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VORTEX PINNING WITH BOUNDED FIELDS FOR THE GINZBURG–LANDAU EQUATION

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ABSTRACT. – We investigate vortex pinning in solutions to the Ginzburg–Landau equation. The coefficient, $a(x)$, in the Ginzburg–Landau free energy modeling non-uniform superconductivity is nonnegative and is allowed to vanish at a finite number of points. For a sufficiently large applied magnetic field and for all sufficiently large values of the Ginzburg–Landau parameter $\kappa = 1/\varepsilon$, we show that minimizers have nontrivial vortex structures. We also show the existence of local minimizers exhibiting arbitrary vortex patterns, pinned near the zeros of $a(x)$.

RÉSUMÉ. – On étudie la localisation des vortex des solutions de l'équation de Ginzburg–Landau. Dans l'énergie libre de Ginzburg–Landau, le coefficient $a(x)$ modélise la supraconductivité non uniforme. Ce coefficient est positif et s'annule en un nombre fini de points. On montre que, pour un champ magnétique assez grand et pour toutes les valeurs du paramètre de Ginzbug–Landau $\kappa = 1/\varepsilon$ assez grandes, les minimiseurs présentent des structures de vortex non triviales. On montre aussi l'existence de minimiseurs locaux présentant une structure prescrite de vortex situés au voisinage des zéros de $a(x)$.

Introduction

In this paper we analyze several aspects of vortex pinning in superconductivity using the Ginzburg–Landau theory as our model. To describe these phenomena consider the energy

$$J_\varepsilon(\psi, A) = \int_\Omega \left[\left|(\nabla - iA)\psi\right|^2 + |\nabla \times A - h_e \mathbf{e}_3|^2 + \frac{1}{2\varepsilon^2}\left(a - |\psi|^2\right)^2 \right] \tag{1}$$

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for $\varepsilon > 0$. Here Ω is a bounded simply connected domain in $\mathbb{R}^2$ with a smooth ($C^{2,1}$) boundary and $a:\Omega \to \mathbb{R}$. The domain Ω represents the cross-section of an infinite cylindrical body with $\mathbf{e}_3$ as its generator. The body is subjected to an applied magnetic field, $h_e\mathbf{e}_3$ where $h_e \geqslant 0$ is constant. The function $A:\Omega \to \mathbb{R}^2$ is the magnetic potential and $\nabla \times A = \nabla \times (A^1, A^2, 0)$ is the induced magnetic field in the cylinder. The function ψ is complex-valued where $|\psi|^2 = \psi^*\psi$ represents the density of superconducting election pairs and

$$j = -\frac{i}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2 A \tag{2}$$

denotes the superconducting current density circulating in the cross-section Ω. The parameter $\varepsilon = 1/\kappa$ is a positive number where κ is the Ginzburg–Landau parameter associated to the material. We analyze the small ε (large κ) regime. It is here that vortex dominated current patterns are expected in stable equilibria for J_ε. The prototypical picture of this phenomenon is that of a finite number of non-superconducting points in Ω (at which $\psi = 0$, called vortices), each of which is surrounded by a ring of the super current j.

If the material is homogeneous, the function a in J_ε is taken to be a constant, proportional to $T_c - T$. Here T is the body's temperature and T_c is the material's critical temperature. For $T \geqslant T_c$ ($a \leqslant 0$), it is easy to show that the only equilibria for J_ε are completely non-superconducting and have $\psi \equiv 0$, $\nabla \times A \equiv h_e\mathbf{e}_3$. For $T < T_c$ ($a > 0$), superconducting minimizers exist if the applied field strength h_e is not too large. There are a number of mathematical investigations of the relationship between h_e and the nature of stable superconducting states for this case. In [11] Sandier and Serfaty showed that there exists a constant H_{c_1} proportional to $|\log(\varepsilon)|$ as $\varepsilon \to 0$, such that if $h_e \leqslant H_{c_1}$, then minimizers for J_ε are purely superconducting, satisfying $|\psi| > 0$ in Ω. In [12] they showed that for h_e slightly greater than H_{c_1} and such that $h_e \ll \varepsilon^{-2}$, minimizers are in a mixed state having a vortex-like structure. It was shown by Giorgi and Phillips in [5] that for $h_e \geqslant C\varepsilon^{-2}$ for some constant C, superconductivity is completely suppressed, in that all equilibria for J_ε have $\psi \equiv 0$.

Inhomogeneous superconducting materials can arise naturally due to material defects or the presence of grain boundaries. Inhomogeneities can be inserted intentionally, as well, by adding non-superconducting (normal) impurities to the material. (See [3] and [4].) A consequence of having material inhomogeneities is that they tend to pin or stabilize supercurrent patterns. The classical Ginzburg–Landau theory can be modified to take normal inclusions into account. This is done by having the critical temperature, T_c, depend on position which is equivalent to having $a = a(x)$. (See [10].) It is possible that $a(x)$ may vanish or change sign within the domain.

A mathematical study for the Ginzburg–Landau equations corresponding to the energy (1) with variable $a(x)$ was done by Aftalion, Sandier, and Serfaty in [1] where the case $\frac{1}{2} \leqslant a(x) \leqslant 1$ was considered. They proved among other things, that H_{c_1} remains of order $|\log(\varepsilon)|$ as $\varepsilon \to 0$. In this paper we consider the case where Ω contains a finite number of point impurities, $\{x_1, \ldots, x_n\}$, and that $a(x)$ vanishes at these normal sites. In this instance, the strong pinning enables us to show that the transition threshold for h_e, denoted by $H_{c_1} = H_{c_1}(\varepsilon)$, separating the presence or absence of vortices, is of order 1 as $\varepsilon \to 0$. (See Corollary 4.4.) In addition, for each h_e and all ε sufficiently small, we show that there are local minimizers for J_ε with prescribed vortex structure about each of the x_i corresponding to the homotopy classes in $\Omega \setminus \{x_1, \ldots, x_n\}$. (See Theorem 4.6.) In this way we are able to pin supercurrent patterns near the zeros of $a(x)$. (See Theorem 4.6.)

Another way of introducing inhomogeneities is by making holes (voids) in the body. In [8,9,13] J_ε was studied with $a = 1$, $h_e = 0$ but with Ω multiply connected by Jimbo and Morita, Jimbo and Zhai, and Rubinstein and Sternberg, respectively. In that setting, local minimizers with prescribed vortex structures associated to the homotopy classes of Ω were shown to exist.

We require that $a(x)$ satisfy the following.

Assume: $a \in C^1(\overline{\Omega}\backslash\{x_1, \ldots, x_n\}) \cap C^\beta(\Omega)$ for some $\beta > 0$, $\sqrt{a} \in H^1(\Omega)$, $a(x) \geqslant 0$ for all x in $\overline{\Omega}$, and $a(x) = 0$ iff $x \in \{x_1, \ldots, x_n\}$ where $x_1, \ldots, x_n$ are distinct points in Ω and $n \in \mathbb{N}$. Moreover, assume that there are positive constants m_i, M_i and α_i so that $m_i|x - x_i|^{\alpha_i} \leqslant a(x) \leqslant M_i|x - x_i|^{\alpha_i}$ in some neighborhood $\mathcal{U}_i$ of x_i for $1 \leqslant i \leqslant n$.

DEFINITION. – *Let $\varepsilon > 0$ and let $(\psi_\varepsilon, A_\varepsilon) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \equiv \mathcal{M}$. Then $(\psi_\varepsilon, A_\varepsilon)$ is an equilibrium for J_ε if and only if $(\psi_\varepsilon, A_\varepsilon)$ is a weak solution of the Euler–Lagrange equations and natural boundary conditions for critical points of J_ε in $\mathcal{M}$, namely*:

$$\begin{gathered} -(\nabla - iA_\varepsilon)^2\psi_\varepsilon = \frac{1}{\varepsilon^2}\big(a - |\psi_\varepsilon|^2\big)\psi_\varepsilon \quad \text{in } \Omega, \\ (\nabla - iA_\varepsilon)\psi_\varepsilon \cdot n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3}$$

and

$$\begin{gathered} \nabla \times \nabla \times A_\varepsilon = -\frac{i}{2}\big(\psi_\varepsilon^*\nabla\psi_\varepsilon - \psi_\varepsilon\nabla\psi_\varepsilon^*\big) - |\psi_\varepsilon|^2A_\varepsilon \equiv j_\varepsilon \quad \text{in } \Omega, \\ \nabla \times A_\varepsilon = h_e\mathbf{e_3} \quad \text{on } \partial\Omega. \end{gathered} \tag{4}$$

For $\varepsilon = 0$ we set

$$J_0(\psi, A) = \int_\Omega \big[\big|(\nabla - iA)\psi\big|^2 + |\nabla \times A - h_e\mathbf{e}_3|^2\big]. \tag{5}$$

Denote

$$H_a^1 = \big\{\psi \in H^1(\Omega; \mathbb{C}) \text{ such that } |\psi| = \sqrt{a} \text{ almost everywhere}\big\}.$$

Note that H_a^1 is nonempty, since $\sqrt{a} \in H_a^1$ by our assumptions on a. We prove in Section 1 (see Theorem 1.4) that each $\psi \in H_a^1$ can be written as $\psi = \sqrt{a}\,\mathrm{e}^{i\theta(x)}$, where $\theta(x) = \theta_0(x) + \sum_{i=1}^n d_i\theta_i(x)$, θ_0 is a measurable function determined up to an additive constant, $2\pi k$ for $k \in Z$, satisfying $\int_\Omega a|\nabla\theta_0|^2 < \infty$, $D = (d_1, \ldots, d_n) \in Z^n$ is uniquely determined, and $\theta_i(x)$ is the azimuthal angle about x_i for $1 \leqslant i \leqslant n$ (so that $(\cos\theta_i(x), \sin\theta_i(x)) = (x - x_i)/|x - x_i|$ for all $x \neq x_i$ in $\mathbb{R}^2$). Thus ψ corresponds to a unique $D \in Z^n$ describing a homotopy class for ψ in $\Omega\backslash\{x_1, \ldots, x_n\}$. We write

$$H_a^1 = \bigcup_{D\in Z^n} H_{a,D}^1.$$

We note that $H^1_{a,D}$ is both open and closed in H^1_a and that if $\{u_n\} \subset H^1_{a,D}$ such that $u_n \rightharpoonup u$ in H^1 then $u \in H^1_{a,D}$. (See Theorem 1.5.)

DEFINITION. – *Let $(\psi_0, A_0) \in H^1_a \times H^1(\Omega; \mathbb{R}^2) \equiv \mathcal{M}_0$. Then (ψ_0, A_0) is an equilibrium for J_0 if and only if (ψ_0, A_0) is a weak solution of the Euler–Lagrange equations and natural boundary conditions for critical points of J_0 in $\mathcal{M}_0$, namely*:

$$
\begin{aligned}
&\operatorname{div}\left[-\frac{i}{2}\left(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right) - |\psi_0|^2 A_0\right] = 0 \quad \textit{in } \Omega,\\
&\left[-\frac{i}{2}\left(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right) - |\psi_0|^2 A_0\right]\cdot \vec{n} = 0 \quad \textit{on } \partial\Omega,
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
&\nabla\times\nabla\times A_0 = \left[-\frac{i}{2}\left(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right) - |\psi_0|^2 A_0\right] \equiv j_0 \quad \textit{in } \Omega,\\
&\nabla\times A_0 = h_e \mathbf{e}_3 \quad \textit{on } \partial\Omega.
\end{aligned}
\tag{7}
$$

The functionals J_ε, for $\varepsilon \geqslant 0$, are gauge invariant. By this we mean that if $(\psi, A) \in \mathcal{M}$ $(\mathcal{M}_0)$ and if $\phi \in H^2(\Omega)$, then the gauge transformation, $(\psi', A') = G_\phi(\psi, A)$ defined by

$$
\begin{aligned}
\psi' &\equiv \psi\, \mathrm{e}^{i\phi},\\
A' &\equiv A + \nabla\phi,
\end{aligned}
$$

satisfies $(\psi', A') \in \mathcal{M}$ $(\mathcal{M}_0)$, $J_\varepsilon(\psi, A) = J_\varepsilon(\psi', A')$, and (ψ', A') is an equilibrium for $J_\varepsilon(J_0)$ if (ψ, A) is one. In this paper we will fix a gauge by requiring (without loss of generality) that A satisfy

$$
\begin{aligned}
\operatorname{div} A &= 0 \quad \text{in } \Omega,\\
A\cdot n &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{8}
$$

since this can be accomplished by an appropriate gauge transformation. With this choice of gauge (the Coulomb gauge), A is determined from the value of $\nabla\times A = (\partial_x A_1 - \partial_y A_2)\mathbf{e}_3 \equiv h\mathbf{e}_3$ by first solving

$$
\begin{aligned}
\Delta\xi &= h \quad \text{in } \Omega,\\
\xi &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\tag{9}
$$

From (8), (9), and the fact that Ω is simply connected we have $A = \nabla^\perp\xi$ where $(\partial_x, \partial_y)^\perp \equiv (-\partial_y, \partial_x)$. An important feature of the gauge choice (8) is that the boundary conditions in (3) and (6) can be replaced by

$$
\nabla\psi\cdot\vec{n} = 0 \quad \text{on } \partial\Omega
$$

and, since $\nabla\times\nabla\times A = -\Delta A + \nabla(\operatorname{div} A)$, the term $\nabla\times\nabla\times A$ in Eqs. (4) and (7) is equal to $-\Delta A$.

