Ann. Inst. Henri Poincaré,
Vol. 3, n° 4, 1986, p. 331-343. *Analyse non linéaire*

A finiteness result in the free boundary value problem for minimal surfaces

by

Friedrich TOMI
Mathematisches Institut der Universität, Im Neuenheimer Feld 288,
D-6900 Heidelberg, W-Germany

ABSTRACT. — It is proved that if an analytic H-convex body M in $\mathbb{R}^3$ admits an infinite number of area minimizing disc type surfaces interior to M and supported by ∂M then M must be diffeomorphic to a solid torus. Moreover, the set of these minimal surfaces then forms an analytic one-parameter family which foliates M.

Key-words: Free boundary value problem, minimal surfaces.

RÉSUMÉ. — On prouve le résultat suivant : si une partie M analytique et H-convexe de $\mathbb{R}^3$ admet dans son intérieur un nombre infini de surfaces minimisantes de type disques et supportées par ∂M, il faut que M soit difféomorphe à un tore solide. En plus, l'ensemble des solutions forme une famille analytique de dimension 1 qui constitue un feuilletage de M.

Besides the widely known problem of Plateau there is a variety of other geometrically appealing boundary value problems for minimal surfaces including problems with so called free boundary conditions. For background information we refer the reader to the books of Courant [*3*] and Nitsche [*8*]. Courant poses a totally free boundary value problem in the following manner. Given a compact surface S in $\mathbb{R}^3$ and a closed Jordan curve Γ in $\mathbb{R}^3\backslash S$ which is not contractible in $\mathbb{R}^3\backslash S$ one asks for a surface of minimal (or stationary) area in the class of all disc type surfaces whose boundary curve lies on S and is linked with Γ. Assuming sufficient differentiability, solutions of this problem will of course be minimal surfaces in the sense of differential geometry (i. e. surfaces of mean curvature zero) and they will also sit orthogonally on S along their boundary. In the given formulation of the problem the surfaces are allowed to penetrate the supporting surface S what is of course unrealistic from the view point of an experimentor who wants to produce such minimal surfaces in the form of soap films using a supporting surface S made from a solid material. In a physically realistic model one should therefore restrict all comparison surfaces to a component of $\mathbb{R}^3\backslash S$. In the present paper we decide to choose a bounded component. Imposing such an additional side condition we have to put up with the fact that minimizing surfaces will possibly touch the supporting surface S along interior portions of arbitrary size. Since free variations of the surface are no longer admissible along such portions the minimizing surface will in general not be a differential geometric minimal surface. There is however a now well known geometric condition on a 3-dimensional manifold with boundary M which prevents interior contact between minimizing surfaces in M and the boundary of M—unless the surface is totally contained in ∂M. This is the condition that ∂M has non-negative inward mean curvature, a property which we shall call « H-convexity », for short.

Meeks and Yau [7] have used the free boundary value problem for minimal surfaces in the study of certain questions in 3-dimensional topology. For this purpose they proved the existence of minimizing discs in a compact 3-dimensional Riemannian manifold M with convex boundary ∂M where ∂M has nontrivial fundamental group. The main result of Meeks and Yau from the view point of minimal surface theory refers however to the embedded character of any solution provided by their existence theorem. Moreover, they show that any two different solutions are disjoint.

It is not difficult to see and was already remarked by Meeks and Yau themselves that their results remain valid if one replaces the convexity condition by the weaker condition of H-convexity.

In the present paper we shall deal with the question concerning the number of solutions to the free boundary value problem. Let us first consider two simple examples in which continua of solutions exist.

EXAMPLE 1. — Take any simple closed curve α in $\mathbf{R}^3$ and let M be a thin regular closed neighborhood of α, i. e. $M = \{ x \in \mathbb{R}^3 \mid \operatorname{dist}(x, \alpha) \leq \varepsilon \}$ for sufficiently small ε. The torus M can be generated by moving the center of a circular disc of radius ε along α, keeping the disc always perpendicular to α. A short calculation shows that this S^1-family of plane discs solves the free boundary value problem for M. Let us remark that $M \in C^\infty$ if $\alpha \in C^\infty$ and $M \in C^\omega$ if $\alpha \in C^\omega$.

