} = -c_7 \frac{\partial}{\xi_{11}} \sum_{j\neq 1} U(\xi_j - \xi_1) + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right),$$

and estimate (5.6) thus follows.

Let us continue now with I_1. Using Lemma 4.2 we get

$$I_1 = -\int \sum_{i=1}^k U(x-\xi_i)^2 \left\{ \sum_{j=1}^k \tau_\sigma H\left(|x-\xi_j|\right) + \frac{c_5}{k} \sum_{j\neq l} \delta_*\left(|\xi_j - \xi_l|\right) \right\}$$
$$\times \frac{\partial U}{\partial x_1}(x-\xi_1)\,dx + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right)$$

where H is given by (4.4) and δ_* by (4.5). Let us first estimate

$$g_{ijl} = \int U(x-\xi_i)^2 \delta_*\left(|\xi_j - \xi_l|\right) \frac{\partial U}{\partial x_1}(x-\xi_1)\,dx,$$

with $j \neq l$. For $i = 1$ this term is zero, while for $i \neq 1$ we can estimate, using (4.6) and Lemma 5.2,

$$|g_{ijl}| \leqslant |\xi_j - \xi_l|^2 e^{-|\xi_j - \xi_l| - |\xi_i - \xi_1|} = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right).$$

Let us consider now the terms

$$I_{ij} = \int U(x - \xi_i)^2 H\big(|x - \xi_j|\big) \frac{\partial U}{\partial x_1}(x - \xi_1)\, dx.$$

First we observe that the term corresponding to $i = j = 1$ vanishes, by symmetry. If i and j are both different, and both different from 1, then the resulting term is of lower order, more precisely

$$\begin{aligned} I_{ij} &= \int U(x - \xi_i)^2 H\big(|x - \xi_j|\big) \frac{\partial}{\partial x_1} U(x - \xi_1)\, dx \\ &= \int U^2 \frac{\partial}{\partial \xi_{11}} \int U\big(x - (\xi_i - \xi_1)\big)^2 \log\big(\big|x - (\xi_j - \xi_1)\big|\big) U(x)\, dx + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{\gamma}}\right) \\ &= C \frac{\partial}{\partial \xi_{11}} \log\big|(\xi_j - \xi_1)\big| U(\xi_i - \xi_1) + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{\gamma}}\right) = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{\gamma}}\right). \end{aligned}$$

On the other hand, if $i = 1$,

$$\begin{aligned} I_{1j} &= -\frac{1}{3} \frac{\partial}{\partial \xi_{11}} \int U(x)^3 H\big(\big|x - (\xi_j - \xi_1)\big|\big)\, dx \\ &= \frac{1}{3} \int U^2 \frac{\partial}{\partial \xi_{11}} \int U(x)^3 \log\big(\big|x - (\xi_j - \xi_1)\big|\big)\, dx + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{\gamma}}\right) \\ &= c_8 \frac{\partial}{\partial \xi_{11}} \log\big(|\xi_j - \xi_1|\big)\, dx + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{\gamma}}\right). \end{aligned}$$

Now, as for I_{i1}, we get

$$I_{i1} = \int U\big(x - (\xi_i - \xi_1)\big)^2 H\big(|x|\big) \frac{\partial U}{\partial x_1}(x)\, dx = \mathrm{O}(e^{-|\xi_i - \xi_1|}) = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{\gamma}}\right).$$

Combining the above estimates immediately yields (5.7).

Hence we have found that

$$\int \mathcal{S} Z_{11} = \sum_{j \neq 1} \frac{\partial}{\partial \xi_{11}} \left[c_7 U(\xi_j - \xi_1) + c_8 \frac{\log|\xi_j - \xi_1|}{\log \sigma}\right] + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right).$$

Thus, we obtain the following result:

LEMMA 5.3. – *There exists a* $\gamma > 0$ *such that for all points* ξ_i *satisfying* (4.1) *and* (4.2) *we have,*

$$\int \mathcal{S} Z_{j\alpha} = \sum_{m \neq j} \frac{\partial F(|\xi_j - \xi_m|)}{\partial \xi_{j\alpha}} + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right),$$

where

$$F(r) = c_8 \frac{\log r}{\log 1/\sigma} + c_7 U(r)$$

and c_7, c_8 *are universal constants.*

6. The finite-dimensional reduction

We will carry out the finite-dimensional reduction process sketched in the first part of the paper. As in the previous section, we shall assume that the points ξ_i satisfy (4.1) and (4.2). Recall from Section 2 that the original problem was recast in the form

$$-\Delta u + u = \frac{u^2}{T(u^2)}. \tag{6.1}$$

Rather than solving this directly we consider instead the problem of finding A such that for certain constants $c_{i\alpha}$ one has

$$-\Delta A + A = \frac{A^2}{T(A^2)} + \sum c_{i\alpha} Z_{i\alpha} \tag{6.2}$$

and $\langle A - W, Z_{i\alpha}\rangle = 0$ for all i, α. Rewriting $A = W + \phi$ we get that this problem is equivalent to

$$\begin{aligned} &-\Delta\phi + \phi - 2W\phi + 2W^2\omega \int W\phi \\ &\quad = \Delta W - W + \frac{W^2}{V}\frac{(W+\phi)^2}{T((W+\phi)^2)} - \frac{W^2}{V} - 2W\phi + 2W^2\omega \int W\phi + \sum c_{i\alpha} Z_{i\alpha} \\ &\quad = \mathcal{S} + N(\phi) + \sum c_{i\alpha} Z_{i\alpha} \end{aligned} \tag{6.3}$$

and

$$\langle \phi, Z_{i\alpha}\rangle = 0 \quad \text{for all } i, \alpha. \tag{6.4}$$

Using the operator $\mathcal{T}$ introduced in Proposition 3.1, we see that the problem is then equivalent to finding a $\phi \in \mathcal{H}$ so that

$$\phi = \mathcal{T}\big(\mathcal{S} + N(\phi)\big) \equiv Q(\phi).$$

We will show that this fixed point problem has a unique solution in a region of the form

$$\mathcal{B} = \left\{ \phi \in \mathcal{H} \,\middle|\, \|\phi\|_* \leqslant \frac{1}{(\log 1/\sigma)^{1/2+\gamma}} \right\}, \tag{6.5}$$

for some $\gamma > 0$, provided that σ is sufficiently small.

