A MORSE LEMMA AT INFINITY FOR YAMABE TYPE PROBLEMS ON DOMAINS

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ABSTRACT. – In this paper we consider the following nonlinear elliptic problem (P): $-\Delta u = u^p$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded and smooth domain in $\mathbb{R}^n$, $n \geq 4$, $p + 1 = 2n/(n-2)$ is the critical Sobolev exponent. We prove a version of Morse lemmas at infinity for this problem. As application of these lemmas we will give a characterization of the critical points at infinity of the functional corresponding to (P).

RÉSUMÉ. – Dans cet article nous considérons le problème elliptique non linéaire (P) : $-\Delta u = u^p$, $u > 0$ dans Ω, $u = 0$ sur $\partial\Omega$, où Ω est un domaine borné et régulier de $\mathbb{R}^n$, $n \geq 4$ et $p + 1 = 2n/(n-2)$ est l'exposent critique de Sobolev. Nous prouvons une version des lemmes de Morse à l'infini pour le problème (P) et nous appliquerons ces lemmes afin de donner une caractérisation des points critiques à l'infini de la fonctionnelle correspondante au problème (P).

1. Introduction and the main results

We prove in this paper a version of a Morse Lemma at infinity for Yamabe-type problems on domains in R^n, $n \geq 4$. These problems are related to the study of the equation

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}}, \ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

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where $\Omega \subset R^n$, $n \geqslant 4$, Ω is a bounded and regular set. This problem has a variational structure. The related functional is

$$J(u) = \left(\int_\Omega |u|^{\frac{2n}{n-2}} \right)^{-\frac{n-2}{n}}$$

defined on

$$\Sigma = \left\{ u \in H_0^1(\Omega) \mid \int_\Omega |\nabla u|^2 = 1 \right\}.$$

The problem (P) is delicate from a variational viewpoint because the functional J does not satisfy the Palais–Smale condition (P.S. for short). This means that there exist sequences along which J is bounded, its gradient goes to zero and which do not converge. The P.S. condition fails for J on

$$\Sigma^+ = \{ u \in \Sigma \text{ s.t. } u \geqslant 0 \}.$$

Its failure has been analyzed throughout the works of [13,14,16]. The analysis carried out in [11] and [15] comes out here virtually without any change. These various studies have led to the characterization of the sequences failing the Palais–Smale condition. In order to describe this characterization, we need to introduce some notations.
Let, for $a \in \Omega$ and $\lambda > 0$ given

$$\delta_{(a,\lambda)}(x) = c_0 \left(\frac{\lambda}{1 + \lambda^2 |x - a|^2} \right)^{\frac{(n-2)}{2}} \tag{1.1}$$

c_0 is chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the Yamabe problem on R^n. Let P be the projection from $H^1(\Omega)$ onto $H_0^1(\Omega)$ i.e. $u = Pf$ is the solution of

$$\Delta u = \Delta f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Let, for $\varepsilon > 0$, $p \in N^*$ and w either a solution of (P) or zero

$$V(p, \varepsilon, w) = \left\{ u \in \Sigma \text{ s.t. } \exists (a_1, \ldots, a_p) \in \Omega^p, \ \exists (\lambda_1, \ldots, \lambda_p) \in (]\varepsilon^{-1}, +\infty[)^p, \right.$$

$$\exists (\alpha_0, \alpha_1, \ldots, \alpha_p) \in (]0, +\infty[)^{p+1} \text{ s.t. } \lambda_i d(a_i, \partial\Omega) > \varepsilon^{-1},$$

$$\left. \left| u - \alpha_0 w - \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \right|_{H_0^1} < \varepsilon, \ \left| \frac{\alpha_i}{\alpha_j} - 1 \right| < \varepsilon, \varepsilon_{ij} < \varepsilon \right\} \tag{1.2}$$

where, for $i \neq j$,

$$\varepsilon_{ij}^{-1} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{\frac{n-2}{2}}.$$

The failure of Palais–Smale condition can be described as follows.

PROPOSITION 1.1. – *Let $(u_k) \in \Sigma^+$ be a sequence such that $(\partial J(u_k))$ tends to zero and $(J(u_k))$ is bounded. Then, after possibly having extracted a subsequence, there exists $p \in N^*$, a sequence (ε_k), ε_k tends to zero, and w (either a solution of* (P) *or zero) such that $u_k \in V(p, \varepsilon_k, w)$.*

Thinking of these sequences which do not satisfy the (P.S.) condition as "critical points", a natural idea is to try to find suitable parameters in order to complete a Morse Lemma at infinity. For manifolds without boundary, this program has been completed in [4] and [9]. We would here extend the proof of the existence of a Morse Lemma at infinity to the case of Dirichlet boundary conditions. We introduce the following parameterization of the set $V(p, \varepsilon, w)$ (where w is a solution of (P)). We denote by $W_u(w)$ the unstable manifold of w for a decreasing pseudogradient of J. If a function u belongs to $V(p, \varepsilon, w)$ then, for $\varepsilon > 0$ small enough, the minimization problem

$$\min\left\{ \left\| u - \alpha_0(w + h) - \sum_{i=1}^{p} \alpha_i P\delta(a_i, \lambda_i) \right\|_{H_0^1}, \ \alpha_i > 0, \ a_i \in \Omega, \ \lambda_i > 0, \right.$$
$$\left. h \in T_w\big(W_u(w)\big) \right\} \tag{1.3}$$

has an unique solution, up to permutation (see [4–6]).

Therefore, for $\varepsilon > 0$ sufficiently small, any u in $V(p, \varepsilon, w)$ can be uniquely written as

$$u = \alpha_0(w + h) + \sum_{i=1}^{p} \alpha_i P\delta(a_i, \lambda_i) + v$$

where v satisfies the following conditions

$$(V_0) \quad \begin{cases} (v, P\delta(a_i, \lambda_i))_{H_0^1(\Omega)} = (v, \frac{\partial}{\partial \lambda_i} P\delta(a_i, \lambda_i))_{H_0^1(\Omega)} = 0, \\ (v, \frac{\partial}{\partial a_i} P\delta(a_i, \lambda_i)) = (v, w)_{H_0^1(\Omega)} = (v, h)_{H_0^1(\Omega)} = 0, \end{cases} \tag{1.4}$$

and the α_i's satisfy

$$\frac{\alpha_i}{\alpha_j} \in [1 - \varepsilon, 1 + \varepsilon] \quad \forall i, \ \forall j.$$

We denote by G the Green's function of the Laplacian with Dirichlet boundary condition on Ω and by H its regular part i.e.

$$G(x, y) = |x - y|^{2-n} - H(x, y) \quad \text{for } (x, y) \in \Omega^2,$$

$$\Delta_x H = 0 \quad \text{in } \Omega^2, \qquad G = 0 \quad \text{on } \partial(\Omega^2).$$

For $q \in N^*$, and $x = (x_1, \ldots, x_q) \in \Omega^q$, such that $x_i \neq x_j \ \forall i \neq j$, we denote by $M(x) = (m_{ij})_{1 \leqslant i, j \leqslant q}$ the matrix defined by

$$m_{ii} = H(x_i, x_i), \qquad m_{ij} = -G(x_i, x_j) \quad \text{for } i \neq j, \tag{1.5}$$

by $\rho(x)$ its least eigenvalue and by $e(x)$ the eigenvector corresponding to $\rho(x)$ whose norm is 1 and whose components are strictly positive (see [8] and [9]).

In this paper we assume that zero is a regular value of ρ for each $q \leqslant p$ (such assumption is true if, for example, Ω is a thin or a large annuli (see [1,2])). Our main results are the following Morse lemmas at infinity.

THEOREM 1.2. – *For $\varepsilon > 0$ sufficiently small given, there exists a change of variables, such that for any $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i, \lambda_i)} + v$ belongs to $V(p, \varepsilon)$, $(a_i, \lambda_i, v) \to (a_i', \lambda_i', V)$ where V belongs to a neighborhood of zero in a fixed Hilbert space so that*

$$J\left(\sum_{i=1}^{p} \alpha_i P\delta_{(a_i, \lambda_i)} + v\right) = J\left(\sum_{i=1}^{p} \alpha_i P\delta_{(a_i', \lambda_i')}\right) + |V|^2.$$

Furthermore, if each a_i belongs to a neighborhood of x_i such that $\rho(X) > 0$ and $\rho'(X) = 0$, where $X = (x_1, \ldots, x_p)$, then we can find another change of variables $(a_i, \lambda_i) \to (a_i', \lambda_i')$ such that, for η a fixed small constant

$$J\left(\sum_{i=1}^{p} \alpha_i P\delta_{(a_i, \lambda_i)}\right) = \Psi(\alpha, a', \lambda')$$

$$:= \frac{(S_n)^{2/n} \sum_{i=1}^{p} \alpha_i^2}{(\sum_{i=1}^{p} \alpha_i^{2n/(n-2)})^{(n-2)/n}} \left(1 + \left(\frac{c_1'}{pS_n} - \eta\right)\rho(a')\sum_{i=1}^{p} \frac{1}{\lambda_i'^{n-2}}\right)$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$, $a' = (a_1', \ldots, a_p')$, $\lambda' = (\lambda_1', \ldots, \lambda_p')$ and c_1' is a positive constant and where $S_n = \int_{\mathbb{R}^n} \delta^{2n/(n-2)}$.

THEOREM 1.3. – *For $\varepsilon > 0$ sufficiently small given, there exists a change of variables, such that for any $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i, \lambda_i)} + \alpha_0(w + h) + v$ belongs to $V(p, \varepsilon, w)$, $(a_i, \lambda_i, h, v) \to (a_i', \lambda_i', H, V)$ where each of H and V belongs to a neighborhood of zero in a fixed Hilbert space so that*

$$J\left(\sum_{i=1}^{p} \alpha_i P\delta_{(a_i, \lambda_i)} + \alpha_0(w + h) + v\right) = J\left(\sum_{i=1}^{p} \alpha_i P\delta_{(a_i', \lambda_i')} + \alpha_0 w\right) + |V|^2 - |H|^2.$$

The proof of these theorems is of course quite difficult and extremely technical. In principle, it relies on the construction of a suitable pseudogradient Z at infinity, as in [4,5,9], which in turn relies on very delicate expansion of J and ∂J near infinity.

This construction is even more difficult when there is a boundary, because the distance to the boundary appears (in the denominator) in the expansion (see Section 2). We need to complete much more delicate and careful expansion.

However, these Theorems are useful. They should be useful for the study of the existence of multiple solutions to (P). At this point, we will illustrate their usefulness through the following three results.

PROPOSITION 1.4. – *Assume that zero is a regular value of ρ. There are no critical points at infinity and no critical points in a neighborhood of $V(p, \varepsilon, w)$ for $w \neq 0$ i.e. we can define a pseudogradient which satisfies the (P.S.) condition on decreasing flow lines and has no asymptotes in this neighborhood.*

PROPOSITION 1.5. – *Assume that zero is a regular value of ρ. Then we have*:

(i) *For ε small enough, J does not have any critical point in $V(p, \varepsilon)$.*

(ii) *The only critical points at infinity of J correspond to $\sum_{i=1}^{p} \delta_{(x_i,\infty)}$ where $p \in N^*$ and the x_i's satisfy*

$$\rho(x_1, \ldots, x_p) > 0 \quad and \quad \rho'(x_1, \ldots, x_p) = 0.$$

(iii) *There is $p_0 \in N^*$ such that, above $(p_0 S_n)^{2/n}$, J does not have any critical point at infinity.*

We also give a new proof, based on these theorems, of the formula for the difference of topology between the sets W_p and W_{p-1} where

$$W_p = \{u \in \Sigma^+ \text{ s.t. } J(u) < ((p+1)S_n)^{2/n}\}$$

due to $V(p, \varepsilon)$. This formula was stated in [6] and a proof was provided in [8]. Namely, we prove

PROPOSITION 1.6. – *We assume that J has no critical point in Σ^+*

$$H_*(W_p, W_{p-1}) = H_*\big(\Omega^p \times_{\sigma_p} \Delta_{p-1}, \ \Omega^p \times \partial \Delta_{p-1} \cup_{\sigma_p} I_p \times \Delta_{p-1}\big)$$

where $I_p = \{x \in \Omega^p \text{ s.t.} \rho(x) \leqslant 0\}$, $\Delta_{p-1} = \{(\alpha_1, \ldots, \alpha_p) \text{ s.t. } \alpha_i \geqslant 0, \ \sum \alpha_i = 1\}$ and σ_p is the permutation group.

The remainder of the present paper is organized as follow. Section 2 will be devoted to the expansion of J and its gradient. In Section 3 we will study the v-part of u. In Section 4 we will construct a suitable pseudogradient and then we will prove Theorem 1.2, Proposition 1.5 and Proposition 1.6. The proof of Theorem 1.3 and Proposition 1.4 are given in Section 5. Lastly, the proofs require some technical estimates which may be found in appendices.

2. The expansion of the functional and its gradient

In this section, we will give the expansion of $J(\sum_{i=1}^{p} \alpha_i P\delta_i + v)$, $(\partial J(\sum_{i=1}^{p} \alpha_i P\delta_i), \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i})$, $(\partial J(\sum_{i=1}^{p} \alpha_i P\delta_i), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i})$.