EXAMPLE 2. — Let α be a simple closed curve on S^2, parametrized by arclength, $\alpha = \alpha(s)$, $0 \leq s \leq l$. We choose an orthonormal basis e_1, e_2 of the normal space of α,

$$e_1(s) = \alpha(s), \quad e_2(s) = \alpha(s) \times \alpha'(s).$$

Now let $\rho = \rho(\varphi)$, $0 \leq \varphi \leq 2\pi$, be a smooth, positive, but small function which describes the boundary of the plane domain

$$B = \{ (r \cos \varphi, r \sin \varphi) \mid 0 \leq r \leq \rho(\varphi), 0 \leq \varphi \leq 2\pi \}.$$

Now we move B along α according to

$$B(s) = \{ \alpha(s) + r \cos \varphi e_1(s) + r \sin \varphi e_2(s) \mid 0 \leq r \leq \rho(\varphi), 0 \leq \varphi \leq 2\pi \}, 0 \leq s \leq l.$$

If ρ is sufficiently small then the S^1-family of plane surfaces $B(s)$ generates a smooth torus M and one easily verifies that $B(s)$ always sits orthogonal on ∂M. Therefore each $B(s)$ solves the free boundary value problem for M. Let us remark that this construction yields rotationally symmetric tori as a special case.

By glueing together pieces of tori of the kind constructed in the above examples we can produce smooth H-convex bodies of arbitrary genus which possess continua of solutions. It is the content of our theorem that this is not possible for analytic H-convex bodies.

THEOREM. — If a compact analytic H-convex body M in $\mathbb{R}^3$ which is not simply connected admits infinitely many minimizing solutions to the free boundary value problem then it must be homeomorphic to a solid torus. More precisely, the set of all solutions can be represented as an analytic S^1-family of minimal embeddings $F(\xi, .) : D \to M$ where D is the closed unit disc in $\mathbb{R}^2$ and the map $F : S^1 \times D \to M$ is an analytic diffeomorphism.

It would be interesting to say more about the geometry of S^1-families of surfaces minimally spanning some torus. As a first step in this direction we can make the following.

REMARK. — If the torus M is foliated by a smooth S^1-family $\mathcal{F}$ of plane, disc type surfaces, each one being orthogonal on ∂M, then all surfaces in the family are congruent.

Proof. — Let us denote by X the vector field of unit normal vectors of the surfaces in $\mathcal{F}$. We choose an arbitrary $B_0 \in \mathcal{F}$ and pick a point $p_0 \in \text{int}(B_0)$. Let then $\alpha = \alpha(s)$ be the integral curve of X with $\alpha(0) = p_0$. We can now (at least locally) construct an orthonormal basis $e_1(s)$, $e_2(s)$ of the normal space of α at $\alpha(s)$ which additionally satisfies $e_1' \cdot e_2 = e_1 \cdot e_2' = 0$. Since α, by construction, intersects all surfaces in $\mathcal{F}$ orthogonally, it follows that $e_1(s)$ and $e_2(s)$ span the plane of that surface in $\mathcal{F}$ which passes through $\alpha(s)$. Therefore in a neighborhood of B_0 we can parametrize the boundaries of our family of surfaces in the form

$$\partial B(s) = \{ \alpha(s) + (x_1(t) + u(s,t)x_2'(t))e_1(s) + (x_2(t) - u(s,t)x_1'(t))e_2(s) \mid 0 \le t \le l \},$$

where $\alpha(0) + x_1(t)e_1(0) + x_2(t)e_2(0)$, $0 \le t \le l$, is a parametrization of ∂B_0 with respect to arclength and u is a smooth function with $u(0,t) = 0$. A short calculation shows that the condition of orthogonality of $B(s)$ to ∂M (generated by the family $\partial B(s)$) reads

$$u_s(1 + u(x_1'x_2'' - x_2'x_1'')) = 0 .$$

By continuity, u stays small for small values of s and therefore $u_s = 0$, i. e. $u(s,t) = u(0,t) = 0$. This proves our remark.

Our examples of S^1-families minimally spanning some torus consisted in families of plane surfaces. It would be interesting to know if also tori admitting non-flat families exist and, if this is the case, wether all surfaces in such a family must be congruent or at least isometric.

The author is indebted to William H. Meeks for a very stimulating discussion on the subject of the paper.