We recall that from Lemma 4.1,

$$\|\mathcal{S}\|_{**} \leqslant \frac{1}{(\log 1/\sigma)^{1/2+\gamma}},$$

for some $\gamma > 0$. On the other hand, $N(\phi)$ admits the estimate provided by the following lemma.

LEMMA 6.1. – *Assume that* $\|\phi\|_* \leqslant \frac{1}{(\log 1/\sigma)^{1/2+\gamma}}$ *for some* $\gamma > 0$. *Then*

$$\|N(\phi)\|_{**} \leqslant C\left[\frac{1}{(\log 1/\sigma)^{2(1-2\mu)/3}} + \|\phi\|_*\right]\|\phi\|_*$$

provided that μ *in the definition of* $*$- *and* $**$-*norm and* b *in* (4.1), (4.2) *are such that* $\mu < \frac{1}{8}$ *and* $\frac{1}{\mu} < b$, *and* σ *is taken sufficiently small.*

Proof. – Let us assume first $|x| > 10b \log\log\sigma$, where μ is as in the definitions of the $*$- and $**$-norms. We observe that using a suitable barrier one can show that in this range of x we have

$$T\big((W+\phi)^2\big) \geqslant \frac{C}{\log 1/\sigma} e^{\sigma|x|}.$$

Let $\min j \leqslant k|x-\xi_j| = |x-\xi_i|$. Then

$$\begin{aligned}
&e^{3\mu|x-\xi_i|}\big|N(\phi)\big| \\
&\quad\leqslant e^{3\mu|x-\xi_i|}\left[\frac{2W\phi V + \phi^2 - 2W^2T(W\phi) - WT(\phi^2)}{VT((W+\phi)^2)} - 2W\phi + 2\omega W^2\int W\phi\right] \\
&\quad\leqslant C\log\frac{1}{\sigma}e^{3\mu|x-\xi_i|}e^{2\sigma|x|}\big(W|\phi| + \phi^2 + W^2\|\phi\|_* + W\|\phi\|_*^2\big) \\
&\quad\leqslant C\log\frac{1}{\sigma}e^{(3\mu+2\sigma)|x-\xi_i|}\big[e^{-|x-\xi|}\big(|\phi| + |\phi|_* + \|\phi\|_*^2\big) + \phi^2\big] \\
&\quad\leqslant C\log\frac{1}{\sigma}e^{(-3\mu+2\sigma)|x-\xi_i|}\big(\|\phi\|_* + \|\phi\|_*^2\big) \\
&\quad\leqslant \frac{1}{(\log 1/\sigma)^2}\big(\|\phi\|_* + \|\phi\|_*^2\big).
\end{aligned} \tag{6.6}$$

Provided that $\frac{1}{\mu} < b$ and σ is taken sufficiently small.

Let us consider now the case $|x| < 10b \log\log\sigma$. We decompose $N(\phi)$ in the form

$$N(\phi) = N_1(\phi) + N_2(\phi),$$

where

$$N_1(\phi) = (W+\phi)^2\left[\frac{1}{T((W+\phi)^2)} - \frac{1}{V} + \frac{2T(W\phi)}{V^2}\right] - \left[(2W+\phi)\phi\frac{2T(W\phi)}{V^2}\right]$$

and

$$N_2(\phi) = -2\phi W\left(1 - \frac{1}{V}\right) + 2W^2\left[\omega\int W\phi - \frac{T(W\phi)}{V^2}\right] + \frac{\phi^2}{V}.$$

We have that $T((W+\phi)^2) = V + 2T(W\phi) + T(\phi^2)$. On the other hand, $V(x) = 1 + \mathrm{O}(\frac{1}{(\log 1/\sigma)^{1-\delta}})$ in this range, for any $\delta > 0$. Also,

$$T(W\phi) = \omega \int W\phi + \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1-\delta}}\right)\|\phi\|_*$$

and in particular $|T(W\phi)| = \mathrm{O}(\|\phi\|_*)$. Likewise, and $T(\phi^2) = \mathrm{O}(\|\phi\|_*^2)$. Combining these facts we obtain

$$\begin{aligned}\left|N_1(\phi)\right| &\leqslant \left(W^2+\phi^2\right)T(\phi^2) + C\left[\left(2W\phi+\phi^2\right)\left|T(W\phi)\right|\right]\\ &\leqslant Ce^{-4\mu \min_{i\leqslant k}|x-\xi_i|}\|\phi\|_*^2.\end{aligned}$$

A similar analysis yields

$$\begin{aligned}\left|N_2(\phi)\right| &\leqslant \frac{C}{(\log 1/\sigma)^{1-\delta}}\left(|\phi|W + W^2\|\phi\|_*\right) + C|\phi|^2\\ &\leqslant Ce^{-4\mu \min_{i\leqslant k}|x-\xi_i|}\left(\|\phi\|_*^2 + \frac{1}{(\log 1/\sigma)^{2(1-2\mu)/3}}\|\phi\|_*\right),\end{aligned}$$

hence

$$Ce^{3\mu \min_{i\leqslant k}|x-\xi_i|}\left|N(\phi)\right| \leqslant \left(\|\phi\|_*^2 + \frac{1}{(\log 1/\sigma)^{2(1-2\mu)/3}}\|\phi\|_*\right)$$

for $|x| < 10b\log\log\frac{1}{\sigma}$. Combining this estimate with (6.6), yields the result of the lemma. □