We establish the following main results in this paper.

THEOREM 1. – *Fix $h_e \geqslant 0$. For each $D \in Z^n$, J_0 has an equilibrium (with our choice of gauge), (ψ_D, A_D), in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. Moreover, (ψ_D, A_D) is unique up to uniform rotations of ψ_D in Ω, $\psi_D \to \psi_D\, \mathrm{e}^{ic}$ for $c \in \mathbb{R}$.* (*See Theorem* 3.2.)

We remark that $(\psi, A) \to (\psi\, \mathrm{e}^{ic}, A)$ is a gauge transformation in $\mathcal{M}$ ($\mathcal{M}_0$), and thus $J_\varepsilon(\psi, A) = J_\varepsilon(\psi\, \mathrm{e}^{ic}, A)$ for all $c \in \mathbb{R}$ and $\varepsilon \geqslant 0$.

THEOREM 2. – *Fix $h_e \geqslant 0$. Let $(\psi_{\varepsilon_k}, A_{\varepsilon_k})$ be an equilibrium for J_{ε_k} for $k = 1, 2, \ldots$ such that $\varepsilon_k \to 0^+$ and*

$$\liminf_{k\to\infty} J_{\varepsilon_k}(\psi_{\varepsilon_k}, A_{\varepsilon_k}) \leqslant c < \infty. \tag{10}$$

There exists a finite subset $\mathcal{D} = \mathcal{D}(c, h_e)$ of Z^n, a subsequence $\{\varepsilon_{k_\ell}\}$, and $(\psi^0, A^0) \in H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$ for some $D \in \mathcal{D}$ such that

$$(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) \rightharpoonup (\psi^0, A^0) \quad \text{in } \mathcal{M}.$$

Moreover (ψ^0, A^0) is an equilibrium for J_0. (*See Theorem* 4.1.)

Note that

$$J_\varepsilon(\sqrt{a}) = |\Omega| h_e^2 + \int_\Omega |\nabla \sqrt{a}|^2 \quad \text{for } \varepsilon \geqslant 0. \tag{11}$$

Thus, given h_e, it follows from Theorem 2 that a sequence of minimizers with $\varepsilon_k \to 0^+$ will satisfy (10).

THEOREM 3. – *Fix $h_e \geqslant 0$. Let $(\psi_{\varepsilon_k}, A_{\varepsilon_k})$ be a minimizer of J_{ε_k} in $\mathcal{M}$ for $k = 1, 2, \ldots$ with $\varepsilon_k \to 0+$. Then a subsequence $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) \to (\psi_D, A_D)$ in $\mathcal{M}$, where (ψ_D, A_D) is a minimizer of J_0 in $\mathcal{M}_0$ and $(\psi_D, A_D) \in H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. Moreover, if $R > 0$ and $\overline{B_R(x_i)}$ are disjoint subsets of Ω for $i = 1, \ldots, n$, then for all ℓ sufficiently large, $|\psi_{\varepsilon_{k_\ell}}| > 0$ outside $\bigcup_{i=1}^n B_R(x_i)$ and the degree of $\psi_{\varepsilon_{k_\ell}}$ in $\overline{B_R(x_i)}$ is d_i for all $i \in \{1, \ldots, n\}$ where $D = (d_1, \ldots, d_n)$.* (*See Theorem* 4.2.)

We prove in Corollary 3.6 that for $h_e \geqslant 0$ fixed, the set of all D in Z^n such that $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$ contains a minimizer of J_0 in $\mathcal{M}_0$ is a nonempty finite set (depending only on Ω, $a(x)$, and h_e), which we denote by $\mathcal{D}_0 = \mathcal{D}_0(h_e)$.

THEOREM 4. – *Let $(\psi_\varepsilon, A_\varepsilon)$ be a minimizer of J_ε for each $\varepsilon > 0$. Fix $R > 0$ as in Theorem 3 and $h_e \geqslant 0$. There exists $\varepsilon_0 = \varepsilon_0(R, h_e) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $|\psi_\varepsilon| > 0$ outside $\bigcup_{i=1}^n B_R(x_i)$ and the degree of ψ_ε in $\overline{B_R(x_i)}$ for $i = 1, \ldots, n$, denoted by $D_\varepsilon = (d_{1,\varepsilon}, \ldots, d_{n,\varepsilon})$, is in $\mathcal{D}_0$. Moreover, there exists $\overline{h_e} > 0$ (depending only on Ω and $a(x)$) such that if $h_e > \overline{h_e}$ and $0 < \varepsilon < \varepsilon_0(R, h_e)$, then $D_\varepsilon \neq \vec{0}$.* (*See Theorem* 4.3.)

We remark that Theorem 4 implies that $\{H_{c_1}(\varepsilon)\}$ is uniformly bounded in ε as $\varepsilon \to 0^+$. (See Corollary 4.4.)

The equilibrium found in Theorem 1 is (by uniqueness) the minimizer for J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. Since $H^1_{a,D}$ is open in H^1_a, it is also a local minimizer for J_0 in $\mathcal{M}_0$. Given $h_e \geqslant 0$, let (ψ_D, A_D) be such a solution. For local minimizers of J_ε in $\mathcal{M}$, we have (in contrast to Theorem 4) that all degrees in Z^n near $x_1, \ldots, x_n$ are attainable:

THEOREM 5. – *Fix $h_e \geqslant 0$ and any D in Z^n. For each $\varepsilon > 0$ sufficiently small, there exists a local minimizer, $(\psi_\varepsilon, A_\varepsilon)$, of J_ε in $\mathcal{M}$ such that $(\psi_\varepsilon, A_\varepsilon) \to (\psi_D, A_D)$ in $\mathcal{M}$ as $\varepsilon \to 0$. In addition, for any $R > 0$ as in Theorem 3, there exists $\varepsilon_1(R, h_e) > 0$ such that $|\psi_\varepsilon| > 0$ outside $\bigcup_{i=1}^n B_R(x_i)$, and the degree of ψ_ε in $\overline{B_R(x_i)}$ is d_i for all $\varepsilon < \varepsilon_1$, where $D = (d_1, \ldots, d_n)$. (See Theorem 4.6.)*

1. Preliminaries

It is well known that if $(\psi, A) \in \mathcal{M}$ and $\psi = \rho\, e^{i\theta}$, then $\nabla\theta$ is uniquely determined almost everywhere in $\{\rho > 0\}$, $\rho \in W^{1,2}(\Omega)$, $\rho\nabla\theta \in L^2(\Omega; \mathbb{R}^2)$,

$$\begin{aligned} &\left|(\nabla - iA)\psi\right|^2 = |\nabla\rho|^2 + \left|\rho(\nabla\theta - A)\right|^2 \\ &\text{and} \quad j \equiv -\frac{i}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2 A = \rho^2(\nabla\theta - A) \text{ a.e. in } \Omega. \end{aligned} \tag{12}$$

If $(\psi_\varepsilon, A_\varepsilon) \in \mathcal{M}$ and $(\psi_\varepsilon, A_\varepsilon)$ is an equilibrium for J_ε with $\varepsilon > 0$, then from (3) we can derive the equations

$$\begin{aligned} -\operatorname{div}(\rho_\varepsilon\nabla\rho_\varepsilon) + |\nabla\rho_\varepsilon|^2 + \frac{|j_\varepsilon|^2}{\rho_\varepsilon^2} &= \frac{1}{\varepsilon^2}(a - \rho_\varepsilon^2)\rho_\varepsilon^2 \quad \text{in } \Omega, \\ \rho_\varepsilon\nabla\rho_\varepsilon \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega, \\ \operatorname{div} j_\varepsilon &= 0 \quad \text{in } \Omega, \quad \text{and} \\ j_\varepsilon \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{13}$$

where $\psi_\varepsilon = \rho_\varepsilon\, e^{i\theta_\varepsilon}$ and $j_\varepsilon = \rho_\varepsilon^2(\nabla\theta_\varepsilon - A_\varepsilon)$. These equations are obtained by using test functions of the form $\varphi = \psi_\varepsilon^*\phi$ in the formulation (3) such that $\phi \in L^\infty(\Omega)$ and $(1 + |\psi_\varepsilon|)|\nabla\phi| \in L^2(\Omega)$. Moreover, if we define h_ε by $\nabla \times A_\varepsilon = h_\varepsilon\mathbf{e}_3$ then (4) can be rewritten as

$$\begin{aligned} -\nabla^\perp h_\varepsilon \equiv (\partial_y, -\partial_x)h_\varepsilon &= j_\varepsilon \quad \text{in } \Omega, \\ h_\varepsilon - h_{\mathbf{e}} &\quad \text{on } \partial\Omega. \end{aligned} \tag{14}$$

Similarly, if $(\psi^0, A^0) \in \mathcal{M}_0$ and (ψ^0, A^0) is an equilibrium for J_0 then (6) and (7) can be rewritten as

$$\begin{aligned} \operatorname{div} j_0 &= 0 \quad \text{in } \Omega, \\ j_0 \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{15}$$

and

$$\begin{aligned} -\nabla^\perp h_0 &= j_0 \quad \text{in } \Omega, \\ h_0 &= h_e \quad \text{on } \partial\Omega, \end{aligned} \tag{16}$$

where $\psi^0 = \rho_0\, e^{i\theta_0} = \sqrt{a}\, e^{i\theta_0}$, h_0 is defined by $\nabla \times A^0 = h_0\mathbf{e}_3$, and $j_0 = \rho_0^2(\nabla\theta_0 - A^0)$.

The following three results concern maximum principles and regularity for equilibria of J_ε. The proofs are only a slight variation of the proofs for the case in which $a \equiv 1$ in Ω. (See [5] and [6].)

LEMMA 1.1. – *If* $(\psi_\varepsilon, A_\varepsilon) \in \mathcal{M}(\mathcal{M}_0)$ *and* $(\psi_\varepsilon, A_\varepsilon)$ *is an equilibrium for* $J_\varepsilon(J_0)$ *where* $\varepsilon \geqslant 0$, *then* $|\psi_\varepsilon| \leqslant \sup_\Omega \sqrt{a}$.

Proof. – For $(\psi_0, A_0) \in \mathcal{M}_0$, we have $|\psi_0| = \sqrt{a}$ in Ω and hence the result is trivial in this case. If $\varepsilon > 0$ and $(\psi_\varepsilon, A_\varepsilon)$ is an equilibrium for J_ε, the result follows by using

$$\phi_\varepsilon \equiv \max\{0, |\psi_\varepsilon(x)| - \sup_\Omega \sqrt{a}\}/|\psi_\varepsilon(x)| = \left(\rho_\varepsilon(x) - \sup_\Omega \sqrt{a}\,\right)^+/\rho_\varepsilon(x)$$

as a test function in the weak formulation of the first two equations in (13), which yields

$$0 \leqslant \int_E |\nabla \rho_\varepsilon|^2 = \int_E (-\phi_\varepsilon) \cdot \frac{|j_\varepsilon|^2}{\rho_\varepsilon^2} + \frac{1}{\varepsilon^2} \int_E \left(a - \rho_\varepsilon^2\right)\rho_\varepsilon^2 \phi_\varepsilon \leqslant 0,$$

where $E = \{x \in \Omega\colon \phi_\varepsilon(x) > 0\}$. It follows that E has zero measure. Thus $\phi_\varepsilon \leqslant 0$ a.e. in Ω which proves the lemma. □

LEMMA 1.2. – *For* $\varepsilon > 0$ *equilibria are of class,* $C^{2,\beta}(\overline{\Omega})$ *for some* $\beta > 0$.