Note added in proof. Recently R. Gulliver and S. Hildebrandt have constructed a torus foliated by an S^1-family of non-flat minimal discs.

1. *A PRIORI* BOUNDS FOR MINIMIZING MAPS

Compactness of the set of minimizing maps is a basic ingredient in the proof of our theorem. This compactness is established by showing the existence of uniform bounds for derivatives of minimizing maps. This is done by an indirect argument which is linked with the proof of the existence of solutions. We must therefore start by defining the class of admissible mappings underlying an existence proof. We assume that M is a three-dimensional compact, H-convex submanifold with boundary of $\mathbb{R}^3$ of class $C^{k,\alpha}$, $k \ge 3$, $0 < \alpha < 1$, which is not simply connected. We choose a homotopically non-trivial Jordan curve Γ in int(M) and denote by D the closed unit disc in R^2 and by $d(p)$ the distance of a point $p \in M$ from ∂M. Then we define $\mathscr{C}(M, \Gamma)$ to be the set of all mappings f of Sobolev class $H_2^1(D, \mathbb{R}^3)$ such that

(1) $f(z) \in M$ for almost all $z \in D$

(2) $\lim_{z \to w} d(f(z)) = 0$ for all $w \in \partial D$

(3) there exists a radius r_0, $0 < r_0 < 1$, depending on f, such that for almost all $r \in]r_0, 1[$ the path $f \mid C_r$, $C_r = \{ z \in \mathbb{C} \mid |z| = r \}$, is not contractible in $M \backslash \Gamma$.

As for condition (3) we remark that from $f \in H_2^1(D, \mathbb{R}^3)$ it follows that $f \mid C_r \in H_2^1(C_r, \mathbb{R}^3)$ for almost all r and therefore $f \mid C_r$ is continuous for such r.

As usual in classical minimal surface theory, instead of area one minimizes Dirichlet's integral

$$D(f) = \frac{1}{2} \iint (|f_{x_1}|^2 + |f_{x_2}|^2) dx_1 dx_2 .$$

It can be shown that under the above conditions on M any minimizing $f \in \mathscr{C}(M, \Gamma)$ is a conformal harmonic map which maps int (D) into int (M); furthermore $f \in C^{k,\alpha}(D, \mathbb{R}^3)$ and $f(D)$ is orthogonal on ∂M along the boundary. For our *a priori* estimate below as well as for our structure investigation in the following sections we may therefore assume that minimizing maps enjoy all those properties just listed.

In view of the non-compactness of the conformal group of D and the invariance of our problem under this group it is obviously necessary to impose a normalization condition on minimizing maps in order to get estimates. A suitable such condition is

$$(4) \qquad f(0) \in \Gamma$$

for $f \in \mathscr{C}(M, \Gamma)$. We have then

PROPOSITION 1. — Let M be an H-convex body of class $C^{k,\alpha}$ with $k \geq 3$. Then all maps f which minimize Dirichlet's integral in $\mathscr{C}(M, \Gamma)$ and additionally satisfy (4) are uniformly bounded in $C^{k,\alpha}$-norm.

Proof. — Let us denote by $\mathscr{C}^*$ the set of all minimizing maps which fulfill (4). We first show that the Dirichlet integrals of these maps satisfy a uniform boundary strip condition, i. e. for each $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$(5) \qquad \iint_{R < |x| < 1} |\nabla f|^2 dx < \varepsilon$$

for all $f \in \mathscr{C}^*$. If this were not true then there would be an $\varepsilon > 0$ and a sequence $f_n \in \mathscr{C}^*$ such that

$$\iint_{1 - \frac{1}{n} < |x| < 1} |\nabla f_n|^2 dx \geq \varepsilon \qquad (n = 1, 2, \ldots).$$

Since (f_n) is a sequence of bounded harmonic maps with

$$D(f_n) = \delta := \inf \{ D(g) \mid g \in \mathscr{C}(M, \Gamma) \}$$

we may assume that the sequence (f_n) converges to a harmonic map f together with all derivatives locally uniformly in the interior of D. It follows that

$$D(f) \leq \delta - \varepsilon$$

and that f satisfies (4). As in the proof of Satz 3.1 in [4] one can then conclude that $f \in \mathscr{C}(M, \Gamma)$ and hence $D(f) \geq \delta$. This contradiction proves (5) and from Satz 2.1 in [4] we may then infer that the family $\{ d \circ f \mid f \in \mathscr{C}^* \}$ is equicontinuous, where d is distance from ∂M. Using this information we see from the proof of boundary continuity of minimizing surfaces that all $f \in \mathscr{C}^*$ satisfy a uniform Hölder condition [5]. The estimates for the derivatives now follow from [6].