Using the definition of the corresponding norms, splitting different ranges of x as in the above proof, it is readily checked that the following holds: If

$$\|\phi_i\|_* < \frac{1}{(\log 1/\sigma)^{1/2+\gamma}}, \quad i = 1, 2,$$

then, given $\varepsilon > 0$, for all σ sufficiently small one has that

$$\left\|N(\phi_1) - N(\phi_2)\right\|_{**} \leqslant \varepsilon\|\phi_1 - \phi_2\|_*.$$

Proposition 3.1 implies that the operator Q is a contraction mapping in the set $\mathcal{B}$ defined in (6.5). On the other hand, taking $\gamma = \frac{1}{8} - \mu$, we also get from the above lemma that Q maps $\mathcal{B}$ into itself. Banach fixed point theorem, then yields the existence of a unique fixed point of Q in this region, which depends continuously in the $*$-norm on the points ξ_i. We summarize this result in the following proposition:

PROPOSITION 6.1. – *There is a number $\gamma > 0$ such that for all sufficiently small σ and all points ξ_i satisfying* (4.1), (4.2) *we have the existence of a unique solution to* (6.3), (6.4), *$\phi = \phi(\xi_1,\dots,\xi_k)$ and $\mathbf{c} = \mathbf{c}(\xi_1,\dots,\xi_k)$ which satisfies $\|\phi_i\|_* < (\log\frac{1}{\sigma})^{-1/2-\gamma}$. Besides, $(\phi,\mathbf{c})$ depend continuously on the ξ_i's.*

In addition the following formula holds for the components $c_{j\alpha}$ of $\mathbf{c}$:

$$c_{j\alpha} = b_{j\alpha} + \varepsilon_{j\alpha}, \quad j = 1,\dots,k,\ \alpha = 1, 2, \tag{6.7}$$

with

$$b_{j\alpha} = \sum_{m \neq j} \frac{\partial F(|\xi_j - \xi_m|)}{\partial \xi_{j\alpha}},$$

the error terms $\varepsilon_{j\alpha}$ *which satisfy*

$$\varepsilon_{j\alpha} = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right),$$

and

$$F(r) = c_7 U(r) + c_8 \frac{\log r}{\log \sigma}.$$

Proof. – We only need to prove the formula for $c_{j\alpha}$'s. Let us observe that the $c_{i\alpha}$ satisfy the relations

$$\sum_{i,\alpha} c_{\imath\alpha} \langle Z_{i\alpha}, Z_{j\beta} \rangle = -\langle \mathcal{S}, Z_{j\beta} \rangle - \langle N(\phi), Z_{j\beta} \rangle + \langle \phi, L^*(Z_{j\beta}) \rangle,$$

which define an "almost diagonal" system, from which the $c_{j\alpha}$'s can be solved for uniquely. The main term in the above expansion is given by $\langle \mathcal{S}, Z_{j\beta} \rangle$. To obtain estimates for these numbers, which will equal the $c_{j\beta}$'s at leading order, we observe that

$$\left| \langle \phi, L^*(Z_{j\beta}) \rangle \right| \leqslant \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{\frac{2}{3}}}\right) \|\phi\|_* = \mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right).$$

Formula (6.7) is now an immediate corollary of Lemma 5.3, Lemma 6.1, and the expressions found for the $c_{i\alpha}$'s. □

In the following section we will find that points ξ_i that make all $c_{i\alpha}$'s vanish indeed exist, satisfying the conditions in Theorems 1.1–1.3.

7. The reduced problem

7.1. Invariance of c under permutations of $\hat{\xi}$

In the remainder of this paper we will denote $\hat{\xi} = (\xi_1, \ldots, \xi_k)$, and $\mathbf{c} = \mathbf{c}(\hat{\xi}) = (c_1, \ldots, c_k)$, where $\xi_j = \xi_{j1} + i\xi_{j2}$, $c_j = c_{j1} + ic_{j2}$ and i is the imaginary unit.

In this section we will study the effect of permutating the components of $\hat{\xi}$ on the values of function $\mathbf{c}$. We think of the components of $\hat{\xi}$ as complex numbers and consider only such permutations which act on ξ_j's. As before we assume that ξ_j's satisfy conditions (4.1), (4.2).

LEMMA 7.1. – *Let* Π *be a permutation of the components of a vector* $\hat{\xi} = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^{2k}$. *The following statements hold*:

(i) $L(\hat{\xi}) = L(\Pi\hat{\xi})$.
(ii) $\mathcal{S}(\hat{\xi}) = \mathcal{S}(\Pi\hat{\xi})$.

