Quantum nonlinear Schrödinger equation.
I. Intertwining operators

To the memory of Mark Kac

by

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ABSTRACT. — We consider the quantum nonlinear Schrödinger equation (NLS) as a model of the quantum (nonrelativistic) field theory in $1 + 1$ dimensions. In this paper we develop a calculus of intertwining operators for the NLS. This calculus will be used in subsequent publications to solve explicitly an initial value problem for the NLS.

RÉSUMÉ. — On s'occupe ici de l'équation de Schrödinger non linéaire quanitique (NLS) comme un modèle de la théorie quantique (non relativiste) des champs en $1 + 1$ dimensions. Dans ce travail nous présentons un calcul des opérateurs entrelaçants pour le NLS. Ce calcul sera utilisé dans les publications suivantes à donner la solution explicite d'un problème de valeur initiale pour le NLS.

§ 0. INTRODUCTION

This is the first in the series of papers on the quantum Nonlinear Schrödinger equation (NLS)

$$\sqrt{-1}\,\Psi_t = -\Psi_{xx} + 2c\Psi^+(x)\Psi^2(x) \tag{0.1}$$

where $\Psi^+(x, t)$, $\Psi(x, t)$ are time dependent operator valued distributions (fields) in the Fock space $\mathscr{H} = \overset{\infty}{\underset{N=0}{\oplus}} \mathscr{H}_N$. On the jargon of physics literature $\Psi^+(x, t)$, $\Psi(x, t)$ are the creation, annihilation operators, respectively, in the position representation. NLS is the evolution equation given by the unitary group $e^{\sqrt{-1}t\hat{H}}$ where

$$\hat{H} = \int_{-\infty}^{\infty} dx \left[- \Psi^+(x)\Psi_{xx} + c\Psi^{+\,2}(x)\Psi^2(x) \right] \tag{0.2}$$

is the selfadjoint Hamiltonian, which preserves the N-particle sectors $\mathscr{H}_N = L_2^{sym}(\mathbb{R}^N)$ and $H_N = \hat{H}\,|_{\mathscr{H}_N}$ is the N-body Hamiltonian with δ-potential

$$H_N = - \Delta_N + c \sum_{i \neq j} \delta(x_i - x_j) \tag{0.3}$$

The problem of solving NLS explicitely is thus equivalent to the problem of constructing the group $e^{\sqrt{-1}t\hat{H}}$ which is the direct sum of the groups $e^{\sqrt{-1}tH_N}$, $N = 0, 1, \ldots$

Faddeev and his collaborators (cf. [15] [16]) and independently Thacker with his collaborators (cf. [12]) claimed a solution of (0.1) by the quantum inverse scattering method. This method yields a quantization prescription for solving NLS. More precisely one considers the classical version of NLS (or the Zakharov-Shabat equation [14])

$$\sqrt{-1}\varphi_t = - \varphi_{xx} + 2c\,|\,\varphi\,|^2\varphi \tag{0.4}$$

where $\varphi(x, t)$ is a complex valued function. After solving (0.4) by the inverse scattering method (cf. [17]) one « quantizes » the solution replacing functions by fields written in the normal order. The quantized solution of (0.4) is supposed to solve (0.1).

The quantization prescription has serious drawbacks as B. Davies ([2]) pointed out. Due to the singularities in (0.1) it is impossible to check whether the obtained expression solves the NLS. I noticed some time ago that these singularities arise as a result of formal manipulations with the Hamiltonian $\hat{H}$. The Hamiltonian $\hat{H}$ is not the sum of $- \int_{-\infty}^{\infty} dx\Psi^+(x)\Psi_{xx}$ and $c\int_{-\infty}^{\infty} dx\Psi^{+\,2}(x)\Psi^2(x)$ because the highly singular operator density $\Psi^{+\,2}(x)\Psi^2(x)$ does not define an operator on $\mathscr{H}$. This problem with $\hat{H}$ is not at all different from the problem with its N-body restrictions H_N where (0.3) is not the sum of the Laplacean and the δ-potential $c \sum_{i \neq j} \delta(x_i - x_j)$.

The problem is actually not a problem because one can make and does make sense out of H_N and therefore $\hat{H}$ in a few equivalent ways. The one

we work with here defines H_N as a boundary value problem (see § 1). The authors of [15] and [12] are of course aware of the fact that H_N is defined as a boundary value problem but proceed to work formally with $\hat{H}$ which disguises the singularities. The reader can easily see for instance that the square of (0.3) involves expressions $\delta^2(x_i - x_j)$ which don't make sense even as distributions. Nevertheless H_N^2 is well defined and one can write it down using the definition of H_N as a boundary value problem.

In view of the above one should take the quantization prescription for NLS with a grain of salt. In this series of papers we solve NLS explicitly using the approach of intertwining operators which started in [3] and has been applied since to NLS [4] and other (unrelated) problems ([5] [6]).

In § 1 we develop the calculus of intertwining operators P_N, P_N^*, P_N^{*-1}, P_N^{-1} which (for $c \geqslant 0$) produce an equivalence of H_N and the free Hamiltonian $-\Delta_N$. The material of § 1 can be viewed as a far reaching extension of the Bethe Ansatz for the delta Bose gas (0.3) (cf. [8] [13]). The main purpose of § 1 is to establish convenient formulas for intertwining operators on $\mathcal{H}_N$, $N = 1, 2, \ldots$ which will be later on put together for all N to produce formulas in the Fock space $\hat{\mathcal{H}} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N$. The exposition in § 1 is completely self-contained and on the way we derive formulas for the Bethe Ansatz eigenstates from the calculus of intertwining operators. As another application of our techniques we obtain at the end of § 1 the scattering matrix for H_N.

In § 2 we pass to the Fock space $\hat{\mathcal{H}}$ recalling the basic definitions for the reader's convenience. The intertwining operator on $\hat{\mathcal{H}}$ is the direct sum of the corresponding operators on $\mathcal{H}_N$, i. e.

$$\hat{P} = \bigoplus_{N=0}^{\infty} P_N \tag{0.5}$$

and it yields an equivalence ($c \geqslant 0$) of the interacting Hamiltonian

$$\hat{H} = \bigoplus_{N=0}^{\infty} H_N \tag{0.6}$$

with the free Hamiltonian

$$\hat{H}_0 = \bigoplus_{N=0}^{\infty} (-\Delta_N) \tag{0.7}$$

These intertwining operators conjugate solutions of NLS (0.1) with the solutions of the linear equation

$$\sqrt{-1} \Psi_t = -\Psi_{xx} \tag{0.8}$$

In order to convert this general remark into an explicit formula for solutions $\Psi(x, t)$ of the NLS we need to expand $\hat{P}$ in terms of the « standard fields » $\Psi_0(x), \Psi_0^+(x)$ (see § 2 for definitions). This will be done in a forthcoming paper [7]. In § 2 besides preparing the ground for [7] we use the second

quantized intertwining operators $\hat{P}$, $\hat{P}^{-1}$, $\hat{P}*$, $\hat{P}*^{-1}$ to construct the creation and annihilation operators $b^+(k)$, $b(l)$ for the Bethe Ansatz eigenstates and their companion fields $a^+(k)$, $a(l)$. Our formulas for $a(k)$, $a^+(k)$, $b(k)$, $b^+(k)$ easily yield the commutation relations for these operators (Theorem 2.1) which were written earlier ([12] [16]) on the basis of analogy with the Zakharov-Shabat equation.

§ 1. INTERTWINING OPERATORS
FOR THE N-BODY PROBLEM

Throughout this section, $N \geqslant 2$ is fixed. We denote by W the group of permutations of N items, denote by $x \to wx$ its natural action on $\mathbb{R}^N$ and let $\mathcal{H} \subset L_2(\mathbb{R}^N)$ be the subspace of symmetric L_2-functions $f(x_1, \ldots, x_N)$. Let $C_+ = \{ x_1 \geqslant \ldots \geqslant x_N \}$ be the positive « octant » and denote by

$$\Theta(x) = \Theta(x_1, \ldots, x_N) = \prod_{i < j} \theta(x_i - x_j)$$ the indicator function of C_+ (θ is

the indicator function of $\mathbb{R}_+$). Multiplication by Θ is an isometry of $\mathcal{H}$ on $L_2(C_+)$ and the symmetrization operator

$$(Sf)(x) = (N!)^{-1} \sum_{w \in W} f(wx)$$

is the Hermitian projection of $L_2(\mathbb{R}^N)$ on $\mathcal{H}$.

Denote by H_0 the positive Laplacean on H, fix a real number c and set

$$H = H_0 + c \sum_{i \neq j} \delta(x_i - x_j). \qquad (1.1)$$

We define the operator corresponding to (1.1) as the positive Laplacean on $L_2(C_+)$ with boundary conditions

$$(\partial/\partial x_i - \partial/\partial x_{i+1})f = cf \qquad (1.2)$$

on the walls $C_{+,i} = \{ x_1 \geqslant \ldots \geqslant x_i = x_{i+1} \geqslant \ldots \geqslant x_N \}$ of C_+, $i = 1, \ldots, N-1$.

Let $L \subset L_2(\mathbb{R}^N)$ be the dense subspace of smooth rapidly decaying at infinity functions and denote by ∂_i the operator $\partial/\partial x_i$ $i = 1, \ldots, N$. The imaginary unit will be denoted by $\sqrt{-1}$. We use the shorthand $\exp(\sqrt{-1} \langle k \mid x \rangle)$ for $\exp[\sqrt{-1}(k_1 x_1 + \ldots + k_N x_N)]$. The inverse Fourier transform $\mathcal{F}^{-1}$ of $f \in L_2(\mathbb{R}^N)$ is denoted by $\hat{f}(k)$ and we use the convention that

$$(\mathcal{F}^{-1}f)(k) = \hat{f}(k) = (2\pi)^{-N} \int_{-\infty}^{\infty} f(x) \exp(-\sqrt{-1} \langle x \mid k \rangle) d^N x. \quad (1.3)$$

We denote the Fourier transform $\mathscr{F} f$ by $\check{f}$ and we have

$$(\mathscr{F} \hat{f})(x) = f(x) = \int_{-\infty}^{\infty} \hat{f}(k) \exp{(\sqrt{-1} \langle k \mid x \rangle)} d^N k . \tag{1.4}$$

For any $i \neq j$ we define the operator A_{ij} on L by

$$(A_{ij}f)(x_1, \ldots, x_N) = \int_0^{\infty} f(x_1, \ldots, x_i + y, \ldots, x_j - y, \ldots, x_N) e^{-cy} dy \tag{1.5}$$

Operators A_{ij} are instrumental in our construction of the intertwining operator P.