PROPOSITION 2.1. – *For $p \in N^*$, $\varepsilon > 0$, small and $u = \sum_{i=1}^{p} \alpha_i P\delta_i + v \in V(p, \varepsilon)$, we have the following expansion*

$$
\begin{aligned}
J(u) = {} & \frac{(\sum_{i=1}^{p} \alpha_i^2) S_n^{2/n}}{\big(\sum_{i=1}^{p} \alpha_i^{2n/(n-2)}\big)^{(n-2)/n}} \Bigg[1 + \frac{c_1'}{S_n} \sum_{i=1}^{p} \frac{H(a_i, a_i)}{\lambda_i^{n-2}} \bigg(\frac{2\alpha_i^{2n/(n-2)}}{\sum \alpha_j^{2n/(n-2)}} - \frac{\alpha_i^2}{\sum \alpha_j^2} \bigg) \\
& - \frac{c_1'}{S_n} \sum_{i \neq j} \bigg(\varepsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \bigg) \bigg(\frac{2\alpha_i^{(n+2)/(n-2)} \alpha_j}{\sum \alpha_j^{2n/(n-2)}} - \frac{\alpha_i \alpha_j}{\sum \alpha_k^2} \bigg) + \frac{1}{\sum \alpha_j^2 S_n} Q(v, v) \\
& - (f, v) + O\bigg(\sum_{k=1}^{p} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) \bigg) + O\big(|v|^{\inf(\frac{2n}{n-2}, 3)}\big) \Bigg]
\end{aligned}
$$

where c_1' is a positive constant,

$$Q(v, v) = |v|_{H_0^1}^2 - \frac{n+2}{n-2} \frac{\sum \alpha_k^2}{\sum \alpha_k^{2n/(n-2)}} \int \left(\sum \alpha_i P\delta_i\right)^{\frac{4}{n-2}} v^2$$

and

$$(f, v) = 2\left(\sum \alpha_j^{\frac{2n}{n-2}} S_n\right)^{-1} \int \left(\sum \alpha_i P\delta_i\right)^{\frac{n+2}{n-2}} v.$$

Proof. –

$$J(u) = \frac{|\sum \alpha_i P\delta_i + v|^2}{\left(\int (\sum \alpha_i P\delta_i + v)^{2n/(n-2)}\right)^{(n-2)/n}} = \frac{N}{D^{(n-2)/n}},$$

$$N = \left|\sum \alpha_i P\delta_i + v\right|^2 = \sum \alpha_i^2 |P\delta_i|^2 + |v|^2 + \sum_{i \neq j} \alpha_i \alpha_j (P\delta_i, P\delta_j).$$

From Lemmas A.1 and A.3 (in Appendix A), we obtain

$$N = \sum \alpha_i^2 \left(S_n - c_1' \frac{H(a_i, a_i)}{\lambda_i^{n-2}}\right) + \sum_{i \neq j} \alpha_i \alpha_j c_1' \left(\varepsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}}\right)$$
$$+ |v|^2 + R_2, \tag{2.1}$$

where R_2 satisfies

$$R_2 = O\left(\sum_k \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1})\right). \tag{2.2}$$

We are left for D. Observe that, for $q = 2n/(n-2)$, we have

$$D = \int \left(\sum \alpha_i P\delta_i\right)^q + q \int \left(\sum \alpha_i P\delta_i\right)^{q-1} v$$
$$+ \frac{q(q-1)}{2} \int \left(\sum \alpha_i P\delta_i\right)^{q-2} v^2 + O(|v|^{\inf(3,q)}).$$

We have also

$$\int \left(\sum_{i=1}^p \alpha_i P\delta_i\right)^q = \sum_{i=1}^p \alpha_i^q \int P\delta_i^q + q \sum_{i \neq j} \alpha_i^{q-1} \alpha_j \int P\delta_i^{q-1} P\delta_j$$
$$+ O\left(\sum_{i \neq j} \int \sup(\alpha_i P\delta_i, \alpha_j P\delta_j)^{\frac{4}{n-2}} \inf(\alpha_i P\delta_i, \alpha_j P\delta_j)^2\right).$$

Using Lemmas A.2, A.3 and A.4, and the fact that $4/(n-2) \leqslant 2$, we get

$$D = \sum \alpha_i^q \left(S_n - c_1' q \frac{H(a_i, a_i)}{\lambda_i^{n-2}}\right) + q c_1' \sum_{i \neq j} \alpha_i^{q-1} \alpha_j \left(\varepsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}}\right)$$
$$+ q \int \left(\sum \alpha_i P\delta_i\right)^{q-1} v + \frac{q(q-1)}{2} \int \left(\sum \alpha_i P\delta_i\right)^{q-2} v^2$$
$$+ R_2 + O(|v|^{\inf(3,q)}). \tag{2.3}$$

Combining (2.1) and (2.3), the proof of Proposition 2.1 follows.   □

PROPOSITION 2.2. – *Let* $n \geqslant 4$. *For* $u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p, \varepsilon)$, *we have the following expansion*

$$\left(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right) = 2J(u)c_1'\left[-\frac{n-2}{2}\alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}}(1 + o(1))\right.$$
$$\left. - \sum_{j \neq i} \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2}\frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}}\right)(1 + o(1)) + R_2\right],$$

where R_2 *is defined in* (2.2).

Proof. – We have

$$\partial J(u) = 2J(u)\left[u + J(u)^{\frac{n}{n-2}}\Delta^{-1}\left(u^{\frac{n+2}{n-2}}\right)\right]. \tag{2.4}$$

Thus

$$\left(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right) = 2J(u)\left[\sum \alpha_j \left(P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right)\right.$$
$$\left. - J(u)^{\frac{n}{n-2}}\int \left(\sum \alpha_j P\delta_j\right)^{\frac{n+2}{n-2}}\lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right]. \tag{2.5}$$

Observe that, since $n \geqslant 4$

$$\left(\sum \alpha_j P\delta_j\right)^{\frac{n+2}{n-2}} = \sum (\alpha_j P\delta_j)^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\sum_{i \neq j}(\alpha_i P\delta_i)^{\frac{4}{n-2}}\alpha_j P\delta_j$$
$$+ O\left[\sum_{i \neq j}(\alpha_i P\delta_i)^{\frac{2}{n-2}}(\alpha_j P\delta_j)^{\frac{n}{n-2}}\right]$$
$$+ O\left[\sum_{k \neq j;\ k, j \neq i}(\alpha_j P\delta_j)^{\frac{4}{n-2}}(\alpha_k P\delta_k)\right]. \tag{2.6}$$

Combining (2.6), (2.5), Lemmas A.5–A.9 and the following facts: $|\lambda_i \partial P\delta_i/\partial \lambda_i| \leqslant c\delta_i$, $P\delta_k \leqslant \delta_k$ and $J(u)^{n/(n-2)}\alpha_j^{4/(n-2)} = 1 + o(1)$, $\forall j = 1, \ldots, p$, Proposition 2.5 follows.   □

PROPOSITION 2.3. – *Let* $n \geqslant 4$. *For* $u = \sum_{i=1}^{p} \alpha_i P\delta_i$ *belonging to* $V(p, \varepsilon)$, *we have the following expansion*

$$\left(\partial J(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right) = J(u)c_1'\left[\frac{\alpha_i}{\lambda_i^{n-1}}\frac{\partial H(a_i, a_i)}{\partial a_i}(1 + o(1))\right.$$
$$+ 2\sum_{j \neq i}\alpha_j \left(\frac{1}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{\frac{\partial H(a_i, a_j)}{\partial a_i}}{\lambda_i(\lambda_i \lambda_j)^{(n-2)/2}}\right)$$
$$\left.\times \left(1 - J(u)^{\frac{n}{n-2}}\left(\alpha_j^{\frac{4}{n-2}} + \alpha_i^{\frac{4}{n-2}}\right)\right)\right] + R_3,$$

where R_3 *satisfies*

$$R_3 = \mathrm{O}\left(\sum_k \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \sum_{k \neq j} \varepsilon_{kj}^{\frac{n}{n-2}} \log(\varepsilon_{kj}^{-1}) + \sum_{j \neq i} \lambda_j |a_i - a_j| \varepsilon_{ij}^{\frac{n+1}{n-2}} \right). \qquad (2.7)$$

As in the proof of Proposition 2.2, we get (2.5) but with $\lambda_i \partial P\delta_i / \partial\lambda_i$ changed by $\lambda_i^{-1} \partial P\delta_i / \partial a_i$. Thus, using Lemmas A.10–A.14, the proposition follows.

PROPOSITION 2.4. – *Let $n \geqslant 4$, for $u = \sum_{i=1}^p \alpha_i P\delta_i + \alpha_0(w+h)$ in $V(p, \varepsilon, w)$, we have*

$$(-\partial J(u), h) \geqslant c|h|^2 + \mathrm{O}\left(\sum \lambda_i^{2-n} \right).$$

Proof. – Let $u = \sum \alpha_i P\delta_i + \alpha_0(w+h)$, using (2.4) and the fact that

$$\left(\sum \alpha_i P\delta_i + \alpha_0(w+h) \right)^{\frac{n+2}{n-2}} = \alpha_0^{\frac{n+2}{n-2}} \left(w^{\frac{n+2}{n-2}} + \frac{n+2}{n-2} w^{\frac{4}{n-2}} h \right) + \mathrm{O}(|h|^{\frac{n+2}{n-2}}) \qquad (2.8)$$

$$+ \mathrm{O}\left(|h|^{\inf(\frac{n+2}{n-2}, 2)} + \sum (w^{\frac{4}{n-2}} + |h|^{\frac{4}{n-2}}) P\delta_i + P\delta_i^{\frac{n+2}{n-2}} \right)$$

we need to estimate

$$\int \delta_i^{\frac{n+2}{n-2}} |h| \leqslant |h|_{L^\infty} \int \delta_i^{\frac{n+2}{n-2}} = \mathrm{O}(\lambda_i^{-\frac{n-2}{2}} |h|_{L^\infty}) = \mathrm{o}(|h|_{L^\infty}^2) + \mathrm{O}\left(\frac{1}{\lambda_i^{n-2}} \right), \qquad (2.9)$$

$$\int_\Omega \delta_i |h| \leqslant |h|_{L^\infty} \int_B \delta_i = \mathrm{O}\left(|h|_{L^\infty} \lambda_i^{\frac{2-n}{2}} \right) = \mathrm{o}(|h|_{L^\infty}^2) + \mathrm{O}\left(\frac{1}{\lambda_i^{n-2}} \right), \qquad (2.10)$$

where $\Omega \subset B = B(a_i, R)$.

The function h belongs to $T_w(W_u(w))$ which has a dimension equal to the index of w. Thus

$$|h|_{L^\infty} = \mathrm{O}(|h|_{H_0^1}); \qquad \int w^{\frac{n+2}{n-2}} h = (w, h) = 0. \qquad (2.11)$$

Therefore

$$(\partial J(u), h) = 2J(u) \left[\alpha_0 |h|_{H_0^1}^2 - \frac{n+2}{n-2} J(u)^{\frac{n}{n-2}} \alpha_0^{\frac{n+2}{n-2}} \int w^{\frac{4}{n-2}} h^2 \right.$$

$$\left. + \mathrm{o}(|h|_{H_0^1}^2) + \mathrm{O}\left(\sum \frac{1}{\lambda_i^{n-2}} \right) \right].$$

Since $u = \sum \alpha_i P\delta_i + \alpha_0(w+h) \in V(p, \varepsilon, w)$, we have

$$J(u)^{\frac{n}{n-2}} \alpha_0^{\frac{4}{n-2}} = 1 + \mathrm{o}(1), \qquad |h|^2 - \frac{n+2}{n-2} \int w^{\frac{4}{n-2}} h^2 \leqslant -\beta_0 |h|^2.$$

Thus the result follows. $\square$

PROPOSITION 2.5. – *Let $n \geqslant 4$, for $u = \sum_{i=1}^p \alpha_i P\delta_i + \alpha_0(w+h) \in V(p, \varepsilon, w)$ we have*

$$\left(\partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial\lambda_i} \right)$$

$$= 2J(u)c_1' \left[\frac{n-2}{2} \alpha_0 \frac{w(a_i)}{\lambda_i^{(n-2)/2}} (1 + \mathrm{o}(1)) - \frac{n-2}{2} \alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} (1 + \mathrm{o}(1)) \right.$$

$$- \sum_{j \neq i} \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right) (1 + o(1))$$

$$+ o \left(|h|^2 + \sum \frac{1}{\lambda_k^{n/2}} + \sum \varepsilon_{kr}^{\frac{n-1}{n-2}} \right) + R_2 + O \left(\frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} \right) \Bigg],$$

where R_2 is defined in (2.2).

Proof. – We have

$$\int \left(\sum \alpha_j P\delta_j + \alpha_0(w+h) \right)^{\frac{n+2}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}$$

$$= \sum_{k=1}^{p} \alpha_k^{\frac{n+2}{n-2}} \int P\delta_k^{\frac{n+2}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} + \alpha_0^{\frac{n+2}{n-2}} \int w^{\frac{n+2}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}$$

$$+ \frac{n+2}{n-2} \alpha_i^{\frac{4}{n-2}} \int \left(P\delta_i^{\frac{4}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right) \left(\sum_{j \neq i} \alpha_j P\delta_j + \alpha_0 w \right)$$

$$+ O \left(\sum_{j=1}^{p} \int \delta_j^{\frac{4}{n-2}} \delta_i |h| + \int \delta_i |h|^{\frac{n+2}{n-2}} + \int w^{\frac{4}{n-2}} |h| \delta_i \right.$$

$$+ \sum_{j \neq i} \int_{\delta_j \leqslant w} w^{\frac{4}{n-2}} \delta_j \delta_i + \int_{w \leqslant \delta_j} \delta_j^{\frac{4}{n-2}} w \delta_i + \int_{B(a_i, d_i)} \delta_i^{\frac{n}{n-2}}$$

$$+ \int_{B^c(a_i, d_i)} \delta_i^{\frac{n+2}{n-2}} + \int_{B^c(a_i, d_i)} \delta_i^2 + \sum \varepsilon_{kr}^{\frac{n}{n-2}} \log(\varepsilon_{kr}^{-1}) \Bigg).$$

Observe that

$$\int_{B(a_i, d_i)} \delta_i^{\frac{n}{n-2}} = O \left(\frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} \right),$$

$$\int_{B^c(a_i, d_i)} \delta_i^{\frac{n+2}{n-2}} \leqslant \frac{c}{\lambda_i^{(n-2)/2}} \frac{1}{(\lambda_i d_i)^2} = o \left(\frac{1}{\lambda_i^{n/2}} \right) + O \left(\frac{1}{(\lambda_i d_i)^n} \right),$$

$$\int_{B^c(a_i, d_i)} \delta_i^2 = (\text{if } n \geqslant 5) \left(o \left(\frac{1}{\lambda_i^{n/2}} \right) + O \left(\frac{1}{(\lambda_i d_i)^n} \right) \right) + (\text{if } n = 4) o \left(\frac{d_i}{\lambda_i} + \frac{1}{(\lambda_i d_i)^2} \right).$$

We also have, for $n \geqslant 5$, using Holder's inequality

$$\int \delta_j \delta_i \leqslant \left(\varepsilon_{ij}^{\frac{n-1}{n-2}} (\log \varepsilon_{ij}^{-1})^{\frac{n-1}{n}} \right)^{\frac{n-4}{n-1}} \left(\frac{\log(\lambda_j)}{\lambda_j^{n/2}} + \frac{\log(\lambda_i)}{\lambda_i^{n/2}} \right)^{\frac{4}{n}}$$

$$= o \left(\varepsilon_{ij}^{\frac{n-1}{n-2}} + \lambda_j^{-\frac{n}{2}} + \lambda_i^{-n/2} \right) \tag{2.12}$$

(since $(n-4)/(n-1) + 4/n > 1$). For $n = 4$,

$$\int \delta_i \delta_j \leqslant \frac{1}{\lambda_i} (\log \lambda_i)^{1/2} \frac{1}{\lambda_j} (\log \lambda_j)^{1/2} = o \left(\frac{d_i}{\lambda_i} + \frac{1}{(\lambda_i d_i)^2} + \frac{1}{\lambda_j^2} \right).$$