(iii) *$N(W(\hat{\xi}), \psi) = N(W(\Pi\hat{\xi}), \psi)$, for any function $\psi \in H^1$.*
(iv) *$\mathbf{c}(\Pi\hat{\xi}) = \Pi\mathbf{c}(\hat{\xi})$ and $\phi(\Pi\hat{\xi}) = \phi(\hat{\xi})$.*
(v) *Suppose that Π_j is a permutation which leaves the jth coordinate of $\hat{\xi}$ invariant. Then $c_j(\Pi_j\hat{\xi}) = c_j(\hat{\xi}) = (\Pi_j\mathbf{c}(\hat{\xi}))_j$.*

Proof. –
(i) Observe that $W(\hat{\xi}) = W(\Pi\hat{\xi})$ and $W^2(\hat{\xi}) = W^2(\Pi\hat{\xi})$. It follows that for any function ψ

$$L(\hat{\xi})\psi = -\Delta\psi + \left[1 - W(\hat{\xi})\right]\psi + 2\omega W^2(\hat{\xi}) \int W(\hat{\xi})\psi = L(\Pi\hat{\xi})\psi.$$

Observing that $V(\hat{\xi}) = V(\Pi\hat{\xi})$ and using explicit formulas for $\mathcal{S}$ and N we easily prove (ii) and (iii). We omit the details.
(iv) For each $\hat{\xi}$ there is a unique solution $(\phi(\hat{\xi}), \mathbf{c}(\hat{\xi}))$ to

$$L(\hat{\xi})\phi(\hat{\xi}) = R\big(W(\hat{\xi})\big) + N\big(W(\hat{\xi}), \phi\big) + \sum_{j,\alpha} c_{j\alpha}(\hat{\xi}) Z_{j\alpha}(\hat{\xi})$$
$$\big\langle \phi(\hat{\xi}), Z_{j\alpha}(\hat{\xi})\big\rangle = 0.$$

Let $\tilde{\phi}(\hat{\xi}) = \phi(\Pi\hat{\xi})$. Then, by (i)–(iii) $(\tilde{\phi}(\hat{\xi}), \mathbf{c}(\Pi\hat{\xi}))$ satisfies

$$L(\hat{\xi})\tilde{\phi}(\hat{\xi}) = R\big(W(\hat{\xi})\big) + N\big(W(\hat{\xi}), \tilde{\phi}\big) + \sum_{j,\alpha} c_{j\alpha}(\Pi\hat{\xi}) Z_{j\alpha}(\Pi\hat{\xi})$$
$$\big\langle \tilde{\phi}(\hat{\xi}), Z_{j\alpha}(\Pi\hat{\xi})\big\rangle = 0.$$

Since vectors $\mathbf{Z}(\hat{\xi})$ and $\mathbf{Z}(\Pi\hat{\xi})$ differ only by the permutation Π of their components therefore by uniqueness we obtain that $\mathbf{c}(\hat{\xi})$ and $\mathbf{c}(\Pi\hat{\xi})$ also differ only by the same permutation Π of their components, namely $\Pi\mathbf{c}(\hat{\xi}) = \mathbf{c}(\Pi\hat{\xi})$. This completes the proof of (iv).

The last statement is an easy consequence of (iv). The proof of the lemma is complete. □

7.2. Reducing number of equations for concentric polygons

We recall that speaking of components of $\mathbf{c}$ we mean the complex numbers c_j.

We begin with a corollary which shows that if we impose certain symmetries on the set of spikes then the number of equations can be reduced.

COROLLARY 7.1. –
(i) *Let $\vartheta \in (0, 2\pi)$ be given and suppose that $\hat{\xi}$ is such that $\hat{\xi}$ seen as a subset of $\mathbf{C}$ is invariant under the rotation by ϑ (we will denote the resulting vector by $e^{i\vartheta}\hat{\xi}$). Then, knowing $k-1$ components of $\mathbf{c}(\hat{\xi})$ suffices to determine all k components of $\mathbf{c}(\hat{\xi})$.*

(ii) *Let* $\hat{\xi}$ *be given and let* $\overline{\hat{\xi}}$ *denote a vector whose components are complex conjugates of* $\hat{\xi}$. *Assume that there is a permutation* Π *such that* $\Pi\hat{\xi} = \overline{\hat{\xi}}$ *and that for some* j *we have* $\bar{\xi}_j = \xi_j$. *Then* $\bar{c}_j(\hat{\xi}) = c_j(\hat{\xi})$. *A similar statement holds if* $\Pi\hat{\xi} = i\overline{\hat{\xi}}$.

Proof. –

(i) Observe that if $\hat{\xi}$ satisfies the assumptions of the corollary then rotation of the components of $\hat{\xi}$ by angle ϑ is also a permutation of $\hat{\xi}$. By Lemma 7.1 we then have $e^{i\vartheta}\mathbf{c}(\hat{\xi}) = \mathbf{c}(e^{i\vartheta}\hat{\xi})$ and (i) follows.

(ii) By the assumption Π is a permutation of $\hat{\xi}$ which leaves ξ_j invariant. Using (v) of Lemma 7.1 we know then that taking complex conjugates of the components $\mathbf{c}$ is a permutation of $\mathbf{c}$, which leaves c_j invariant. The second part of the statement follows by the same argument. □

7.3. Proof of Theorem 1.1

If $k = 1$ and $\hat{\xi} = \xi_1 = 0$ then from Corollary 7.1(ii) we obtain $\mathbf{c} = c_1 = 0$.
We assume that $k > 1$ and let P be a regular k polygon

$$P = \{(z_1, \dots, z_k) \mid z_j = e^{i\theta_j},\ \theta_j = 2\pi(j-1)/k,\ j = 1, \dots, k\}.$$

We will denote

$$\beta_1 = |1 - e^{i\theta_1}|.$$

Consider set

$$\mathcal{M}_k = \left\{(\xi_1, \dots, \xi_k) \,\middle|\, \xi_j = r z_j,\ \frac{2}{3}\log\log\frac{1}{\sigma} < r\beta_1 < \frac{b}{2}\log\log\frac{1}{\sigma}\right\}.$$

We want to find r such that

$$\mathbf{c}(rz_1, \dots, rz_k) = 0.$$

Observe that (7.3) is a system of $2k$ equations with just one unknown r. However we claim that since z_j's are vertices of a regular k polygon therefore by the results of previous subsection we can reduce the number of equations to just 1. We will presently prove this claim.