PROPOSITION 1.1. — i) Operators A_{ij} commute with each other and with operators ∂_k for all k and we have

$$(\partial_i - \partial_j)A_{ij} = -1 + cA_{ij} \tag{1.6}$$

ii) For any $c \neq 0$

$$(A_{ij}f)(x) = \int_{-\infty}^{\infty} [c - \sqrt{-1}(k_i - k_j)]^{-1} \hat{f}(k) \exp{(\sqrt{-1} \langle k \mid x \rangle)} d^N k \tag{1.7}$$

Proof. — Operators A_{ij} are convolution operators (of a special type) therefore they commute with each other and with (infinitesimal) translations. Any convolution operator A is conjugate by the Fourier transform to the operator of multiplication by a function $m(k)$ which is called the Fourier multiplier of A. If $A \exp{(\sqrt{-1} \langle k \mid x \rangle)}$ is defined then

$$A \exp{(\sqrt{-1} \langle k \mid x \rangle)} = m(k) \exp{(\sqrt{-1} \langle k \mid x \rangle)}$$

which yields (1.7). Formula (1.6) is obtained by an elementary computation. It is equivalent to

$$A_{ij} = [c - (\partial_i - \partial_j)]^{-1} \tag{1.8}$$

For $n = 1, \ldots$ define operators $H_0^{(n)}$ on $\mathscr{H}$ by

$$H_0^{(n)} = \sum_{i=1}^{N} (-\sqrt{-1}\partial/\partial x_i)^n \tag{1.9}$$

As differential operators with constant coefficients, $H_0^{(n)}$ commute with each other and $H_0^{(2)} = H_0$ in earlier notation. Define operators $H^{(n)}$ on $\mathscr{H}$ as $\sum_{i=1}^{N} (-\sqrt{-1}\partial/\partial x_i)^n$ on C_+ with boundary conditions on the walls $C_{+,i}$

$$(\partial/\partial x_i - \partial/\partial x_{i+1})^{2k+1} f = c(\partial/\partial x_i - \partial/\partial x_{i+1})^{2k} f \tag{1.10}$$

for $1 \leqslant 2k + 1 \leqslant n - 1$ and $i = 1, \ldots, N - 1$.

Denote by Θ the operator of multiplication by the function Θ and define the operator P on $\mathscr{H}$ by

$$P = N! S\Theta \left[\prod_{i<j} (1 - cA_{ij}) \right] S \qquad (1.11)$$

THEOREM 1.1. — Operator P intertwines $H_0^{(n)}$ with $H^{(n)}$, i.e.

$$H^{(n)}P = PH_0^{(n)} \qquad (1.12)$$

for $n \geqslant 1$.

Proof. — The domain $D_0^{(n)}$ of $H_0^{(n)}$ consists of n times differentiable symmetric functions on $\mathbb{R}^N$ or, equivalently, n times differentiable functions on C_+ with boundary conditions

$$(\partial/\partial x_i - \partial/\partial x_{i+1})^{2k+1} f = 0 \qquad (1.13)$$

for $1 \leqslant 2k + 1 \leqslant n - 1$ on the walls of C_+. The domain $D^{(n)}$ of $H^{(n)}$ is given by the boundary conditions (1.10).

Regarding P as an operator from the symmetric part of $L_2(\mathbb{R}^N)$ to $L_2(C_+)$ we see from (1.11) that Pf is obtained by applying the integral operator $\prod_{i<j} (1 - cA_{ij})$ to f and restricting the result to C_+. By Proposition 1.1, $\prod_{i<j} (1 - cA_{ij})$ commutes with all differential operators with constant coefficients. Thus it remains to show that $PD_0^{(n)} \subset D^{(n)}$.

Fix an index i, $1 \leqslant i \leqslant N - 1$ and represent the operator $\prod_{i<j} (1 - cA_{ij})$ as

$$\prod_{i<j} (1 - cA_{ij}) = (1 - cA_{i,i+1}) \prod_{i<j}{}' (1 - cA_{ij}) \qquad (1.14)$$

where $\prod_{i<j}{}'$ means the product over all $i < j$ except $i < i + 1$. Since the operators $1 - cA_{ij}$ commute, by Proposition 1.1, the order of factors in the product (1.14) does not matter. Denote the functions $\left[\prod_{i<j} (1 - cA_{ij}) \right] f$ and $\left[\prod_{i<j}{}' (1 - cA_{ij}) \right] f$ by g and φ respectively. Then

$$g = (1 - cA_{i,i+1})\varphi . \qquad (1.15)$$

The product $\prod_{i<j}{}' (1 - cA_{ij})$ is invariant with respect to the transposition s_i:

$i \to i + 1$, therefore φ is symmetric in variables x_i and x_{i+1}. Since φ is differentiable at least the same number of times as f, we have

$$(\partial/\partial x_i - \partial/\partial x_{i+1})^{2k+1}\varphi = 0 \qquad (1.16)$$

on $C_{+,i}$ for $1 \leqslant 2k + 1 \leqslant n - 1$. Using (1.6) we obtain

$$
\begin{aligned}
(\partial/\partial x_i - \partial/\partial x_{i+1})^{2k+1}g &= (\partial_i - \partial_{i+1})^{2k}(\partial_i - \partial_{i+1})(1 - cA_{i,i+1})\varphi \\
&= (\partial_i - \partial_{i+1})^{2k+1}\varphi + c(\partial_i - \partial_{i+1})^{2k}(1 - cA_{i,i+1})\varphi \\
&= (\partial_i - \partial_{i+1})^{2k+1}\varphi + c(\partial_i - \partial_{i+1})^{2k}g .
\end{aligned}
\qquad (1.17)
$$

In view of (1.16), g satisfies the boundary conditions (1.10). The Theorem is proved.

Let $\mathrm{D} = \bigcap_{n \geqslant 1} \mathrm{D}^{(n)}$ be the common domain of operators $\mathrm{H}^{(n)}$, $n \geqslant 1$. It consists of infinitely differentiable functions on C_+ satisfying boundary conditions (1.10) for all $k \geqslant 0$. From now on we consider $\mathrm{H}^{(n)}$ as operators on D.

COROLLARY 1.1. — Operators $\mathrm{H}^{(n)}$, $n \geqslant 1$ commute.

Proof. — Let $\mathrm{D}_0 \subset \mathscr{H}$ be the space of smooth functions with boundary conditions (1.16) for all $k \geqslant 0$ and rapidly decaying at infinity. The argument of Theorem 1.1 shows that $\mathrm{PD}_0 \subset \mathrm{D}$ and for $f \in \mathrm{D}_0$, $g = \mathrm{P}f$ and any m, n, we have, by Theorem 1.1,

$$
\begin{aligned}
\mathrm{H}^{(m)}\mathrm{H}^{(n)}g = \mathrm{H}^{(m)}\mathrm{H}^{(n)}\mathrm{P}f = \mathrm{H}^{(m)}\mathrm{PH}_0^{(n)}f &= \mathrm{PH}_0^{(m)}\mathrm{H}_0^{(n)}f \\
&= \mathrm{PH}_0^{(n)}\mathrm{H}_0^{(m)}f = \mathrm{H}^{(n)}\mathrm{H}^{(m)}g .
\end{aligned}
$$

PROPOSITION 1.2. — Let P^* be the formal adjoint of P. Then

$$\mathrm{P}^* = \mathrm{N}\,!\,\mathrm{S}\left[\prod_{i<j}(1 - cA_{ji})\right]\Theta\mathrm{S} \qquad (1.18)$$

and P^* intertwines $\mathrm{H}^{(n)}$ with $\mathrm{H}_0^{(n)}$ for $n \geqslant 1$, i.e.

$$\mathrm{H}_0^{(n)}\mathrm{P}^* = \mathrm{P}^*\mathrm{H}^{(n)} . \qquad (1.19)$$

Proof. — Using (1.7) it is elementary to see that

$$A_{ij}^* = A_{ji} \qquad (1.20)$$

which implies (1.18). Applying $*$ to (1.12) and using that $\mathrm{H}^{(n)}$, $\mathrm{H}_0^{(n)}$ are symmetric, we get (1.19).

For $i \neq j$ set

$$B_{ij} = (A_{ij} + A_{ji})/2 . \qquad (1.21)$$

From the definition we have

$$B_{ij}f(x_1, \ldots, x_N) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-c|y|} f(x_1, \ldots, x_i + y, \ldots, x_j - y, \ldots, x_N) dy \quad (1.22)$$

and, by (1.20),

$$B_{ij}^* = B_{ij} = B_{ji}. \quad (1.23)$$

For $i \neq j$ define the operator D_{ij} by

$$(D_{ij}f)(x_1, \ldots, x_N) = \int_0^{\infty} f(x_1, \ldots, x_i + y, \ldots, x_j - y, \ldots, x_N) dy \quad (1.24)$$

From the definition we have

$$D_{ij}^* = -D_{ij} = D_{ji} \quad (1.25)$$

and from (1.8)

$$1 + cD_{ij} = (1 - cA_{ij})^{-1}. \quad (1.26)$$

For $k = (k_1, \ldots, k_N) \in C_+$ let $f_0(x \mid k)$ denote the symmetrized plane wave

$$f_0(x \mid k) = (N!)^{-1/2} \sum_{w \in W} \exp(\sqrt{-1} \langle wk \mid x \rangle). \quad (1.27)$$

The functions $f_0(x \mid k)$ are generalized eigenfunctions (eigenstates) of $H_0^{(n)}$, $n \geqslant 1$

$$H_0^{(n)} f_0(. \mid k) = (k_1^n + \ldots + k_N^n) f_0(. \mid k) \quad (1.28)$$

and they are normalized to δ-function, i.e.

$$\int_{-\infty}^{\infty} f_0(x \mid k') f_0(x \mid k) dx^N = (2\pi)^N \delta(k - k') \quad (1.29)$$

and

$$\int_{C_+} f_0(k \mid x) f_0(k \mid y) dk^N = (2\pi)^N \delta(x - y). \quad (1.30)$$

The following theorem is crucial for the calculus of intertwining operators.

THEOREM 1.2. — i) The operator P*P commutes with all symmetric differential operators with constant coefficients and

$$P^*P = S\left[\prod_{i<j} (1 - cB_{ij})\right] S. \quad (1.31)$$

ii) The operator P has a left inverse P^{-1} which is the inverse of P for $c \geqslant 0$ and

$$P^{-1} = N! \, S\left[\prod_{i<j} (1 + cD_{ij})\right] \Theta S. \quad (1.32)$$

iii) The operator P* has a right inverse $(P^*)^{-1}$ which is the inverse of P* for $c \geqslant 0$ and

$$(P^*)^{-1} = N! S\Theta \left[\prod_{i<j} (1 - cD_{ij}) \right] S. \qquad (1.33)$$

Proof. — By Theorem 1.1 and Proposition 1.2, P*P commutes with operators $H_0^{(n)}$ for all n. By the classical invariant theorem, $H_0^{(n)}$, $1 \leqslant n \leqslant N$, generate the algebra of invariant with respect to permutations differential operators with constant coefficients.