Observe also, using Holder's inequality, we obtain

$$
\int\limits_{w\leqslant\delta_j} \delta_j^{\frac{4}{n-2}} w\delta_i \leqslant c\int \delta_j^{\frac{n}{n-2}}\delta_i \leqslant \left(\varepsilon_{ij}^{\frac{n-1}{n-2}}\left(\log\varepsilon_{ij}^{-1}\right)^{\frac{n-1}{n}}\right)^{\frac{n-2}{n-1}}\left(\frac{\log(\lambda_j)}{\lambda_j^{n/2}}\right)^{\frac{2}{n}} = o\left(\varepsilon_{ij}^{\frac{n-1}{n-2}}+\lambda_j^{-\frac{n}{2}}\right)
$$

(since $(n-2)/(n-1)+2/n > 1$ for $n\geqslant 4$). Using the Lemmas A.5–A.9, A.15, A.16, and the fact that $J(u)^{n/4}\alpha_i = 1 + o(1)$ for each $i=0,\dots,p$, the result follows.   □

PROPOSITION 2.6. – *For* $n\geqslant 4$ *and* $u = \sum_{i=1}^p \alpha_i P\delta_i + \alpha_0(w+h)$ *belonging to* $V(p,\varepsilon,w)$, *we have*

$$
\left(\partial J(u),\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right) = J(u)c_1'\left[\left(\frac{\alpha_i}{\lambda_i^{n-1}}\frac{\partial H(a_i,a_i)}{\partial a_i}-\frac{\alpha_0}{\lambda_i^{n/2}}Dw(a_i)\right)(1+o(1))\right.
$$

$$
+ o\left(\sum\frac{1}{\lambda_k^{n/2}}\right) + 2\sum_{j\neq i}\alpha_j\left(\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}-\frac{\frac{\partial H(a_i,a_j)}{\partial a_i}}{\lambda_i(\lambda_i\lambda_j)^{(n-2)/2}}\right)
$$

$$
\left.\times\left(1-J(u)^{\frac{n}{n-2}}\left(\alpha_j^{\frac{4}{n-2}}+\alpha_i^{\frac{4}{n-2}}\right)\right) + R_3 + o\left(|h|^2\right)\right],
$$

where R_3 *is defined in Proposition 2.3.*

Proof. –

$$
\left(\partial J(u),\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right) = 2J(u)\left[\sum_{j=1}^p\alpha_j\left(P\delta_j,\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right)+\alpha_0\left(w+h,\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right)\right.
$$

$$
- J(u)^{\frac{n}{n-2}}\left(\int\left(\sum_{j=1}^p\alpha_j P\delta_j+\alpha_0 w\right)^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right)
$$

$$
+ (\text{if } n\geqslant 5)O\left(\sum\frac{1}{\lambda_k^{n-2}}\right) + o\left(|h|^2\right)
$$

$$
\left.+ (\text{if } n=4)o\left(\frac{1}{\lambda_i^2}+\frac{1}{(\lambda_i d_i)^3}+\sum_{j\neq i}\varepsilon_{ij}^{3/2}\right)\right].
$$

We consider now the term

$$
\int\left(\sum_{j=1}^p\alpha_j P\delta_j+\alpha_0 w\right)^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}
$$

$$
= \sum_{j=1}^p\alpha_j^{\frac{n+2}{n-2}}\int P\delta_j^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}+\alpha_0^{\frac{n+2}{n-2}}\int w^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}
$$

$$
+ \frac{n+2}{n-2}\alpha_i^{\frac{4}{n-2}}\int P\delta_i^{\frac{4}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\left(\sum_{j\neq i}\alpha_j P\delta_j+\alpha_0 w\right)
$$

$$
+ O\left(\sum_{k\neq r}\varepsilon_{kr}^{\frac{n}{n-2}}\log\left(\varepsilon_{kr}^{-1}\right)+\int\limits_{P\delta_i\leqslant w} w^{\frac{4}{n-2}}\delta_i\left|\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right|+\int\limits_{w\leqslant\delta_j} w\delta_j^{\frac{4}{n-2}}\delta_i\right)
$$

$$+ \text{(if } n = 4\text{)} \left(3 \sum_{j=1}^{p} \alpha_0^2 \alpha_j \int w^2 P\delta_j \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} + O\left(\sum_{j \neq i} \int \delta_j^2 \left| \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right| \right) \right)$$

$$+ \text{(if } n \geqslant 5\text{)} O\left(\int_{w \leqslant \delta_i} \delta_i^{\frac{4}{n-2}-1} w^2 \left| \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right| + \int_{\delta_j \leqslant w} w^{\frac{4}{n-2}} \delta_j \delta_i \right.$$

$$+ \left. \int_{\delta_i \leqslant w} w \delta_i^{\frac{4}{n-2}} \left| \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right| \right).$$

Observe that, we have the following estimates

$$\int_{\delta \leqslant w} w^{\frac{4}{n-2}} \delta \left| \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} \right| \leqslant c \int_{\delta \leqslant w} \delta \left| \frac{1}{\lambda} \frac{\partial \delta}{\partial a} \right| + c \int_{\delta \leqslant w} \delta \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right|$$

$$\leqslant c \int |x-a| \delta^{\frac{n}{n-2}} \delta^{\frac{1}{n-2}} + c \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right|_{H_0^1} \left(\int_{\delta \leqslant w} \delta^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}}.$$

For $n \geqslant 6$, we have $2n/(n+2) \geqslant n/(n-2)$, thus

$$\int_{\delta \leqslant w} \delta^{\frac{2n}{n+2}} = \text{(if } n \geqslant 6\text{)} O\left(\frac{\log(\lambda)}{\lambda^{n/2}} \right) + \text{(if } n = 4,5\text{)} o\left(\frac{1}{\lambda} \right).$$

Therefore

$$\int_{\delta \leqslant w} w^{\frac{4}{n-2}} \delta \left| \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} \right| = o\left(\frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^{n-1}} \right),$$

$$\int_{w \leqslant \delta} \delta^{\frac{4}{n-2}-1} w^2 \left| \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} \right| \leqslant \int_{w \leqslant \delta} \delta^{\frac{4}{n-2}-1} w^2 \left| \frac{1}{\lambda} \frac{\partial \delta}{\partial a} \right| + \int_{w \leqslant \delta} \delta^{\frac{4}{n-2}-1} w^2 \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right|$$

$$\leqslant \int_{w \leqslant \delta} w^2 |x-a| \delta^{\frac{6}{n-2}} + \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right|_{L^\infty} \int_\Omega \delta^{\frac{n}{n-2}}$$

$$\leqslant \int |x-a| \delta^{\frac{n+1}{n-2}} + \frac{c}{(\lambda d)^{n/2} d^{(n-2)/2}} \frac{\log(\lambda)}{\lambda^{n/2}}$$

$$= o\left(\frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^{n-1}} \right).$$

By the same way,

$$\int_{\delta \leqslant w} w \delta^{\frac{4}{n-2}} \left| \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} \right| \leqslant \int |x-a| \delta^{\frac{n+1}{n-2}} + \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right|_{L^\infty}^{\frac{2}{n-2}} \frac{\log(\lambda)}{\lambda^{n/2}}$$

$$= o\left(\frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^{n-1}} \right).$$

For $n = 4$,

$$\int w^2 P\delta \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} = \int w^2 \delta \frac{1}{\lambda} \frac{\partial \delta}{\partial a} + O\left(\int \delta \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right| + \int \theta \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right| \right)$$

$$= o\left(\frac{1}{\lambda^2} + \frac{1}{(\lambda d)^3} \right)$$

and

$$\int w^2 P\delta_j \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} = \int w^2 P\delta_j \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} + O\left(\int \delta_j \left| \frac{1}{\lambda_i} \frac{\partial \theta_i}{\partial a_i} \right| \right)$$

$$= O\left(\int \delta_j |x - a_i| \delta_i^2 + \frac{1}{\lambda_j (\lambda_i d_i)^2} \right)$$

$$= o\left(\varepsilon_{ij}^{3/2} + \frac{1}{\lambda_j^2} + \frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^3} \right).$$

Using Lemmas A.10–A.14, A.17, A.18, the result follows. □

3. The v-part of u

In this section we deal with the v-part of u, in order to show that it is negligible to the concentration phenomenon.

LEMMA 3.1. – *There is a C^1-map which to each (α, a, λ) such that $\sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)}$ belongs to $V(p, \varepsilon)$ associates $\bar{v} = \bar{v}(\alpha, a, \lambda)$ satisfying*

$$J\left(\sum \alpha_i P\delta_i + \bar{v} \right) = \min\left\{ J\left(\sum \alpha_i P\delta_i + v \right), \ v \ \text{satisfies} \ (V_0) \right\}$$

and we have the following estimate

$$|\bar{v}| \leqslant \begin{cases} \sum_i \frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{(n+2)/2}} + \sum_{j \neq i} \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} (\log(\varepsilon_{ij}^{-1}))^{\frac{n+2}{2n}} & \text{if } n \geqslant 6, \\ \sum_i \frac{1}{(\lambda_i d_i)^{n-2}} + \sum_{j \neq i} \varepsilon_{ij} (\log(\varepsilon_{ij}^{-1}))^{\frac{n-2}{n}} & \text{if } n \leqslant 5. \end{cases}$$

Proof. – Since $\alpha_i / \alpha_j = 1 + o(1)$ then the quadratic form Q defined in Proposition 2.1 is close to

$$|v|_{H_0^1}^2 - \frac{n+2}{n-2} \sum \int P\delta_i^{\frac{4}{n-2}} v^2$$

and therefore it is a quadratic positive form on v (see [3]). Since $\bar{v}$ minimize J in the v-space, it is easy to check the following claim (see [3])

$$\exists \alpha > 0 \ \text{s.t.}, \quad \alpha |\bar{v}|_{H_0^1}^2 \leqslant |(f, \bar{v})| \leqslant |f| |\bar{v}|_{H_0^1}.$$

Thus, it is sufficient to estimate $|f|$ where f is defined in Proposition 2.1. We have

$$(f, v) = 2\left(\sum \alpha_j^q S_n \right)^{-1} \sum_{i=1}^p \alpha_i^{q-1} \int P\delta_i^{q-1} v + O\left(\sum_{j \neq i} \int_{P\delta_j \leqslant P\delta_i} P\delta_i^{q-2} P\delta_j |v| \right).$$

Observe that

$$\int P\delta_i^{q-1}v = \int \delta_i^{q-1}v + \mathrm{O}\left(\int_{B_i \cup B_i^c} \delta_i^{q-2}\theta_i|v| \right),$$

where $B_i = \{x, |x - a_i| < d_i\}$. Then, using the Holder's inequality, we need to estimate

$$\int_{B_i^c} (\delta_i^{q-2}\theta_i)^{\frac{2n}{n+2}} \leqslant \int_{B_i^c} \delta_i^q = \mathrm{O}\left(\frac{1}{(\lambda_i d_i)^n} \right) \tag{3.1}$$

and

$$|\theta_i|_{L^\infty}\left[\int_{B_i} \delta_i^{\frac{4q}{n+2}} \right]^{\frac{n+2}{2n}}$$

$$= \mathrm{O}\left(\frac{1}{(\lambda_i d_i)^{(n+2)/2}} + (\text{if } n = 6)\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} + (\text{if } n \leqslant 5)\frac{1}{(\lambda_i d_i)^{n-2}} \right). \tag{3.2}$$

See also

$$\int_{P\delta_j \leqslant P\delta_i} P\delta_i^{\frac{4}{n-2}} P\delta_j|v| \leqslant |v|\left[\int_{P\delta_j \leqslant P\delta_i} \left(P\delta_i^{\frac{4}{n-2}} P\delta_j \right)^{\frac{2n}{n+2}} \right]^{\frac{n+2}{2n}}. \tag{3.3}$$

If $n \geqslant 6$, we have $\frac{2n}{n+2} \geqslant \frac{n}{n-2}$ and thus

$$\int_{P\delta_j \leqslant P\delta_i} \left(P\delta_i^{\frac{4}{n-2}} P\delta_j \right)^{\frac{2n}{n+2}} \leqslant \int (P\delta_i P\delta_j)^{\frac{n}{n-2}} = \mathrm{O}\left(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) \right). \tag{3.4}$$

If $n \leqslant 5$, we have $1 < \frac{4}{n-2}$, thus

$$\int \left(P\delta_i^{\frac{4}{n-2}} P\delta_j \right)^{\frac{2n}{n+2}} \leqslant \left[\int (\delta_i \delta_j)^{\frac{n}{n-2}} \right]^{\frac{2(n-2)}{n+2}} \leqslant \varepsilon_{ij}^{\frac{2n}{n+2}} \left(\log(\varepsilon_{ij}^{-1}) \right)^{\frac{2(n-2)}{n+2}}. \tag{3.5}$$

Thus the estimate of $|f|$ follows.   □

LEMMA 3.2. – *There is a C^1-map which to each (α, a, λ, h) such that $u = \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} + \alpha_0(w + h)$ belongs to $V(p, \varepsilon, w)$ associates $\bar{v} = \bar{v}(\alpha, a, \lambda, h)$ satisfying*

$$J(u + \bar{v}) = \min\{J(u + v), \ v \text{ satisfies } (V_0)\}.$$

We have the following estimate

$$|\bar{v}| \leqslant c \begin{cases} \sum_i \left(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{(n+2)/2}} + \frac{\log(\lambda_i)}{\lambda_i^{(n+2)/4}} \right) \\ \quad + \sum_{i \neq j} \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} (\log \varepsilon_{ij}^{-1})^{\frac{n+2}{n-2}} + |h|^{\frac{n+2}{n-2}} & \text{if } n \geqslant 6, \\ \sum_i \left(\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{1}{\lambda_i^{(n-2)/2}} \right) + \sum_{i \neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{n-2}{n}} + |h|^2 & \text{if } n \leqslant 5. \end{cases}$$

The proof is similar to the proof of Lemma 3.1. So we will omit it.

4. Construction of a pseudogradient

In this section we construct a pseudogradient Z near infinity as in the Proposition A.2 of [4]. The new fact here comes from the boundary. We need a new technical point showing:

(1) how the expansion behave due to the boundary. We will show that terms of the type $O((\lambda_i d_i)^{2-n})$ are added.

(2) we can construct Z so that, on decreasing flow-lines, the minimal distance to the boundary only increases if it is small enough.

This new property requires a careful study of the behavior of the Green's function and its regular part near the boundary.

We begin by giving the following main result.