2. THE LOCAL STRUCTURE OF THE SET OF SOLUTIONS

From now on we shall assume that our manifold M is analytic. The first lemma states for the analytic case the existence of a regular neighborhood of a disc in M which is transversal to ∂M. The proof requires only standard techniques and can therefore be omitted.

LEMMA 1. — Let $f : D \to M$ be an analytic embedding such that $f(\text{int}(D)) \subset \text{int}(M)$, $f(\partial D) \subset \partial M$ and, furthermore $f(D)$ is orthogonal on ∂M along $f(\partial D)$. Then there exist $\delta > 0$ and an analytic diffeomorphism Φ from $D \times]-\delta, \delta[$ onto some neighborhood of $f(D)$ in M, $\Phi = \Phi(z, t)$, such that

$$\Phi(z, 0) = f(z), \quad \Phi_t(z, 0) = N(z) \tag{6}$$

where N is a unit normal of f.

The following lemma states the positivity up to the boundary of the first eigenfunction of a second order elliptic operator with the so called third boundary condition.

LEMMA 2. — Let

$$\beta(u, v) = \int_D (a_{kl} u_{x_k} v_{x_l} + cuv) dx^1 dx^2 + \oint_{\partial D} buv ds$$

be an elliptic bilinear form with analytic coefficients a_{kl}, c, b. Assume furthermore that $\beta(u, u) \geq 0$ for all $u \in H_2^1(D)$ and that $\beta(u_0, u_0) = 0$ for some $u_0 \in H_2^1(D)$, $u_0 \neq 0$. Then u_0 is either strictly positive or strictly negative in D.

Proof. — As in the case of the Dirichlet boundary condition which is classical one shows that u_0 does not change sign in the interior of D, say

$u_0 > 0$. Let us assume that $u_0(z_0) = 0$ at some boundary point z_0. Clearly, u_0 is a solution of the boundary value problem

$$(7) \qquad - \frac{\partial}{\partial x_l}(a_{kl} u_{x_k}) + cu = 0 \quad \text{in D},$$

$$(8) \qquad u_r + bu = 0 \quad \text{on } \partial \text{D}.$$

It follows at once from (8) that $\frac{\partial}{\partial r} u_0(z_0) = 0$ and since z_0 is a minimum of u_0 on D we also have $\frac{\partial}{\partial \theta} u_0(z_0) = 0$ where θ denotes arclength along ∂D. Therefore u_0 has a zero at z_0 of at least second order. By our analycity assumptions u_0 can however be extended into an open neighborhood of D as a solution of (7) and it follows from the nodal line structure of such solutions that one of the nodal lines of u_0 emanating from z_0 had to enter the interior of D, contradicting the positivity of u_0.

Before proving the main result of this section let us recall the formulas for the first and second variation of surface area [2, §§ 109, 116]. If X is any immersed surface with boundary and Y an arbitrary variation vector field along X then we have

$$(9) \qquad \text{DA(X)(Y)} = -2 \iint \text{HN} \cdot \text{Y} d\omega + \oint n \cdot \text{Y} ds$$

where N is a unit normal of X and H the corresponding mean curvature, n is the outer unit normal of the boundary ∂X in the surface X, and integration is performed with respect to surface area in the first and with respect to arclength along ∂X in the second integral. For normal variations $\text{Y} = v\text{N}$ we obtain

$$(10) \qquad \text{D}^2\text{A(X)(Y, Y)} = \iint (g^{kl} v_{x_k} v_{x_l} + 2\text{K}v^2) d\omega$$

where (g^{kl}) is the inverse of the metric tensor of X and K the Gauss curvature.