By (1.29) and (1.30), the functions $f_0(. \mid k)$ form a complete family of generalized eigenvectors of $H_0^{(n)}$, $n \geqslant 1$. Besides, by (1.28), the multiplicity of $\{ f_0(. \mid k) : k \in C_+ \}$ is one, i. e. $f_0(. \mid k)$ and $f_0(. \mid k')$ belong to the same « eigenvalue » of $H_0^{(n)}$, $n \geqslant 1$ if and only if $k = k'$. Therefore any operator A commuting with $H_0^{(n)} \geqslant 1$ diagonalizes on $\{ f_0(. \mid k) : k \in C_+ \}$. In other words there exists a function (multiplier) $a(k)$, $k \in C_+$ such that

$$(A\varphi)(x) = \int_{C_+} a(k)\hat{\varphi}(k)f_0(x \mid k)dk^N \qquad (1.34)$$

for all $\varphi \in \mathcal{H}$.

Consider first the case $c > 0$. The operator P is defined on bounded functions, in particular on $f_0(. \mid k)$. If $f(x \mid k)$, $x, k \in \mathbb{R}^N$ is a function of two variables and $w \in W$ we use notation $wf(x \mid k)$ for $f(x \mid wk)$. Let $f(x \mid k) = Pf_0(x \mid k)$. An elementary computation shows that

$$f(x \mid k)|_{C_+} = (N!)^{-1/2} \sum_W w \left\{ \left[\prod_{i<j} \frac{\sqrt{-1}(k_j - k_i)}{c + \sqrt{-1}(k_j - k_i)} \right] \exp(\sqrt{-1}\langle k \mid x \rangle) \right\}. \qquad (1.35)$$

Denote by C_w the domain wC_+. Then $\mathbb{R}^N = \bigcup_{w \in W} C_w$ and $f(x \mid k)|_{C_w}$ is determined by (1.35) and the symmetry. Thus, $f(x \mid k)$ is bounded and P* is defined on bounded functions, by (1.18). We will calculate $P^*f(x \mid k)$. Denote $\Theta \exp(\sqrt{-1}\langle k \mid x \rangle)$ by $e(x \mid k)$. It suffices to calculate

$$\left[\prod_{i<j} (1 - cA_{ji}) \right] e(x \mid k).$$

From (1.8) using that A_{ij} commute between themselves and with $\hat{c}_k$ we get

$$\prod_{i<j} (1 - cA_{ji}) = \prod_{i<j} (\hat{\partial}_i - \partial_j) \prod_{i<j} A_{ji}. \qquad (1.36)$$

An elementary computation shows that for any $i < j$

$$
A_{ji}e(x \mid k) = \begin{cases}
[c - \sqrt{-1}(k_j - k_i)]^{-1} \exp(\sqrt{-1}\langle k \mid x \rangle) + a_1 e^{-c(x_i - x_{i+1})} \\
\qquad \exp(\sqrt{-1}\langle k \mid x \rangle) \quad \text{for} \quad x \in D_1 \\
[c - \sqrt{-1}(k_j - k_i)]^{-1} \exp(\sqrt{-1}\langle k \mid x \rangle) + a_2 e^{-c(x_{j-1} - x_j)} \\
\qquad \exp(\sqrt{-1}\langle k \mid x \rangle) \quad \text{for} \quad x \in D_2 \\
0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{for} \quad x \in \mathbb{R}^N \backslash D
\end{cases}
$$

$$(1.37)$$

where $D = \{ x_1 \geqslant \ldots \geqslant x_{i-1} \geqslant x_{i+1} \geqslant \ldots \geqslant x_{j-1} \geqslant x_{j+1} \geqslant \ldots \geqslant x_N \}$ and where $D_1 \subset D$ is given by $x_i - x_{i+1} < x_{j-1} - x_j$, $D_2 \subset D$ is given by $x_{j-1} - x_j < x_i - x_{i+1}$. For $k \in C^N$ we use the self explanatory notation $\operatorname{Re} k, \operatorname{Im} k \in \mathbb{R}^N$ and $\langle k \mid x \rangle = \langle \operatorname{Re} k \mid x \rangle + \sqrt{-1}\langle \operatorname{Im} k \mid x \rangle$. Then (1.37) can be written as

$$
A_{ji}e(x \mid k) = [c - \sqrt{-1}(k_j - k_i)]^{-1} e(x \mid k) + \varphi(x) \tag{1.38}
$$

where

$$
\varphi(x) = \begin{cases}
a_1 \exp(\sqrt{-1}\langle k_1 \mid x \rangle) & x \in D_1 \subset D \\
a_2 \exp(\sqrt{-1}\langle k_2 \mid x \rangle) & x \in D_2 \subset D \\
0 & x \in \mathbb{R}^N \backslash D
\end{cases} \tag{1.39}
$$

and $\operatorname{Im} k_1, \operatorname{Im} k_2 \neq 0$. Iterating (1.38) and (1.39) we obtain

$$
\left[\prod_{i < j} A_{ji} \right] e(x \mid k) = \left[\prod_{i < j} c - \sqrt{-1}(k_j - k_i) \right]^{-1} e(x \mid k) + \Phi(x) \tag{1.40}
$$

where $\mathbb{R}^N$ is the union of a finite number of polyhedral domains D_p and

$$
\Phi(x)\big|_{D_p} = \sum_{q=1}^{q(p)} a_{p,q} \exp(\sqrt{-1}\langle k_{p,q} \mid x \rangle) \tag{1.41}
$$

with $\operatorname{Im} k_{p,q} \neq 0$. Applying $\prod_{i < j} (\partial_i - \partial_j)$ to (1.40) we get

$$
\left[\prod_{i < j} (1 - c A_{ji}) \right] e(x \mid k) = \left[\prod_{i < j} \frac{\sqrt{-1}(k_i - k_j)}{c + \sqrt{-1}(k_i - k_j)} \right] e(x \mid k) + \Phi'(x) + \psi \tag{1.42}
$$

where $\Phi'(x)$ has the form (1.41) only with other constants $a'_{p,q}$ and ψ is a distribution supported on hyperplanes separating domains D_p. Since

$\Theta f(x \mid k)$ is given by (1.35) we see that the function $P^* f(x \mid k)$ has the form

$$P^* f(x \mid k) = (N!)^{-1/2} \sum_{\mathbf{w}} w \left\{ \left[\prod_{i<j} \frac{(k_i - k_j)^2}{c^2 + (k_i - k_j)^2} \right] \exp \left(\sqrt{-1} \langle k \mid x \rangle \right) \right\} + \tilde{\Phi}(x) + \tilde{\psi} \quad (1.43)$$

where $\tilde{\Phi}(x)$ is given by (1.41) with different constants $\tilde{a}_{p,q}$ and $\tilde{\psi}$ is supported on a finite union of hyperplanes. On the other hand, $P^* f(. \mid k)$ must be proportional to $f_0(. \mid k)$.

Therefore $\tilde{\Phi} = \tilde{\psi} = 0$ and we have

$$P^* f(. \mid k) = \left[\prod_{i<j} \frac{(k_i - k_j)^2}{c^2 + (k_i - k_j)^2} \right] f_0(. \mid k) . \quad (1.44)$$

The last expression can be rewritten in the form (1.34) as

$$(P^* P \varphi)(x) = \int_{C_+} \left[\prod_{i<j} \frac{(k_i - k_j)^2}{c^2 + (k_i - k_j)^2} \right] \hat{\varphi}(k) f_0(x \mid k) dk^N . \quad (1.45)$$

A straightforward computation of the Fourier transform shows that $\prod_{i<j} (k_i - k_j)^2 / [c^2 + (k_i - k_j)^2]$ is the Fourier multiplier of the convolution operator $S \left[\prod_{i<j} (1 - cB_{ij}) \right] S$ which proves assertion $i)$ for $c > 0$.

By analytic continuation in c, formula (1.35) remains valid for arbitrary c. The operator P^* is always defined on bounded functions because it involves integration on bounded domains only. Since $f(x \mid k)$ is always bounded, the second part of the argument above remains valid and proves the assertion (i) for arbitrary c.

The polar decomposition of the operator P

$$P = [P(P^* P)^{-1/2}](P^* P)^{1/2} \quad (1.46)$$

where $Q = P(P^* P)^{-1/2}$ is an isometry and $R = (P^* P)^{1/2}$ is positive definite (by (1.11), Ker $P = 0$) implies the polar decomposition of the left inverse P^{-1}

$$P^{-1} = (P^* P)^{-1/2} [(P^* P)^{-1/2} P^*] = (P^* P)^{-1} P^* . \quad (1.47)$$

From (1.18) and (1.31) we have

$$P^{-1} = N! S \left[\prod_{i<j} (1 - cB_{ij})^{-1} \right] S^2 \left[\prod_{i<j} (1 - cA_{ji}) \right] \Theta S . \quad (1.48)$$

By (1.23), the operator $\prod\limits_{i<j} (1 - cB_{ij})^{-1}$ is symmetric, i. e. it commutes with S, thus

$$P^{-1} = N! S\left[\prod_{i<j} (1 - cB_{ij})^{-1}(1 - cA_{ji})\right] \odot S. \qquad (1.49)$$

According to earlier calculations, the Fourier multipliers of $(1 - cB_{ij})^{-1}$ and $1 - cA_{ji}$ are $[c^2 + (k_i - k_j)^2]/(k_i - k_j)^2$ and $\sqrt{-1}(k_i - k_j)/[c + \sqrt{-1}(k_i - k_j)]$ respectively. Hence, the Fourier multiplier of the product is

$$[c + \sqrt{-1}(k_j - k_i)]/\sqrt{-1}(k_j - k_i)$$

which is the Fourier multiplier of $(1 - cA_{ij})^{-1}$. In view of (1.26), this implies (1.32). The proof of (1.33) is completely analogous and we spare the details.