THEOREM 4.1. – *There exists a pseudogradient Z so that the following holds. There is a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i P\delta_i$ in $V(p, \varepsilon)$ so that*:

(i) $\left(-\partial J(u), Z\right) \geqslant c\left(\sum_i \dfrac{1}{(\lambda_i d_i)^{n-1}} + \sum_{k \neq r} \varepsilon_{kr}^{\frac{n-1}{n-2}}\right).$

(ii) $\left(-\partial J(u + \bar{v}), Z + \dfrac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z)\right) \geqslant c\left(\sum_i \dfrac{1}{(\lambda_i d_i)^{n-1}} + \sum_{k \neq r} \varepsilon_{kr}^{\frac{n-1}{n-2}}\right).$

(iii) $|Z|$ *is bounded.*

(iv) Z *satisfies the Palais–Smale condition away from the critical points at infinity.*

(v) *The minimal distance to the boundary only increases if it is small enough.*

(vi) *The λ_i's are bounded away from the case where $\rho(X) > 0$ and $\rho'(X) = o(1)$ where $X = (a_1, \dots, a_p)$.*

We will prove Theorem 4.1 at the end of the section. We now prove Theorem 1.2 and Proposition 1.5.

Proof of Theorem 1.2. – Using Theorem 4.1, the proof is similar to some argument in Appendix 2 [4] (see also [9]).   □

Proof of Proposition 1.5. – On the one hand, from the normal forme of J and claims (ii) and (vi) of Theorem 4.1, we immediately derive (i) and (ii) of Proposition 1.5. On the other hand, since Ω is bounded, we easily deduce claim (iii) of Proposition 1.5.   □

Next we will prove some technical results needed for the proof of Theorem 4.1.

For $u = \sum_{i=1}^p \alpha_i P\delta_i \in V(p, \varepsilon)$, we introduce the following condition: for $i \in \{1, \dots, p\}$

$$\frac{1}{2^{p+1}} \sum_{i \neq k} \varepsilon_{ki} \leqslant \sum_{j=1}^p \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}}. \tag{4.1}$$

We divide the set $\{1, \dots, p\}$ into $T_1 \cup T_2$ where

$$T_1 = \{i \text{ s.t. } i \text{ satisfies (4.1)}\},$$
$$T_2 = \{i \text{ s.t. } i \text{ does not satisfy (4.1)}\}.$$

In T_2, we order the λ_i's: $\lambda_{i_1} \leqslant \lambda_{i_2} \leqslant \cdots \leqslant \lambda_{i_s}$.

LEMMA 4.2. – *There exists a vector-field X_1 defined on the variables λ_j's, for $j \in T_2$, as follows*

$$X_1 = -\sum_{k=1}^{s} 2^k \alpha_{i_k} \lambda_{i_k} \frac{\partial P \delta_{i_k}}{\partial \lambda_{i_k}}.$$

This vector-field satisfies

$$(-\partial J(u), X_1) \geqslant c \sum_{i \in T_2} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n-2}} \right) + R_2$$

where R_2 is defined in (2.2).

Proof. – Using Proposition 2.2, we derive

$$(-\partial J(u), X_1) = 2J(u)c_1' \sum_{k=1}^{s} \left[-\sum_{j \neq i_k} 2^k \alpha_j \alpha_{i_k} \lambda_{i_k} \frac{\partial \varepsilon_{ji_k}}{\partial \lambda_{i_k}} (1 + o(1)) \right.$$
$$\left. - \frac{n-2}{2} \sum_{j=1}^{p} 2^k \alpha_j \alpha_{i_k} \frac{H(a_j, a_{i_k})}{(\lambda_j \lambda_{i_k})^{(n-2)/2}} (1 + o(1)) + R_2 \right]. \quad (4.2)$$

Observe that

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \frac{n-2}{2} \varepsilon_{ij} \left(1 - 2 \frac{\lambda_j}{\lambda_i} \varepsilon_{ij}^{\frac{2}{n-2}} \right). \quad (4.3)$$

Thus, if $\lambda_i \geqslant \lambda_j$

$$-2\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geqslant -\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \geqslant \frac{n-2}{4} \varepsilon_{ij} \quad (4.4)$$

and for $j \in T_1$ and $i \neq j$,

$$\frac{\lambda_j}{\lambda_i} \varepsilon_{ij}^{\frac{2}{n-2}} = o(1) \quad (4.5)$$

indeed, if $\frac{1}{2}d_j \leqslant d_i \leqslant 2d_j$, we use the fact that j satisfies (4.1) and $H(a_i, a_k) \leqslant (d_i d_k)^{-(n-2)/2}$ (see Lemma B.2) and in the other case we use the inequality $|a_i - a_j| \geqslant \frac{1}{2} \max(d_i, d_j)$. Thus

$$(-\partial J(u), X_1) \geqslant \frac{n-2}{4} c \sum_{i \in T_2} \left(\sum_{j \neq i} \varepsilon_{ij} - 2^p \sum_{j=1}^{p} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} + R_2 \right). \quad (4.6)$$

Since $i \in T_2$ and $H(a_i, a_i) \geqslant d_i^{2-n}$ (see Lemma B.2), the Lemma follows. □

We order all the $\lambda_i d_i$: $\lambda_1 d_1 \leqslant \lambda_2 d_2 \leqslant \cdots \leqslant \lambda_p d_p$. Let us define

$$I_1 = \{1\} \cup \{i \text{ s.t. } \forall k \leqslant i, \ c_1 \lambda_k d_k \leqslant \lambda_{k-1} d_{k-1} \leqslant \lambda_k d_k\}$$

where c_1 is a constant chosen small enough.

COROLLARY 4.3. – *If $T_2 \cap I_1 \neq \emptyset$ and $I_1 \neq \{1, -, p\}$, the vector-field X_1 satisfies the condition* (i) *of Theorem* 4.1.

Proof. – The lower-bound of the estimate of Lemma 4.2 is limited to the indices of T_2, thus if $T_2 \cap I_1 \neq \emptyset$, we can make the term $(\lambda_1 d_1)^{-(n-2)}$ appears in this lower-bound and therefore we can also make all the $(\lambda_i d_i)^{-(n-2)}$ appear in this formula. For $i \in T_1$ (i.e. i satisfies (4.1)), we have

$$\frac{1}{2^{p+1}} \sum_{j \neq i} \varepsilon_{ij} \leqslant \sum_{j=1}^{p} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \leqslant \frac{p}{(\lambda_1 d_1)^{n-2}}. \tag{4.7}$$

Therefore, from $(\lambda_1 d_1)^{-(n-2)}$, we can make the term $\sum_{i \in T_1} \varepsilon_{ij}$ appear in this lower bound. Hence, the proof of the corollary. $\quad\square$

If $I_1 = \{1, -, p\}$, we change the vector-field X_1, so that along its flow-lines, I_1 is defined continuously even though there could be a switch in the relative ordering of the $\lambda_i d_i$'s. To this effect, let $i_1 = \min\{i \text{ s.t. } i \in T_2\}$, we define

$$F_{i_1} = \{j \notin T_2 \text{ s.t. } \lambda_{i_1} d_{i_1} \leqslant 2c_1 \lambda_j d_j\}.$$

LEMMA 4.4. – *If $T_2 \cap I_1 \neq \emptyset$ and $I_1 = \{1, \ldots, p\}$, the following vector-field*

$$X_2 = - \sum_{i \in T_2 \cup F_{i_1}} \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i}$$

satisfies the condition (i) *of Theorem* 4.1.

Proof. – Using Proposition 2.2, we obtain (4.2) but with indices in $T_2 \cup F_{i_1}$. Since $I_1 = \{1, \ldots, p\}$, we have for each $i \neq k$

$$\varepsilon_{ik} = \left(\frac{1}{\lambda_i \lambda_k |a_i - a_k|^2} \right)^{\frac{n-2}{2}} \left(1 + O\left(\frac{1}{\lambda_i^2 |a_i - a_k|^2} + \frac{1}{\lambda_k^2 |a_i - a_k|^2} \right) \right)$$

$$= \left(\frac{1}{\lambda_i \lambda_k |a_i - a_k|^2} \right)^{\frac{n-2}{2}} + O\left(\frac{1}{(\lambda_1 d_1)^n} + \varepsilon_{ik}^{\frac{n}{n-2}} \right). \tag{4.8}$$

Indeed if $d_i \leqslant \frac{1}{2} d_k$ or $d_i \geqslant 2d_k$, we have $|a_i - a_k| \geqslant \frac{1}{2} \max(d_i, d_k)$ and the result follows. In the other case i.e. $\frac{1}{2} d_k \leqslant d_i \leqslant 2d_k$, using that $i, k \in I_1$, thus λ_i / λ_k and λ_k / λ_i are bounded. Therefore $(\lambda_r |a_i - a_k|)^{-2} = O(\varepsilon_{ik}^{2/(n-2)})$, for $r = i, k$. Thus, for $i \in F_{i_1}$, using (4.8) and (4.3)

$$\sum_{j \neq i} \left(-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right)$$

$$= \frac{n-2}{2} \sum_{j \neq i} \frac{G(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} + O\left(\frac{1}{(\lambda_1 d_1)^n} + \sum_{j \neq i} \varepsilon_{ij}^{\frac{n}{n-2}} \right). \tag{4.9}$$

Furthermore, if $i \in F_{i_1}$

$$\frac{H(a_i, a_i)}{\lambda_i^{n-2}} = O\left(\frac{1}{(\lambda_i d_i)^{n-2}} \right) = O\left(\frac{(2c_1)^{n-2}}{(\lambda_{i_1} d_{i_1})^{n-2}} \right) = o\left(\frac{1}{(\lambda_{i_1} d_{i_1})^{n-2}} \right).$$

For $i \in T_2$, using (4.3), (4.8) and the fact that i does not satisfy (4.1) then

$$\sum_{j \neq i} -\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \sum_{j=1}^{p} \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \geq c \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n-2}} \right).$$

Therefore, using the fact that the Green's function is positive, thus we derive

$$(-\partial J(u), X_2) \geq c \sum_{i \in T_2} \left(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n-2}} \right) + R_2.$$

Using the fact that $T_2 \cap I_1 \neq \emptyset$, the result follows as in the Proof of Corollary 4.3. □

Now, let us define, for c_2 a fixed small constant

$$L = \{ j \in T_1 \text{ s.t. } \exists i \in T_1 \text{ s.t. } c_2 \max(d_i, d_j) \geq |a_i - a_j| \}. \tag{4.10}$$

For $i \in L$, we denote i_0 the index such that

$$c_2 \max(d_i, d_{i_0}) \geq |a_i - a_{i_0}|. \tag{4.11}$$

LEMMA 4.5. – *If there exist two indices i and i_0 belonging to T_1 and satisfying (4.11), then, we can assume that $\lambda_i \geq \lambda_{i_0}$, there exists a vector-field X_3, defined on the variable λ_i as follows*

$$X_3 = -\alpha_i \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i}.$$

This vector-field satisfies

$$(-\partial J(u), X_3) \geq c \left(\varepsilon_{ii_0} + \frac{1}{(\lambda_i d_i)^{n-2}} \right) + R_2 + O \left(\sum_{k \in T_2} \varepsilon_{ik} \right).$$

Proof. – Using Proposition 2.2, we derive (4.2) (for $s = 1$). The terms with $j \in T_2$ can be seen like $O(\varepsilon_{ij})$. We are interesting now with the indices $j \in T_1$. Observe that, for $i, k \in T_1$, we have (4.8). Indeed, if $2d_i \leq d_k$ or $d_i \geq 2d_k$, we have $2|a_i - a_k| \geq \max(d_i, d_k)$ and the result follows. In the other case i.e. $\frac{1}{2} d_k \leq d_i \leq 2d_k$, using that $i, k \in T_1$, we have, as in (4.5)

$$\varepsilon_{ik} = o \left(\left(\frac{\lambda_i}{\lambda_k} \right)^{\frac{n-2}{2}} + \left(\frac{\lambda_k}{\lambda_i} \right)^{\frac{n-2}{2}} \right).$$

Therefore

$$\frac{1}{\lambda_k^2 |a_i - a_k|^2} = \frac{\lambda_i}{\lambda_k} \frac{1}{\lambda_i \lambda_k |a_i - a_k|^2} \leq c \frac{\lambda_i}{\lambda_k} \varepsilon_{ik}^{\frac{2}{n-2}} \leq c \frac{\lambda_i}{\lambda_k} \frac{1}{(\lambda_i d_i)(\lambda_1 d_1)} = O \left(\frac{1}{(\lambda_1 d_1)^2} \right)$$

and we use the same argument for $(\lambda_i |a_i - a_k|)^{-2}$. Thus we obtain (4.9) with the indices $j \in T_1$. Using the fact that the Green's function is positive and that $H(a_i, a_i) \leq d_i^{2-n}$ (see Lemma B.2), thus

$$\left(\partial J(u), \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right) \geq c \left(-\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{G(a_i, a_{i_0})}{(\lambda_i \lambda_{i_0})^{(n-2)/2}} \right) + O \left(R_2 + \sum_{k \in T_2} \varepsilon_{ik} \right) \tag{4.12}$$

where i and i_0 are the indices satisfying (4.11). Using (4.11) and $\lambda_i \geqslant \lambda_{i_0}$ thus

$$\frac{1}{(\lambda_i d_i)^{n-2}} + \frac{H(a_i, a_{i_0})}{(\lambda_i \lambda_{i_0})^{(n-2)/2}} \leqslant \left(\frac{c_2^2}{\lambda_i \lambda_{i_0} |a_i - a_{i_0}|^2} \right)^{\frac{n-2}{2}}. \tag{4.13}$$

Using (4.8) and (4.12), the lemma follows. $\quad\square$

COROLLARY 4.6. – *If $T_2 \cap I_1 = \emptyset$ and there are two indices i and i_0 belonging to I_1 such that (4.11) is satisfied with i and i_0. Thus, for m_1 is a fixed large constant, the vector-field $X_3 + m_1 X_1$ satisfies the condition* (i) *of Theorem 4.1.*

Proof. – Since i belongs to I_1 and the term $(\lambda_i d_i)^{(2-n)}$ appears in the lower-bound of the estimate of Lemma 4.5, as in the proof of Corollary 4.3, we can make all the $(\lambda_k d_k)^{(2-n)}$ and $\sum_{k \in T_1} \varepsilon_{kr}$ appear in this lower-bound. Thus, using Lemma 4.2 and Lemma 4.5, for m_1 a fixed large constant, the corollary follows. $\quad\square$