First observe that for each j, $1 < j \leqslant k$, rotation of the components of $\hat{\xi}$ by θ_j is a permutation of the components of $\hat{\xi}$ therefore the same is true for the components of $\mathbf{c}$. It follows that it suffices to know just one of c_j's to determine the rest.

Let's say that we want to find $c_1 = \operatorname{Re} c_1 + i \operatorname{Im} c_1$. As $\operatorname{Im}\xi_1 = 0$, from Corollary 7.1 we know that $\operatorname{Im} c_1 = 0$. Thus it suffices to solve a single scalar equation

$$\operatorname{Re} c_1(rz_1, \dots, rz_k) = 0.$$

We know that

$$\operatorname{Re} c_1(rz_1, \dots, rz_k) = \sum_{j=2}^{k} \frac{\partial F(|\xi_1 - \xi_j|)}{\partial \xi_{11}} + \varepsilon_{11},$$

where, for $\xi_j = rz_j$, we have

$$\sum_{j=2}^{k} \frac{\partial F(|\xi_1 - \xi_j|)}{\partial \xi_{11}} \sum_{j=2}^{k} \left[\frac{c_8}{r \log 1/\sigma |1 - e^{i\theta_j}|} + c_7 U'\left(r|1 - e^{i\theta_j}|\right) \right] \frac{1 - \cos\theta_j}{|1 - e^{i\theta_j}|}.$$

It is easy to see from the asymptotic formulas for U and the above that as $r\beta_1$ varies between $\frac{2}{3}\log\log\frac{1}{\sigma} < r\beta_1 < \frac{3}{2}\log\log\frac{1}{\sigma}$ the expression for $\operatorname{Re} c_1$ changes sign. Thus there exists ρ^σ such that

$$\operatorname{Re} c_1\left(\rho^\sigma z_1, \ldots, \rho^\sigma z_k\right) = 0.$$

Since the remaining components of $\mathbf{c}$ can be obtained by rotating c_1, therefore we have that $\mathbf{c}(\rho^\sigma z_1, \ldots, \rho^\sigma z_k) = 0$.

Using the asymptotic formulas for $U'(r)$ for large r we can easily show that ρ^σ satisfies

$$\rho^\sigma = \log\log\frac{1}{\sigma} + \frac{1}{2}\log\log\log\frac{1}{\sigma}\left[1 + \mathrm{o}(1)\right].$$

This ends the proof of the first part of the theorem.

To prove the second part we define

$$z_1 = 0, \quad z_j = e^{2\pi i(j-2)/(k-1)}, \quad j = 2, \ldots, k,$$

and

$$\mathcal{M}_{k,1} = \left\{ (\xi_1, \ldots, \xi_k) \,\Big|\, \xi_j = rz_j, \frac{2}{3}\log\log\frac{1}{\sigma} < r\beta_1 < \frac{b}{2}\log\log\frac{1}{\sigma} \right\}.$$

Observe that by Corollary 7.1(iii) we have $c_1(\xi_1, \ldots, \xi_k) \equiv 0$ if $\hat{\xi} \in \mathcal{M}_{k,1}$.

In order to show that $c_j(rz_1, \ldots, rz_k) = 0$, $j = 2, \ldots, k$, we use the invariance of the set $\{\xi_2, \ldots, \xi_k\}$ with respect to rotations to reduce the number of equations to one and then we use basically the same argument as in the case considered above. The details are omitted.

7.4. Proof of Theorem 1.2

Let $k = 2n$ be a positive integer, $k > 2$ and $Q_{2n} \subset \mathbb{R}^2$ be a set of points defined by

$$Q_{2n} = \left\{ \hat{\xi} \mid \xi_j = \begin{cases} rz_j, & j = 1, \ldots, n, \ \frac{2}{3}\log\log\frac{1}{\sigma} < r\beta_1 < \frac{b}{2}\log\log\frac{1}{\sigma} \\ Rz_j, & j = n+1, \ldots, 2n, \ r + 1 < R \end{cases} \right\},$$

where $z_j = e^{2\pi i(j-1)/k}$, $j = 1, \ldots, k$.

We want to show that there exists $\hat{\xi} \in Q_{2n}$ such that

$$\mathbf{c}(\hat{\xi}) = 0.$$

The system we need to solve now is a system of $2k = 4n$ equations with two variables r, R. First we will show that this system can be reduced to a system of two equations with two unknowns.

To this end observe that because of the invariance of the "inner" polygon (i.e., set $\{\xi_1,\dots,\xi_n\}$) with respect to rotations by θ_j we only need to know one of the components among $c_1,\dots,c_n$. Similarly because of the invariance of the outer polygon we only need to know one of the components among $c_{n+1},\dots,c_{2n}$. This reduces the number of equations to 4. In addition symmetry of Q_{2n} with respect to x axis and the fact that $\operatorname{Im}\xi_1 = 0 = \operatorname{Im}\xi_{n+1}$ implies $\operatorname{Im} c_1 = 0 = \operatorname{Im} c_{n+1}$. Consequently it suffices to solve

$$\operatorname{Re} c_1(\hat{\xi}) = 0,$$
$$\operatorname{Re} c_{n+1}(\hat{\xi}) = 0,$$

where $\hat{\xi} \in Q_{2n}$ depends on r, R.