The generalized eigenfunctions $f(x \mid k) = Pf_0(x \mid k)$ of $H^{(n)}$, $n \geqslant 1$, given by (1.35) are called the Bethe Ansatz eigenstates (cf. [12]). It is known (cf. [10]) that for $c \geqslant 0$ the Bethe Ansatz eigenstates are complete orthogonal (but not normalized) in $\mathscr{H}$. That is $\int \overline{\varphi}(x) f(x \mid k) dx^N = 0$ for all k implies $\varphi = 0$. Completeness of $f(. \mid k)$ means that Ker $P^* = 0$ hence P is invertible, thus P^{-1} given by (1.47) is the inverse of P. Theorem 1.2 is proved.

The following Corollary was proved in the course of proof of Theorem 1.2.

COROLLARY 1.2. — Let $\mathscr{H}_c$ denote the closure of $P\mathscr{H}$ in $\mathscr{H}$. Then $\mathscr{H}_c$ is the subspace of absolutely continuous spectrum of the Hamiltonian H (for $c \geqslant 0$, $\mathscr{H}_c = \mathscr{H}$). The operator P^{-1} given by (1.32) intertwines $H^{(n)}$ with $H_0^{(n)}$ for $n \geqslant 1$ and Ker $P^{-1} = \mathscr{H}_c^{\perp}$. The operator $(P^*)^{-1}$ given by (1.33) intertwines $H_0^{(n)}$ with $H^{(n)}$ for $n \geqslant 1$ and the closure of $(P^*)^{-1}\mathscr{H}$ is $\mathscr{H}_c$.

Sometimes it is convenient to use another « normalization » of Bethe Ansatz eigenstates. Namely set $g(. \mid k) = (P^*)^{-1} f_0(. \mid k)$, $k \in C_+$.

COROLLARY 1.3. — The functions $g(. \mid k)$ form a complete orthogonal in $\mathscr{H}_c$ family of common eigenstates of $H^{(n)}$, $n \geqslant 1$. We have

$$g(x \mid k)|_{C_+} = (N!)^{-1/2} \sum_w w\left\{\prod_{i<j} \frac{c + \sqrt{-1}(k_i - k_j)}{\sqrt{-1}(k_i - k_j)} \exp(\sqrt{-1}\langle k \mid x \rangle)\right\}. \qquad (1.50)$$

The two families of Bethe Ansatz eigenstates are related by

$$g(. \mid k) = \left[\prod_{i<j} \frac{c^2 + (k_i - k_j)^2}{(k_i - k_j)^2}\right] f(. \mid k). \qquad (1.51)$$

Proof. — The Fourier multiplier of the convolution operator $\prod_{i<j}(1-c\mathbf{D}_{ij})$

is $\prod_{i<j}[c+\sqrt{-1}(k_i-k_j)]/\sqrt{-1}(k_i-k_j)$. Using this and (1.33) we obtain

(1.50) the same way as we obtained (1.35) in the proof of Theorem 1.2.
To prove (1.51) we use that $g(.\,|\,k) = P(P^*P)^{-1}f_0(.\,|\,k)$ and, by (1.44),

$$(P^*P)f_0(.\,|\,k) = \left[\prod_{i<j}\frac{(k_i-k_j)^2}{c^2+(k_i-k_j)^2}\right]f_0(.\,|\,k). \qquad (1.52)$$

Throughout this section we have used the convolution operators on $L_2(\mathbb{R}^N)$ of a special type. Let g be a distribution on R with its Fourier transform

$$\breve{g}(k) = \int_{-\infty}^{\infty} g(x)e^{\sqrt{-1}kx}dx \qquad (1.53)$$

and the inverse Fourier transform

$$\hat{g}(k) = (2\pi)^{-1}\int_{-\infty}^{\infty} g(x)e^{-\sqrt{-1}kx}dx. \qquad (1.54)$$

For every pair $i \neq j$ of indices we associate with g the convolution operator G_{ij} on $L_2(\mathbb{R}^N)$ by

$$(G_{ij}f)(x) = \int_{-\infty}^{\infty} g(t)f(x_1, \ldots, x_i+t, \ldots, x_j-t, \ldots, x_N)dt. \qquad (1.55)$$

A straightforward computation shows that

$$(G_{ij}f)(x) = \int_{\mathbb{R}^N} \breve{g}(k_i-k_j)\hat{f}(k)\exp(\sqrt{-1}\langle k\,|\,x\rangle)dk^N \qquad (1.56)$$

that is the Fourier multiplier of G_{ij} is $g(k_i-k_j)$. Set

$$G = \prod_{i<j}G_{ij}. \qquad (1.57)$$

Then G is the convolution operator with the Fourier multiplier $\prod_{i<j}\breve{g}(k_i-k_j)$.

We call $\breve{g}(k)$, $k \in \mathbb{R}$ the elementary multiplier of G and $g(x)$, $x \in \mathbb{R}$ the elementary kernel of G. Using notation A_{ij} for the operator which was earlier denoted by $1 - cA_{ij}$ (see (1.5)) we have

$$P = N!S\Theta AS \qquad (1.58)$$

where A is the convolution operator of type (1.50) with the elementary kernel

$$a(x) = \delta(x) - c\theta(x)e^{-cx} \tag{1.59}$$

and the elementary multiplier

$$\breve{a}(k) = -\sqrt{-1}k/(c - \sqrt{-1}k). \tag{1.60}$$

Analogous formulas hold for the other intertwining operators P*, P*$^{-1}$, P^{-1}.

The operator

$$Q = P(P^*P)^{-1/2} \tag{1.61}$$

is the normalized intertwing operator. We calculate Q explicitly in the following Theorem.

THEOREM 1.3. — *i*) The operator Q is an isometry of $\mathscr{H}$ on the space $\mathscr{H}_c$ of the absolutely continuous spectrum of H and Q is unitary for $c \geqslant 0$.
ii) We have

$$Q = N! S\Theta US \tag{1.62}$$

where U is the unitary convolution operator of type (1.57) with the elementary multiplier

$$\breve{u}(k) = -\sqrt{-1} \operatorname{sgn} k \frac{c + \sqrt{-1}k}{(c^2 + k^2)^{1/2}} \tag{1.63}$$

and the elementary kernel (for $c \neq 0$)

$$u(x) = \frac{1}{2}\left(c - \frac{d}{dx}\right)[L_0(cx) - I_0(cx)] \tag{1.64}$$

where $L_0(x)$ and $I_0(x)$ are the modified Struve and Bessel functions respectively.

Proof. — Assertion *i*) was proved in the course of proof of Theorem 1.2. By (1.52),

$$P^*P = STS \tag{1.65}$$

where T is the convolution operator of type (1.57) with the elementary multiplier $k^2/(c^2 + k^2)$. Therefore

$$(P^*P)^{-1/2} = ST^{-1/2}S \tag{1.66}$$

where T$^{-1/2}$ has the same type and its elementary multiplier $\sqrt{c^2 + k^2}/|k|$ is invariant under $k \to -k$. This property of the elementary multiplier of an operator G of type (1.57) is equivalent to the W-invariance of the Fourier multiplier of G which means that G commutes with the permutation group W, hence with S. In particular T and T$^{-1/2}$ commute with S. Now

$$Q = N! S\Theta ASST^{-1/2}S = N! S\Theta AT^{-1/2}S$$

and we set $U = AT^{-1/2}$. Since the set of convolution operators of type (1.57) is closed under multiplication, U has form (1.57) with the elementary multiplier

$$\breve{u}(k) = \frac{-\sqrt{-1}k(c^2+k^2)^{1/2}}{(c-\sqrt{-1}k)|k|} = -\sqrt{-1}\,\operatorname{sgn} k \, \frac{c+\sqrt{-1}k}{(c^2+k^2)^{1/2}}.$$

Since $|\breve{u}(k)| = 1$, the absolute value of the Fourier multiplier $\displaystyle\prod_{i<j}\breve{u}(k_i - k_j)$ of U is 1, thus U is unitary. The kernel $u(x)$ is the inverse Fourier transform of $\breve{u}$ (see (1.54)) and after elementary transformations we have

$$u(x) = \pi^{-1} \int_0^\infty \frac{k \cos kx - c \sin kx}{(c^2 + k^2)^{1/2}} \, dk. \tag{1.67}$$

Using tables of Fourier sine transform (cf. [9], p. 416) we obtain (1.64). Theorem 1.3 is proved.

Now we can normalize the Bethe Ansatz eigenstates.

COROLLARY 1.4. — Set $\varphi(k\,|\,.) = Qf_0(k\,|\,.)$, $k \in C_+$. i) $\varphi(k\,|\,x)$ are symmetric functions and

$$\varphi(k\,|\,x)\,|_{C_+} = (-\sqrt{-1})^{N(N-1)/2}(N\,!)^{-1/2}$$

$$\sum_{\mathbf{w}} w \left\{ \prod_{i<j} \operatorname{sgn}(k_i - k_j) \frac{c + \sqrt{-1}(k_i - k_j)}{|\,c + \sqrt{-1}(k_i - k_j)\,|} \exp(\sqrt{-1}\langle k\,|\,x\rangle) \right\}. \tag{1.68}$$

ii) $\{\varphi(k\,|\,.), k \in C_+\}$ is the complete in $\mathscr{H}_c$ ($\mathscr{H}$ for $c \geqslant 0$) family of normalized to δ-function simultaneous eigenstates of operators $H^{(n)}$, $n \geqslant 1$.

Proof. — By (1.27)-(1.30), functions $f_0(k\,|\,.)$, $k \in C_+$ form a complete in $\mathscr{H}$ family of normalized simultaneous eigenstates of $H_0^{(n)}$, $n \geqslant 1$, and $Q : \mathscr{H} \to \mathscr{H}_c$ is an intertwining isometry, hence ii). We can calculate $Qf_0(k\,|\,.)$ using (1.62) and (1.63) which yields (1.68).

In order to formulate the following Proposition we indicate by subscript c the dependence of functions and operators on parameter c.

PROPOSITION 1.3. — i) The operator valued function $c \to Q_c$ is continuous in the uniform operator topology for c bounded away from zero and infinity and in the strong topology for all c. $Q_0 = 1$.

ii) Strong limits $\underset{c \to \pm\infty}{s\text{-lim}} Q_c = Q_{\pm\infty}$ exist and $Q_{-\infty} = (-)^{N(N-1)/2}Q_{-\infty}$.

iii) We have

$$Q_{\pm\infty} = N\,! \, S\Theta U_{\pm\infty} S \tag{1.69}$$

where $U_{+\infty}$ is the operator of form (1.57) with the elementary kernel

$$u_{+\infty}(x) = -(\pi x)^{-1} \tag{1.70}$$

and $U_{-\infty} = (-)^{N(N-1)/2} U_{+\infty}$. For $i \neq j$ the operator

$$(U_{+\infty})_{i,j} f(x) = -\pi^{-1} \int_{-\infty}^{\infty} y^{-1} f(x_1, \ldots, x_i + y, \ldots, x_j - y, \ldots, x_N) dy \quad (1.71)$$

is defined by the Cauchy principal value, i. e. $(U_\infty)_{ij}$ is the Hilbert transform in variable $x_i - x_j$.