For d_0 a fixed small constant, we introduce the set

$$I_1' = \{i \in I_1 \text{ s.t. } d_i < d_0\}.$$

LEMMA 4.7. – *If $T_2 \cap I_1 = \emptyset$ and, for each i and j belonging to I_1, (4.11) is not satisfied, if we have also $I_1' \neq \emptyset$, then there exists a vector-field X_4, defined on the variables a_i's, for $i \in I_1'$, as follows*

$$X_4 = \sum_{i \in I_1'} \alpha_i \frac{\partial P\delta_i}{\partial a_i} \left(-\frac{n_i}{\lambda_{j_0}} \right)$$

where $\lambda_{j_0} = \max\{\lambda_i, \ i \in I_1'\}$ and n_i is the outward normal to $\partial \Omega_{d_i} = \{x \in \Omega \text{ s.t. } d(x, \partial\Omega) = d_i\}$ at a_i. This vector-field satisfies

$$(-\partial J(u), X_4) \geqslant c \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n-1}} + O \left(\sum_{i \in I_1', j \in T_2 \cup L_i} \varepsilon_{ij} \right) + R_3$$

where R_3 is defined in Proposition 2.3 and

$$L_i = \{j \in L \text{ s.t. } i \text{ and } j \text{ satisfy (4.11)}\}.$$

Proof. – Using the Proposition 2.3, we derive

$$(-\partial J(u), X_4) = \frac{c_1'}{\lambda_{j_0}} J(u) \sum_{i \in I_1'} \left[\frac{\alpha_i^2}{\lambda_i^{n-2}} \frac{\partial H(a_i, a_i)}{\partial n_i} (1 + o(1)) + O \left(\sum_{j \in T_2 \cup L_i} \lambda_i \varepsilon_{ij} \right) \right.$$

$$+ 2 \sum_{j \in T_1 \setminus L_i} \alpha_i \alpha_j \left(\frac{\partial \varepsilon_{ij}}{\partial n_i} - \frac{\partial H(a_i, a_j)}{\partial n_i} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \right)$$

$$\times \left. \left(1 - J(u)^{\frac{n}{n-2}} \left(\alpha_i^{\frac{4}{n-2}} + \alpha_j^{\frac{4}{n-2}} \right) \right) + \lambda_i R_3 \right] \tag{4.14}$$

where R_3 is the remainder term in Proposition 2.3. For $i \in I_1'$ and $j \in T_1 \setminus (I_1 \cup L_i)$, using Lemma B.2

$$\frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{\partial H}{\partial n_i}(a_i, a_j) \leqslant \frac{cH(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2} d_i} \leqslant \frac{c}{d_i (\lambda_i d_i \lambda_j d_j)^{(n-2)/2}} = \mathrm{o}\left(\frac{1}{d_i (\lambda_i d_i)^{n-2}}\right).$$

For i and j in I_1', if d_i/d_j, d_j/d_i and $|a_i - a_j|/d_i$ are bounded, we use the estimate of $(\partial H/\partial n_i)(a_i, a_j)$ provided in the Appendix B and proved in the appendix of [8], we derive $(\partial H/\partial n_i)(a_i, a_j) > 0$. In the other case, we have

$$\frac{\partial H}{\partial n_i}(a_i, a_j) \leqslant \frac{H(a_i, a_j)}{d_i} \leqslant \frac{1}{d_i \max(d_i, d_j, |a_i - a_j|)^{n-2}} = \mathrm{o}\left(\frac{1}{(d_i d_j)^{(n-1)/2}}\right)$$

and therefore

$$(\lambda_i \lambda_j)^{\frac{-(n-2)}{2}} \frac{\partial H}{\partial n_i}(a_i, a_j) = \mathrm{o}\left(\frac{1}{d_i (\lambda_i d_i)^{n-2}} + \frac{1}{d_j (\lambda_j d_j)^{n-2}}\right). \tag{4.15}$$

We also have for $i \in I_1'$ (see Lemma B.2)

$$\frac{\partial H(a_i, a_i)}{\partial n_i} = \frac{n-2}{2^{n-2}} \frac{1}{d_i^{n-1}} \left(1 + \mathrm{o}(1)\right).$$

For i and j belong to I_1', $n_i - n_j = \mathrm{O}(|a_i - a_j|)$ therefore

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} n_i + \frac{\partial \varepsilon_{ij}}{\partial a_j} n_j = \frac{n-2}{2} \lambda_i \lambda_j (a_i - a_j) \varepsilon_{ij}^{\frac{n}{n-2}} (n_j - n_i) = \mathrm{O}(\varepsilon_{ij}).$$

For each $i \in I_1'$ and $j \in T_1 \setminus (I_1 \cup L_i)$ we have $c_2 \max(d_i, d_j) \leqslant |a_i - a_j|$, therefore

$$\left|\frac{\partial \varepsilon_{ij}}{\partial a_i}\right| = (n-2)\lambda_i \lambda_j |a_i - a_j| \varepsilon_{ij}^{\frac{n}{n-2}} \leqslant \frac{c}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{1}{|a_i - a_j|^{n-1}}$$

$$= \mathrm{O}\left(\frac{1}{c_2^{n-1}(\lambda_i d_i \lambda_j d_j)^{(n-2)/2} d_i}\right) = \mathrm{O}\left(\frac{c_1^{(n-2)/2}}{c_2^{n-1}(\lambda_i d_i)^{n-2} d_i}\right)$$

$$= \mathrm{o}\left(\frac{1}{d_i (\lambda_i d_i)^{n-2}}\right) \tag{4.16}$$

if we choose c_1 and c_2 so that $c_1^{(n-2)/2} = \mathrm{o}(c_2^{n-1})$. For $i \in I_1'$, $j \in I_1 \setminus I_1'$, we claim

$$\frac{\partial H(a_i, a_j)}{\partial n_i} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} - \frac{\partial \varepsilon_{ij}}{\partial a_i} n_i$$

$$= -\frac{\partial G(a_i, a_j)}{\partial n_i} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} + \mathrm{o}\left(\frac{\lambda_i}{(\lambda_1 d_1)^{n+1}}\right). \tag{4.17}$$

Indeed, since $T_2 \cap I_1 = \emptyset$, then i and j belong to T_1. Using (4.8) and the fact that (4.11) is not satisfied, thus

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} = (n-2)(a_j - a_i)\lambda_i \lambda_j \varepsilon_{ij}^{\frac{n}{n-2}} = \frac{(n-2)(a_j - a_i)}{(\lambda_i \lambda_j)^{(n-2)/2}|a_i - a_j|^n}\left(1 + \mathrm{O}\left(\frac{1}{(\lambda_1 d_1)^2}\right)\right).$$

Therefore

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} = \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{\partial}{\partial a_i} \left(\frac{1}{|a_i - a_j|^{n-2}} \right) + O\left(\frac{\lambda_i}{c_2^{n-1} (\lambda_1 d_1)^{n+1}} \right)$$

and our claim follows. Thus

$$(-\partial J(u), X_4) \geqslant \frac{c}{\lambda_{j_0}} \sum_{i \in I_1'} \left[\frac{1}{(\lambda_i d_i)^{n-2} d_i} - \sum_{j \in I_1 \setminus I_1'} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \frac{\partial G(a_i, a_j)}{\partial n_i} \right.$$

$$\left. + O\left(\sum_{j \in I_1'} \varepsilon_{ij} \right) + O\left(\sum_{j \in T_2} \lambda_i \varepsilon_{ij} \right) + \lambda_i R_3 + O\left(\lambda_i \sum_{j \in L_i} \varepsilon_{ij} \right) \right].$$

Observe that, for $i \in I_1'$ and $j \in I_1 \setminus I_1'$ we have $d_i \leqslant d_j$ then, using Lemma B.3, $-\partial G(a_i, a_j)/\partial n_i > 0$. For $i, j \in I_1'$, using the fact that (4.11) does not satisfied, thus $\varepsilon_{ij} = O((\lambda_i d_i)^{(2-n)} + (\lambda_j d_j)^{(2-n)})$ and therefore, since d_i and d_j are small, we have $\varepsilon_{ij} = o((\lambda_i d_i)^{2-n} d_i^{-1} + (\lambda_j d_j)^{2-n} d_j^{-1})$. Using the fact that $j_0 \in I_1'$ and that $\lambda_{j_0} d_{j_0}$ and $\lambda_1 d_1$ are of the same order, we can make all the $(\lambda_i d_i)^{1-n}$, for $i \in T_1$, appear in the lower bound. The result follows. $\square$

LEMMA 4.8. – *There exists a vector-field X_5 satisfying*

$$(-\partial J(u), X_5) \geqslant c \sum_{i \in I_1', j \in L_i} \varepsilon_{ij} + o\left(\sum_{i \in I_1', \, j \in T_2} \varepsilon_{ij} \right) + R_2.$$

Proof. – For $i \in I_1'$ and $j \in L_i$, using Lemma 4.5, we can find a vector-field $X_3(i, j)$ depending on the indices i and j and satisfying the estimate of Lemma 4.5. The vector-field X_5 will be defined as follows

$$X_5 = \sum_{i \in I_1', \, j \in L_i} X_3(i, j).$$

The result follows. $\square$

COROLLARY 4.9. – *If $T_2 \cap I_1 = \emptyset$ and, for each i and j belonging to I_1, (4.11) is not satisfied, if we have also $I_1' \neq \emptyset$, then, for m_1 and m_2 two fixed large constants, the vector-field $X_4 + m_1 X_1 + m_2 X_5$ satisfies the condition* (i) *of Theorem* 4.1.

Proof. – For m_1 and m_2 two fixed large constant, we derive by Lemma 4.2, Lemma 4.7 and Lemma 4.8

$$(-\partial J(u), X_4 + m_1 X_1 + m_2 X_5) \geqslant c \left(\sum_{i=1}^p \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{j \in T_2} \varepsilon_{ij} \right) + R_2 + R_3. \quad (4.18)$$

As in the proof of Corollary 4.3, we can make $\sum_{i \in T_1} \varepsilon_{ij}^{(n-1)/(n-2)}$ appears in the lower-bound of (4.18). Therefore the corollary follows. $\square$

LEMMA 4.10. – *If $T_2 \cap I_1 = \emptyset$ and, for each i and j belonging to I_1, (4.11) is not satisfied, if we have also $I_1' = \emptyset$, then there exists a vector-field X_6 depending on the*

values of ρ and ρ'. This vector-field satisfies

$$(-\partial J(u), X_6) \geqslant c \sum_{i \in I_1} \left(\frac{1}{(\lambda_i d_i)^{n-1}} + O\left(\sum_{j \in T_2 \cup L_i} \varepsilon_{ij} \right) \right) + R_2 + R_3.$$

Proof. – Since $I_1' = \emptyset$, thus, for each $i \in I_1$ we have $d_i \geqslant d_0$. We denote by $M = M(a_i, i \in I_1)$ the matrix defined in (1.5), by ρ its least eigenvalue and by e the eigenvector associated to ρ. Let $\eta > 0$ be such that for any x belongs to a neighborhood $C(e, \eta)$ of e, where

$$C(e, \eta) \subset \left\{ y \in (R_+^*)^{\operatorname{card} I_1} \text{ s.t. } \left| \frac{y}{|y|} - e \right| < \eta \right\}$$

we have

$$^t x M x - \rho |x|^2 \leqslant \frac{1}{2} |\rho| |x|^2 \quad \text{and} \quad {}^t x \frac{\partial M}{\partial a_i} x = \left(\frac{\partial \rho}{\partial a_i} + o(1) \right) |x|^2 \qquad (4.19)$$

and for each $x \in C(e, \eta)^c$ we have

$$^t x M x - \rho |x|^2 > c |x|^2.$$

Let $\Lambda = {}^t(\lambda_{j_1}^{-(n-2)/2}, \dots, \lambda_{j_s}^{-(n-2)/2})$ ($\{j_1, \dots, j_s\} = I_1$). If Λ belongs to the set $C(e, \eta)^c$, then we move the vector Λ to $C(e, \eta)$, as in [9], along

$$\Lambda(t) = |\Lambda| \frac{(1-t)\Lambda + t|\Lambda|e}{|(1-t)\Lambda + t|\Lambda|e|}.$$

Using Proposition 2.2, we derive

$$(-\partial J(u), X_6) = c \left[-\frac{d}{dt} ({}^t \Lambda(t) M \Lambda(t)) + o\left(\sum_{i,j \in I_1} \varepsilon_{ij} + \sum_{i \in I_1} \frac{1}{\lambda_i^{n-2}} \right) \right.$$
$$\left. + R_2 + O\left(\sum_{i \in I_1, \ j \in T_2 \cup T_1 \setminus I_1} \varepsilon_{ij} \right) \right]. \qquad (4.20)$$

As in [9], we have

$$\frac{d}{dt} ({}^t \Lambda(t) M \Lambda(t)) < -c |\Lambda|^2 = -c \sum_{i \in I_1} \frac{1}{\lambda_i^{n-2}}.$$

For $i \in I_1$, $j \in T_1 \setminus I_1$, using (4.1), we obtain

$$\varepsilon_{ij} = O\left(\left(\frac{1}{(\lambda_1 d_1)(\lambda_j d_j)} \right)^{\frac{n-2}{2}} \right) = O\left(\frac{c_1^{(n-2)/2}}{d_0^{n-2}} \frac{1}{\lambda_1^{n-2}} \right) = o\left(\frac{1}{\lambda_1^{n-2}} \right) \qquad (4.21)$$

if we assume that $c_1 = o(d_0^2)$. Therefore, the lemma follows in this case. In the case where $\Lambda \in C(e, \eta)$, the construction of the vector-field depends on the value of ρ and

$|\rho'|$. Since zero is a regular value of ρ then there exists a constant $\rho_0 > 0$ such that either $|\rho| > \rho_0$ or $|\rho'| > \rho_0$.

If $\rho < -\rho_0$ we decrease all the λ_i's, for $i \in I_1$. If we assume that $c_1^{\frac{n-2}{2}} = o(\rho_0 d_0^{n-2})$, using Proposition 2.2, (4.3), (4.5) and (4.21), the result follows as in the above case.