PROPOSITION 2. — Let M be analytic and H-convex and let f be a minimizing embedded solution to the free boundary value problem for M. We assume furthermore that f is not an isolated solution, i. e. there exists a sequence of stationary solutions which are geometrically different from f but tend to f in the C^0-norm. Then there is a one parameter family $\text{F} = \text{F}(t)$ of area minimizing surfaces in $\mathscr{C}(\text{M}, \Gamma)$, $|t| < \delta$, which is analytic in t with respect to the $\text{C}^{2+\alpha}$-norm and has the following properties: *i*) $\text{F}(0) = f$, *ii*) $\text{F}'(0)$ is a non-zero normal vector along f, *iii*) every solution to the free boundary value problem sufficiently close to f in the C^0-norm after suitable reparametrization belongs to the family F.

Proof. — We can clearly assume that f satisfies (4). To some fixed compact neighborhood V of the identity in the conformal group of D we can find a neighborhood U of f such that to each $g \in U$ there corresponds $\tau \in V$ such that $g.\tau$ also satisfies (4). It follows then from Proposition 5 that any sequence of solutions converging to f in C^0 in fact converges in $C^{k+\alpha}$. We shall work with $k = 2$. Using the analytic diffeomorphism Φ constructed in Lemma 2 we see that any surface $g \in C^{2+\alpha}(D, M)$ sufficiently close to f can be represented in the form

$$g = \Phi(.\,, u) \circ \tau$$

where u is a real function and τ a diffeomorphism of D of class $C^{2+\alpha}(D)$. Giving up the requirement of conformal parameters we need therefore only consider surfaces of the form $g(z) = \Phi(z, u(z))$ with $u \in C^{2+\alpha}(D)$. We want to set up the conditions on u that such a surface is a minimal surface sitting orthogonally on ∂M along its boundary. For any immersion $g : D \to \mathbb{R}^3$ let $H[g]$ denote the mean curvature of g. We can then define the nonlinear second order differential operator

$$h : U^{2+\alpha} \to C^\alpha(D)\,,$$
$$h(u) = 2H[\Phi(.\,, u)]$$

where $U^{2+\alpha}$ is some neighborhood of 0 in $C^{2+\alpha}(D)$. In view of the analycity of Φ the operator h is also analytic with respect to the corresponding norms and from (6) and a classical formula in differential geometry [2, § 117] we obtain

(11) $$Dh(0)(v) = 2DH[f](vN) = \Delta_f v - 2K_f v$$

where Δ_f and K_f are the Laplace-Beltrami operator and the Gauss curvature of the induced metric of f, respectively.

Using polor coordinates (r, θ) on D we can write the condition that an immersion $g : D \to M$ is orthogonal to ∂M along $g(\partial D)$ as

(12) $$g_r \cdot \Phi_u - \frac{g_r \cdot g_\theta}{|g_\theta|^2} g_\theta \cdot \Phi_u = 0\,.$$

Inserting the expressions

$$g_r = \Phi_r + \Phi_u u_r,\ g_\theta = \Phi_\theta + \Phi_u u_\theta$$

in the left hand side of (12) we obtain the analytic boundary operator

$$b : U^{2+\alpha} \to C^{1+\alpha}(\partial D)\,,$$
$$b(u) = |\Phi_u|^2 u_r + \Phi_r \cdot \Phi_u + \frac{(\Phi_r + \Phi_u u_r)\cdot(\Phi_\theta + \Phi_u u_\theta)}{|\Phi_\theta + \Phi_u u_\theta|^2}(|\Phi_u|^2 u_\theta + \Phi_\theta \cdot \Phi_u).$$

Using (6) and the conformality of f one calculates

(13) $$Db(0)(v) = v_r + f_r . \Phi_{uu}(.\,, 0)v\,.$$

All minimal surfaces in M close to f which sit orthogonal on ∂M can therefore be described as the zero set of the nonlinear elliptic operator

$$T : U^{2+\alpha} \to C^{\alpha}(D) \times C^{1+\alpha}(\partial D),$$
$$T(u) = (h(u), b(u)).$$

In order to study the zeros of T we first consider the linear elliptic operator $L = DT(0)$ together with its corresponding bilinear form

$$\beta_L(u, v) = \int_D - Dh(0)(u)v\,d\omega_f + \oint_{\partial D} Db(0)(u)v\,d\theta.$$