Proof. – With our choice of gauge (8), we have $\nabla \times \nabla \times A_\varepsilon = -\Delta A_\varepsilon$. The system (3) and (4) is thus uniformly elliptic and regularity follows from the classical theory. (See [6].) □

LEMMA 1.3. – *Fix* $h_e \geqslant 0$. *Assume* $\varepsilon \geqslant 0$ *and* $(\psi_\varepsilon, A_\varepsilon)$ *is an equilibrium for* J_ε.
Set $M = \max(J_\varepsilon(\psi_\varepsilon, A_\varepsilon), J_\varepsilon(\sqrt{a}, 0), \max_\Omega a)$. *Then*

$$\|A_\varepsilon\|_{2,2} \leqslant C(M, \Omega), \tag{17}$$

$$\|\psi_\varepsilon\|_{1,2} \leqslant C(M, \Omega), \tag{18}$$

and if $\varepsilon > 0$

$$|\nabla \psi_\varepsilon| \leqslant C(M, \Omega)/\varepsilon \quad \text{in } \Omega, \tag{19}$$

where $C(M, \Omega)$ *denotes a constant depending only on* $M, a(x)$, *and* Ω, *and the subscript* $k, 2$ *denotes the norm in* $W^{k,2}(\Omega)$.

Proof. – We argue for $\varepsilon > 0$. The proofs of (17) and (18) for the case $\varepsilon = 0$ are identical.

We write (using (12))

$$J_\varepsilon(\psi, A) = \int_\Omega \left[|\nabla|\psi||^2 + |\psi|^2 |\nabla\theta - A|^2 + \frac{1}{2\varepsilon^2}\left(a - |\psi|^2\right)^2 + |\nabla \times A - h_e \mathbf{e}_3|^2 \right].$$

Recall that $j_\varepsilon = |\psi_\varepsilon|^2(\nabla\theta_\varepsilon - A_\varepsilon)$ and h_ε is defined by

$$h_\varepsilon \mathbf{e}_3 = \nabla \times A_\varepsilon. \tag{20}$$

From this and (12), we have

$$J_\varepsilon(\psi_\varepsilon, A_e) = \int_\Omega \left[|\nabla|\psi_\varepsilon||^2 + |\psi_\varepsilon|^{-2}|j_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\left(a - |\psi_\varepsilon|^2\right)^2 + |h_\varepsilon - h_e|^2 \right]. \tag{21}$$

Thus $\|j_\varepsilon\|_2^2 \leqslant \sup_\Omega a \cdot J_\varepsilon(\psi_\varepsilon, A_\varepsilon) \leqslant C(M, \Omega)$, where $\|j_\varepsilon\|_2$ denotes the L^2 norm of j_ε in Ω. Then from (14), we have

$$\|\nabla h_\varepsilon\|_2 \leqslant C(M.\Omega).$$

Using this estimate together with (9) we see that $\|\nabla\xi\|_{2,2} \leqslant C(M, \Omega)$. Thus

$$\|A_\varepsilon\|_{2,2} = \|\nabla\xi\|_{2,2} \leqslant C(M, \Omega).$$

Note that this implies

$$\|A_\varepsilon\|_{C^\gamma(\overline{\Omega})} \leqslant C(M, \Omega, \gamma) \quad \text{for each } \gamma \in (0, 1). \tag{22}$$

Now

$$\|\nabla\psi_\varepsilon\|_2^2 \leqslant C\left(\left\|(\nabla - iA_\varepsilon)\psi_\varepsilon\right\|_2^2 + \|A_\varepsilon\psi_\varepsilon\|_2^2\right).$$

So we see

$$\|\nabla\psi_\varepsilon\|_2^2 \leqslant C(M, \Omega).$$

This proves (17) and (18) for $\varepsilon > 0$ (and $\varepsilon = 0$).

To prove (19) let $y = x/\varepsilon$, $\Omega_\varepsilon = \Omega/\varepsilon$, $\widetilde{\psi}_\varepsilon(y) = \psi_\varepsilon(\varepsilon y)$, and $\tilde{A}_\varepsilon = \varepsilon A_\varepsilon$. We have from the Ginzburg–Landau equation

$$\begin{aligned} \Delta_y\widetilde{\psi}_\varepsilon - 2i\tilde{A}_\varepsilon \cdot \nabla_y\widetilde{\psi}_\varepsilon - |\tilde{A}_\varepsilon|^2\widetilde{\psi}_\varepsilon &= \left(a(\varepsilon y) - |\widetilde{\psi}_\varepsilon|^2\right)^2\widetilde{\psi}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \partial_n\widetilde{\psi}_\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

Here we have used the choice of gauge. From (22) we see that $|\tilde{A}_\varepsilon(y)| = |\varepsilon A_\varepsilon(y)| \leqslant \varepsilon C(M, \Omega)$. It follows from local elliptic estimates and Lemma 1.1 that $\widetilde{\psi}_\varepsilon \in W^{2,p}(\Omega_\varepsilon)$ for $p < \infty$ and

$$|\nabla_y\widetilde{\psi}_\varepsilon| \leqslant C(M, \Omega) \quad \text{in } \Omega_\varepsilon \quad \text{for } 0 < \varepsilon \leqslant 1.$$

(Here we use that $\partial\Omega$ is of class $C^{2,1}$.) Thus $|\nabla\psi_\varepsilon| \leqslant C(M, \Omega)/\varepsilon$ in Ω. □

The remaining results in this section are facts about

$$H_a^1 = \left\{\psi \in H^1(\Omega; \mathbb{C}): \ |\psi| = \sqrt{a} \text{ a.e. in } \Omega\right\}$$

which are used later in this paper.

THEOREM 1.4. – *Each $u \in H_a^1$ can be written as*

$$u(x) = \sqrt{a(x)} \cdot \prod_{j=1}^n \left(\frac{z - z_j}{|z - z_j|}\right)^{d_j} \cdot e^{i\varphi(x)} = \sqrt{a(x)} \cdot e^{i\theta(x)}$$

where $z = z(x) = x^1 + ix^2$ *for* $x = (x^1, x^2)$ *in* Ω*,* $z_j = z(x_j)$, $\varphi \in H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$, $\theta(x) = \varphi(x) + \sum_{j=1}^n d_j \theta_j(x)$, *and* $\theta_j(x)$ *is the azimuthal angle of* x *about* x_j *for* $1 \leqslant j \leqslant n$*. Moreover, for each* $u \in H^1_a$, $D \equiv (d_1, \ldots, d_n) \in Z^n$ *is unique,* $\varphi \in H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$ *is unique up to an additive constant* $2\pi k$ *for* $k \in Z$*, and* φ *satisfies* $\int_\Omega a|\nabla\varphi|^2 \leqslant C(\Omega, a, D) + \int_\Omega |\nabla u|^2$.

Proof. – Fix $u \in H^1_a$ and set $v(x) = u(x)/\sqrt{a(x)} = u(x)/|u(x)|$. Then $v \in H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\}; \mathbb{S}^1)$ where $\mathbb{S}^1 = \{z \in \mathbb{C}\colon |z| = 1\})$. It follows from Schoen and Uhlenbeck [14] that there exists a sequence $\{v_m\}$ such that

$$v_m \in C^2\left(\overline{\Omega} \setminus \bigcup_{j=1}^n B_{\frac{1}{m}}(x_j); \mathbb{S}^1\right)$$

for $m = 1, 2, 3, \ldots$ and

$$v_m \to v \quad \text{in } H^1_{\mathrm{loc}}\big(\Omega \setminus \{x_1, \ldots, x_n\}\big) \text{ as } m \to \infty.$$

(See also [2].) We compute the degree of each v_m near x_j; as follows:

We say that a radius r is *admissible* for a given x_j and v_m if $\overline{B_r(x_j)} \cap \{x_1, \ldots, x_n\} = \{x_j\}$ and $\partial B_r(x_j) \subset \Omega \setminus \bigcup_{i=1}^n B_{\frac{1}{m}}(x_i)$. For any such r, since v_m is smooth and $|v_m| = 1$ in $\Omega \setminus \bigcup_{i=1}^n B_{\frac{1}{m}}(x_1, \ldots, x_n)$, the winding number of v_m on $\partial B_r(x_j)$ is defined by:

$$d_{j,m} = -\frac{i}{2\pi} \int\limits_{\partial B_r(x_j)} v_m^*(v_m)_\tau \tag{23}$$

where $\tau = \nu^\perp = (-\nu_2, \nu_1)$, ν is the exterior unit normal on the boundary of $B_r(x_j)$, and $(v_m)_\tau$ is the derivative of v_m in the direction τ. It is well known from degree theory that $d_{j,m}$ is integer-valued and independent of r for all admissible r with respect to x_j and m. Thus if $0 < r_1 < r_2 < \infty$ and r_2 satisfies $\overline{B_{r_2}(x_j)} \subset \Omega$ and $\overline{B_{r_2}(x_j)} \cap \{x_1, \ldots, x_m\} = \{x_j\}$, then for all m sufficiently large, any $r \in [r_1, r_2]$ is admissible for x_j and v_m, and we may integrate (23) to obtain

$$d_{j,m} = -\frac{i}{2\pi(r_2 - r_1)} \int\limits_{B_{ra}(x_j) \setminus B_{r_1}(x_j)} v_m^*(v_m)_\tau \, dx. \tag{24}$$

Since $d_{j,m}$ is integer-valued and $v_m \to v$ in $H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$ as $m \to \infty$, it follows from (24) that $d_{j,m}$ is independent of m for all m sufficiently large. Thus there exists $d_j \in Z$ such that $d_j = d_{j,m}$ for all m sufficiently large and, letting $m \to \infty$, we have

$$d_j = -\frac{i}{2\pi(r_2 - r_1)} \int\limits_{B_{r_2}(x_j) \setminus B_{r_1}(x_j)} v^* v_\tau \, dx. \tag{25}$$

We may use this to define the degree of v near x_j, since (25) is independent of $r_2 > r_1 > 0$ provided that $\overline{B_{r_2}(x_j)} \subset \Omega$ and $\overline{B_{r_2}(x_j)} \cap \{x_1, \ldots, x_n\} = \{x_j\}$ and it is clear (25) is independent of the particular converging sequence $\{v_m\}$. In particular, we can define the degree of u in $\overline{B_r(x_j)}$ by (25), for $v = u/|u|$ and $r_1 < r_2$ as above. (See also [7].)

Now consider the real two-dimensional vector field

$$F_m = -\sum_{j=1}^{n} d_j \nabla\theta_j - i v_m^* \nabla v_m \tag{26}$$

in $C^1(\overline{\Omega} \setminus \bigcup_{j=1}^n B_{\frac{1}{m}}(x_j))$, where $\theta_j(x)$ is the (multivalued) azimuthal angle of x about x_j and thus $\nabla\theta_j(x)$ is well defined in $\Omega \setminus \{x_j\}$ for $1 \leqslant j \leqslant n$. Since $\nabla \times \nabla\theta_j = 0$ in $\Omega \setminus \{x_j\}$ and v_m is C^2 with $|r_m|^2 = v_m v_m^* = 1$ in $\overline{\Omega} \setminus \bigcup_{j=1}^n B_{\frac{1}{m}}(x_j)$, it follows that $\nabla \times F_m = 0$ in $\overline{\Omega} \setminus \bigcup_{j=1}^n B_{\frac{1}{m}}(x_j)$. Thus if m is sufficiently large so that $d_j = d_{j,m}$ for $1 \leqslant j \leqslant n$, then by Stokes' Theorem and (23), $\oint_C F_m \cdot dr = 0$ for any closed curve, C, in $\overline{\Omega} \setminus \bigcup_{j=1}^n B_{\frac{1}{m}}(x_j)$. Moreover, there exists $\varphi_m \in C^2(\overline{\Omega} \setminus \bigcup_{j=1}^n B_{\frac{1}{m}}(x_j))$ such that $\nabla\varphi_m = F_m$ for $m = 1, 2, \ldots.$ From this and (26) we obtain

$$v_m \nabla\varphi_m = -v_m \left(\sum_{j=1}^n d_j \nabla\theta_j\right) - i \nabla v_m$$

and hence

$$\nabla\left(v_m \,\mathrm{e}^{-i\varphi_m} \cdot \mathrm{e}^{-i\sum_{j=1}^n d_j\theta_j}\right) = \mathrm{e}^{-i\varphi_m} \cdot \mathrm{e}^{-i\sum_{j=1}^n d_j\theta_j} \times \left[\nabla v_m - i\left(v_m \nabla\varphi_m + v_m \sum_{j=1}^n d_j \nabla\theta_j\right)\right] = 0.$$

As a result (adding a constant to φ_m if necessary), we have

$$v_m(x) = \mathrm{e}^{i\varphi_m(x)} \cdot \mathrm{e}^{i\sum_{j=1}^n d_j\theta_j(x)} = \mathrm{e}^{i\varphi_m(x)} \cdot \prod_{j=1}^n \left(\frac{z - z_j}{|z - z_j|}\right)^{d_j}.$$

By (26), $\nabla\varphi_m = -\sum_{j=1}^n d_j \nabla\theta_j - i v_m^* \nabla v_m$ and since $v_m \to v$ in $H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$, we have $\nabla\varphi_m \to -\sum_{j=1}^n d_j \nabla\theta_j - i v^* \nabla v$ and

$$\mathrm{e}^{i\varphi_m} \equiv v_m \cdot \prod_{j=1}^n \left(\frac{z - z_j}{|z - z_j|}\right)^{-d_j} \to v \cdot \prod_{j=1}^n \left(\frac{z - z_j}{|z - z_j|}\right)^{-d_j}$$

in $L^2_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_m\})$. It follows that $\{\varphi_m\}$ (after possibly subtracting constants $2\pi k_m$ where $k_m \in Z$) converges in $H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$, to some $\varphi \in H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$, and $u = \sqrt{a}v = \sqrt{a}\,\mathrm{e}^{i(\varphi + \sum_{j=1}^n d_j\theta_j)}$ a.e. in Ω.