If $|\rho'| > \rho_0$ and $\rho > -\rho_0$, then we move the points a_i's along $\lambda_{j_0}\dot{a}_i = -\partial\rho/\partial a_i$ for each $i \in I_1$ where $\lambda_{j_0} = \max\{\lambda_i,\ i \in I_1\}$. Using Proposition 2.3, we derive

$$\left(-\partial J(u), X_6\right) = \frac{1}{\lambda_{j_0}} \sum_{i\in I_1} \left(-\partial J(u), \frac{\partial P\delta_i}{\partial a_i}\right)\left(-\frac{\partial\rho}{\partial a_i}\right)$$

$$= \frac{1}{\lambda_{j_0}} \sum_{i\in I_1} \left[c\frac{\partial\rho}{\partial a_i}\left({}^t\Lambda\frac{\partial M}{\partial a_i}\Lambda\right) + O\left(\sum_{j\in T_2\cup L_i}\lambda_i\varepsilon_{ij}\right)\right.$$

$$\left. + \sum_{j\in T_1\setminus(I_1\cup L_i)} O\left(\frac{\partial H}{\partial a_i}(a_i, a_j)\frac{1}{(\lambda_i\lambda_j)^{(n-2)/2}} + \frac{\partial\varepsilon_{ij}}{\partial a_i}\right) + R_3\right], \quad (4.22)$$

where R_3 is defined in Proposition 2.3. For $i \in I_1$ and $j \in T_1 \setminus (I_1 \cup L_i)$, (4.11) is not satisfied and therefore we have

$$\frac{\partial H}{\partial a_i}(a_i, a_j)\frac{1}{(\lambda_i\lambda_j)^{(n-2)/2}} \leqslant \frac{1}{d_i^{n-1}}\frac{1}{(\lambda_i\lambda_j)^{(n-2)/2}} \leqslant \frac{(c_1 D)^{(n-2)/2}}{d_0^{(3n-4)/2}}\frac{1}{\lambda_1^{n-2}},$$

where D is the diameter of Ω. As in (4.16)

$$\left|\frac{\partial\varepsilon_{ij}}{\partial a_i}\right| \leqslant (\lambda_i\lambda_j)^{\frac{2-n}{2}}|a_i - a_j|^{1-n} \leqslant \frac{(c_1 D)^{(n-2)/2}}{c_2^{n-1}d_0^{(3n-4)/2}}\frac{1}{\lambda_1^{n-2}}.$$

We can choose the constants c_1 and c_2 so that

$$\frac{\partial H}{\partial a_i}(a_i, a_j)(\lambda_i\lambda_j)^{\frac{2-n}{2}} + \frac{\partial\varepsilon_{ij}}{\partial a_i} = o\left(\frac{1}{\lambda_1^{n-2}}\right). \quad (4.23)$$

Using that Λ belongs to $C(e, \eta)$, thus (4.19) is satisfied. Therefore (4.22) and (4.23) imply

$$\left(-\partial J(u), X_6\right) \geqslant \frac{1}{2}\frac{c}{\lambda_{j_0}}|\rho'|^2|\Lambda|^2 + O\left(\sum_{i\in I_1,\ j\in T_2\cup L_i}\varepsilon_{ij}\right) + R_3$$

$$\geqslant \frac{1}{2}c\rho_0^2\frac{|\Lambda|^2}{\lambda_{j_0}} + O\left(\sum_{i\in I_1,\ j\in T_2\cup L_i}\varepsilon_{ij}\right) + R_3.$$

Therefore, the lemma follows. □

COROLLARY 4.11. – *If $T_2 \cap I_1 = \emptyset$ and, for each i and j belonging to I_1, (4.11) is not satisfied, if we have also $I_1' = \emptyset$, then, for m_1 and m_2 two fixed large constants, the vector-field $X_6 + m_1 X_1 + m_2 X_5$ satisfies the condition* (i) *of Theorem* 4.1

Proof. – The proof of this corollary is similar to the proof of Corollary 4.9. □

Proof of Theorem 4.1. – The vector field Z required in Theorem 4.1 will be defined by a convex combination of all these cases defined above. Combining Corollaries 4.3,

4.6, 4.9, 4.11 and Lemma 4.4, we easily derive (i), (iii), (iv), (v) and (vi) of Theorem 4.1. Now, we are going to prove claim (ii). We follow the argument of [4] and [9]. Let $u = \sum_{i=1}^{p} \alpha_i P\delta_i$ and $\bar{u} = u + \bar{v}$. Thus

$$\left(-\partial J(\bar{u}), Z + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z) \right) = (-\partial J(\bar{u}), Z) - \left(\partial J(\bar{u}), \frac{\partial \bar{v}}{(\alpha_i, a_i, \lambda_i)}(Z) \right).$$

We need to prove that the terms contained $\bar{v}$ are $o(\sum \varepsilon_{kr}^{(n-1)/(n-2)} + \sum (\lambda_i d_i)^{1-n})$. We consider $(\partial J(\bar{u}), \partial \bar{v}/\partial(\alpha_i, a_i, \lambda_i)(Z))$. Let

$$E = \operatorname{span}_i \{ P\delta_i, \partial P\delta_i/\partial \lambda_i, \partial P\delta_i/\partial a_i \}, \qquad F = \operatorname{span}_i \{ \partial P\delta_i/\partial \lambda_i, \partial P\delta_i/\partial a_i \}$$

and Q_E (respectively Q_F) be the orthogonal projection onto E (respectively F). Since $\bar{v}$ extremizes $J(\sum \alpha_i P\delta_i + v)$ in the v-space, $\partial J(\bar{u})$ belongs to E. Thus

$$\left(\partial J(\bar{u}), \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z) \right) = \left(\partial J(\bar{u}), Q_E \left(\frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z) \right) \right).$$

Since $\bar{v}$ satisfies (V_0) then $(\bar{v}, P\delta_i) = 0$ and therefore

$$\left(\frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z), P\delta_i \right) = -\left(\bar{v}, \frac{\partial P\delta_i}{\partial(\alpha_i, a_i, \lambda_i)}(Z) \right) = 0.$$

Thus $Q_E(\partial \bar{v}/\partial(\alpha_i, a_i, \lambda_i)(Z)) = Q_F(\partial \bar{v}/\partial(\alpha_i, a_i, \lambda_i)(Z))$ and therefore

$$\left(\partial J(\bar{u}), \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z) \right) = \left(Q_F(\partial J(\bar{u})), Q_F \left(\frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z) \right) \right).$$

By Proposition A.2 of [4], we have, for each $\phi \in F$,

$$\left(Q_F \left(\frac{\partial \bar{v}}{(\alpha_i, a_i, \lambda_i)}(Z) \right), \phi \right) \leqslant |\bar{v}||Z||\phi|.$$

Thus

$$\left(Q_F(\partial J(\bar{u})), Q_F \left(\frac{\partial \bar{v}}{(\alpha_i, a_i, \lambda_i)}(Z) \right) \right) \leqslant |\bar{v}||Z| \sup_{\phi \in F} \left(\frac{(\partial J(\bar{u}), \phi)}{|\phi|} \right).$$

Using Propositions 2.2 and 2.3, we get

$$\sup_{\phi \in F} \left(\frac{(\partial J(\bar{u}), \phi)}{|\phi|} \right) = O \left(\sum \varepsilon_{ij} + \sum \frac{1}{(\lambda_i d_i)^{n-2}} + |\bar{v}| \right).$$

Using the estimate of $\bar{v}$ (see Lemma 3.1) and that $|Z| \leqslant c$, we derive

$$\left(\partial J(\bar{u}), \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(Z) \right) = o \left(\sum \varepsilon_{kr}^{\frac{n-1}{n-2}} + \sum \frac{1}{(\lambda_i d_i)^{n-1}} \right).$$

We, now, left for $(\partial J(\bar{u}), Z)$. Observe that

$$(\partial J(\bar{u}), Z) = 2J(\bar{u})\left((\bar{u}, Z) - J(\bar{u})^{\frac{n}{n-2}}\int \bar{u}^{\frac{n+2}{n-2}}Z\right)$$

$$= 2J(u)\left((u, Z) - J(u)^{\frac{n}{n-2}}\int \bar{u}^{\frac{n+2}{n-2}}Z\right)$$

$$+ O\left(|\bar{v}|\left(\sum \varepsilon_{kr} + \sum \frac{1}{(\lambda_i d_i)^{n-2}} + |\bar{v}|\right)\right)$$

$$= \left(\partial J(u), Z\right) - 2\frac{n+2}{n-2}J(u)^{\frac{2(n-1)}{n-2}}\int u^{\frac{4}{n-2}}\bar{v}Z + O(|\bar{v}|^2)$$

$$+ O\left(\int\limits_{u\leqslant\bar{v}} |\bar{v}|^{\frac{n+2}{n-2}}|Z|\right) + O\left(|\bar{v}|\left(\sum \varepsilon_{kr} + \sum \frac{1}{(\lambda_i d_i)^{n-2}} + |\bar{v}|\right)\right).$$

Observe that, for ϕ_i equal to $\lambda_i^{-1}\partial P\delta_i/\partial a_i$ or $\lambda_i\partial P\delta_i/\partial\lambda_i$,

$$\int u^{\frac{4}{n-2}}\bar{v}\phi_i = \alpha_i^{\frac{4}{n-2}}\int P\delta_i^{\frac{4}{n-2}}\phi_i\bar{v} + O\left(\sum_{j\neq i}\left(\int\limits_{\delta_i\leqslant\delta_j} \delta_j^{\frac{4}{n-2}}|\bar{v}|\delta_i + \int\limits_{\delta_j\leqslant\delta_i} \delta_i^{\frac{4}{n-2}}\delta_j|\bar{v}|\right)\right)$$

$$\leqslant \int P\delta_i^{\frac{4}{n-2}}\phi_i\bar{v} + c|\bar{v}|\left(\sum_{k\neq j}\int\limits_{\delta_k\leqslant\delta_j} (\delta_j^{\frac{4}{n-2}}\delta_k)^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}}.$$

Observe that, for $n \geqslant 6$ and $k \neq j$,

$$\int\limits_{\delta_j\leqslant\delta_k} (\delta_k^{\frac{4}{n-2}}\delta_j)^{\frac{2n}{n+2}} \leqslant \int (\delta_k\delta_j)^{\frac{n}{n-2}} = O(\varepsilon_{kj}^{\frac{n}{n-2}}\log(\varepsilon_{kj}^{-1}))$$

and for $n \leqslant 5$,

$$\left[\int (\delta_k^{\frac{4}{n-2}}\delta_j)^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} \leqslant \left[\int (\delta_k\delta_j)^{\frac{2n}{n+2}}\delta_k^{\frac{6-n}{n-2}\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} = O\left(\varepsilon_{kj}\left(\log(\varepsilon_{kj}^{-1})\right)^{\frac{n-2}{n}}\right).$$

Using the same argument than (3.2), we have

$$\int P\delta_i^{\frac{4}{n-2}}\phi_i\bar{v} \leqslant c|\bar{v}|\left(\frac{1}{(\lambda_i d_i)^{(n+2)/2}} + (\text{if } n = 6)\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} + (\text{if } n \leqslant 5)\frac{1}{(\lambda_i d_i)^{n-2}}\right).$$

Since Z belongs to F thus

$$\int u^{\frac{4}{n-2}}\bar{v}Z = o\left(\sum \varepsilon_{kr}^{\frac{n-1}{n-2}} + \sum \frac{1}{(\lambda_i d_i)^{n-1}}\right).$$

Since $|Z|(x) \leqslant c\sum \delta_i(x)$, thus

$$\int\limits_{u\leqslant\bar{v}} |\bar{v}|^{\frac{n+2}{n-2}}|Z| \leqslant c\int\limits_{u\leqslant\bar{v}} |\bar{v}|^{\frac{n+2}{n-2}}u + \sum_i \int |\bar{v}|^{\frac{n+2}{n-2}}(\delta_i - P\delta_i)$$

$$\leqslant c\left(|\bar{v}|^{\frac{2n}{n-2}} + \sum_i \frac{1}{(\lambda_i d_i)^n}\right). \tag{4.24}$$

The proof of (ii) is therefore completed.   □

Before ending this section, we give a new proof, based on Theorem 1.2 and 4.1, of the formula for the difference of topology of the critical points at infinity of the functional J. This formula was proved in [8].

Proof of Proposition 1.6. – From Proposition 1.5, we have the only critical points at infinity for J are $\sum \delta(x_i, \infty)$ where $X = (x_1, \ldots, x_p)$ satisfies $\rho(X) > 0$ and $\rho'(X) = 0$. Moreover, Theorem 1.2 gives the normal form of J in the neighborhood of these critical points at infinity. Thus, the homology required is the product of the homologies defined by each variables. Therefore, using the fact that in the α-space is an unique maximum and in the λ-space, there is an unique minimum, our proposition follows.   □

5. Proof of Theorem 1.3

We consider now the Morse Lemma when there is a solution w of (P). The proof of the Theorem 1.3 is similar to Theorem 1.2 and it relies also on the existence of a pseudo-gradient Z' which satisfies a proposition similar to the Theorem 4.1. We have the following main result which the proof is given at the end of this section.

THEOREM 5.1. – *There exists a pseudogradient Z' so that the following holds*: *there is a constant $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i P\delta_i + \alpha_0(w + h)$ in $V(p, \varepsilon, w)$ such that*
(i)

$$(-\partial J(u), Z') \geqslant c \sum_{i=1}^{p} (\lambda_i^{-n/2} + (\lambda_i d_i)^{1-n}) + c \sum_{k \neq i} \varepsilon_{ki}^{\frac{n-1}{n-2}} + c|h|^2,$$

(ii)

$$\left(-\partial J(u + \bar{v}), Z' + \frac{\partial \bar{v}}{\partial (\alpha_i, a_i, \lambda_i, h)}(Z') \right)$$

$$\geqslant c \left(\sum_{i=1}^{p} (\lambda_i^{-n/2} + (\lambda_i d_i)^{1-n}) + \sum_{k \neq i} \varepsilon_{ki}^{\frac{n-1}{n-2}} + |h|^2 \right).$$

This pseudogradient satisfies the (P.S.) *condition and it increases the least distance to the boundary along any flow line.*

First, we immediately derive from claim (ii) of this theorem, that ∂J cannot be zero in $V(p, \varepsilon, w)$ and hence the Proposition 1.6 follows.

Now, we state the following lemma, which the proof may be deduce from [4], pp. 354–355.

LEMMA 5.2. – *There is a C^1-map which to each (α, a, λ) such that $\sum_{i=1}^{p} \alpha_i P\delta_{(a_i, \lambda_i)} + \alpha_0 w$ belongs to $V(p, \varepsilon, w)$ associates $\overline{h} = \overline{h}(\alpha, a, \lambda)$ satisfying*

$$J\left(\sum \alpha_i P\delta_i + \alpha_0 w + \overline{h} \right) = \max\left\{ J\left(\sum \alpha_i P\delta_i + \alpha_0(w + h) \right), \; h \in T_w(W_u(w)) \right\}.$$

We now prove Theorem 1.3.

Proof of Theorem 1.3. – Arguing as in Appendix 2 of [4], we immediately derive from Theorem 5.1 that for each $u = \sum_{i=1}^{p} \alpha_i P\delta_i + \alpha_0(w + h)$ belongs to $V(p, \varepsilon, w)$ we can find a change of variables $(a, \lambda, h) \to (\tilde{a}, \tilde{\lambda}, \tilde{h})$ such that

$$J\left(\sum \alpha_i P\delta_i + \alpha_0(w + h) + \bar{v}\right) = J\left(\sum \alpha_i P\tilde{\delta}_i + \alpha_0(w + \tilde{h})\right).$$

From Lemma 5.2, we deduce that there is a change of variables $h - \bar{h} \to H$ such that

$$J\left(\sum \alpha_i P\delta_i + \alpha_0(w + h)\right) = J\left(\sum \alpha_i P\delta_i + \alpha_0 w + \overline{h}\right) - |H|^2.$$

Arguing as in the proof of Lemma 3.1, we obtain the following estimate

$$|\bar{h}| = O\left(\sum \lambda_i^{-\frac{n-2}{2}}\right).$$

Thus, by the same argument used to prove Theorem 1.2, we can find another change of variables $(a, \lambda) \to (\tilde{a}, \tilde{\lambda})$ such that

$$J\left(\sum \alpha_i P\delta_i + \alpha_0 w + \bar{h}\right) = J\left(\sum \alpha_i P\tilde{\delta}_i + \alpha_0 w\right).$$

The proof of the Theorem 1.3 is therefore completed under Theorem 5.1.   □

We, now, left on the proof of Theorem 5.1.