Proof. — Since the operator $N! \Theta S$ does not depend on c, by (1.62), it suffices to prove assertion i) for the function $c \to U_c$. Since $U_c = \displaystyle\prod_{i < j} (U_c)_{ij}$

and $(U_c)_{ij}$ is equivalent to the multiplication operator in one variable by the elementary multiplier (1.63), everything boils down to the analysis of the following function of two variables

$$\breve{u}_c(k) : (c, k) \to -\sqrt{-1} \operatorname{sgn} k(c + \sqrt{-1}k)/(c^2 + k^2)^{1/2}. \quad (1.72)$$

The function $\breve{u}_c(k)$ is continuous everywhere on the extended plane except when $c, k \to 0$ and $|c|, |k| \to \infty$ where the limits do not exist. On any part of the (c, k)-plane where c and k can not go simultaneously to 0 or ∞ the function $\breve{u}_c(k)$ is even uniformly continuous.

The elementary inequality

$$| \breve{u}_c(k) - \breve{u}_d(k) | \leqslant f(c/d) \quad (1.73)$$

where f is a continuous function with $f(1) = 0$ implies

$$| \breve{u}_c(k) - \breve{u}_d(k) | \leqslant \text{const} \, | c - d | \quad (1.74)$$

if $0 < \varepsilon \leqslant c, d \leqslant E < \infty$ which gives the continuity of U_c in the uniform topology.

To show the continuity of $c \to U_c$ in the strong topology it suffices to estimate the L_2-norm of $(\breve{u}_c - \breve{u}_d)f$ for $f \in L_2(\mathbb{R})$. For a fixed $f \in L_2(\mathbb{R})$ and any $\varepsilon > 0$ there exists $n > 1$ such that

$$\int_{-\infty}^{-n} | f(k) |^2 dk + \int_{-n^{-1}}^{n^{-1}} | f(k) |^2 dk + \int_{n}^{\infty} | f(k) |^2 dk < \varepsilon.$$

For $| k |$ bounded away from 0 and ∞ the inequality (1.74) holds. Thus

$$\| (\breve{u}_c - \breve{u}_d)f \|^2 = \int_{|k| < n^{-1}} | \breve{u}_c(k) - \breve{u}_d(k) |^2 | f(k) |^2 dk$$

$$+ \int_{|k| > n} | \breve{u}_c(k) - \breve{u}_d(k) |^2 | f(k) |^2 dk + \int_{n^{-1} \leqslant |k| \leqslant n} | \breve{u}_c(k) - \breve{u}_d(k) |^2 | f(k) |^2 dk \quad (1.75)$$

$$\leqslant 4\varepsilon + \text{const} \, | c - d | \, \| f \|^2.$$

Estimate (1.75) gives the uniform continuity of $c \to U_c f$ which implies the strong continuity of $c \to U_c$.

The argument above also shows the existence of the limits $\lim\limits_{c \to \pm \infty} U_c f$ which correspond to the limit elementary multipliers

$$u_{\pm}(k) = \lim_{c \to \pm \infty} \left(- \sqrt{-1} \operatorname{sgn} k \, \frac{c + \sqrt{-1}k}{(c^2 + k^2)^{1/2}} \right) = \pm \sqrt{-1} \operatorname{sgn} k . \quad (1.76)$$

The corresponding convolution operator is known to be the Hilbert transform. The details are left to the reader. The proof of the following Corollary is also left to the reader.

COROLLARY 1.5. — The Hamiltonian H_∞ ($c = \infty$, the infinite strength of interaction) is equal to the Dirichlet Laplacian in C_+. The unitary intertwining operator Q_∞ transforms the Neumann Laplacian H_0 into the Dirichlet Laplacian H_∞. The normalized Bethe Ansatz eigenstates $f_\infty(k\,|\,.) = Q_\infty f_0(k\,|\,.)$ of H_∞ are given by

$$f_\infty(k\,|\,x)\,|_{C_+} = (-\sqrt{-1})^{N(N-1)/2}(N!)^{-1/2} \sum_{w} \det(w) \exp(\sqrt{-1} \langle wk\,|\,x \rangle). \quad (1.77)$$

Calculus of intertwining operators allows to obtain explicitely the wave operators W_{in}, W_{out} of the scattering theory and the scattering operator $W_{out} W_{in}^{-1}$ for the Hamiltonian (1.1). We recall the basic notions of the scattering theory. By general definition

$$W_{in} = \lim_{t \to -\infty} [e^{-\sqrt{-1}tH_0} e^{\sqrt{-1}tH}]$$
$$W_{out} = \lim_{t \to +\infty} [e^{-\sqrt{-1}tH_0} e^{\sqrt{-1}tH}]. \quad (1.78)$$

The wave operators are isometries from the space $\mathscr{H}_c$ of absolutely continuous spectrum onto $\mathscr{H}$ and we denote their right inverses by $W_{in}^{-1}, W_{out}^{-1}$ respectively. Both operators W_{in}^{-1} and W_{out}^{-1} intertwine H_0 with H

$$W_{in}^{-1} H_0 = H W_{in}^{-1}, \qquad W_{out}^{-1} H_0 = H W_{out}^{-1} . \quad (1.79)$$

For the rest of this section we denote the scattering operator $W_{out} W_{in}^{-1}$ by S and the symmetrization operator by Sym. By (1.79), S commutes with the free Hamiltonian

$$SH_0 = H_0 S . \quad (1.80)$$

Let for $k \in C_+$

$$f_{in}(k\,|\,.) = W_{in}^{-1} f_0(k\,|\,.)$$
$$f_{out}(k\,|\,.) = W_{out}^{-1} f_0(k\,|\,.) \quad (1.81)$$

be the incoming and outgoing scattering states. Then

$$f_{in}(k\,|\,x_1 < x_2 \ldots < x_N) = (N!)^{-1/2}\left[e^{\sqrt{-1} \langle k|x \rangle} + \sum_{w \neq 1} p(w, k) e^{\sqrt{-1} \langle wk|x \rangle} \right] \quad (1.82)$$

and

$$f_{\text{out}}(k \mid x_1 > \ldots > x_N) = (N !)^{-1/2} \left[e^{\sqrt{-1} \langle k \mid x \rangle} + \sum_{w \neq 1} q(w, k) e^{\sqrt{-1} \langle wk \mid x \rangle} \right].$$

(1.83)

Let $w_0 \in W$ be the longest permutation i.e. w_0:

$$1 \to N, \quad 2 \to N - 1, \ldots, N \to 1.$$

Then $w_0 C_+ = C_- = \{ x : x_1 < \ldots < x_N \}$ and (1.82) implies by symmetry that in C_+ we have

$$f_{\text{in}}(k \mid x_1 > \ldots > x_N) = (N !)^{-1/2} \left[e^{-\sqrt{-1} \langle w_0 k \mid x \rangle} + \sum_{w \neq w_0} p(w, k) e^{\sqrt{-1} \langle wk \mid x \rangle} \right].$$

(1.84)

Now for every $k = (k_1 > \ldots > k_N)$ we have 5 Bethe Ansatz eigenstates $f(k \mid .)$, $g(k \mid .)$, $\varphi(k \mid .)$, $f_{\text{in}}(k \mid .)$ and $f_{\text{out}}(k \mid .)$ which coincide up to scalar factors.

Denote by V the convolution operator on $L_2(\mathbb{R}^N)$ of type (1.57) with the elementary multiplier

$$\check{v}(k) = -\sqrt{-1} \frac{c + \sqrt{-1} \mid k \mid}{(c^2 + k^2)^{1/2}}.$$

(1.85)

THEOREM 1.4. — i) The inverse wave operators are equal to

$$W_{\text{in}}^{-1} = N ! \, \text{Sym} \, \Theta UV \, \text{Sym}$$

(1.86)

and

$$W_{\text{out}}^{-1} = N ! \, \text{Sym} \, \Theta UV^{-1} \, \text{Sym}$$

(1.87)

respectively.

ii) The unitary operators UV and UV^{-1} are convolution operators of type (1.57) with elementary multipliers

$$\tilde{w}_{\text{in}}(k) = \begin{cases} -\dfrac{(c + \sqrt{-1}k)^2}{c^2 + k^2} & k > 0 \\ 1 & k < 0 \end{cases}$$

(1.88)

and

$$\tilde{w}_{\text{out}}(k) = \begin{cases} 1 & k > 0 \\ -\dfrac{(c + \sqrt{-1}k)^2}{c^2 + k^2} & k < 0 \end{cases}$$

(1.89)

respectively. The corresponding elementary kernels are equal to

$$w_{\text{in}}(x) = (1 - c)\delta(x) - \frac{1}{2} c \, (\text{sgn } x + \text{sgn } c) e^{-|c|x}$$
$$+ \frac{c\sqrt{-1}}{2\pi} (1 + \text{sgn } c) e^{-|c|x} \text{Ei}(|c|x) + \frac{c\sqrt{-1}}{2\pi} (1 - \text{sgn } c) e^{|c|x} \text{Ei}(-|c|x) \quad (1.90)$$

and

$$w_{out}(x) = \overline{w}_{in}(x) \tag{1.91}$$

where the exponential integral $Ei(x)$ (cf. [9])

$$Ei(x) = \int_{-\infty}^{x} t^{-1}e^t dt \tag{1.92}$$

is given by the Cauchy principal value for $x > 0$.

iii) The scattering operator is equal to

$$S = Sym\ V^2\ Sym \tag{1.93}$$

where the convolution operator V^2 has elementary multiplier

$$\breve{s}(k) = \breve{v}^2(k) = -\frac{(c + \sqrt{-1}\,|\,k\,|)^2}{c^2 + k^2} \tag{1.94}$$

and the elementary kernel

$$s(x) = (1 - 2c)\delta(x) - |\,c\,|\,e^{-|c|\,|x|}$$
$$-\frac{c\sqrt{-1}}{\pi}\,[e^{|c|x}Ei(-\,|\,c\,|\,x) + e^{-|c|x}Ei(|\,c\,|\,x)]\,. \tag{1.95}$$

Proof. — Since $|\,\breve{v}(k)\,| = 1$, the operator V is unitary and, since $\breve{v}(-k) = \breve{v}(k)$, V commutes with Sym. Compairing (1.68) with (1.83) and (1.84) we have

$$f_{in}(k\,|\,.) = \left[\prod_{i<j} \breve{v}(k_i - k_j)\right]\varphi(k\,|\,.)$$
$$f_{out}(k\,|\,.) = \left[\prod_{i<j} \breve{v}^{-1}(k_i - k_j)\right]\varphi(k\,|\,.) \tag{1.96}$$

which, in view of the above, implies

$$W_{in}^{-1} = QV = N!\ Sym\ \Theta U\ Sym\ V = N!\ Sym\ \Theta UV\ Sym$$

and

$$W_{out}^{-1} = QV^{-1} = N!\ Sym\ \Theta U Sym\ V^{-1} = N!\ Sym\ \Theta UV^{-1}Sym\,.$$

Formulas (1.88) and (1.89) are obvious. (1.90) is obtained from (1.88) using the tables of Fourier sine and cosine transforms (cf. [9]). (1.91) follows from

$$\breve{w}_{out}(k) = \breve{w}_{in}(-\,k)^-\,.$$

For the scattering operator S we have

$$S = W_{out}W_{in}^{-1} = Sym\ VQ^{-1}\ QVSym$$

which proves (1.93). The set of convolution operators of type (1.57) is

closed under multiplication and the elementary multiplier of a product is the product of elementary multipliers of the factors. This proves (1.94). Formula (1.95) is obtained using tables of the Fourier transform (cf. [9]).