Setting $\theta(x) = \varphi(x) + \sum_{j=1}^n d_j\theta_j(x)$, we have

$$|\nabla u|^2 = |\nabla\sqrt{a}|^2 + a|\nabla\theta|^2 \geqslant a|\nabla\theta|^2.$$

Since $|\nabla\theta_j(x)| = c(d_j)|x - x_j|^{-1}$ and $a(x) \leqslant c|x - x_j|^{\alpha_j}$ where $\alpha_j > 0$ for $1 \leqslant j \leqslant n$,

$$\int_\Omega a(x)\left|\nabla\theta_j(x)\right|^2 \leqslant C(\Omega, a, D) < \infty \tag{27}$$

where $D = (d_1, \ldots, d_n)$. Thus

$$\int_\Omega a|\nabla\varphi|^2 \leqslant \int_\Omega |\nabla u|^2 + C \cdot \sum_{j=1}^n \int_\Omega a|\nabla\theta_j|^2 \leqslant \int_\Omega |\nabla u|^2 + C(\Omega, a, D).$$

Finally, to show that $D \in Z^n$ is unique and $\varphi \in H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$ is unique (up to an additive constant $2\pi l$ where $l \in Z$) for each $u \in H^1_a$, assume that $\widetilde{D} = (\tilde{d}_1, \ldots, \tilde{d}_n) \in Z^n$ and $\tilde{\varphi} \in H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$ such that $u = \sqrt{a}\,\mathrm{e}^{i[\tilde{\varphi} + \sum_{k=1}^n \tilde{d}_k\theta_k]}$. Then $v \equiv u/\sqrt{a}$ satisfies $-iv^*\nabla v = \nabla\tilde{\varphi} + \sum_{k=1}^n \tilde{d}_k \nabla\theta_k$ in $\Omega \setminus \{x_1, \ldots, x_n\}$. Fixing $j \in \{1, \ldots, n\}$ and integrating over $B_{r_2}(x_j) \setminus B_{r_1}(x_j)$ for $0 < r_1 < r_2$ as in (25), we have

$$\begin{aligned}
d_j &= -\frac{i}{2\pi(r_2 - r_1)} \int_{B_{r_2}(x_j)\setminus B_{r_1}(x_j)} v^* v_\tau \, dx \\
&= \frac{1}{2\pi(r_2 - r_1)} \int_{B_{r_2}(x_j)\setminus B_{r_1}(x_j)} \left(\tilde{\varphi}_\tau + \sum_{k=1}^n \tilde{d}_k(\theta_k)_\tau\right) dx \\
&= \frac{1}{2\pi(r_2 - r_1)} \cdot \left[0 + \tilde{d}_j \cdot 2\pi(r_2 - r_1)\right] \\
&= \tilde{d}_j
\end{aligned}$$

where $\tau = \tau(x) = (x - x_j)^\perp/|x - x_j|$ for all $j \in \{1, \ldots, n\}$. Thus $\mathrm{e}^{i(\varphi - \tilde{\varphi})} = 1$ in $\Omega \setminus \{x_1, \ldots, x_n\}$ with $\varphi - \tilde{\varphi}$ in $H^1_{\mathrm{loc}}(\Omega \setminus \{x_1, \ldots, x_n\})$ and it follows that $\varphi - \tilde{\varphi} = 2\pi l$ for some $l \in Z$. □

For each $D \in Z^n$; we define

$$H^1_{a,D} = \left\{u \in H^1_a \colon u = \sqrt{a}\,\mathrm{e}^{i[\varphi + \sum_{j=1}^n d_j\theta_j]} \quad \text{where } \varphi \in H^1_{\mathrm{loc}}\big(\Omega \setminus \{x_1, \ldots, x_n\}\big)\right\}.$$

By Theorem 1.4, it follows that

$$H^1_a = \bigcup_{D \in Z^n} H^1_{a,D}$$

and $H^1_{a,D} \cap H^1_{a,D'} = \emptyset$ for $D \neq D'$ in Z^n. We will need the following additional properties of $H^1_{a,D}$:

THEOREM 1.5. – *For each $D \in Z^n$, $H^1_{a,D}$ is a nonempty, open and closed subset of H^1_a. In addition, $H^1_{a,D}$ is sequentially weakly closed in $H^1(\Omega; \mathbb{C})$, i.e. if $\{u_k\} \subset H^1_{a,D}$ and $u_k \to u$ weakly in $H^1(\Omega; \mathbb{C})$, then $u \in H^1_{a,D}$.*

Proof. – Our hypotheses on a in Ω ensure that $\sqrt{a} \in H^1(\Omega)$ and $\sqrt{a}\nabla\theta_j \in L^2(\Omega)$ for each $j \in \{1,\dots,n\}$ (see (27)); hence $\sqrt{a}\,\mathrm{e}^{i[\sum_{j=1}^n d_j\theta_j]} \in H^1_{a,D}$ and $H^1_{a,D} \neq \emptyset$. To prove that $H^1_{a,D}$ is open in H^1_a, assume that $u_0 = \sqrt{a}\,\mathrm{e}^{i[\varphi+\sum_{j=1}^n d_j\theta_j]} \in H^1_{a,D}$ and let

$$B_R(u_0) = \left\{u \in H^1_a\colon\ \|u-u_0\|_{H^1(\Omega;\mathbb{C})} < R\right\}$$

where $R > 0$. Since $u \in H^1_a$, there exists $\tilde\varphi \in H^1_{\mathrm{loc}}(\Omega \setminus \{x_1,\dots,x_n\})$ and $\widetilde{D} \in Z^n$ such that $u = \sqrt{a}\,\mathrm{e}^{i[\tilde\varphi+\sum_{j=1}^n \tilde d_j\theta_j]}$. Set $v_0 = u_0/|u_0| = u_0/\sqrt{a}$ and $v = u/|u| = u/\sqrt{a}$. By (25), there exist positive numbers $r_1 < r_2$ such that for each $j \in \{1,\dots,n\}$,

$$d_j = -\frac{i}{2\pi(r_2-r_1)}\int_{S_j} v_0^*(v_0)_\tau\,dx \tag{28}$$

and

$$\tilde d_j = -\frac{i}{2\pi(r_2-r_1)}\int_{S_j} v^*(v)_\tau\,dx$$

where $S_j = B_{r_2}(x_j) \setminus B_{r_1}(x_j)$. Since a is C^1 and $|a| > 0$ on $\overline{S_j}$ for each $j \in \{1,\dots,n\}$, we have

$$\begin{aligned}\left\|v_0^*\nabla v_0 - v^*\nabla v\right\|_{L^1(S_j)} &\leqslant \left\|v^*(\nabla v_0-\nabla v)\right\|_{L^1(S_j)} + \left\|(v_0^*-v^*)\nabla v_0\right\|_{L^1(S_j)}\\ &\leqslant C(a,r_1,r_2,v_0)\cdot\left[1+\|u_0\|_{H^1(S_j)}\right]\cdot\|u-u_0\|_{H^1(S_j)}.\end{aligned}$$

From this and (28), it follows that if R is sufficiently small (depending on r_1, r_2, Ω, a, and u_0), we have $d_j = \tilde d_j$ and $u \in H^1_{a,D}$. Thus $B_R(u_0) \subset H^1_{a,D}$ for R sufficiently small and we conclude that $H^1_{a,D}$ is an open subset of H^1_a. Now since $H^1_a = \bigcup_{D\in Z^n} H^1_{a,D}$ and $H_{a,D} \cap H^1_{a,D'} = \emptyset$ for $D \neq D'$ in Z^n, $H^1_{a,D}$ is also a closed subset of H^1_a.

Finally, to prove that $H^1_{a,D}$ is weakly sequentially closed in H^1_a, assume that $\{u_k\} \subset H^1_{a,D}$ and $u_k \to u$ weakly in $H^1(\Omega;\mathbb{C})$. By compactness, a subsequence (which we relabel as $\{u_k\}$) satisfies $u_k \to u$ in $L^2(\Omega)$. Thus $|u| = \sqrt{a}$ a.e. in Ω and hence $u \in H^1_{a,\widetilde{D}}$ for some $\widetilde{D} \in Z^n$. It follows from (28) (with v_0 replaced by $u_k/\sqrt{a}$ and v replaced by $u/\sqrt{a}$ and the weak convergence of u_k to u that $D = \widetilde{D}$ and $u \in H^1_{a,D}$. □

2. A weighted Sobolev space

Set

$$V \equiv \left\{g \in H^1(\Omega)\colon \int_\Omega a^{-1}|\nabla g|^2 < \infty\right\}.$$

Then V is a Hilbert space with norm

$$\|g\|_V = \left(\int_\Omega \left[a^{-1}|\nabla g|^2 + g^2\right]\right)^{1/2}.$$

We prove in Section 3 that if (ψ_D, A_D) is an equilibrium for J_0 in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$ and h_D is defined by $h_D\mathbf{e}_3 = \nabla \times A_D$, then $h_D \in V$ and

$$\operatorname{div}\bigl(a^{-1}\nabla h_D\bigr) - h_D = -2\pi \sum_{j=1}^{n} d_j \delta_{x_j}$$

in the weak sense in V; moreover, we can evaluate the minimum energy of J_0 in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$ and $H^1_a \times H^1(\Omega;\mathbb{R}^2)$ using Hilbert space properties of $V \cap H^1_0(\Omega)$. We need:

LEMMA 2.1. – *The map $g \in V \to g(x_i)$ is continuous on V for each $1 \leqslant i \leqslant n$ where $g(x_i)$ is defined by*

$$g(x_i) \equiv \lim_{r\to 0} \fint_{\partial B_r(x_i)} g. \tag{29}$$

Moreover, any $g \in V$ satisfies:

$$\lim_{r\to 0} \fint_{\partial B_r(x_i)} \bigl(g - g(x_i)\bigr)^2 a^{-1} = 0. \tag{30}$$

Proof. – Let $0 < s < r$. Then g has a trace on $\partial B_r(x_i)$ and $\partial B_s(x_i)$, and

$$\begin{aligned}\left| \fint_{\partial B_r(x_i)} g - \fint_{\partial B_s(x_i)} g \right| &\leqslant \frac{1}{2\pi} \int_{B_r(x_i)\setminus B_s(x_i)} \bigl[|\nabla g|/|x|\bigr] \cdot a^{-1/2} \cdot a^{1/2} \\ &\leqslant \frac{1}{2\pi} \Biggl(\int_{B_r(x_i)\setminus B_x(x_i)} a^{-1}|\nabla g|^2 \Biggr)^{1/2} \cdot \Biggl(\int_{B_r(x_i)\setminus B_s(x_i)} |x|^{\alpha_i - 2} \Biggr)^{1/2} \\ &\leqslant \frac{1}{2\pi} \|g\|_V \cdot (r^{\alpha_i} - s^{\alpha_i})^{1/2}.\end{aligned}$$

Thus, the limit in (29) exists and $g(x_i)$ is well defined by (29). Letting $s \to 0$, we have

$$\left| g(x_i) - \fint_{\partial B_r(x_i)} g \right| \leqslant \frac{1}{2\pi} \Biggl(\int_{B_r(x_i)} a^{-1}|\nabla g|^2 \Biggr)^{1/2} \cdot r^{\alpha_i/2}$$

and hence

$$\lim_{r\to 0} r^{-\alpha_i/2} \left| g(x_i) - \fint_{\partial B_r(x_i)} g \right| = 0. \tag{31}$$

Multiplying the above inequality by r and integrating from 0 to R for $R < \operatorname{dist}(x_i, \partial\Omega)$, we obtain

$$\bigl|g(x_i)\bigr| \leqslant C \Biggl(\int_{B_R} g + \|g\|_V \Biggr) \leqslant C\bigl(\|g\|_{L^2(B_R)} + \|g\|_V\bigr) \leqslant C\|g\|_v$$

where B_R denotes the ball of radius R centered at x_i, and C is a constant depending only on R and a. This proves the continuity of the map $g \to g(x_i)$ on V.