Proof of Theorem 5.1. – The new fact here is the terms $(-w(a)/\lambda^{(n-2)/2})$ and $|h|^2$ where w is the solution of (P) and h belongs to $T_w(W_u(w))$. By Proposition 2.4, we get

$$(-\partial J(u), h) \geqslant c|h|^2 + O\left(\sum_i \frac{1}{\lambda_i^{n-2}}\right).$$

We need to add another vector-field on the variables (a_i, λ_i). It will be defined as in Theorem 4.1.

Let $d_0 > 0$ be a small constant. We divide the set $\{1, \ldots, p\}$ into $T_1 \cup T_2 \cup T_3$ where:

$$T_1 = \{i \text{ s.t. } i \text{ satisfies (4.1) and } d_i < d_0\},$$
$$T_2 = \{i \text{ s.t. } i \text{ does not satisfy (4.1) and } d_i < d_0\},$$
$$T_3 = \{i \text{ s.t. } d_i \geqslant d_0\}.$$

In $T_2 \cup T_3$, we order the λ_i 's: $\lambda_{i_1} \leqslant \lambda_{i_2} \leqslant \cdots \leqslant \lambda_{i_s}$. On these variables, we define a vector-field as follows

$$X_1' = -\sum_{k=1}^{s} 2^k \alpha_{i_k} \lambda_{i_k} \frac{\partial P\delta_{i_k}}{\partial \lambda_{i_k}}.$$

By Proposition 2.5, we derive, as in (4.2)

$$\left(-\partial J(u), X_1'\right) \geqslant \sum_{k=1}^{s} \left[\frac{w(a_{i_k})}{\lambda_{i_k}^{(n-2)/2}} - \sum_{j \neq i_k} 2^k \alpha_{i_k} \alpha_j \lambda_{i_k} \frac{\partial \varepsilon_{j i_k}}{\partial \lambda_{i_k}} \left(1 + o(1)\right) + O\left(\frac{\log(\lambda_{i_k} d_{i_k})}{\lambda_{i_k}^{n/2}}\right) \right.$$
$$\left. - \frac{n-2}{2} \sum_j 2^k \alpha_{i_k} \alpha_j \frac{H(a_j, a_{i_k})}{(\lambda_j \lambda_{i_k})^{(n-2)/2}} \left(1 + o(1)\right) + R_4 \right],$$

where R_4 is the remainder term in Proposition 2.5, it satisfies

$$R_4 = o\left(|h|^2 + \sum \frac{1}{\lambda_k^{n/2}} + \sum \varepsilon_{kr}^{\frac{n-1}{n-2}} \right) + R_2.$$

Observe that
- for $i \in T_3$, $w(a_i) > c(d_0)$ and therefore

$$\frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} = o\left(\frac{w(a_i)}{\lambda_i^{(n-2)/2}} \right) \quad \text{and} \quad \sum_j \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} = o\left(\frac{w(a_i)}{\lambda_i^{(n-2)/2}} \right);$$

- for $i \in T_2$, we have $w(a_i) \geqslant c d_i$. Therefore:

$$\frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} = o\left(\frac{d_i}{\lambda_i^{(n-2)/2}} \right) = o\left(\frac{w(a_i)}{\lambda_i^{(n-2)/2}} \right).$$

Observe that, for $i \in T_2 \cup T_3$ and $j \in T_1$, using (4.3) and (4.5), thus we get as in (4.6)

$$\left(-\partial J(u), X_1'\right) \geqslant c \sum_{i \in T_2 \cup T_3} \left[\frac{1}{\lambda_i^{n/2}} + \sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^{n-2}} \right] + R_4.$$

We need to add some more terms in our lower-bound. Thus, we will define another vector field using the set T_1.

First, we order the $\lambda_i d_i$'s for $i \in T_1$ (for simplicity, we will assume that $T_1 = \{1, \ldots, q\}$): $\lambda_1 d_1 \leqslant \lambda_2 d_2 \leqslant \cdots \leqslant \lambda_q d_q$ and we define the set

$$I_1 = \{1\} \cup \left\{ i \in T_1 \text{ s.t. } \forall k \leqslant i \ c_1 \lambda_k d_k \leqslant \lambda_{k-1} d_{k-1} \right\},$$

where c_1 is a fixed small constant. For $i \in I_1$, we introduce

$$L_i = \left\{ j \in T_1 \text{ s.t. } c_2 \max(d_i, d_j) \geqslant |a_i - a_j| \right\},$$

where c_2 is a fixed small constant. We define the following two vector-fields using the variables a_i's and λ_i's, for $i \in T_1$

$$X_2' = -\sum_{i \in T_1} \frac{1}{(\lambda_i d_i)} \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \quad \text{and} \quad X_3' = \sum_{i \in I_1} \alpha_i \frac{\partial P \delta_i}{\partial a_i} \left(\frac{-n_i}{\lambda_{i_0}} \right)$$

where $\lambda_{i_0} = \max\{\lambda_i, \ i \in I_1\}$. The vector-field X_2' satisfies

$$\big(\partial J(u), X_2'\big) = 2J(u) \sum_{i \in T_1} \frac{c_1'}{\lambda_i d_i} \left[\frac{n-2}{2} \left(\alpha_0 \frac{w(a_i)}{\lambda_i^{(n-2)/2}} - \alpha_i \frac{H(a_i, a_i)}{\lambda_i^{n-2}} \right) (1 + o(1)) \right.$$

$$\left. + R_4 + O\left(\frac{\log(\lambda_i d_i)}{\lambda_i^{n/2}} + \sum_{j \neq i} \varepsilon_{ij} \right) \right].$$

Using the fact that $w(a_i) \geqslant cd_i$ and, by (4.1), $\sum \varepsilon_{ij} = O(\sup((\lambda_k d_k)^{2-n}))$, we derive

$$\big(\partial J(u), X_2'\big) \geqslant c \sum_{i \in T_1} \frac{1}{\lambda_i^{n/2}} + O\left(\sup \frac{1}{(\lambda_k d_k)^{n-1}} \right) + R_4.$$

Furthermore, X_3' satisfies, using Proposition 2.6

$$\big(-\partial J(u), X_3'\big) = \frac{c_1'}{\lambda_{i_0}} J(u) \sum_{i \in I_1} \left[\frac{\alpha_i^2}{\lambda_i^{n-2}} \frac{\partial H(a_i, a_i)}{\partial a_i} (1 + o(1)) - \frac{\alpha_0 \alpha_i}{\lambda_i^{(n-2)/2}} \frac{\partial w(a_i)}{\partial n_i} (1 + o(1)) \right.$$

$$+ 2 \sum_{j \in T_1 \setminus L_i} \alpha_i \alpha_j \left(\frac{\partial \varepsilon_{ij}}{\partial n_i} - \frac{\partial H(a_i, a_j)}{\partial n_i} \frac{1}{(\lambda_i \lambda_j)^{(n-2)/2}} \right)$$

$$\times \left[1 - J(u)^{\frac{n}{n-2}} (\alpha_i^{\frac{4}{n-2}} + \alpha_j^{\frac{4}{n-2}}) \right]$$

$$+ O\left(\sum_{j \in T_2 \cup T_3 \cup L_i} \lambda_i \varepsilon_{ij} \right) + \lambda_i R_5 \right], \tag{5.1}$$

where R_5 is the remainder term in Proposition 2.6. (5.1) is similar to (4.14). Using (4.15), (4.16) and the fact that $\partial w(a_i)/\partial n_i \leqslant 0$, thus we derive

$$\big(-\partial J(u), X_3'\big) \geqslant \sum_{i \in T_1} \left(\frac{c}{(\lambda_i d_i)^{n-1}} + O\left(\sum_{j \in T_2 \cup T_3 \cup L_i} \varepsilon_{ij} \right) \right) + R_5.$$

For $i \in I_1$ and $j \in L_i$, as in Lemma 4.5, we can find a vector-field $X_4'(i, j)$ satisfying

$$\big(-\partial J(u), X_4'(i, j)\big) \geqslant c\varepsilon_{ij} + R_4 + O\left(\sum_{k \in T_2 \cup T_3} (\varepsilon_{ki} + \varepsilon_{kj}) \right)$$

and therefore, we can define

$$X_4' = \sum_{i \in I_1, j \in L_i} X_4'(i, j).$$

Thus

$$\big(-\partial J(u), X_4'\big) \geqslant c \sum_{i \in I_1, j \in L_i} \varepsilon_{ij} + R_4 + O\left(\sum_{k \in T_2 \cup T_3, \, r \in T_1} \varepsilon_{kr} \right).$$

We can choose $m_1 > m_2 > m_3$, three fixed large constants, so that the vector-field Z', required in Theorem 5.1, will be defined as follows

$$Z' = m_1 X_1' + m_2 X_4' + m_3 X_3' + X_2' + h$$

it satisfies the following required estimate

$$(-\partial J(u), Z') \geqslant c \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n/2}} + \frac{1}{(\lambda_i d_i)^{n-1}} + \sum_{k \neq r} \varepsilon_{kr}^{\frac{n-1}{n-2}} \right).$$

Therefore the proof of (i) is completed. Using the estimate of $\bar{v}$ (see Lemma 3.2), the proof of (ii) follows as in Theorem 4.1. Thus Theorem 5.1 is completed. $\quad\square$

Acknowledgements

Appendix A

In this appendix, we collect the estimates of the different integral quantities which occur in the paper. These estimates are originally introduced by Bahri [3] and Bahri and Coron [6]. For the proof, we refer the interested reader to [3,6,12]. In this appendix, we suppose that $\lambda_i d_i$ is large enough and ε_{ij} is small enough. We have the following estimates

LEMMA A.1. –

$$|P\delta|^2 = S_n - c_1' \frac{H(a, a)}{\lambda^{n-2}} + O\left(\frac{\log(\lambda d)}{(\lambda d)^n} \right)$$

where $c_1' = c_0 c_2'$, $c_2' = c_0^{\frac{n+2}{n-2}} \int_{R^n} \frac{dx}{(1+|x|^2)^{(n+2)/2}}$ and c_0 is defined in (1.1).

LEMMA A.2. –

$$\int_{\Omega} P\delta^{\frac{2n}{n-2}} = S_n - \frac{2n}{n-2} c_1' \frac{H(a, a)}{\lambda^{n-2}} + O\left(\frac{\log(\lambda d)}{(\lambda d)^n} \right).$$

LEMMA A.3. – For $i \neq j$

$$(P\delta_i, P\delta_j) = c_1' \left(\varepsilon_{ij} - \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{(n-2)/2}} \right) + O\left(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} \right).$$

LEMMA A.4. – For $i \neq j$

$$\int_{\Omega} P\delta_i^{\frac{n+2}{n-2}} P\delta_j = (P\delta_i, P\delta_j) + O\left(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} \right).$$

LEMMA A.5. –

$$\left(P\delta, \lambda \frac{\partial P\delta}{\partial \lambda} \right) = \frac{n-2}{2} c_1' \frac{H(a, a)}{\lambda^{n-2}} + O\left(\frac{\log(\lambda d)}{(\lambda d)^n} \right).$$

LEMMA A.6. –

$$\int P\delta^{\frac{n+2}{n-2}}\lambda\frac{\partial P\delta}{\partial\lambda} = 2\left(P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}\right) + \mathrm{O}\left(\frac{\log(\lambda d)}{(\lambda d)^n}\right).$$

LEMMA A.7. – *For $i \neq j$*

$$\left(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\right) = c_1'\left(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H(a_i, a_j)}{(\lambda_i\lambda_j)^{(n-2)/2}}\right)$$
$$+ \mathrm{O}\left(\sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}\log(\varepsilon_{ij}^{-1})\right).$$

LEMMA A.8. – *For $i \neq j$*

$$\int P\delta_j^{\frac{n+2}{n-2}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i} = \left(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\right) + \mathrm{O}\left(\sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}\log(\varepsilon_{ij}^{-1})\right).$$

LEMMA A.9. – *For $i \neq j$*

$$\frac{n+2}{n-2}\int P\delta_j\left(P\delta_i^{\frac{4}{n-2}}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\right) = \left(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\right) + \mathrm{O}\left(\sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^n}\right)$$
$$+ \mathrm{O}\left(\varepsilon_{ij}^{\frac{n}{n-2}}\log(\varepsilon_{ij}^{-1})\right).$$

LEMMA A.10. –

$$\left(P\delta, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a}\right) = -\frac{1}{2}\frac{c_1'}{\lambda^{n-1}}\frac{\partial H}{\partial a}(a, a) + \mathrm{O}\left(\frac{1}{(\lambda d)^n}\right).$$

LEMMA A.11. –

$$\int P\delta^{\frac{n+2}{n-2}}\frac{1}{\lambda}\frac{\partial P\delta}{\partial a} = 2\left(P\delta, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a}\right) + \mathrm{O}\left(\frac{\log(\lambda d)}{(\lambda d)^n}\right).$$

LEMMA A.12. – *For $i \neq j$*

$$\left(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right) = -\frac{c_1'}{(\lambda_i\lambda_j)^{(n-2)/2}}\frac{1}{\lambda_i}\frac{\partial H}{\partial a_i}(a_i, a_j) + c_1'\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}$$
$$+ \mathrm{O}\left(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n+1}{n-2}}\lambda_j|a_i - a_j|\right).$$

LEMMA A.13. – *For $i \neq j$*

$$\int P\delta_j^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = \left(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right) + \mathrm{O}\left(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^n} + \varepsilon_{ij}^{\frac{n}{n-2}}\log(\varepsilon_{ij}^{-1})\right).$$

LEMMA A.14. – *For $i \neq j$*

$$\frac{n+2}{n-2} \int P\delta_j \left(P\delta_i^{\frac{4}{n-2}} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right)$$

$$= \left(P\delta_j, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \right) + O\left(\sum_{k=i,j} \frac{1}{(\lambda_k d_k)^n} \right) + O\left(\varepsilon_{ij}^{\frac{n}{n-2}} \log(\varepsilon_{ij}^{-1}) \right).$$

LEMMA A.15. –

$$\left(w, \lambda \frac{\partial P\delta}{\partial \lambda} \right) = -\frac{n-2}{2} c_2' \frac{w(a)}{\lambda^{(n-2)/2}} \left(1 + O\left(\frac{1}{(\lambda d)^2} \right) \right) + O\left(\frac{1}{\lambda^{n/2}} + \frac{1}{(\lambda d)^n} \right).$$