§ 2. FOCK SPACE

We start by recalling generalities about the second quantization of a many body problem. We restrict our exposition to the case of interacting particles on the line with the pair potential interaction

$$H_N = - \Delta_N + \sum_{i \neq j} v(x_i - x_j) \tag{2.1}$$

although the formalism holds for more general interactions. Denote by $\mathscr{H}_N$ the space $L_2^{sym}(\mathbb{R}^N)$ and set

$$\mathscr{H} = \overset{\prime}{\underset{N=0}{\oplus}} \mathscr{H}_N$$

$$\hat{H} = \overset{\infty}{\underset{N=0}{\oplus}} H_N \tag{2.2}$$

where $\mathscr{H}_0 = \mathbb{R}$ and $H_0 = 0$. The subspace $\mathscr{H}_N$ of $\mathscr{H}$ is called the N-particle sector. One chooses a generator $\Omega \in \mathscr{H}_0$, $\|\Omega\| = 1$. Then $\mathscr{H}_0 = \mathbb{C}\Omega$ and Ω is called the vacuum vector. The space $\mathscr{H}$ is called the Fock space and $\hat{H}$ is the second quantized Hamiltonian (2.1). As is customary in the second quantization, we denote by X^+ the operator adjoint to X and call the operator valued distributions on $\mathscr{H}$ the fields. For $x \in \mathbb{R}$ we define operators $\psi_0(x)$, $\psi_0^+(x)$ on $\mathscr{H}$ by

$$(\psi_0(x) f)(x_1, \ldots, x_N) = \sqrt{N+1} f(x, x_1, \ldots, x_N) \tag{2.3}$$

where $f \in \mathscr{H}_{N+1}$ and

$$(\psi_0^+(x) f)(x_1, \ldots, x_{N+1}) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \delta(x - x_i) f(x_1, \ldots, \hat{x}_i, \ldots, x_{N+1}) \tag{2.4}$$

where $f \in \mathscr{H}_N$. The operators $\psi_0(x)$, $\psi_0^+(y)$ satisfy the canonical commutation relations

$$[\psi_0(x), \psi_0(y)] = [\psi_0^+(x), \psi_0^+(y)] = 0$$
$$[\psi_0(x), \psi_0^+(y)] = \delta(x - y) \tag{2.5}$$

and

$$\psi_0(x)\Omega = 0. \tag{2.6}$$

Any system $\psi(x)$, $\psi^+(y)$ of fields in $\mathscr{H}$ satisfying (2.5), (2.6) is unitarily equivalent to $\psi_0(x)$, $\psi_0^+(y)$. The fields $\psi^+(x)$, $\psi(y)$ are called the creation,

annihilation operators respectively. We will call $\psi_0(x)$, $\psi_0^+(y)$ the standard fields. We refer the reader to [1] or [11] for more details on the formalism of second quantization.

Any reasonable operator on $\mathcal{H}$ can be expressed in terms of the standard fields. In particular

$$\hat{H} = \int_{-\infty}^{\infty} dx[-\psi_0^+(x)\psi_{0xx}] + \int_{-\infty}^{\infty}\int dxdy\psi_0^+(x)\psi_0^+(y)v(x-y)\psi_0(x)\psi_0(y)$$

$$(2.7)$$

where, as usual in the physics literature on second quantization, we put differentials in front. The second quantized Hamiltonian $\hat{H}$ generates a one-parameter unitary group $e^{\sqrt{-1}t\hat{H}}$. The corresponding evolution equation in $\mathcal{H}$

$$-\sqrt{-1}\frac{\partial}{\partial t}f = \hat{H}f \tag{2.8}$$

is equivalent to the operator evolution equation

$$\sqrt{-1}\frac{\partial}{\partial t}X = [X, \hat{H}]. \tag{2.9}$$

Applying the evolution to the standard fields we obtain the time dependent fields

$$\psi(x, t) = e^{\sqrt{-1}t\hat{H}}\psi_0(x)e^{-\sqrt{-1}t\hat{H}}, \qquad \psi^+(y, t) = e^{\sqrt{-1}t\hat{H}}\psi_0^+(y)e^{-\sqrt{-1}t\hat{H}} \quad (2.10)$$

which for any t satisfy the canonical (equal time) commutation relations (2.5), (2.6). Applying the time evolution to (2.7) we see that

$$\hat{H} = \int_{-\infty}^{\infty} dx[-\psi^+(x, t)\psi_{xx}(x, t)]$$
$$+ \int_{-\infty}^{\infty}\int dxdy\psi^+(x, t)\psi^+(y, t)v(x - y)\psi(x, t)\psi(y, t) \quad (2.11)$$

which simply means that the Hamiltonian $\hat{H}$ is an integral of the evolution (2.9). From (2.11) and the equal time commutation relations we obtain the evolution equation for the fields

$$\sqrt{-1}\frac{\partial}{\partial t}\psi(x, t) = -\psi_{xx}(x, t)$$
$$+ 2\int_{-\infty}^{\infty} dy\psi^+(y, t)v(x - y)\psi(x, t)\psi(y, t) \quad (2.12)$$

Although the evolution equation (2.12) is nonlocal and nonlinear it is equivalent to the infinite sequence of local linear evolution equations

$$-\sqrt{-1}\frac{\partial}{\partial t}f_N = H_N f_N \tag{2.13}$$

where $f_N \in \mathcal{H}_N$, $N = 1, 2 \ldots$

Now we specialize to the case $v(x - y) = c\delta(x - y)$. Equations (2.11) and (2.12) become formally

$$\hat{H} = \int_{-\infty}^{\infty} dx \left[- \psi^+(x)\psi_{xx} + c\psi^{+\,2}(x)\psi^2(x) \right] \qquad (2.14)$$

and

$$\sqrt{-1}\,\psi_t = - \psi_{xx} + 2c\psi^+\psi^2\,. \qquad (2.15)$$

Equation (2.15) is the quantum Nonlinear Schrödinger equation and we will refer to it as NLS.

When solving the NLS one should not forget that expressions (2.14) and (2.15) are formally obtained from (2.11) and (2.12), which make sense literally for smooth potentials $v(x)$, by setting $v(x) = c\delta(x)$. It is clear from the preceeding exposition that the Hamiltonian $\hat{H}$ of (2.14) is the direct sum of N-body Hamiltonians

$$H_N = - \Delta_N + c \sum_{i \neq j} \delta(x_i - x_j) \qquad (2.16)$$

which are defined by the boundary conditions (1.2). With this understanding and our earlier conventions on the fields $\psi(x, t)$, $\psi^+(y, t)$, the NLS (2.15) is well defined and we proceed to solve it explicitly. By that we mean an explicit expansion of the time dependent fields $\psi(x, t)$, $\psi^+(y, t)$ in terms of the initial data—the standard fields $\psi_0(x) = \psi(x, 0)$, $\psi_0^+(y) = \psi^+(y, 0)$. The fields $\psi(x, t)$, $\psi^+(y, t)$ are called interacting as opposed to the free fields $\psi_0(x, t)$, $\psi_0^+(y, t)$ which correspond to the case $c = 0$.

Our solution of NLS is based on the explicit equivalence of the Hamiltonian $\hat{H}$ which depends on c and the free Hamiltonian $\hat{H}_0$ which corresponds to $c = 0$. Quoting the results of § 1 we will use the subscript $N = 1, 2...$ in formulas established there. Set

$$\hat{P} = \bigoplus_{N=0}^{\infty} P_N$$
$$(\hat{P}^*)^{-1} = \bigoplus_{N=0}^{\infty} (P_N^*)^{-1}\,. \qquad (2.17)$$

In what follows we denote by H the restriction $H|_{\mathcal{H}_c}$. Then

$$\hat{H} = \hat{P}\hat{H}_0\hat{P}^{-1} \qquad (2.18)$$

which implies

$$e^{\sqrt{-1}\,t\hat{H}} = \hat{P}e^{\sqrt{-1}\,t\hat{H}_0}\hat{P}^{-1} \qquad (2.19)$$

i. e. $\hat{P}$ intertwines the unitary groups $e^{\sqrt{-1}\,t\hat{H}_0}$ and $e^{\sqrt{-1}\,t\hat{H}}$. Analogous formulas hold with the other intertwining operator $(\hat{P}^*)^{-1}$. The following Proposition is obvious.

PROPOSITION 2.1. — i) Let $A_0(t)$ be an integral curve of the free (operator) evolution, i. e.

$$\sqrt{-1} \frac{\partial}{\partial t} A_0(t) = [A_0(t), \hat{H}_0].$$

Then $A(t) = \hat{P} A_0(t) \hat{P}^{-1}$ is an integral curve of the interacting evolution, i. e.

$$\sqrt{-1} \frac{\partial}{\partial t} A(t) = [A(t), \hat{H}].$$

ii) Let $A(t) = e^{\sqrt{-1}t\hat{H}} A e^{-\sqrt{-1}t\hat{H}}$ and $A_0(t) = e^{\sqrt{-1}t\hat{H}_0} A e^{-\sqrt{-1}t\hat{H}_0}$ be an interacting and a free evolution respectively. Then

$$A(t) = \hat{P} B_0(t) \hat{P}^{-1} \tag{2.20}$$

where

$$B_0(0) = \hat{P}^{-1} A(0) \hat{P}. \tag{2.21}$$

In view of Proposition 2.1, in order to obtain an explicit formula for the interacting fields $\psi(x, t)$, it suffices to express the intertwining operators $\hat{P}$ and $\hat{P}^{-1}$ in terms of the standard fields $\psi_0(x)$, $\psi_0^+(y)$. Before doing it, let us establish a direct connection between the intertwining operators $\hat{P}$, $(\hat{P}*)^{-1}$ and the Bethe Ansatz eigenstates.