Next set $\tilde{g} = g - g(x_i)$. We have for $s < r < \operatorname{dist}(x_i, \partial\Omega)$

$$\begin{aligned} \left| \fint_{\partial B_r} \tilde{g}^2 - \fint_{\partial B_s} \tilde{g}^2 \right| &\leqslant \int_{B_r \setminus B_s} |\tilde{g}||\nabla g|/|x| \\ &\leqslant \Bigg(\int_{B_r \setminus B_s} |\nabla g|^2 a^{-1} \Bigg)^{1/2} \Bigg(\int_{B_r \setminus B_s} \tilde{g}^2 |x|^{\alpha_i - 2} \Bigg)^{1/2} \\ &\leqslant \|g\|_V \Bigg(\int_{B_r \setminus B_s} \tilde{g}^2 |x|^{\alpha_i - 2} \Bigg)^{1/2} \end{aligned} \tag{32}$$

where B_r and B_s are centered at x_i. Set

$$F(s) = \int_{B_r \setminus B_s} \tilde{g}^2 |x|^{\alpha_i - 2} = 2\pi \int_s^r \fint_{\partial B_\tau} \tilde{g}^2 \tau^{\alpha_i - 1} \, d\tau.$$

CLAIM 1. – *$\lim_{s\to 0} F(s)$ is finite.*

To prove this, note that $F'(s) = -s^{\alpha_i - 2} \int_{\partial B_s} \tilde{g}^2$ and thus (32) can be rewritten as

$$\left| \frac{1}{2\pi} r^{1-\alpha_i} F'(r) - \frac{1}{2\pi} s^{1-\alpha_i} F'(s) \right| \leqslant C_1(r) \cdot F(s)^{1/2}$$

where $C_1(r) = (\int_{B_r} |\nabla g|^2 a^{-1})^{1/2} \leqslant \|g\|_V < \infty$ for $0 < r \leqslant R$. Thus

$$-F'(s) \leqslant s^{\alpha_i - 1} \big[C_1(r) \cdot F(s)^{1/2} + C_2(r) \big]$$

for $0 < s < r$. Since $F(s)$ is monotone nonincreasing on $(0, r)$, we obtain the claimed result if we prove that F is bounded on $(0, r)$. Without loss of generality, we can assume that $F(s) \geqslant c_0 > 0$ for $0 < s < \frac{r}{2}$ (if not, the result follows easily). Dividing the above inequality by $F^{1/2}(s)$, we have

$$-\big(F^{1/2}(s)\big)' \leqslant C\big(s^{\alpha_i - 1} + 1\big).$$

Integrating from s to $r/2$ we find that

$$F^{1/2}(s) \leqslant C(r) < \infty \quad \text{for } s < r/2,$$

which proves Claim 1.

It follows from Claim 1 and (32) that $\{\fint_{\partial B_r} \tilde{g}^2\}$ is Cauchy as $r \to 0$.

CLAIM 2. – $\lim_{r\to 0} \fint_{\partial B_r(x_i)} \tilde{g}^2 = 0$.

To prove this, set $\gamma \equiv \lim_{r\to 0} \fint_{\partial B_r(x_i)} \tilde{g}^2$. Integrating (29), we have $g(x_i) = \lim_{r\to 0} \fint_{B_r(x_i)} g$. Thus from (31),

$$\lim_{r\to 0} \fint_{B_r} \left| g - \fint_{B_r} g \right|^2 = \lim_{r\to 0} \fint_{B_r} \left| g - g(x_i) \right|^2 = \lim_{r\to 0} 2r^{-2} \int_0^r s \fint_{\partial B_s} \tilde{g}^2 \, ds = \gamma$$

where $B_r = B_r(x_i)$. By the Sobolev inequality in two dimensions we have

$$\fint_{B_r} \left| g - \fint_{B_r} g \right|^2 \leqslant C \int_{B_r} |\nabla g|^2 \to 0 \quad \text{as } r \to 0$$

and thus $\gamma = 0$, which proves Claim 2.

We are now in a position to prove (30). Letting $s \to 0$ in (32) and using Claim 2, we obtain

$$\fint_{\partial B_r} \tilde{g}^2 \leqslant d(r) \left(\int_{B_r} \tilde{g}^2 |x|^{\alpha_i - 2} \right)^{1/2}$$

where $d(r) \equiv (\int_{B_r} |\nabla g|^2 a^{-1})^{1/2}$ and $\lim_{r\to 0} d(r) = 0$ since $g \in V$. Set $G(r) = \int_0^r \tau^{\alpha_i - 1} \fint_{\partial B_\tau} \tilde{g}^2 \, d\tau$. Then $G(0) = 0$ and from the above inequality,

$$r^{\alpha_i - 1} \fint_{\partial B_r} \tilde{g}^2 = G'(r) \leqslant d(r) G^{1/2}(r) r^{\alpha_i - 1}.$$

Separating and integrating and using the fact that $d(r)$ is monotone nondecreasing, we get

$$G^{1/2}(r) \leqslant C d(r) r^{\alpha_i}.$$

Inserting this in the estimate on $G'(r)$ gives

$$r^{-\alpha_i} \fint_{\partial B_r} \tilde{g}^2 = r^{1 - 2\alpha_i} G'(r) \leqslant C d^2(r).$$

Since $a(x) \sim |x - x_i|^{\alpha_i}$ in a neighborhood of x_i, we see that (30) follows. □

3. Analysis of the case $\varepsilon = 0$

Recall that $J_0(\psi, A) = \int_\Omega [|(\nabla - iA)\psi|^2 + |\nabla \times A - h_e \mathbf{e}_3|^2]$.

In this section, we prove the results stated as Theorem 1 in the introduction concerning equilibria of J_0. (See Theorem 3.2.) We also establish a formula for $J_0(\psi_D, A_D)$ where (ψ_D, A_D) is an equilibrium for J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. We shall need the following results concerning $\nabla \times A_D$:

LEMMA 3.1. – *Let (ψ_D, A_D) be an equilibrium for J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. Define h_D by $\nabla \times A_D = h_D \mathbf{e}_3$. Then $h = h_D$ is the unique solution of*:

$$\begin{aligned} &\int_\Omega a^{-1} \nabla h \cdot \nabla \zeta + \int_\Omega h\zeta = 2\pi \sum_{i=1}^n d_i \zeta(x_i) \quad \forall \zeta \in V \cap H^1_0(\Omega), \\ &h - h_e \in V \cap H^1_0(\Omega). \end{aligned} \tag{33}$$

Proof. – Since $\psi_D \in H^1_{a,D}$, we have $\psi_D = \sqrt{a}\,\mathrm{e}^{i\theta}$ and $\sqrt{a}\nabla\theta \in L^2(\Omega)$. In addition, since $\sqrt{a} \in H^1(\Omega)$ and $a \in C^1(\overline{\Omega}\backslash\{x_1,\dots,x_n\}) \cap C^\beta(\Omega)$ by assumption, it follows from (7), (12), and (14) that $J_0(\psi_D, A_D) < \infty$, $j_D \in L^2(\Omega)$, and $h_D - h_e \in H^1_0(\Omega)$. By (12), (16), and (17), $-\nabla^\perp h_D = j_D \equiv (j_D^1, j_D^2) = a(\nabla\theta - A_D)$. Thus $|\nabla h_D|/\sqrt{a} = \sqrt{a}|\nabla\theta - A_D| \in L^2(\Omega)$ and $h_d - h_e \in V \cap H^1_0(\Omega)$. Moreover,

$$\operatorname{div}\left(\frac{1}{a}\nabla h_D\right) = -\nabla\times\left(\frac{1}{a}j_D\right) = -\nabla\times(\nabla\theta - A_D) = h_D \quad \text{in } \Omega\backslash\{x_1,\dots,x_n\}$$

in the sense of distributions. Since a is C^1 and positive in $\Omega\backslash\{x_1,\dots,x_n\}$, it follows that $h_D \in H^2_{\mathrm{loc}}(\Omega\backslash\{x_1,\dots,x_n\})$ and hence $\theta \in H^2_{\mathrm{loc}}(\Omega\backslash\{x_1,\dots,x_n\})$.

Now let $\zeta \in V \cap H^1_0(\Omega)$ and consider

$$\int_\Omega a^{-1}\nabla h_D\cdot\nabla\zeta = \lim_{r\to 0}\int_{\Omega\backslash\bigcup_{i=1}^n B_r(x_i)} -\nabla\theta\cdot\nabla^\perp\zeta - \int_\Omega h_D\zeta.$$

For $r > 0$ fixed and small we can integrate by parts to obtain

$$\begin{aligned}\int_{\Omega\backslash\bigcup_{i=1}^n B_r(x_i)} -\nabla\theta\nabla^\perp\zeta &= \sum_{i=1}^n \int_{\partial B_r(x_i)} \zeta\theta_\tau\\ &= 2\pi\sum_{i=1}^n \zeta(x_i)d_i + \sum_{i=1}^n\int_{\partial B_r(x_i)} \big(\zeta-\zeta(x_i)\big)\theta_\tau, \qquad (34)\end{aligned}$$

where $\tau = \nu^\perp$ and ν is the outward pointing unit normal to $\partial B_r(x_i)$. By (12) and (16), the last term can be written as

$$\sum_{i=1}^n\int_{\partial B_r(x_i)} \big(\zeta-\zeta(x_i)\big)\big[-\partial_\nu h_D a^{-1} + A_D\cdot\tau\big].$$

Using (22) and (29) for each i,

$$\lim_{r\to 0}\int_{\partial B_r(x_i)}\big(\zeta-\zeta(x_i)\big)A_D\cdot\tau = 0.$$

Now

$$\int_{\partial B_r(x_i)} \big|\zeta-\zeta(x_i)\big||\nabla h_D|a^{-1} \leqslant C\left(\fint_{\partial B_r}\big|\zeta-\zeta(x_i)\big|^2 a^{-1}\right)^{1/2}\left(r\int_{\partial B_r(x_i)}|\nabla h_D|^2a^{-1}\right)^{1/2}.$$

By (30) the first term in the product tends to zero as $r\to 0$. As for the second term, we claim that $\liminf_{r\to 0} r\cdot\sum_{i=1}^n\int_{\partial B_r(x_i)}|\nabla h_D|^2a^{-1} = 0$. Indeed if not, there exists a constant $C > 0$ and $r_0 > 0$ such that

$$\infty = \int_0^{r_0}\frac{C}{r}\,dr \leqslant \sum_{i=1}^n\int_{B_{r_0}(x_i)}|\nabla h_D|^2a^{-1},$$

which contradicts the fact that $h_D - h_e \in V$. It follows that there is a sequence $r_j \to 0$ such that the last term on the right side of (34) tends to zero as $r = r_j \to 0$, and hence $h = h_D$ satisfies (33).

Now to prove that solutions of (33) are unique, assume that h_1 and h_2 are solutions. Then $h_1 - h_2 \in V \cap H_0^1(\Omega)$ and

$$0 = \int_\Omega \left[a^{-1}\left|\nabla(h_1 - h_2)\right|^2 + |h_1 - h_2|^2\right],$$

whence $h_1 - h_2 = 0$. □

THEOREM 3.2. – *Fix* $h_e \geqslant 0$. *For each* $D \in Z^n$, J_0 *has an equilibrium* (*with our choice of gauge* (8)), *denoted by* (ψ_D, A_D), *in* $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$, *which is unique up to uniform rotations of* ψ_D *in* Ω, $\psi_D \to \psi_D\,\mathrm{e}^{ic}$ *where* $c \in \mathbb{R}$.