After partial integration and using the conformality of f we can write

$$\beta_L(u, v) = \int_D (\nabla u \cdot \nabla v + 2K\sqrt{g_f}uv)dx^1dx^2 + \oint_{\partial D} f_r . \Phi_{uu}(.,0)uv\,d\theta.$$

It follows now from the standard theory of elliptic boundary value problems that L has Fredholm index 0. We obtain more information on L by considering the area functional

$$a(u) = A(\Phi(.,u)).$$

Since $f = \Phi(.,0)$ is area minimizing we conclude at once that

$$Da(0) = 0,\ D^2a(0) \geq 0. \tag{14}$$

From

$$Da(u)(v) = DA(\Phi(.,u))(v\Phi_u(.,u))$$

and (9) and (10) we obtain the expression

$$D^2a(0)(v, v) = \int_D (|\nabla v|^2 + 2K\sqrt{g_f}v^2)dx + \oint_{\partial D} \xi \cdot \Phi_{uu}(.,0)v^2 |f_\theta| d\theta,$$

where ξ is the outer unit normal of $f(\partial D)$ in $f(D)$. Since f has conformal parameters we have

$$\xi = |f_r|^{-1}f_r, \quad |f_\theta| = |f_r|$$

and we see that

$$D^2a(0)(v, v) = \beta_L(v, v).$$

From (14) and Lemma 2 we can now conclude that ker L is one-dimensional. From now on we can argue as in [*10*]: Let

$$P : C^{\alpha}(D) \times C^{1+\alpha}(\partial D) \to \text{range } L$$

be a projection. It follows from the implicit function theorem that for a sufficiently small neighborhood $V^{2+\alpha} \subset U^{2+\alpha}$ of $0 \in C^{2+\alpha}(D)$ the set $V^{2+\alpha} \cap (PT)^{-1}(0)$ is an embedded analytic arc. Obviously one has

$$V^{2+\alpha} \cap T^{-1}(0) = \{ u \in V^{2+\alpha} \cap (PT)^{-1}(0) \mid (I - P)T(u) = 0 \}$$

and we know from our hypotheses that there is a sequence (u_n) such that $u_n \neq 0$ but $u_n \to 0$ in $C^{2+\alpha}(D)$ and $T(u_n) = 0$, i. e. the analytic function $(I - P)T$ has zeros on the 1-dimensional manifold $V^{2+\alpha} \cap (PT)^{-1}(0)$ which accumulate at the point 0. It follows of course that $(I - P)T$ is identically 0 on $V^{2+\alpha} \cap (PT)^{-1}(0)$ and therefore $V^{2+\alpha} \cap T^{-1}(0) = V^{2+\alpha} \cap (PT)^{-1}(0)$, which is an analytic arc. Parametrizing this arc as $u = u(t)$, $|t| < \delta$, we clearly have

$$0 = DT(0)u'(0) = Lu'(0)$$

and it follows from Lemma 2 that $u'(0)$ is non-zero everywhere in D. If we now set

$$F(t) = \Phi(\,.\,, u(t))\,,$$

then

$$F(0) = f,\quad F'(0) = \Phi_u(\,.\,, 0)u'(0) = u'(0)N$$

which is a non-zero normal vector along f. Moreover, it follows from the construction of the family $u(t)$ and formula (9) that $\frac{d}{dt}A(F(t)) = 0$ and hence $A(F(t)) = A(F(0)) = A(f)$. Since f has minimal area in $\mathscr{C}(M, \Gamma)$ we see that each $F(t)$ is also area minimizing in $\mathscr{C}(M, \Gamma)$. This finishes the proof of the proposition.

3. PROOF OF THE THEOREM

From the assumption that M possesses infinitely many solutions to the free boundary value problem we shall deduce that M is the union of a disjointed S^1-family of embedded discs where each disc corresponds to a solution of our free boundary value problem. Then we shall show that the unit normals to these discs form an analytic vector field on M. By means of this vector field we shall finally construct the diffeomorphism from $S^1 \times D$ onto M stipulated in our theorem.

We denote by $\mathscr{D}$ the set of all differentiably embedded discs in M and define a metric d on $\mathscr{D}$ by

$$d(\Delta_1, \Delta_2) = \inf \{ \| f_1 - f_2 \|_0 \mid f_k : D \to \Delta_k \text{ is a diffeomorphism}; k = 1, 2 \},$$

where $\| \ \|_0$ denotes the C^0-norm.