Define for $k \in \mathbb{R}$ the operators $\psi_0(k)$ by

$$\psi_0(k) = \int_{-\infty}^{\infty} dx e^{-\sqrt{-1}kx} \psi_0(x). \tag{2.22}$$

Then

$$\psi_0^+(k) = \int_{-\infty}^{\infty} dx e^{\sqrt{-1}kx} \psi_0^+(x) \tag{2.23}$$

and the commutation relations are

$$[\psi_0(k), \psi_0(k')] = [\psi_0^+(k), \psi_0^+(k')] = 0$$
$$[\psi_0(k), \psi_0^+(k')] = 2\pi\delta(k - k'). \tag{2.24}$$

Operators $\psi_0^+(k)$ are creation operators for the eigenstates of the free Hamiltonian $\hat{H}_0$, i. e. for any $k = (k_1, \ldots, k_N)$

$$\psi_0^+(k_1) \ldots \psi_0^+(k_N)\Omega = f_0(. \,|\, k) \tag{2.25}$$

where $f_0(. \,|\, k)$ given by (1.27) satisfies

$$\hat{H}_0 f_0(. \,|\, k) = \| k \|^2 f_0(. \,|\, k). \tag{2.26}$$

For obvious reasons $\psi_0^+(k)$, $\psi_0(k)$ are called (the standard) momentum creation, annihilation operators as opposed to $\psi_0^+(x)$, $\psi_0(x)$ which are position creation, annihilation operators. Letting $\psi_0(k)$, $\psi_0^+(k)$ evolve under the free time evolution $e^{\sqrt{-1}t\hat{H}_0}$ we obtain the time dependent free fields

$\psi_0(k, t)$, $\psi_0^+(k, t)$. From the definition of $\psi_0(k)$ we see immediately that

$$\psi_0(k, t) = e^{-\sqrt{-1}k^2 t}\psi_0(k), \quad \psi_0^+(k, t) = e^{\sqrt{-1}k^2 t}\psi_0^+(k). \qquad (2.27)$$

Define the operators $b^+(k, t)$ by

$$b^+(k, t) = \hat{P}^{*-1}\psi_0^+(k, t)\hat{P}^*. \qquad (2.28)$$

PROPOSITION 2.2. — $b^+(k, t)$ is an integral curve of the interacting evolution (2.9) and

$$b^+(k, t) = e^{\sqrt{-1}k^2 t}b^+(k, 0). \qquad (2.29)$$

Denote $b^+(k, 0)$ by $b^+(k)$. For any $k = (k_1, \ldots, k_N) \in C_+$

$$b^+(k_1) \ldots b^+(k_N)\Omega = g(.\,|\,k) \qquad (2.30)$$

where $g(.\,|\,k)$ are defined by (1.50).

Proof. — The operator $(\hat{P}^*)^{-1}$ intertwines the free evolution with the interacting evolution. By Proposition 2.1, (i), $b^+(k, t)$ is an integral curve of the interacting evolution (Proposition 2.1 obviously holds with $(\hat{P}^*)^{-1}$ in place of $\hat{P}$). Thus, (2.29) follows from (2.27). We have

$$b^+(k_1) \ldots b^+(k_N)\Omega = \hat{P}^{*-1}\psi_0^+(k_1)\hat{P}^* \ldots \hat{P}^{*-1}\psi_0^+(k_N)\hat{P}^*\Omega$$
$$= \hat{P}^{*-1}\psi_0^+(k_1) \ldots \psi_0^+(k_N)\Omega = \hat{P}^{*-1}f_0(.\,|\,k_1, \ldots, k_N) = g(.\,|\,k).$$

In this sequence of equalities we have used that $\hat{P}^*\Omega = \Omega$, (2.25), that $f_0(.\,|\,k_1, \ldots, k_N) \in \mathscr{H}_N$ and $\hat{P}^{*-1}|_{\mathscr{H}_N} = P_N^{*-1}$.

We summarize the meaning of Proposition 2.2 in the following Corollary.

COROLLARY 2.1. — The fields $b^+(k)$ create Bethe Ansatz eigenstates $g(.\,|\,k_1, \ldots, k_N)$ of the Hamiltonian $\hat{H}$,

Remark. — If we used $\hat{P}$ instead of $\hat{P}^{*-1}$ in (2.28) we would get creation operators for eigenstates $f(.\,|\,k)$ (see (1.35)). Both $f(.\,|\,k)$ and $g(.\,|\,k)$ are Bethe Ansatz eigenstates in different « normalizations » and both options are equivalent. The choice of $\hat{P}^{*-1}$ over $\hat{P}$ is made to facilitate comparison with some formulas in the literature (cf. [12]). Using $\hat{U} = \oplus U_n$ instead of $\hat{P}^{*-1}$ in (1.28) we get creation operators for the normalized Bethe Ansatz eigenstates $\varphi(.\,|\,k)$.

Let $b(k, t)$ be the adjoint to $b^+(k, t)$.

COROLLARY 2.2. — $b(k, t)$ is an integral curve of the interacting evolution and

$$b(k, t) = e^{-\sqrt{-1}k^2 t}b(k) \qquad (2.31)$$

where we set $b(k) = b(k, 0)$.

Proof. — Immediate from Proposition 2.2.

We will now compute the commutation relations for operators $b(k)$, $b^+(k')$. This requires some preparation. Define the operators $a_0(k)$ on $\mathscr{H}$ by

$$a_0(k) = \exp\left[(2\pi)^{-1} \int_{-\infty}^{\infty} dr \log\left(\frac{c + \sqrt{-1}(k-r)}{\sqrt{-1}(k-r)} \right) \psi_0^+(r)\psi_0(r) \right]. \quad (2.32)$$

Then

$$a_0^+(k) = \exp\left[(2\pi)^{-1} \int_{-\infty}^{\infty} dr \log\left(\frac{c + \sqrt{-1}(r-k)}{\sqrt{-1}(r-k)} \right) \psi_0^+(r)\psi_0(r) \right]. \quad (2.33)$$

The basic properties of fields $a_0(k)$, $a_0^+(k)$ are summarized in the following Lemma.

LEMMA 2.1. — For all k and l the fields $a_0(k)$, $a_0^+(l)$ commute with each other and with the Hamiltonian $\hat{H}_0$. The fields $a_0(k)$ and $\psi_0(k)$ satisfy the commutation relations

$$a_0(k)\psi_0^+(l) = \frac{c + \sqrt{-1}(k-l)}{\sqrt{-1}(k-l)} \, \psi_0^+(l)a_0(k) \quad (2.34)$$

$$a_0(k)\psi_0(l) = \frac{\sqrt{-1}(k-l)}{c + \sqrt{-1}(k-l)} \, \psi_0(l)a_0(k) \quad (2.35)$$

$$a_0^+(k)\psi_0^+(l) = \frac{c + \sqrt{-1}(l-k)}{\sqrt{-1}(l-k)} \, \psi_0^+(l)a_0^+(k) \quad (2.36)$$

$$a_0^+(k)\psi_0(l) = \frac{\sqrt{-1}(l-k)}{c + \sqrt{-1}(l-k)} \, \psi_0(l)a_0^+(k). \quad (2.37)$$

For any $(k_1, \ldots, k_N) \in C_+$ we have

$$a_0(k)f_0(. \mid k_1, \ldots, k_N) = \prod_{i=1}^{N} \frac{c + \sqrt{-1}(k-k_i)}{\sqrt{-1}(k-k_i)} \, f_0(. \mid k_1, \ldots, k_N) \quad (2.38)$$

$$a_0^+(k)f_0(. \mid k_1, \ldots, k_N) = \prod_{i=1}^{N} \frac{c + \sqrt{-1}(k_i-k)}{\sqrt{-1}(k_i-k)} \, f_0(. \mid k_1, \ldots, k_N). \quad (2.39)$$

Proof. — Set

$$\alpha(k) = (2\pi)^{-1} \int_{-\infty}^{\infty} dr \log\left(\frac{c + \sqrt{-1}(k-r)}{\sqrt{-1}(k-r)} \right) \psi_0^+(r)\psi_0(r). \quad (2.40)$$

An elementary computation with the commutation relations (2.24) shows that

$$[\alpha(k), \psi_0^+(l)] = \log\left(\frac{c + \sqrt{-1}(k - l)}{\sqrt{-1}(k - l)}\right)\psi_0^+(l) \qquad (2.41)$$

and

$$[\alpha(k), \psi_0(l)] = -\log\left(\frac{c + \sqrt{-1}(k - l)}{\sqrt{-1}(k - l)}\right)\psi_0(l). \qquad (2.42)$$

Since $a_0(k) = \exp\alpha(k)$, we have from (2.41)

$$a_0(k)\psi_0^+(l)a_0(k)^{-1} = \frac{c + \sqrt{-1}(k - l)}{\sqrt{-1}(k - l)}\,\psi_0^+(l)$$

which is equivalent to (2.34). (2.42) implies (2.35) in the same way. Formulas (2.36) and (2.37) follow from (2.35) and (2.34) respectively.

Since

$$f_0(.\,|\,k_1, \ldots, k_N) = \psi_0^+(k_1)\ldots\psi_0^+(k_N)\Omega$$

the commutation relations (2.34)-(2.37) imply (2.38) and (2.39). The latter equations mean that the fields $a_0(k)$, $a_0^+(l)$ are diagonalized by the eigenstates $f_0(.\,|\,k_1, \ldots, k_N)$ of $\hat{H}_0$. Hence they commute with $\hat{H}_0$ and with each other.

Define the fields $a(k)$ by

$$a(k) = \hat{P}^{*-1}a_0(k)\hat{P}^* \qquad (2.43)$$

and let

$$a^+(k) = \hat{P}a_0^-(k)\hat{P}^{-1} \qquad (2.44)$$

be the adjoint fields.

Next Theorem establishes the commutation relations between the fields $b(k)$, $b^+(k)$, $a(l)$, $a^+(l)$.