Proof. – First, we note that J_0 has a minimizer, (ψ_D, A_D), in

$$\left\{(\psi, A) \in H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)\colon \operatorname{div} A = 0 \text{ in } \Omega \text{ and } A \cdot \vec{n} = 0 \text{ on } \partial\Omega\right\}$$

(and hence in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$ by gauge equivalence), by the direct method in the calculus of variations since $H^1_{a,D}$ is sequentially weakly closed in $H^1(\Omega; \mathbb{C})$ by Theorem 1.5. Such a minimizer is an equilibrium for J_0 by considering variations of the form $(\psi_D, A_D) \to (\psi_D\,\mathrm{e}^{\varepsilon i f}, A_D + \varepsilon B)$, which yields (6) and (7) as Euler–Lagrange equations.

Now to prove uniqueness of equilibria for J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$ satisfying the gauge condition (8), assume that (ψ, A) and (ψ', A') are two such equilibria. By Lemma 3.1, we must have $\nabla \times A = \nabla \times A'$. By the choice of gauge, this implies that $A = A'$. From (12), (16), and Lemma 3.1, it follows that $j = j'$ and hence $\nabla\theta = \nabla\theta'$ and $\psi = \psi'\,\mathrm{e}^{ic}$ for some $c \in \mathbb{R}$. □

We next evaluate J_0 on equilibria. Consider the $n + 1$ functions in $V \cap H_0^1(\Omega)$, $\{\eta_0, \ldots, \eta_n\}$, solving

$$\operatorname{div}\left(a^{-1}\nabla\eta_0\right) = \eta_0 + 1, \tag{35}$$

$$\operatorname{div}\left(a^{-1}\nabla\eta_i\right) = \eta_i - 2\pi\delta(x_i) \quad \text{for } 1 \leqslant i \leqslant n. \tag{36}$$

Note that by Lemma 2.1, $\delta(x_i) \in V'$, the dual space of V, and clearly $1 \in V'$. Thus the existence and uniqueness of solutions follows from the Lax–Milgram lemma. Using $\min(\eta_i, 0)$, as test functions in (36) we see that $\eta_i > 0$ in Ω for $i = 1, \ldots, n$. Set

$$a_{ij} = a_{ji} \equiv \int_\Omega \left[a^{-1}\nabla\eta_i\nabla\eta_j + \eta_i\eta_j\right] \quad \text{for } 1 \leqslant i, j \leqslant n \tag{37}$$

and

$$b_i = \int_\Omega \left[a^{-1}\nabla\eta_0\nabla\eta_i + \eta_0\eta_i\right] \quad \text{for } 0 \leqslant i \leqslant n. \tag{38}$$

Then $b_i = -\int_\Omega \eta_i < 0$ for $1 \leqslant i \leqslant n$.

For any $\vec{C} = (c_1, \dots, c_n) \in \mathbb{R}^n$ set $\eta_{\vec{C}} = \sum_{i=1}^n c_i \eta_i$. Then

$$\int_\Omega \left[a^{-1} \nabla \eta_{\vec{C}} \nabla \eta_{\vec{C}} + \eta_{\vec{C}} \eta_{\vec{C}}\right] = \sum a_{ij} c_i c_j \geqslant 0 \tag{39}$$

with equality iff $\eta_{\vec{C}} \equiv 0$. Moreover, we have:

LEMMA 3.3. – *The matrix* $[a_{ij}]$ *defined by* (37) *is positive definite.*

Proof. – By (39), it is sufficient to prove that $\eta_{\vec{C}} \equiv 0$ in Ω implies $\vec{C} = 0$. If not, let $\vec{C}$ be a nonzero vector in $\mathbb{R}^n$ satisfying $\vec{\eta}_C \equiv 0$ in Ω. Then $\eta_q = \sum_{i \neq q}^n (c_i/c_q) \eta_i$ for some q satisfying $c_q \neq 0$. Whence

$$\begin{aligned} 2\pi \delta(x_q) &= -\operatorname{div}\left(a^{-1} \eta_q\right) + \eta_q = -\operatorname{div}\left(a^{-1} \left(\sum_{i \neq q} (c_i/c_q) \eta_i \right)\right) + \sum_{i \neq q} (c_i/c_q) \eta_i \\ &= 2\pi \sum_{i \neq q} (c_i/c_q) \delta(x_i), \end{aligned}$$

which is impossible. □

Set

$$\mathcal{E}(\vec{C}, h_e) \equiv \sum a_{ij} c_i c_j + 2 \sum_{i=1}^n b_i c_i h_e + b_0 h_e^2. \tag{40}$$

THEOREM 3.4. – *Fix* $h_e \geqslant 0$. *If* (ψ_D, A_D) *is an equilibrium for* J_0 *with* $\psi_D \in H^1_{a,D}$, *then* $h_D = \sum_{i=1}^n d_i \eta_i + h_e \eta_0 + h_e$ *and*

$$J_0(\psi_D, A_D) = \int_\Omega |\nabla \sqrt{a}|^2 + \mathcal{E}(D, h_e). \tag{41}$$

Thus (ψ_D, A_D) is a minimizer of J_0 in $\mathcal{M}_0$ if and only if $\mathcal{E}(D, h_e) = \inf\{\mathcal{E}(\vec{C}, h_e)\colon \vec{C} \in Z^n\}$.

Proof. – Indeed,

$$J_0(\psi_D, A_D) = \int_\Omega \left[|\nabla \sqrt{a}|^2 + a|\nabla \theta_D - A_D|^2 + |\nabla \times A_D - h_e \mathbf{e}_3|^2\right].$$

Since $h_D \mathbf{e}_3 = \nabla \times A_D$ we see from (12) and (16) that

$$J_0(\psi_D, A_D) = \int_\Omega \left[|\nabla \sqrt{a}|^2 + a^{-1} |\nabla h_D|^2 + |h_D - h_e|^2\right].$$

Now h_D is the unique solution to (33). Thus

$$h_D = \sum_{i=1}^n d_i \eta_i + h_e \eta_0 + h_e.$$

Using (39) and (40) we see that (41) holds. □

Using Theorem 1.4 and Theorem 3.4, we can conclude that minimizers of J_0 in $\mathcal{M}_0$ (or families of equilibria of J_0 in $\mathcal{M}_0$ with uniformly bounded energies) are contained in a finite number of the spaces $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$. More precisely, we have:

THEOREM 3.5. – *Fix $h_e \geqslant 0$. Let $c \geqslant \inf_{(\psi,A)\in\mathcal{M}_0} J_0(\psi,A)$ and let $\mathcal{F} = \mathcal{F}(c,h_e)$ be the family of all equilibria, (ψ,A), of J_0 in $\mathcal{M}_0$ satisfying $J_0(\psi,A) \leqslant c$. Then there exists a nonempty, finite subset $\mathcal{D}$ of Z^n (depending on c and h_e) such $\mathcal{F} \subset \bigcup_{D\in\mathcal{D}}[H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)]$.*

Proof. – If $(\psi,A) \in \mathcal{F}$, then by Theorem 1.4, (ψ,A) is an equilibrium for J_0 in $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$ for some D in Z^n. By Theorems 3.2 and 3.4, we have

$$c \geqslant J_0(\psi,A) = J_0(\psi_D,A_D) = \int_\Omega |\nabla\sqrt{a}|^2 + \mathcal{E}(D,h_e). \tag{42}$$

Now since $[a_{ij}] \geqslant \mu I$ for some $\mu > 0$ by Lemma 3.3, we have (by (40)):

$$\begin{aligned}\mathcal{E}(\vec{C},h_e) &\geqslant \mu|\vec{C}|^2 - 2|b|\cdot|h_e|\cdot|\vec{C}| + b_0h_e^2\\ &= \mu\left(|\vec{C}| - \frac{|b|}{\mu}|h_e|\right)^2 - \frac{|b|^2\cdot|h_e|^2}{\mu^2} + b_0h_e^2\end{aligned}$$

where $b = (b_1,\ldots,b_n)$, for all $\vec{C}$ in Z^n. From this and (42), we obtain

$$c + \frac{|b|^2|h_e|^2}{\mu^2} - b_0h_e^2 - \int_\Omega |\nabla\sqrt{a}|^2 \geqslant \mu\left(|D| - \frac{|b|}{\mu}|h_e|\right)^2.$$

The set of all such D in Z^n is finite, which proves the theorem. □

We remark that when $h_e \geqslant 0$ and $c = \inf_{(\chi,A)\in\mathcal{M}_0} J_0(\chi,A) \equiv c(h_e)$, we have

$$c \leqslant J_0(\sqrt{a},0) = \int_\Omega |\nabla\sqrt{a}|^2 + \mathcal{E}(0,h_e) = \int_\Omega |\nabla\sqrt{a}|^2 + b_0h_e^2.$$

In this case, $\mathcal{F} = \mathcal{F}(c(h_e),h_e)$ is the family of minimizers of J_0 in $\mathcal{M}_0$ (for fixed $h_e \geqslant 0$). Letting $\mathcal{D}_0 \equiv \mathcal{D}_0(h_e)$ be the finite set $\mathcal{D}$ corresponding to $\mathcal{F}$ in this case, it follows from (40), Theorem 3.4, and the above inequality that

$$b_0h_e^2 \geqslant \mu|D|^2 - 2|b||h_e||D| + b_0h_e^2$$

and thus

$$|D| \leqslant \frac{2|b|\cdot|h_e|}{\mu} \tag{43}$$

for all D in $\mathcal{D}_0$. Thus we have:

COROLLARY 3.6. – *Fix $h_e \geqslant 0$. Let $\mathcal{D}_0 = \mathcal{D}_0(h_e)$ be the set of all D in Z^n such that $H^1_{a,D} \times H^1(\Omega;\mathbb{R}^2)$ contains a minimizer of J_0 in $\mathcal{M}_0$. Then $\mathcal{D}_0$ is a finite, nonempty set in Z^n.*

We conclude this section with a result which will be used later to estimate $H_{c_1} = H_{c_1}(\varepsilon)$. Recall that $b_i < 0$ for $1 \leqslant i \leqslant n$.

THEOREM 3.7. – *Set* $\bar{h}_e = \min\{-a_{ii}/2b_i\colon\ i = 1, 2, \ldots, n\}$. *If* $h_e > \bar{h}_e$ *and* (ψ_D, A_D) *is a minimizer of* J_0 *in* $\mathcal{M}_0$ *with* $\psi_D \in H^1_{a,D}$, *then* $D \neq \vec{0}$.

Proof. – Let $j \in \{1, \ldots, n\}$ satisfy $\bar{h}_e = -a_{jj}/2b_j$. Let $\vec{e}_j$ be the vector in Z^n whose ith component is δ_{ij} for $i = 1, \ldots, n$. If $h_e > \bar{h}_e$, then by (40), we have:

$$\mathcal{E}(\vec{e}_j, h_e) = a_{jj} + 2b_j h_e + b_0 h_e^2 = -2b_j(\bar{h}_e - h_e) + b_0 h_e^2 < b_0 h_e^2 = \mathcal{E}(\vec{0}, h_e).$$

By Theorem 3.4, we must have $D \neq \vec{0}$. □

4. Limiting results

In this section, we prove that minimizers, $(\psi_\varepsilon, A_\varepsilon)$ of J_ε exhibit "pinning" of vortices near $\{x_1, \ldots, x_n\}$, the zeroes of $a(x)$, for ε sufficiently small. In addition, the behavior of ψ_ε near vortices (i.e., near the zeroes of ψ_ε) is determined by the set $\mathcal{D}_0(h_e)$ for each $h_e \geqslant 0$. These results were stated as Theorems 2–5 in the introduction.

Throughout this section, we assume without loss of generality that any equilibrium of J_ε in $\mathcal{M}$ (or J_0 in $\mathcal{M}_0$) considered here satisfies our gauge choice (8). For ease of notation in stating the theorems, we let (ψ_D, A_D) denote an (appropriate) equilibrium of J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. Recall that any such (ψ_D, A_D) is unique up to a uniform rotation of ψ_D in Ω.