Let $\mathscr{D}_{\min}$ be the subset of $\mathscr{D}$ consisting of all discs $f(D)$ where f is a minimizing solution of our free boundary value problem. It follows from Proposition 1 that $\mathscr{D}_{\min}$ is compact with respect to the topology just introduced. Let us now assume that $\mathscr{D}_{\min}$ is not finite. Then $\mathscr{D}_{\min}$ possesses some accumulation point $\Delta_0 = f_0(D)$. Let $\mathscr{D}_0$ denote the connected component of Δ_0 in $\mathscr{D}_{\min}$. Obviously all discs in $\mathscr{D}_0$ are then accumulation points. If $\Delta_n \to \Delta$ in $\mathscr{D}_{\min}$ and $\Delta_n = f_n(D)$ where f_n are corresponding solutions of our minimization problem satisfying the normalization condition (4) then it follows from Proposition 1 that, at least for a subsequence (n_k),

$$f_{n_k} \rightarrow f \quad \text{in } C^{2,\alpha}(D)$$

and consequently $f(D) = \Delta$. We conclude therefore from Proposition 2 that locally $\mathscr{D}_0$ is given by analytic one-parameter families of embeddings which make $\mathscr{D}_0$ a one-dimensional differentiable manifold. $\mathscr{D}_0$ is also compact as a component of the compact space $\mathscr{D}_{\min}$ and therefore, by the classification of 1-manifolds, $\mathscr{D}_0$ is homeomorphic to S^1. Consider now the set

$$M_0 = \bigcup \Delta$$

which is obviously a compact subset of M. Since the families $F = F(t)$ representing $\mathscr{D}_0$ locally have the property (cf. Proposition 2) that $F'(0)$ is a non-zero normal vector field along the disc $F(0)$ it follows from the inverse mapping theorem that the map $(t, z) \mapsto F(t)(z)$ is a diffeomorphism from $]-\delta, \delta[\, \times D$ onto a neighborhood of $F(0)(D)$. Therefore M_0 is also open in M and we may conclude that $M_0 = M$. It follows then that $\mathscr{D}_0 = \mathscr{D}_{\min}$ since by the results of Meeks and Yau [7] every disc in $\mathscr{D}_{\min} \backslash \mathscr{D}_0$ had to be disjoint from all discs in $\mathscr{D}_0$. We now orient the local families $F = F(t)$ of Proposition 2 by requiring that they form an oriented atlas of $\mathscr{D}_{\min} \cong S^1$.

Then to each such oriented family $F = F(t)$ there corresponds a unique analytic family of vector fields $N = N(t)$ such that $N(t)$ is a unit normal for $F(t)$ and $N(0) = F'(0)$. We may now define a global vector field on M by setting

$$X(p) = N(t)(z) \quad \text{for} \quad p = F(t)(z).$$

This definition is unambiguous since by the results of Meeks and Yau [7] there is for given $p \in M$ at most one disc $\Delta \in \mathscr{D}_{\min}$ containing p and for $p \in \Delta = F(t)(D)$ exactly one $z \in D$ such that $p = F(t)(z)$. In particular, $X \mid \partial M$ is a nowhere vanishing tangential field on ∂M. Since ∂M must be connected as the union of the boundaries of the discs in $\mathscr{D}_{\min}$ it follows already from the Poincaré index theorem [1] and the classification of compact surfaces [9] that ∂M is homeomorphic to a torus and hence M homeomorphic to a solid torus. In order to prove the more precise statement in our theorem we want to show that the vector field X defined above possesses a closed orbit which intersects each disc $\Delta \in \mathscr{D}_{\min}$ exactly once. For this purpose we fix a disc $\Delta_0 \in \mathscr{D}_{\min}$ and cut M along Δ_0, considering the two sides of Δ_0, denoted by Δ_0^+ and Δ_0^-, as bottom and top of a cylinder. We claim that any flow line of the vector field X starting from Δ_0^+ at time $t = 0$ reaches every disc $\Delta \in \mathscr{D}_{\min}$ different from Δ_0^+ after finite positive time. Since the vector field is always transversal to each disc in $\mathscr{D}_{\min}$ it is obvious that the set of all those discs having the required property is open. Using the local charts for $\mathscr{D}_{\min}$ given in Proposition 2 it is not difficult to see that this set is also closed. Hence our last claim is proved and therefore any flow line starting from Δ_0^+ in particular reaches Δ_0^- after finite time. This however says that the Poincaré map of Δ_0 is well defined, which assigns to each point $p \in \Delta_0$ that point of Δ_0 to which the flow line issueing from p at time $t = 0$ returns for the first time. Being continuous this map must have a fixed point which establishes the existence of a closed orbit γ of X intersecting Δ_0 exactly once. Since γ is transversal to all discs $\Delta \in \mathscr{D}_{\min}$ it intersects all these discs exactly once. Now we are able to construct our distinguished diffeomorphism from $S^1 \times D$ onto M which maps each copy of D onto a solution of our free boundary value problem. Let us first assume that the closed orbit γ is contained in the interior of M and let $\gamma = \gamma(\xi)$, $\xi \in S^1$, be an analytic parametrization of γ. Furthermore we choose a unit vector field Y along γ which is everywhere orthogonal to γ. It is then easily seen that the conditions