THEOREM 2.1. — The fields $a(k)$, $a^+(l)$ commute with each other and with the interacting Hamiltonian $\hat{H}$. They are diagonalized by the Bethe Ansatz eigenstates and

$$a(k)g(.\,|\,k_1, \ldots, k_N) = \prod_{i=1}^{N}\frac{c + \sqrt{-1}(k - k_i)}{\sqrt{-1}(k - k_i)}\,g(.\,|\,k_1, \ldots, k_N) \qquad (2.45)$$

$$a^+(k)g(.\,|\,k_1, \ldots, k_N) = \prod_{i=1}^{N}\frac{c + \sqrt{-1}(k_i - k)}{\sqrt{-1}(k_i - k)}\,g(.\,|\,k_1, \ldots, k_N). \qquad (2.46)$$

The following commutation relations are satisfied

$$b(k)b^+(l) = \frac{c^2 + (k - l)^2}{(k - l)^2} b^+(l)b(k) + 2\pi\delta(k - l)a^+(k)a(k) \quad (2.47)$$

$$a(k)b^+(l) = \frac{c + \sqrt{-1}(k - l)}{\sqrt{-1}(k - l)} b^+(l)a(k) \quad (2.48)$$

$$a^+(k)b(l) = \frac{\sqrt{-1}(l - k)}{c + \sqrt{-1}(l - k)} b(l)a^+(k) \quad (2.49)$$

$$a(k)b(l) = \frac{\sqrt{-1}(k - l)}{c + \sqrt{-1}(k - l)} b(l)a(k) \quad (2.50)$$

$$a^+(k)b^+(l) = \frac{c + \sqrt{-1}(l - k)}{\sqrt{-1}(l - k)} b^+(l)a^+(k) . \quad (2.51)$$

Proof. — The operator $\hat{P}^*\hat{P}$ on $\hat{\mathscr{H}}$ which is given by

$$\hat{P}^*\hat{P} = \overset{\infty}{\underset{N=0}{\oplus}} P_N^*P_N \quad (2.52)$$

is invertible and we have

$$(\hat{P}^*P)\psi_0(k)(\hat{P}^*P)^{-1} = a_0^+(k)a_0(k)\psi_0(k) . \quad (2.53)$$

To establish (2.53) we apply both sides of it to the free eigenstates $f_0(.\,|\,k_1, \ldots, k_N)$ and use (1.45), (2.36) and (2.37) to show that we obtain the same thing. The operator $a_0^+(k)a_0(k)$ is symmetric and taking the adjoint of (2.53) we get

$$(\hat{P}^*P)^{-1}\psi_0^+(k)(\hat{P}^*P) = \psi_0^+(k)a_0^+(k)a_0(k) . \quad (2.54)$$

By (1.45), $\hat{P}^*\hat{P}$ is diagonalized by the eigenstates $f_0(.\,|\,k)$, therefore $\hat{P}^*\hat{P}$ commutes with $a_0(k)$ and $a_0^+(k)$.

From the definition of fields $b(k)$ we have

$$b(k)b^+(l) = \hat{P}\psi_0(k)(\hat{P}^*\hat{P})^{-1}\psi_0^+(l)\hat{P}^* .$$

Switching $\psi_0(k)$ and $(\hat{P}^*\hat{P})^{-1}$ around by (2.53) we get

$$b(k)b^+(l) = \hat{P}^{*-1}a_0^+(k)a_0(k)\psi_0(k)\psi_0^+(l)\hat{P}^* . \quad (2.55)$$

On the other hand

$$b^+(l)b(k) = \hat{P}^{*-1}\psi_0^+(l)(\hat{P}^*\hat{P})\psi_0(k)P^{-1}$$

and switching $\hat{P}^*\hat{P}$ with $\psi_0(k)$ by (2.53) we get

$$b^+(l)b(k) = \hat{P}^{*-1}\psi_0^+(l)a_0^+(k)a_0(k)\psi_0(k)\hat{P}^* . \quad (2.56)$$

From the commutation relations (2.34)-(2.37) follows

$$\psi_0^+(l)a_0^+(k)a_0(k) = \frac{(k-l)^2}{c^2+(k-l)^2}\, a_0^+(k)a_0(k)\psi_0^+(l)\,. \qquad (2.57)$$

Thus

$$b^+(l)b(k) = \frac{(k-l)^2}{c^2+(k-l)^2}\, \hat{P}^{*-1}a_0^+(k)a_0(k)\psi_0^+(l)\psi_0(k)\hat{P}^*$$

$$= \frac{(k-l)^2}{c^2+(k-l)^2}\, \hat{P}^{*-1}a_0^+(k)a_0(k)\psi_0(k)\psi_0^+(l)\hat{P}^*$$

$$- (2\pi)\delta(k-l)\frac{(k-l)^2}{c^2+(k-l)^2}\, \hat{P}^{*-1}a_0^+(k)a_0(k)\hat{P}^*\,. \qquad (2.58)$$

The first summand in the right hand side of (2.58) is $\dfrac{(k-l)^2}{c^2+(k-l)^2}\, b(k)b^+(l)$, by (2.55). The second summand is

$$-\frac{2\pi}{c^2}\delta(k-l)\hat{P}^{*-1}a_0^+(k)a_0(k)\hat{P}^* = -\frac{2\pi}{c^2}\delta(k-l)\hat{P}\hat{P}^{-1}\hat{P}^{*-1}a_0^+(k)a_0(k)\hat{P}^* =$$

$$-\frac{2\pi}{c^2}\delta(k-l)\big[\hat{P}a_0^+(k)\hat{P}^{-1}\big]\big[\hat{P}^{*-1}a_0(k)\hat{P}^*\big] = -\frac{2\pi}{c^2}\delta(k-l)a^+(k)a(k) \qquad (2.59)$$

where we have used that $\hat{P}^{-1}\hat{P}^{*-1} = (\hat{P}^*\hat{P})^{-1}$ commutes with the fields $a_0^+(k)$, $a_0(k)$. From (2.58) and (2.59) we have

$$b^+(l)b(k) = \frac{(k-l)^2}{c^2+(k-l)^2}\, b(k)b^+(l) - \frac{2\pi}{c^2}\delta(k-l)a^+(k)a(k) \qquad (2.60)$$

which immediately implies (2.47).

Formula (2.48) follows directly from the definitions of $a^+(k)$ and $b(l)$ and Lemma 2.1, (2.37). Equation (2.49) is the adjoint of (2.48). To show (2.50) we have

$$a(k)b(l) = \hat{P}^{*-1}a_0(k)(\hat{P}^*\hat{P})\psi_0(l)\hat{P}^{-1} = \hat{P}a_0(k)\psi_0(l)\hat{P}^{-1}$$

since $a_0(k)$ and $\hat{P}^*\hat{P}$ commute. Analogously

$$b(l)a(k) = \hat{P}\psi_0(l)(\hat{P}^{-1}\hat{P}^{*-1})a_0(k)\hat{P}^* = \hat{P}\psi_0(l)a_0(k)\hat{P}^{-1}$$

because $\hat{P}^{-1}\hat{P}^{*-1} = (\hat{P}^*\hat{P})^{-1}$. Now Lemma 2.1, (2.35) implies (2.50). Taking the adjoint we obtain (2.51).

Using that $\hat{P}^*\hat{P}$ commutes with $a_0(k)$ and $a_0^+(l)$ we have

$$a(k)a^+(l) = \hat{P}^{*-1}a_0(k)\hat{P}^*\hat{P}a_0^+(l)\hat{P}^{-1} = \hat{P}a_0(k)a_0^+(l)\hat{P}^{-1}$$

$$= \hat{P}a_0^+(l)a_0(k)\hat{P}^{-1}\hat{P}^{*-1}\hat{P}^* = \hat{P}a_0^+(l)\hat{P}^{-1}\hat{P}^{*-1}a_0(k)\hat{P}^* = a^+(l)a(k)\,.$$

The Hamiltonian $\hat{H}$ commutes with $a(k)$ and $a^-(k)$ because

$$\hat{H} = \hat{P}^{*-1}\hat{H}_0\hat{P}^* = \hat{P}\hat{H}_0\hat{P}^{-1}$$

and $\hat{H}_0$ commutes with $a_0(k)$, $a_0^+(k)$. From (2.38) and (2.43) we have

$$a(k)g(.\,|\,k_1,\,\ldots,k_N) = \hat{P}^{*\,-1}a_0(k)\,\hat{P}^*\,\hat{P}^{*\,-1}f_0(.\,|\,k_1,\,\ldots,k_N)$$

$$= \prod_{i=1}^{N} \frac{c + \sqrt{-1}(k - k_i)}{\sqrt{-1}(k - k_i)}\,\hat{P}^{*\,-1}f_0(.\,|\,k_1,\,\ldots,k_N)$$

which proves (2.45). A parallel argument shows that

$$a^+(k)f(.\,|\,k_1,\,\ldots,k_N) = \prod_{i=1}^{N} \frac{c + \sqrt{-1}(k_i - k)}{\sqrt{-1}(k_i - k)}\,f(.\,|\,k_1,\,\ldots,k_N)\,. \quad (2.61)$$

By the Remark after Corollary 2.1, eigenstates $f(.\,|\,k_1,\,\ldots,k_N)$ and $g(.\,|\,k_1,\,\ldots,k_N)$ are proportional, thus (2.61) implies (2.46). The Theorem is proved.

COROLLARY 2.3. — The action of fields $b(k)$ on the Bethe Ansatz eigenstates $g(.\,|\,k_1,\,\ldots,k_N)$ is given by

$$b(k)g(.\,|\,k_1,\,\ldots,k_N)$$

$$= 2\pi \sum_{i=1}^{N} \delta(k - k_i) \prod_{j \neq i} \frac{c^2 + (k - k_j)^2}{(k - k_j)^2}\,g(.\,|\,k_1,\,\ldots,\hat{k}_i,\,\ldots,k_N)\,. \quad (2.62)$$

Proof. — By (2.30), (2.45), (2.46) and (2.47) we have

$$b(k)g(.\,|\,k_1,\,\ldots,k_N) = \frac{c^2 + (k - k_1)^2}{(k - k_1)^2}\,b^+(k_1)b(k)g(.\,|\,k_2,\,\ldots,k_N)$$

$$+ 2\pi\delta(k - k_1) \prod_{j=2}^{N} \frac{c^2 + (k - k_j)^2}{(k - k_j)^2}\,g(.\,|\,k_2,\,\ldots,k_N) \quad (2.63)$$

Iterating (2.63) and using that $b(k)\Omega = 0$ we obtain (2.62).

In the next paper [7] we will express the intertwining operators $\hat{P}$, etc... and the fields $a(k)$, $a^+(k)$, $b(k)$, $b^+(k)$, $\psi(x, t)$, $\psi^+(x, t)$ in terms of the standard fields which will give us an explicit solution of the NLS.

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