THEOREM 4.1. – *Fix* $h_e \geqslant 0$. *Let* $(\psi_{\varepsilon_k}, A_{\varepsilon_k})$ *be a sequence of equilibria for* J_{ε_k} *such that* $\varepsilon_k \to 0^+$ *and* $\liminf_{k\to\infty} J_{\varepsilon_k}(\psi_{\varepsilon_k, A_{\varepsilon_k}}) < \infty$. *Then there exists a subsequence* $\{\varepsilon_{k_\ell}\}$, *a vector* D *in* Z^n, *and* (ψ_D, A_D) *such that* $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) \rightharpoonup (\psi_D, A_D)$ *in* $\mathcal{M}$.

Proof. – By compactness, Lemmas 1.1 and 1.3, there exists (ψ, A) in $\mathcal{M}$ and a subsequence $\{\varepsilon_{k_\ell}\}$ of $\{\varepsilon_k\}$ satisfying $|\psi_{\varepsilon_{k_\ell}}| \leqslant M$, $\psi_{\varepsilon_{k_\ell}} \rightharpoonup \psi$ in $H^1(\Omega; \mathbb{C})$, $\psi_{\varepsilon_{k_\ell}} \to \psi$ pointwise almost everywhere in Ω, and $A_{\varepsilon_{k_\ell}} \rightharpoonup A$ in $H^1(\Omega; \mathbb{R}^2)$. Furthermore, $\int_\Omega (a - |\psi_{\varepsilon_{k_\ell}}|^2)^2 \leqslant M \cdot \varepsilon_{k_\ell}^2$ where M is a positive number independent of ε_{k_ℓ}. It follows that $|\psi| = \sqrt{a}$ a.e. in Ω and hence $(\psi, A) \in \mathcal{M}_0$. Thus $(\psi, A) = (\psi_D, A_D)$ for some $D \in Z^n$. Since $\{\psi_{\varepsilon_{k_\ell}}\}$ is uniformly bounded and converges pointwise almost everywhere in Ω we have $j_{\varepsilon_{k_\ell}} \rightharpoonup j$ in $L^2(\Omega)$ where j is defined by (2). By (3), (4), and our choice of gauge, we see that $A_{\varepsilon_{k_\ell}} \rightharpoonup A$ in $H^2(\Omega; \mathbb{R}^2)$. Passing to the limit in (6) and (7) we find that (ψ, A) is a weak solution. □

Recall that in Section 3, we defined

$$\mathcal{D}_0(h_e) = \left\{D \in Z^n\colon\ J_0(\psi_D, A_D) = \inf_{(\psi, A)\in\mathcal{M}_0} J_0(\psi, A)\right\}$$

for each fixed $h_e \geqslant 0$. For minimizers of J_ε in $\mathcal{M}$ as $\varepsilon \to 0$, we have:

THEOREM 4.2. – *Fix* $h_e \geqslant 0$. *Let* $\{(\psi_{\varepsilon_k}, A_{\varepsilon_k})\}$ *be a sequence of minimizers of* J_{ε_k} *in* $\mathcal{M}$ *with* $\varepsilon_k \to 0^+$ *as* $k \to \infty$. *Then* $|\psi_{\varepsilon_k}| \to \sqrt{a}$ *in* $C(\overline{\Omega})$, *and there exists a subsequence* $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) \to (\psi_D, A_D)$ *in* $\mathcal{M}$, *where* $D = (d_1, \ldots, d_n) \in \mathcal{D}_0(h_e)$ (*and hence* (ψ_D, A_D) *is a minimizer of* J_0 *in* $\mathcal{M}_0$). *Moreover, if* $R > 0$ *and* $\overline{B_R(x_i)}$ *are disjoint subsets of* Ω *for* $i = 1, \ldots, n$, *then for all* ℓ *sufficiently large,* $|\psi_{\varepsilon_{k_\ell}}|$ *is uniformly positive outside* $\bigcup_{i=1}^n B_R(x_i)$, *and the degree of* $\psi_{\varepsilon_{k_\ell}}$ *in* $\overline{B_R(x_i)}$ *is* d_i.

Proof. – We may apply Theorem 4.1, since

$$J_{\varepsilon_k}(\psi_{\varepsilon_k,A_{\varepsilon_k}}) \leqslant J_{\varepsilon_k}(\sqrt{a},0) \leqslant |\Omega|\cdot h_e^2 + \int_\Omega \left|\nabla(\sqrt{a})\right|^2 < \infty,$$

to obtain a subsequence $\{\varepsilon_{k_\ell}\}$ of $\{\varepsilon_k\}$ such that $\{(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})\}$ converges weakly in $\mathcal{M}$ to (ψ_D, A_D), an equilibrium for J_0 in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ for some D in Z^n. Since $(\psi_{\varepsilon_k}, A_{\varepsilon_k})$ is a minimizer of J_{ε_k} for each k and $(\psi_D, A_D)\in H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ we have:

$$\begin{aligned} J_0(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) + \frac{1}{2\varepsilon_{k_\ell}^2}\int_\Omega \left(a-|\psi_{\varepsilon_{k_\ell}}|^2\right)^2 &= J_{\varepsilon_{k_\ell}}(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) \\ &\leqslant J_{\varepsilon_{k_\ell}}(\psi_D, A_D) = J_0(\psi_D, A_D). \end{aligned}$$

Also, J_0 is weakly lower semicontinuous with respect to the topology on $\mathcal{M}$, and thus

$$\begin{aligned} J_0(\psi_D, A_D) &\leqslant \liminf_{\ell\to\infty} J_0(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) \\ &\leqslant \liminf_{\ell\to\infty} J_0(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) + \liminf_{\ell\to\infty}\frac{1}{2\varepsilon_{k_\ell}^2}\int_\Omega \left(a-|\psi_{\varepsilon_{k_\ell}}|^2\right)^2 \\ &\leqslant \liminf_{\ell\to\infty} J_{\varepsilon_{k_\ell}}(\psi_{\varepsilon_{k_\ell}}, A_{e_{k_\ell}}) \\ &\leqslant J_0(\psi_D, A_D). \end{aligned}$$

In fact both integrals making up J_0 are weakly lower semicontinuous. As a result

$$\lim_{\ell\to\infty}\int_\Omega \left|(\nabla - iA_{\varepsilon_{k_\ell}})\psi_{\varepsilon_{k_\ell}}\right|^2 = \int_\Omega \left|(\nabla - iA_D)\psi_D\right|^2, \tag{44}$$

$$\lim_{\ell\to\infty}\int_\Omega |\nabla\times A_{\varepsilon_{k_\ell}} - h_e\mathbf{e}_3|^2 = \int_\Omega |\nabla\times A_D - h_e\mathbf{e}_3|^2, \tag{45}$$

$$\lim_{\ell\to\infty}\frac{1}{2\varepsilon_{k_\ell}^2}\int_\Omega \left(a-|\psi_{\varepsilon_{k_\ell}}|^2\right)^2 = 0. \tag{46}$$

Eqs. (44), (45), and the weak convergence of $\{(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})\}$ imply that

$$\int_\Omega |\nabla\psi_{\varepsilon_{k_\ell}}|^2 \to \int_\Omega |\nabla\psi_D|^2 \quad \text{and} \quad \int_\Omega |h_{\varepsilon_{k_\ell}}|^2 \to \int_\Omega |h_D|^2$$

as $\ell\to\infty$. Weak convergence and convergence of norms implies strong convergence in a Hilbert space. Thus $\nabla\psi_{\varepsilon_{k_\ell}}\to\nabla\psi_D$ and $h_{\varepsilon_{k_\ell}}\to h_D$ in $L^2(\Omega)$. By our choice of gauge, $A_{\varepsilon_{k_\ell}} = \nabla^\perp\xi_{\varepsilon_{k_\ell}}$ for some $\xi_{\varepsilon_{k_\ell}}$ in $H^1_0(\Omega)$ satisfying $\Delta\xi_{\varepsilon_{k_\ell}} = h_{\varepsilon_{k_\ell}}$ in Ω. As a result, $\xi_{\varepsilon_{k_\ell}}\to\xi_D$ in $H^2(\Omega)$, where $\xi_D\in H^1_0(\Omega)$ and $\Delta\xi_D = h_D$ in Ω, and $A_{\varepsilon_{k_\ell}}\to A_D$ in $H^1(\Omega;\mathbb{R}^2)$. By this and elliptic estimates, $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})\to(\psi_D, A_D)$ in $\mathcal{M}$.

Next, we prove that $|\psi_{\varepsilon_k}|\to\sqrt{a}$ uniformly in $\overline{\Omega}$ for the given sequence, $\{(\psi_{\varepsilon_k}, A_{\varepsilon_k})\}$, as $k\to\infty$. If not, we may choose a subsequence, $\{(\psi_{\varepsilon_{m_\ell}}, A_{\varepsilon_{m_\ell}})\}$, of $\{(\psi_{\varepsilon_k}, A_{\varepsilon_k})\}$, a sequence $\{y_\ell\}\subset\Omega$, and a constant $\gamma>0$ such that

$$\left(a(y_\ell) - \left|\psi_{\varepsilon_{m_\ell}}(y_\ell)\right|^2\right)^2 \geqslant \gamma$$

for $\ell = 1, 2, \ldots$. By passing to a subsequence (which we relabel as $\{(\psi_{\varepsilon_{m_\ell}}, A_{\varepsilon_{m_\ell}})\}$, reasoning as we did for $\{(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})\}$ above, we may assume that $(\psi_{\varepsilon_{m_\ell}}, A_{\varepsilon_{m_\ell}}) \to (\psi_{D'}, A_{D'})$ for some D' in Z^n where $(\psi_{D'}, A_{D'})$ is an equilibrium for J_0 in $H^1_{a,D'} \times H^1(\Omega; \mathbb{R}^2)$ and $(\psi_{\varepsilon_{m_\ell}}, A_{\varepsilon_{m_\ell}})$, $(\psi_{D'}, A_{D'})$ satisfy (44), (45), and (46). By the Hölder continuity of a and (19), we have $(a(x) - |\psi_{\varepsilon_{m_\ell}}(x)|^2)^2 \geqslant \frac{\gamma}{2}$ for all x in $\Omega \cap B_{r\varepsilon_\ell}(y_\ell)$ for some $r > 0$ and all ℓ sufficiently large. This implies that

$$\frac{1}{2\varepsilon_{m_\ell}^2} \int_\Omega \left(a - |\psi_{\varepsilon_{m_\ell}}|^2\right)^2 \geqslant c$$

for some $c > 0$ independent of ℓ, which contradicts (46). Thus $|\psi_{\varepsilon_k}| \to \sqrt{a}$ uniformly in $\overline{\Omega}$ and $k \to \infty$.

Returning to the analysis of $\{(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})\}$ and (ψ_D, A_D), it follows from the uniform convergence of $|\psi_{\varepsilon_{k_\ell}}|$ to $\sqrt{a}$ that for each $0 < \delta \leqslant R$, there exist positive constants t_0 and t_1 depending on δ so that $|\psi_{\varepsilon_{k_\ell}}| \geqslant t_1$ in $\overline{\Omega} \setminus \bigcup_{m=1}^n B_\delta(x_m)$ if $\varepsilon_{k_\ell} \leqslant t_0$. If $R \geqslant r \geqslant \delta$, then

$$\frac{1}{2\pi i} \int_{\partial B_r(x_m)} \frac{\psi_{\varepsilon^*_{k_\ell}}}{|\psi_{\varepsilon_{k_\ell}}|} \left(\frac{\psi_{\varepsilon_{k_\ell}}}{|\psi_{\varepsilon_{k_\ell}}|}\right)_\tau \equiv d_{\varepsilon_{k_\ell}, m}$$

is a well-defined integer independent of r. Since $\psi_{\varepsilon_{k_\ell}} \to \psi_D$ in $H^1(\Omega; \mathbb{C})$, it follows (as in Section 2) that $d_{\varepsilon_{k_\ell}, m} = d_m$ for all $m \in \{1, \ldots, n\}$ and all ℓ sufficiently large, where $D = (d_1, \ldots, d_n)$.