LEMMA A.16. –

$$\int \left(\frac{n+2}{n-2} P\delta^{\frac{4}{n-2}} \lambda \frac{\partial P\delta}{\partial \lambda} \right) w = \left(w, \lambda \frac{\partial P\delta}{\partial \lambda} \right) + o\left(\frac{1}{\lambda^{n/2}} \right) + O\left(\frac{(\log(\lambda d))^{n/2}}{(\lambda d)^n} \right).$$

LEMMA A.17. –

$$\left(w, \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} \right) = -c_1' \frac{Dw(a)}{\lambda^{n/2}} \left(1 + O\left(\frac{1}{(\lambda d)^2} \right) \right) + O\left(\frac{d}{\lambda^{n/2}} + \frac{1}{(\lambda d)^n} \right).$$

LEMMA A.18. –

$$\frac{n+2}{n-2} \int \left(P\delta^{\frac{4}{n-2}} \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} \right) w = \left(w, \frac{1}{\lambda} \frac{\partial P\delta}{\partial a} \right) + o\left(\frac{1}{\lambda^{n/2}} \right) + O\left(\frac{(\log(\lambda d))^{n/2}}{(\lambda d)^n} \right).$$

Appendix B

LEMMA B.1 [3,12]. – *For* $\theta = \theta_{(a,\lambda)} = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$ *and* H *the regular part of the Green's function, we have the following estimates*:

$$\theta(x) = \frac{c_0}{\lambda^{(n-2)/2}} H(a,x) + O\left(\frac{1}{\lambda^{(n+2)/2} d^n} \right), \quad |\theta|_{L^{\frac{2n}{n-2}}} = O\left(\frac{1}{(\lambda d)^{(n-2)/2}} \right),$$

$$\frac{\partial \theta}{\partial a}(x) = \frac{c_0}{\lambda^{(n-2)/2}} \frac{\partial H}{\partial a}(a,x) + O\left(\frac{1}{\lambda^{(n+2)/2} d^{n+1}} \right), \quad \left| \frac{1}{\lambda} \frac{\partial \theta}{\partial a} \right|_{L^{\frac{2n}{n-2}}} = O\left(\frac{1}{(\lambda d)^{n/2}} \right),$$

$$\lambda \frac{\partial \theta}{\partial \lambda}(x) = -\frac{n-2}{2} \frac{c_0}{\lambda^{(n-2)/2}} H(a,x) + O\left(\frac{1}{\lambda^{(n+2)/2} d^n} \right), \quad \left| \lambda \frac{\partial \theta}{\partial \lambda} \right|_{L^{\frac{2n}{n-2}}} = O\left(\frac{1}{(\lambda d)^{(n-2)/2}} \right),$$

where $d = d(a, \partial\Omega)$ *and* c_0 *is the constant defined in* (1.1).

LEMMA B.2 [3,12]. – *For each* $a \in \Omega$, *near the boundary of* Ω, *let* $n_a = n$ *the outward normal to* $\partial\Omega$ *at* a

$$H(a,a) = (2d)^{2-n} + o(d^{2-n}), \qquad H(x,a) \leqslant c \max(d_x, d_a)^{2-n},$$

$$\frac{\partial H}{\partial n}(a,a) = \frac{n-2}{2^{n-2} d^{n-1}} + o\left(\frac{1}{d^{n-1}} \right), \qquad \left| \frac{\partial H}{\partial x} \right|(x,a) \leqslant \frac{c}{d_x} H(x,a).$$

LEMMA B.3. – *Let* x_1 *and* x_2 *be two points of* Ω *such that* $d_1 \leqslant d_2$ *and* $c_2 d_2 \leqslant |x_1 - x_2|$ *where* c_2 *is a fixed constant. If* d_1 *is small enough then* $(\partial G/\partial n_1)(x_1, x_2) \leqslant 0$.

Proof. – We argue by contradiction. We assume that there exists a sequence $(x_1^m, x_2^m) \in \Omega^2$ such that d_1^m tends to zero, $d_1^m \leqslant d_2^m$, $c_2 d_2^m \leqslant |x_1^m - x_2^m|$, $(\partial G / \partial n_1)(x_1^m, x_2^m) > 0$. In the case where $\lim x_1^m \neq \lim x_2^m$, the strong maximum principle yields a contradiction. Suppose, now, that $\lim x_1^m = \lim x_2^m$. Three cases may occur

(1) $d_2^m / |x_1^m - x_2^m|$ tends to zero.
(2) $|x_1^m - x_2^m| / d_2^m$ is bounded and $d_1^m / |x_1^m - x_2^m|$ tends to zero.
(3) $|x_1^m - x_2^m| / d_1^m$ is bounded.

Let us consider the first case. We introduce the transformation T defined on Ω by

$$T : \Omega \to \widetilde{\Omega} = T(\Omega)$$

$$x \to \tilde{x} = \frac{x}{|x_1^m - x_2^m|}.$$

Let G and $\widetilde{G}$ be the Green's functionals associated to Ω and $\widetilde{\Omega}$. We have

$$G(x, y) = \widetilde{G}(\tilde{x}, \tilde{y}) / |x_1^m - x_2^m|^{n-2}.$$

Thus

$$\frac{\partial G}{\partial n_1}(x_1^m, x_2^m) = \frac{1}{|x_1^m - x_2^m|^{n-1}} \frac{\partial \widetilde{G}}{\partial n_1}(\tilde{x}_1^m, \tilde{x}_2^m).$$

Let $\bar{x}_i^m$ be such that $\bar{x}_i^m \in \partial\widetilde{\Omega}$ and $\tilde{d}_i^m = |\tilde{x}_i^m - \bar{x}_i^m|$. For a fixed constant R $(0 < R < 1/2)$, we choose $y_1 \in \widetilde{\Omega}$ such that $B_1(y_1, R)$ is included in $\widetilde{\Omega}$ and contains $\bar{x}_1^m$. We introduce the functional

$$v(x) = e^{-\alpha|x - y_1|^2} - e^{-\alpha R^2}$$

where α is chosen such that $\Delta v \geqslant 0$ on $B_1 \setminus B_2(y_1, r)$ $(0 < r < R/2)$. The functional v satisfies

$$v(x) = 0 \quad \text{on } \partial B_1 \quad \text{and} \quad v(x) > 0 \quad \text{in } B_1.$$

Let A, a set independent of m, be such that

$$B_2(y_1, r) \subset A \subset \widetilde{\Omega}, \ \tilde{x}_2^m \in A \quad \text{and} \quad \bar{x}_2^m \in \partial A.$$

Observe that, for $x \in \partial B_2$, we have $|\bar{x}_2 - x| \geqslant 1/4$ and for each $t \in [\bar{x}_2, \tilde{x}_2]$, $|t - x| \geqslant 1/4$. Thus

$$G_A(x, \tilde{x}_2) = G_A(x, \bar{x}_2) + \frac{\partial G_A}{\partial n_2}(x, \bar{x}_2)\tilde{d}_2 + O\left(\sup_{t \in [\bar{x}_2, \tilde{x}_2]} \frac{\partial^2 G_A}{\partial n_2^2}(x, t)\tilde{d}_2^2\right).$$

Since $\bar{x}_2 \in \partial A$ then $G_A(x, \bar{x}_2) = 0$. Using Hopf Lemma,

$$G_A(x, \tilde{x}_2) \geqslant c(A)\tilde{d}_2 + O(\tilde{d}_2^2) \geqslant c\tilde{d}_2,$$

where c is independent of m. Since $A \subset \widetilde{\Omega}$ then

$$\widetilde{G} \geqslant G_A$$

and therefore for each $x \in \partial B_2$, $v(x) \leqslant M$. Thus, for each $x \in \partial(B_1 \setminus B_2)$

$$\widetilde{G}(x, \tilde{x}_2) - \frac{c\tilde{d}_2}{M} v(x) \geqslant 0.$$

Observe that

$$-\Delta\left(\widetilde{G}(., \tilde{x}_2) - \frac{c\tilde{d}_2}{M} v(.)\right) \geqslant 0 \quad \text{in } B_1 \setminus B_2.$$

Thus, using the maximum principle, we derive

$$\widetilde{G}(., \tilde{x}_2) - \frac{c\tilde{d}_2}{M} v(.) \geqslant 0 \quad \text{in } B_1 \setminus B_2.$$

The value of this functional at $\bar{x}_1$ is zero. Thus

$$\frac{\partial}{\partial n_1}\left(\widetilde{G}(\bar{x}_1, \tilde{x}_2) - \frac{c\tilde{d}_2}{M} v(\bar{x}_1)\right) \leqslant 0.$$

Therefore

$$\frac{\partial \widetilde{G}}{\partial n_1}(\bar{x}_1, \tilde{x}_2) \leqslant \frac{c\tilde{d}_2}{M} \frac{\partial v}{\partial n_1}(\bar{x}_1) \leqslant -c\tilde{d}_2.$$

We need to prove this estimate for $\partial \widetilde{G}/\partial n_1(\tilde{x}_1, \tilde{x}_2)$. Observe that

$$\frac{\partial \widetilde{G}}{\partial n_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial \widetilde{G}}{\partial n_1}(\bar{x}_1, \tilde{x}_2) + \mathrm{O}\left(\sup_{t \in [\bar{x}_1, \tilde{x}_1]} \frac{\partial^2 \widetilde{G}}{\partial n_1^2}(t, \tilde{x}_2)\tilde{d}_1\right).$$

For $t \in [\bar{x}_1, \tilde{x}_1]$, we need to estimate

$$\frac{\partial^2 \widetilde{G}}{\partial n_1^2}(t, \tilde{x}_2) = \frac{\partial^2 \widetilde{G}}{\partial n_1^2}(t, \bar{x}_2) + \mathrm{O}\left(\sup_{z \in [\bar{x}_2, \tilde{x}_2]} \frac{\partial^3 \widetilde{G}}{\partial n_1^2 \partial n_z}(t, z)\tilde{d}_2\right).$$

Since $\partial^2 \widetilde{G}/\partial n_1^2(t, \bar{x}_2) = 0$, a contradiction if we prove that

$$\frac{\partial^{\alpha+\beta} \widetilde{G}}{\partial x^\alpha \partial y^\beta} = \mathrm{O}(1) \quad \text{if } |x - y| \geqslant c.$$

We interest now to this proof. First, we introduce the sets

$$B_3(\bar{x}_1, 1/4) \cap \widetilde{\Omega} \quad \text{and} \quad B_4(\bar{x}_2, 1/4) \cap \widetilde{\Omega}.$$

For each $x \in B_3 \cap \widetilde{\Omega}$ and $y \in B_4 \cap \widetilde{\Omega}$, $\widetilde{G}(x, y)$ is a harmonic functional in $B_3 \cap \widetilde{\Omega}$. Thus

$$\widetilde{G}(x, y) = -\int_{\partial B_3 \cap \widetilde{\Omega}} \frac{\partial G_3}{\partial \nu}(t, x)\widetilde{G}(t, y)\, dt.$$

Therefore

$$\frac{\partial^\alpha \widetilde{G}}{\partial x^\alpha}(x, y) = - \int_{\partial B_3 \cap \widetilde{\Omega}} \frac{\partial^{\alpha+1} G_3}{\partial v \partial x^\alpha}(t, x)\widetilde{G}(t, y)\, dt = \mathrm{O}(1)$$

(depending only on B_3). We have also

$$\widetilde{G}(x, y) = - \int_{\partial B_4 \cap \widetilde{\Omega}} \frac{\partial G_4}{\partial v}(t, y)\widetilde{G}(t, x)\, dt.$$

Therefore

$$\frac{\partial^{\alpha+\beta} \widetilde{G}}{\partial x^\alpha \partial y^\beta}(x, y) = - \int_{\partial B_4 \cap \widetilde{\Omega}} \frac{\partial^{\beta+1} G_4}{\partial v \partial y^\beta}(t, y)\frac{\partial^\alpha \widetilde{G}}{\partial x^\alpha}(t, x)\, dt = \mathrm{O}(1)$$

(depending on B_3 and B_4). The result follows.

We left now for the second case. We introduce the following transformation

$$T : \Omega \to \widetilde{\Omega} = T(\Omega)$$

$$x \to \tilde{x} = x/d_2$$

then $\tilde{d}_2 = 1$, $\tilde{d}_1 = d_1/d_2 = \mathrm{o}(1)$ and $|\tilde{x}_1 - \tilde{x}_2| = |x_1 - x_2|/d_2 \geqslant c_2$. As in the first case, we introduce the set A such that

$$A \subset \widetilde{\Omega}, \quad \bar{x}_1 \in \partial A, \quad \tilde{x}_1 \in A \quad \text{and} \quad \tilde{x}_2 \in A$$

(A is a compact set independent of m). Observe that

$$\widetilde{G} - G_A \geqslant 0 \quad \text{in } A \quad \text{and} \quad \widetilde{G}(\bar{x}_1, \tilde{x}_2) - G_A(\bar{x}_1, \tilde{x}_2) = 0.$$

Thus

$$\frac{\partial}{\partial n_1}\big(\widetilde{G}(\bar{x}_1, \tilde{x}_2) - G_A(\bar{x}_1, \tilde{x}_2)\big) \leqslant 0.$$

Therefore

$$\frac{\partial \widetilde{G}}{\partial n_1}(\bar{x}_1, \tilde{x}_2) \leqslant -c$$

(c is independent of m).

$$\frac{\partial \widetilde{G}}{\partial n_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial \widetilde{G}}{\partial n_1}(\bar{x}_1, \tilde{x}_2) + \mathrm{O}\left(\sup_{t \in [\bar{x}_1, \tilde{x}_1]} \frac{\partial^2 \widetilde{G}}{\partial n_1^2}(t, \tilde{x}_2)\tilde{d}_1\right) \leqslant -c + \mathrm{O}(\tilde{d}_1) \leqslant -c/2$$

a contradiction with the assumptions.

The third case is proved in the appendix of [8]. It relies on the following estimates

$$\frac{\partial}{\partial n_1}\left(\frac{1}{|x_1 - x_2|^{n-2}}\right) = -(n-2)\frac{(x_1 - x_2)n_1}{|x_1 - x_2|^n} = -(n-2)\frac{d_2 - d_1 + \mathrm{o}(d_1)}{|x_1 - x_2|^n}$$

and

$$\frac{\partial H}{\partial n_1}(x_1, x_2) = \frac{-(n-2)}{|x_1 - x_2 + 2d_1 n_1|^n}(x_1 - x_2 + 2n_1.(x_1 - x_2)n_1 - 2d_1 n_1).n_1 + \mathrm{o}(d_1)$$

$$= \frac{(n-2)(d_1 + d_2)}{(|x_1 - x_2|^2 + 4d_1 d_2)^{n/2}} + \mathrm{o}\left(\frac{1}{d_1^{n-1}}\right) > 0. \qquad \square$$

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