Unlike in the case of Theorem 1.1 it is not immediately obvious that system (7.4) has a solution. Because of that we need a preliminary step. Let $r_0, r_1 > 0$ and define a vector field $g(r_0,r_1) \in \mathbb{R}^2$ by

$$g_1(r_0,r_1) = \frac{c_8}{\log 1/\sigma \log\log 1/\sigma}\left[\left(\beta_1 + \frac{1}{1+\beta_1^{-1}}\right)\sum_{j=2}^{k}\frac{1-\cos\theta_j}{|1-e^{i\theta_j}|} + \sum_{j=1}^{k}\frac{(1+2\beta^{-1})(1-\cos\theta_j)}{|1+\beta_1^{-1}(1-e^{i\theta_j})|^2}\right] + 2c_7U'(r_0\beta_1)\frac{1-\cos\theta_1}{\beta_1},$$

$$g_2(r_0,r_1) = \frac{c_8}{\log 1/\sigma \log\log 1/\sigma}\left(\frac{1}{1+\beta_1^{-1}}\sum_{j=2}^{k}\frac{1-\cos\theta_j}{|1-e^{i\theta_j}|} + \sum_{j=1}^{k}\frac{1+\beta_1^{-1}(1-\cos\theta_j)}{|1+\beta_1^{-1}(1-e^{i\theta_j})|^2}\right) + c_7U'(r_1).$$

We will first show that $g(r_0,r_1) = 0$ for some (r_0,r_1) and then use the topological degree argument to solve (7.4).

LEMMA 7.2. – *There exists* $(\tilde r_0,\tilde r_1)$ *such that* $g(\tilde r_0,\tilde r_1) = 0$ *and*

$$\tilde r_0 = \frac{1}{\beta_1}\left(\log\log\frac{1}{\sigma} + \frac{1}{2}\log\log\log\frac{1}{\sigma}\big[1+o(1)\big]\right),$$
$$\tilde r_1 = \log\log\frac{1}{\sigma} + \frac{1}{2}\log\log\log\frac{1}{\sigma}\big[1+o(1)\big].$$

Moreover, for each $M \in (0,1)$ *the topological degree of* $g(r_0,r_1)$ *is well defined in the ball* $B_M = \{[(r_0-\tilde r_0)^2 + (r_1-\tilde r_1)^2]^{1/2} < M\log\log\log\frac{1}{\sigma}\}$ *and we have* $\deg(g,0,B_M) = 1$.

Proof. – Since the equations for (r_0,r_1) are uncoupled the existence and the asymptotic formulas for $(\tilde r_0,\tilde r_1)$ follow by a straightforward calculations using the asymptotic formulas for U'. It is also easy to see that in B_M, $(\tilde r_0,\tilde r_1)$ is a unique zero of g.

We will show now that $\deg(g, 0, B_M) = 1$. First observe that

$$Dg(\tilde r_0, \tilde r_1) = \begin{pmatrix} 2U''(\tilde r_0\beta_1)(1-\cos\theta_1) & 0 \\ 0 & U''(\tilde r_1) \end{pmatrix}.$$

By the asymptotic formulas for U'' and $\tilde r_0, \tilde r_1$ for small σ we have $\det Dg > 0$.

The proof is complete. □

We go back now to solving (7.4). By straightforward calculations we get for $\hat\xi \in Q_{2n}$

$$\begin{aligned}
&\operatorname{Re} c_1(\hat\xi) \\
&\quad = \sum_{j=2}^{k} F'\big(r|1-e^{i\theta_j}|\big)\frac{1-\cos\theta_j}{|1-e^{i\theta_j}|} + \sum_{j=1}^{k} F'\big(|r-Re^{i\theta_j}|\big)\frac{r-R\cos\theta_j}{|r-Re^{i\theta_j}|} + \varepsilon_{11},\\
&\operatorname{Re} c_{n+1}(\hat\xi) \\
&\quad = \sum_{j=2}^{k} F'\big(R|1-e^{i\theta_j}|\big)\frac{1-\cos\theta_j}{|1-e^{i\theta_j}|} + \sum_{j=1}^{k} F'\big(|R-re^{i\theta_j}|\big)\frac{R-r\cos\theta_j}{|R-re^{i\theta_j}|} + \varepsilon_{n+11}.
\end{aligned}$$

We consider $(r, R-r) \in B_M$, where $M > 0$ is to be determined. For $(r, R-r) \in B_M$ there exists $\gamma > 0$ such that

$$\begin{aligned}
U'\big(r|1-e^{i\theta_j}|\big) &= \mathrm{O}\bigg(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\bigg), \quad j = 2, \ldots, n,\\
U'\big(|R-re^{i\theta_j}|\big) &= \mathrm{O}\bigg(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\bigg), \quad j = 2, \ldots, n,\\
U'\big(R|1-e^{i\theta_j}|\big) &= \mathrm{O}\bigg(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\bigg), \quad j = 1, \ldots, n.
\end{aligned}$$

We can write, by using the asymptotic formulas for ε_j,

$$\begin{aligned}
\operatorname{Re} c_1(\hat\xi) = {} & \frac{c_8}{\log 1/\sigma}\bigg(\sum_{j=2}^{k}\frac{1-\cos\theta_j}{r|1-e^{i\theta_j}|^2} + \sum_{j=1}^{k}\frac{r-R\cos\theta_j}{|r-Re^{i\theta_j}|^2}\bigg)\\
& + 2c_7U'\big(r|1-e^{i\theta_j}|\big)\frac{1-\cos\theta_1}{\beta_1}\\
& - c_7U'(R-r) + \mathrm{O}\bigg(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\bigg),\\
\operatorname{Re} c_{n+1}(\hat\xi) = {} & \frac{c_8}{\log 1/\sigma}\bigg(\sum_{j=2}^{k}\frac{1-\cos\theta_j}{R|1-e^{i\theta_j}|^2} + \sum_{j=1}^{k}\frac{R-r\cos\theta_j}{|r-Re^{i\theta_j}|^2}\bigg)\\
& + c_7U'(R-r) + \mathrm{O}\bigg(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\bigg).
\end{aligned}$$