Since $J_0(\psi_D, A_D) = \lim_{\ell \to \infty} J_{\varepsilon_{k_\ell}}(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})$ and $J_0(\psi, A) = J_\varepsilon(\psi, A)$ for all (ψ, A) in $\mathcal{M}_0$, it follows that (ψ_D, A_D) is a minimizer for J_0 in $\mathcal{M}_0$, i.e. $D \in \mathcal{D}_0(h_e)$. Finally the assertion that $|\psi_{\varepsilon_k}| \to \sqrt{a}$ in $C(\overline{\Omega})$ has been proved for a subsequence of an arbitrary sequence. As a result the assertion is true for the original sequence. □

From the above result and Theorem 3.7, we obtain:

THEOREM 4.3. – *Fix $R > 0$ as in Theorem 4.2 and $h_e \geqslant 0$. For each $\varepsilon > 0$, let $(\psi_\varepsilon, A_\varepsilon)$ be a minimizer of J_ε in $\mathcal{M}$. There exists $\varepsilon_0 = \varepsilon_0(R, h_e) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $|\psi_\varepsilon| > 0$ outside $\bigcup_{i=1}^n B_R(x_i)$ and the degree of ψ_ε in $\overline{B_R(x_i)}$ for $i = 1, \ldots, n$, denoted by $D_\varepsilon = (d_{1,\varepsilon}, \ldots, d_{n,\varepsilon})$, is in $\mathcal{D}_0(h_e)$. If, in addition, $h_e > \bar{h}_e$ (for $\bar{h}_e$ defined as in Theorem 3.7), then $D_\varepsilon \neq \vec{0}$.*

Proof. – The first assertion of the theorem follows from Theorem 4.2 and an argument by contradiction, since $|\psi_\varepsilon| \to \sqrt{a}$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0^+$. If $h_e > \bar{h}_e$ and $0 < \varepsilon < \varepsilon_0(R, h_e)$, then $\vec{0} \notin \mathcal{D}_0 \equiv \mathcal{D}_0(h_e)$ by Theorem 3.7. Thus the degree of ψ_ε in $\overline{\Omega}$ for ε sufficiently small is nontrivial. □

Given $\varepsilon > 0$, the lower critical field $H_{c_1}(\varepsilon)$ is defined as the supremum of all nonnegative numbers h_e such that any minimizer, $(\psi_\varepsilon, A_\varepsilon)$, of $J_\varepsilon(\psi, A) = J_\varepsilon(\psi, A; h_e)$ in $\mathcal{M}$ satisfies $|\psi_\varepsilon| > 0$ in $\overline{\Omega}$. (Note that when $h_e = 0$, then every minimizer $(\psi_\varepsilon, A_\varepsilon)$ satisfies $A_\varepsilon = 0$ and $\alpha\psi_\varepsilon > 0$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Thus the set of nonnegative numbers described above is nonempty, and $H_{c_1}(\varepsilon)$ is well-defined for each $\varepsilon > 0$.) From Theorem 4.3, we have:

COROLLARY 4.4. – *Define* $\bar{h}_e > 0$ *as in Theorem* 3.7. *Then* $\limsup_{\varepsilon\to 0} H_{c_1}(\varepsilon) \leqslant \overline{h_e}$.

Proof. – Choose $R > 0$ so that $\overline{B_R(x_i)}$ are disjoint subsets of Ω for $i = 1, \dots, n$. If the inequality is false, there exists $\delta > 0$ and a sequence $\varepsilon_k \to 0^+$ such that $H_{c_1}(\varepsilon_k) > \overline{h_e} + \delta$ for all k. Letting $(\psi_{\varepsilon_k}, A_{\varepsilon_k})$ be a minimizer of J_{ε_k} for $h_e = \overline{h_e} + \delta$ and $k = 1, 2, \dots,$ we have $|\psi_{\varepsilon_k}| > 0$ in $\overline{\Omega}$ for all k, which contradicts Theorem 4.3 since $\varepsilon_k \leqslant \varepsilon_0(R, \overline{h_e} + \delta)$ for all k sufficiently large. □

Our results thus far show that for each $h_e \geqslant 0$, the set of minimizers of J_0 in $\mathcal{M}_0$ are given precisely by the set of minimizers, (ψ_D, A_D), of J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$ with D in $\mathcal{D}_0(h_e)$, where $\mathcal{D}_0(h_e)$ is the finite set of all D in Z^n which minimize $\mathcal{E}(C; h_e)$. Moreover, (ψ_D, A_D) is unique in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$ up to uniform rotations of ψ_D in $\overline{\Omega}$. In addition, for ε sufficiently small, minimizers of J_ε in $\mathcal{M}$ have vortices "pinned" near $x_1, \dots, x_n$ with an order parameter having degrees $(d_{1,\varepsilon}, \dots, d_{n,\varepsilon}) = D_\varepsilon$ near $x_1, \dots, x_n$, respectively, for some D_ε in $\mathcal{D}_0(h_e)$. This proves the results stated as Theorems 1–4 in the introduction.

Our final result shows that in contrast to what we have shown for minimizers, there exist *local minimizers* of J_ε in $\mathcal{M}$ with ε sufficiently small, with arbitrarily prescribed degrees of the order parameter near $x_1, \dots, x_n$, respectively. More precisely, we have:

LEMMA 4.5. – *Fix any* $D = (d_1, \dots, d_n)$ *in* Z^n *and* $h_e \geqslant 0$. *Let* (ψ_D, A_D) *be an equilibrium for* J_0 *in* $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. *For each sequence,* $\varepsilon_k \to 0^+$ *there exists local minimizers* $(\psi_{\varepsilon_k}, A_{\varepsilon_k})$ *of* J_{ε_k} *in* $\mathcal{M}$, *such that* $(\psi_{\varepsilon_k}, A_{\varepsilon_k}) \to (\psi_D, A_D)$ *in* $\mathcal{M}$ *as* $\varepsilon_k \to 0$. *Moreover for each* $R > 0$, *as in Theorem* 4.2, *and all k sufficiently large,* $|\psi_{\varepsilon_k}|$ *is uniformly positive outside* $\bigcup_{i=1}^n B_R(x_i)$ *and the degree of* ψ_{ε_k} *in* $\overline{B_R(x_i)}$ *is* d_i, *for* $i = 1, \dots, n$.

Proof. – Define

$$\mathcal{B}_r \equiv \mathcal{B}_r(\psi_D, A_D) = \bigl\{(\psi, A) \in \mathcal{M}\colon \bigl\|(\psi, A) - (\psi_D, A_D)\bigr\|_{\mathcal{M}} \leqslant r\bigr\}$$

and

$$\mathcal{N} \equiv \bigl\{(\psi, A) \in \mathcal{M}\colon \operatorname{div} A = 0 \text{ in } \Omega \text{ and } A \cdot \vec{n} = 0 \text{ on } \partial\Omega\bigr\}.$$

By Theorem 3.2, $(\psi_D, A_D) \in \mathcal{B}_r \cap \mathcal{N}$ is a minimizer of J_0 in $H^1_{a,D} \times H^1(\Omega; \mathbb{R}^2)$. In addition, for $r > 0$ sufficiently small,

$$\mathcal{B}_r \cap \mathcal{M}_0 = \mathcal{B}_r \cap \bigl[H^1_{a,D} \times H^1\bigl(\Omega; \mathbb{R}^2\bigr)\bigr] \tag{47}$$

since $H^1_{a,D}$ is open in H^1_a by Theorem 1.5. Thus (ψ_D, A_D) is also a local minimizer for J_0 in $\mathcal{M}_0$.

Fix $r > 0$ satisfying (47). For each $\varepsilon > 0$, let $(\psi_\varepsilon, A_\varepsilon)$ be a minimizer for J_ε in $\mathcal{B}_r \cap \mathcal{N}$. Then the sequence $\{\|(\psi_{\varepsilon_k}, A_{\varepsilon_k})\|_{\mathcal{M}}\}$ is bounded. Thus there exists a subsequence $\varepsilon_{k_\ell} \to 0^+$ such that $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) \rightharpoonup (\psi^0, A^0)$ in $\mathcal{M}$. Moreover $(\psi^0, A^0) \in \mathcal{B}_r \cap \mathcal{N} \cap \mathcal{M}_0$ must be a minimizer for J_0 on $\mathcal{B}_r$. (Here we use the weak lower semicontinuity of J_0, the fact that $J_\varepsilon(\psi, A) = J_0(\psi, A)$ for all (ψ, A) in $\mathcal{M}_0$, and that $(\psi_D, A_D) \in \mathcal{B}_r \cap \mathcal{N}$.) As in Theorem 4.2, we obtain

$$\lim_{\ell\to\infty}\int_\Omega \left|(\nabla - iA_{\varepsilon_{k_\ell}})\psi_{\varepsilon_{k_\ell}}\right|^2 = \int_\Omega \left|(\nabla - iA^0)\psi^0\right|^2, \tag{48}$$

$$\lim_{\ell\to\infty}\int_\Omega \left|\nabla\times A_{\varepsilon_{k_\ell}} - h_e\mathbf{e}_3\right|^2 = \int_\Omega \left|\nabla\times A^0 - h_e\mathbf{e}_3\right|^2, \tag{49}$$

$$\lim_{\ell\to\infty}\frac{1}{2\varepsilon_{k_\ell}^2}\int_\Omega \left(a - |\psi_{\varepsilon_{k_\ell}}|^2\right)^2 = 0. \tag{50}$$

Thus just as in Theorem 4.2 we find that

$$\left\|(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}}) - (\psi^0, A^0)\right\|_{\mathcal{M}} \to 0 \tag{51}$$

as $\ell\to\infty$. Since (ψ_D, A_D) is the unique minimizer for J_0 in $\mathcal{B}_r$ up to a uniform rotation of ψ_D (with our gauge assumption), and since (ψ^0, A^0) is in $\mathcal{N}$ as well it follows that $(\psi_D, A_D) = (\psi^0\,\mathrm{e}^{ic}, A^0)$ for some $c\in\mathbb{R}$. From this and (51) we see that $(\psi_{\varepsilon_{k_\ell}}\mathrm{e}^{ic}, A_{\varepsilon_{k_\ell}}) \to (\psi_D, A_D)$ in $\mathcal{M}$. In particular $(\psi_{\varepsilon_{k_\ell}}\mathrm{e}^{ic}, A_{\varepsilon_{k_\ell}})$ is in $\mathcal{B}_r$ for all ℓ sufficiently large. Thus by replacing $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})$ and (ψ^0, A^0) by the gauge equivalent pairs $(\psi_{\varepsilon_{k_\ell}}\mathrm{e}^{ic}, A_{\varepsilon_{k_\ell}})$ and (ψ_D, A_D) respectively we may assume without loss of generality that $(\psi^0, A^0) = (\psi_D, A_D)$. In this case it follows that $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})$ is an interior point of $\mathcal{B}_r$ for all ℓ sufficiently large. As a result, $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})$ is a local minimizer for $J_{\varepsilon_{k_\ell}}$ in $\mathcal{N}$. Since $J_{\varepsilon_{k_\ell}}$ is gauge invarient it follows that $(\psi_{\varepsilon_{k_\ell}}, A_{\varepsilon_{k_\ell}})$ is a local minimizer in $\mathcal{M}$ as well. In particular it is an equilibrium solution. From this and (50) it follows just as in Theorem 4.2 that $|\psi_{\varepsilon_{k_\ell}}| \to \sqrt{a}$ in $C(\overline{\Omega})$ as $\varepsilon_{k_\ell}\to 0$. We see that $\psi_{\varepsilon_{k_\ell}}$ inherits the same degree as ψ_D, namely d_i, in $\overline{B_R(x_i)}$ for $i = 1,\ldots,n$ and all ℓ sufficiently large. Since each sequence $\varepsilon_k\to 0^+$ contains a subsequence for which our assertions hold the same is true for the full sequence. □

THEOREM 4.6. – *Fix any $D = (d_1,\ldots,d_n)$ in Z^n and $h_e \geqslant 0$. Let (ψ_D, A_D) be an equilibrium for J_0 in $H^1_{a,D}\times H^1(\Omega;\mathbb{R}^2)$ and choose $r>0$ satisfying* (47)*. Then for all $\varepsilon>0$ sufficiently small, $\mathcal{B}_r(\psi_D, A_D)$ contains a local minimizer, $(\psi_\varepsilon, A_\varepsilon)$, of J_ε in $\mathcal{M}$ such that for each $R>0$ as in Theorem* 4.2 *and all ε sufficiently small, $|\psi_\varepsilon|$ is uniformly positive outside $\bigcup_{i=1}^n B_R(x_i)$ and the degree of ψ_ε in $\overline{B_R(x_i)}$ is d_i for $i = 1,\ldots,n$.*

Proof. – The theorem follows from Lemma 4.5 using an argument by contradiction. Indeed if the theorem's assertion was false then there would exist a sequence $\varepsilon_k\to 0^+$ for which the lemma could not hold. □

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