$$f(0) = \gamma(\xi), \quad \frac{\partial f}{dx^1}(0) = \left| \frac{\partial f}{\partial x^1}(0) \right| Y(\xi)$$

determine a conformal solution $f = f_\xi$ of our free boundary value problem uniquely. We now define a mapping $F : S^1 \times D \to M$ by $F(\xi, z) = f_\xi(z)$. Using Proposition 2 it is easily verified that F is an analytic diffeomorphism. Let us finally consider the case that the closed orbit is not contained in $\overset{\circ}{M}$ and therefore contained in ∂M.

We choose two analytic parallel curves γ^+, γ^- on ∂M at distance ε from γ. For sufficiently small ε each of the two curves is transversal to all boundaries $\partial\Delta$, $\Delta \in \mathscr{D}_{\min}$, and intersects each such boundary exactly once. We then define our diffeomorphism F by

$$F(\xi, z) = f_\xi(z), \quad \xi \in S^1, \quad z \in D,$$

where f_ξ is the uniquely determined solution of our free boundary value problem satisfying the conditions

$$f(1) = \gamma(\xi), \quad f(-i) \in \gamma^-, \quad f(i) \in \gamma^+ .$$

Thus the theorem is completely proved.

BIBLIOGRAPHY

[1] P. ALEXANDROFF, H. HOPF, *Topologie. Erster Band.*, Springer, Berlin, 1935.
[2] W. BLASCHKE, *Vorlesungen über Differentialgeometrie. I. Elementare Differentialgeometrie.*, Springer, Berlin, 1945.
[3] R. COURANT, *Dirichlet's principle, conformal mapping and minimal surfaces.* Interscience Publ., New York, 1950.
[4] S. HILDEBRANDT, Randwertprobleme für Flächen mit vorgeschriebener mittlerer

Krümmung und Anwendungen auf die Kapillaritätstheorie. II. Freie Ränder. *Arch. Rat. Mech. Analysis*, t. **39**, 1970, p. 275-293.

[*5*] S. HILDEBRANDT, Ein einfacher Beweis für die Regularität der Lösungen gewisser zweidimensionaler Variationsprobleme unter freien Randbedingungen. *Math. Ann.*, t. **194**, 1971, p. 316-331.

[*6*] W. JÄGER, Behavior of minimal surfaces with free boundaries. *Comm. Pure Appl. Math.*, t. **23**, 1970, p. 803-818.

[*7*] W. H. MEEKS, S. T. YAU, Topology of three dimensional manifolds and the embedding problems in minimal surface theory. *Ann. of Math.*, t. **112**, 1980, p. 441-484.

[*8*] J. C. C. NITSCHE, *Vorlesungen über Minimalflächen*. Springer, Berlin-Heidelberg-New York, 1975.

[*9*] H. W. SEIFERT, Threllfall, *Lehrbuch der Topologie*. Chelsea Publ. Co., New York.

[*10*] F. TOMI, On the local uniqueness of the problem of least area. *Archive Rat. Mech. Analysis*, t. **52**, 1973, p. 312-318.

(Manuscrit reçu le 6 mars 1985)

(révisé le 20 février 1986)