We set $r = r_0$, $r_1 = R - r_0$. Then solving system (7.4) is equivalent to solving $f(r_0, r_1) = 0$ where $f = (f_1, f_2)$ and

$$f_1(r_0,r_1)=\frac{c_8}{\log 1/\sigma}\left(\sum_{j=2}^{k}\frac{(2r_0+r_1)(1-\cos\theta_j)}{r_0(r_0+r_1)|1-e^{i\theta_j}|^2}+\sum_{j=1}^{k}\frac{(2r_0+r_1)(1-\cos\theta_j)}{|r_1+r_0(1-e^{i\theta_j})|^2}\right)$$
$$+2U'\big(r_0|1-e^{i\theta_j}|\big)\frac{1-\cos\theta_1}{\beta_1}+\mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right),$$
$$f_2(r_0,r_1)=\frac{c_k}{\log 1/\sigma}\left(\sum_{j=2}^{k}\frac{1-\cos\theta_j}{(r_0+r_1)|1-e^{i\theta_j}|^2}+\sum_{j=1}^{k}\frac{r_1+r_0(1-\cos\theta_j)}{|r_1+r_0(1-e^{i\theta_j})|^2}\right)$$
$$+U'(r_1)+\mathrm{O}\left(\frac{1}{(\log 1/\sigma)^{1+\gamma}}\right).$$

For each $t\in[0,1]$ we consider now a vector field $h^t=tf+(1-t)g$. Using Lemma 7.2 we know that $h^0(\tilde r_0,\tilde r_1)=0$ and $\deg(h^0,0,B_M)=1$. It suffices to show that there exists $M>0$ such that

$$h^t(r_0,r_1)\neq 0,\quad (r_0,r_1)\in\partial B_M.$$

To this end we write $r_m=\tilde r_m+\rho_m$, $m=0,1$. If $(r_0,r_1)\in\partial B_M$ then $|(\rho_0,\rho_1)|=M\log\log\log\frac{1}{\sigma}$ and thus $\max\{|\rho_0|,|\rho_1|\}\geqslant\frac{M}{\sqrt2}\log\log\log\frac{1}{\sigma}$.

We also have

$$\frac{1}{r_0}+\frac{1}{r_0+r_1}=\frac{1}{\log\log 1/\sigma}\left[\beta_1+\frac{1}{1+\beta_1^{-1}}+\mathrm{O}\left(\frac{\log\log\log 1/\sigma}{\log\log 1/\sigma}\right)\right]$$

and therefore

$$\frac{c_8}{\log 1/\sigma}\sum_{j=2}^{k}\frac{(2r_0+r_1)(1-\cos\theta_j)}{r_0(r_0+r_1)|1-e^{i\theta_j}|^2}$$
$$=\frac{c_8}{\log 1/\sigma\,\log\log 1/\sigma}\left(\beta_1+\frac{1}{1+\beta_1^{-1}}\right)\sum_{j=2}^{k}\frac{1-\cos\theta_j}{|1-e^{i\theta_j}|}$$
$$+\mathrm{O}\left(\frac{1}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+\kappa}}\right),$$

with some $\kappa\in(2/3,1)$. Similarly calculating other terms involving (r_0,r_1) in the expressions for (f_1,f_2), using $g(\tilde r_0,\tilde r_1)=0$ and Proposition 3.1 we get

$$h_1^t(r_0,r_1)=2c_7\big[U'\big((\tilde r_0+\rho_0)|1-e^{i\theta_j}|\big)-U'\big(\tilde r_0|1-e^{i\theta_j}|\big)\big]\frac{1-\cos\theta_1}{\beta_1}$$
$$+\mathrm{O}\left(\frac{1}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+\kappa}}\right)$$
$$=2c_7\mu_0e^{-\beta_1(\tilde r_0+\eta\rho_0)}\frac{(1-\cos\theta_1)\rho_0}{\beta_1(\tilde r_0+\eta\rho_0)^{1/2}}+\mathrm{O}\left(\frac{1}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+\kappa}}\right),$$

where $\eta\in[0,1]$ and the last equality follows from the Mean Value Theorem and the asymptotic formula for U''. Likewise (with the same η as above) we have

$$h_2^t(r_0,r_1)=c_7\mu_0e^{-(\tilde r_1+\eta\rho_1)}\frac{\rho_1}{(\tilde r_0+\eta\rho_0)^{1/2}}+\mathrm{O}\left(\frac{1}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+\kappa}}\right).$$

It follows

$$
\begin{aligned}
\left|h^t(r_0, r_1)\right| &\geqslant \frac{C}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+M}}\left|(\rho_0, \rho_1)\right| + \mathrm{O}\left(\frac{1}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+\kappa}}\right) \\
&\geqslant \frac{C \max\{|\rho_0|, |\rho_1|\}}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+M}} + \mathrm{O}\left(\frac{1}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+\kappa}}\right) \\
&\geqslant \frac{C \log\log\log 1/\sigma}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+M}} + \mathrm{O}\left(\frac{1}{\log 1/\sigma\,(\log\log 1/\sigma)^{1+\kappa}}\right) > 0
\end{aligned}
$$

and therefore (7.2) is satisfied provided that $M < \kappa/2$ and σ is sufficiently small. Consequently f, hence $(\operatorname{Re} c_1, \operatorname{Re} c_{n+1})$, has a zero in B_M. The proof is complete. □

REFERENCES

[1] X. Chen, M. Kowalczyk, Dynamics of an interior spike in the Gierer–Meinhardt system, SIAM J. Math. Anal. 33 (1) (2001) 172–193.

[2] M. del Pino, P.L. Felmer, M. Kowalczyk, Boundary spikes in the Gierer–Meinhardt system, Asymptotic Anal., to appear.

[3] M. del Pino, M. Kowalczyk, X. Chen, The Gierer–Meinhardt system: the breaking of symmetry of homoclinics and multi-bump ground states, Comm. Contemp. Math. 3 (3) (2001) 419–439.

[4] A. Doelman, R.A. Gardner, T.J. Kaper, Large stable pulse solutions in reaction–diffusion equations, Indiana Univ. Math. J. 50 (5) (2001) 443–507.

[5] A. Gierer, H. Meinhardt, A theory of biological pattern formation, Kybernetik (Berlin) 12 (1972) 30–39.

[6] C. Ghoussoub, C. Gui, Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev exponent, Math. Z. 229 (1998) 443–474.

[7] C. Ghoussoub, C. Gui, M. Zhu, On a singularly perturbed Neumann problem with the critical exponent, Comm. Partial Differential Equations 26 (2001) 1929–1946.

[8] C. Gui, Multi-peak solutions for a semilinear Neumann problem, Duke Math. J. 84 (1996) 739–769.

[9] C. Gui, C.-S. Lin, Estimates for boundary-bubbling solutions to an elliptic Neumann problem, to appear.

[10] C. Gui, J. Wei, Multiple interior peak solutions for some singular perturbation problems, J. Differential Equations 158 (1999) 1–27.

[11] C. Gui, J. Wei, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, Canad. J. Math. 52 (2000) 522–538.

[12] C. Gui, J. Wei, M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000) 47–82.

[13] D. Iron, M. Ward, J. Wei, The stability of spike solutions to the one-dimensional Gierer–Meinhardt model, Phys. D. 150 (1–2) (2001) 25–62.

[14] M. Kowalczyk, Multiple spike layers in the shadow Gierer–Meinhardt system: existence of equilibria and approximate invariant manifold, Duke Math. J. 98 (1999) 59–111.

[15] J.P. Keener, Activators and inhibitors in pattern formation, Stud. Appl. Math. 59 (1978) 1–23.

[16] C.-S. Lin, W.-M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72 (1988) 1–27.

[17] H. Meinhardt, The Algorithmic Beauty of Sea Shells, 2nd Edition, Springer, Berlin, 1998.

[18] H. Meinhardt, Models of Biological Pattern Formation, Academic Press, London, 1982.
[19] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, Notices of Amer. Math. Soc. 45 (1998) 9–18.
[20] W.-M. Ni, I. Takagi, On the Neumann problem for some semilinear elliptic equations and systems of activator–inhibitor type, Trans. Amer. Math. Soc. 297 (1986) 351–368.
[21] W.-M. Ni, I. Takagi, On the shape of least energy solution to a semilinear Neumann problem, Comm. Pure Appl. Math. 41 (1991) 819–851.
[22] W.-M. Ni, I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, Duke Math. J. 70 (1993) 247–281.
[23] W.-M. Ni, I. Takagi, Point-condensation generated by a reaction–diffusion system in axially symmetric domains, Japan J. Indust. Appl. Math. 12 (1995) 327–365.
[24] W.-M. Ni, X.-B. Pan, I. Takagi, Singular behavior of least-energy solutions of a semilinear Neumann problem involving critical Sobolev exponents, Duke Math. J. 67 (1992) 1–20.
[25] W.-M. Ni, I. Takagi, E. Yanagida, Stability analysis of point-condensation solutions to a reaction–diffusion system proposed by Gierer–Meinhardt, Tohoku Math. J., to appear.
[26] W.-M. Ni, I. Takagi, E. Yanagida, Stability of least energy patterns of the shadow system for an activator–inhibitor model, Japan J. Indust. Appl. Math. 18 (2) (2001) 259–272.
[27] W.-M. Ni, I. Takagi, J. Wei, E. Yanagida, in preparartion.
[28] Y. Nishiura, Global structure of bifurcating solutions of some reaction–diffusion systems, SIAM J. Math. Anal. 13 (1982) 555–593.
[29] I. Takagi, Point-condensation for a reaction–diffusion system, J. Differential Equations 61 (1986) 208–249.
[30] A.M. Turing, The chemical basis of morphogenesis, Philos. Trans. Roy. Soc. London Ser. B 237 (1952) 37–72.
[31] J. Wei, On the boundary spike layer solutions of a singularly perturbed semilinear Neumann problem, J. Differential Equations 134 (1997) 104–133.
[32] J. Wei, Uniqueness and eigenvalue estimates of boundary spike solutions, Proc. Roy. Soc. Edinburgh Sect. A 131 (2001) 1457–1480.
[33] J. Wei, On single interior spike solutions of Gierer–Meinhardt system: uniqueness and spectrum estimates, European J. Appl. Math. 10 (1999) 353–378.
[34] J. Wei, Point-condensations generated by Gierer–Meinhardt system: a brief survey, in: Y. Morita, H. Ninomiya, E. Yanagida, S. Yotsutani (Eds.), New Trends in Nonlinear Partial Differential Equations, 2000, pp. 46–59.
[35] J. Wei, M. Winter, On the two-dimensional Gierer–Meinhardt system with strong coupling, SIAM J. Math. Anal. 30 (1999) 1241–1263.
[36] J. Wei, M. Winter, On multiple spike solutions for the two-dimensional Gierer–Meinhardt system: the strong coupling case, J. Differential Equations 178 (2002) 478–518.
[37] J. Wei, M. Winter, Spikes for the two-dimensional Gierer–Meinhardt system: the weak coupling case, J. Nonlinear Science 6 (2001) 415–458.