Ann. Inst. Henri Poincaré, Anal. non linéaire 18, 2 (2001) 261–270

S0294-1449(00)00065-2/FLA

A PRIORI ESTIMATES FOR SOLUTIONS OF FULLY NONLINEAR SPECIAL LAGRANGIAN EQUATIONS

Yu YUAN

Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA

Received in 26 April 2000, revised 23 October 2000

ABSTRACT. – We derive an a priori $C^{2,\alpha}$ estimate in dimension three for the equation $F(D^2u) = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 = c$, where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the Hessian D^2u. For $-\pi/2 < c < \pi/2$, the c-level set of $F(D^2u)$ fails the convexity condition. Note that for any solution u of the above equation, $(x, \nabla u(x))$ is a minimizing graph in $\mathbb{R}^6$. For $c = 0, \pm\pi$, the equation is equivalent to $\triangle u = \det D^2u$.

AMS classification: 35; 53

RÉSUMÉ. – On déduit une estimation a priori $C^{2,\alpha}$ en dimension trois pour l'équation $F(D^2u) = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 = c$, où $\lambda_1, \lambda_2, \lambda_3$ sont les valuers propres du hessien D^2u. Pour $-\pi/2 < c < \pi/2$, l'ensemble de niveau c du $F(D^2u)$ ne satisfait pas la condition de convexité. Remarquez que pour n'importe qu'elle solution u de l'équation, $(x, \nabla u(x))$ est un graphe qui minimise l'aire dans $\mathbb{R}^6$. Pour $c = 0, \pm\pi$, l'équation est équivalent à $\triangle u = \det D^2u$.

1. Introduction

In this note we derive an *a priori* $C^{2,\alpha}$ estimate in dimension three for solutions to the fully nonlinear elliptic equation

$$F\left(D^2u\right) = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 = c, \tag{1.1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the Hessian D^2u. Notice (1.1) just says that the argument of the complex number $(1 + i\lambda_1)(1 + i\lambda_2)(1 + i\lambda_3)$ is constant c.

For $|c| \geqslant \pi/2$, the c-level set of $F(D^2u)$, $\Sigma_c = \{M \text{ symmetric} \mid F(M) = c\}$ is convex. (This can be seen by computing the second fundamental form of Σ_c, cf. [4] Lemma C.) The $C^{2,\alpha}$ estimate also follows from the well known result of Evans [8] and Krylov [12]. For $|c| < \pi/2$, the c-level set Σ_c fails the convexity condition (cf. [4] Lemma C), it

E-mail address: yyuan@math.utexas.edu (Y. Yuan).

even does not satisfy the assumption in [5], where we prove the $C^{2,\alpha}$ estimate under the assumption that $\Sigma_c \cap \{M \mid \operatorname{Trace} M = t\}$ is convex for all t.

Eq. (1.1) comes from special Lagrangian geometry [10]. The (Lagrangian) graph $(x, \nabla u(x)) \in \mathbb{R}^{2n}$ is called special when the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $D^2 u$ satisfy

$$\arctan \lambda_1 + \cdots + \arctan \lambda_n = c, \tag{1.2}$$

and it is special if and only if $(x, \nabla u(x)) \in \mathbb{R}^{2n}$ is a minimal surface [10, Theorem 2.3, Proposition 2.17]. In the three dimensional case, for $c = 0, \pm\pi$, Eq. (1.1) is equivalent to $\triangle u = \det D^2 u$.

A continuous function $u(x)$ is said to be a viscosity solution of (1.1) if it is both a viscosity subsolution and supersolution. And u is a subsolution (respectively, supersolution), if for $\varphi \in C^2$, $\varphi - u$ has a local minimal at x_0, then $F(D^2\varphi(x_0)) \geqslant c$ (respectively, if $\varphi - u$ has a local maximum at x_0, then $F(D^2\varphi(x_0)) \leqslant c$). Note that if $u \in C^{1,1}$, then u is a viscosity solution of (1.1) if and only if u is a strong solution of (1.1), that is $F(D^2 u) = c$ almost everywhere (cf. [3, p. 367]). Once u is a $W^{2,n}$ strong solution of (1.2), then $(x, \nabla u(x))$ is an absolutely volume-minimizing submanifold in $\mathbb{R}^{2n}$ [10, Theorem 4.2].

THEOREM 1.1. – *Let u be a $C^{1,1}$ viscosity solution of* $F(D^2 u) = \arctan \lambda_1 + \arctan \lambda_2 + \arctan \lambda_3 = c$ *in the unit ball $B_1 \subset \mathbb{R}^3$, where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the Hessian $D^2 u$. Then for* $\alpha \in (0, 1)$ *we have the interior estimates*

$$\left\| D^2 u \right\|_{C^\alpha(B_{1/2})} \leqslant C\left(\alpha, \left\| D^2 u \right\|_{L^\infty(B_1)}\right).$$

The proof of Theorem 1.1 has two steps. In step one (Section 2), by a geometric argument we show that $D^2 u$ is in VMO (vanishing mean oscillation). This means that $D^2 u$ concentrates. In step two (Section 3), we show that u then is very close to a quadratic polynomial (Proposition 3.2), and this "closedenss" improves increasingly as we rescale (Proposition 3.3), thanks to the smoothness of $F(M)$ that makes it look, after rescaling, more and more like a linear operator around $M = D^2 P_k$ (P_k is the kth approximating polynomial).

Once the VMO modulus of $D^2 u$ is available, as in [11] one can get the $C^{2,\alpha}$ estimate by the L^p estimate in [7]. Here we give a different approach, following [1] as in [5]. Since the proof is relatively short, we present it here for completeness.

We close this introduction by the following remark. The reason that we restrict ourselves to the dimension three case is because all 3-d graph like special Lagrangian cones are linear spaces (Lemma 2.1). In higher dimensional case, this assertion remains unclear to us.

2. Preliminary estimates (VMO)

The following lemma follows from a well known result in geometry. Here we give yet another proof.

LEMMA 2.1. – *Let $u \in C^\infty(\mathbb{R}^3 \backslash \{0\})$ be a viscosity solution of* (1.1) *in* $\mathbb{R}^3$, *and homogeneous degree two, that is, $u(tx) = t^2 u(x)$. Then u is a quadratic polynomial.*

Proof. – All we need to do is show that u is C^2 at the origin. Note that $M=(x,\nabla u(x))$ is a three dimensional minimal cone in $\mathbb{R}^6$, and $M\cap S^5$ is homeomorphic to a two sphere S^2. By a result of Calabi [6, Theorem 5.5], $M\subset\mathbb{R}^5\subset\mathbb{R}^6$. So there exits $(a,b)\in\mathbb{R}^3\times\mathbb{R}^3$ such that

$$ax+b\nabla u(x)=0.$$

Without loss of generality, we assume $b=(0,0,-1)$, then

$$\frac{\partial u}{\partial x_3}=a_1x_1+a_2x_2+a_3x_3.$$

Hence $u=a_1x_1x_3+a_2x_2x_3+\frac{1}{2}a_3x_3x_3+v(x_1,x_2)$, and $v\in C^\infty(\mathbb{R}^2\backslash\{0\})$ satisfies $\overline{F}(D^2v)=c$ in the viscosity sense, where $\overline{F}=F|_A$ and

$$A=\left\{\begin{bmatrix} m_{11} & m_{12} & a_1\\ m_{21} & m_{22} & a_2\\ a_1 & a_2 & a_3\end{bmatrix}\,\middle|\, M=\begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\text{ is symmetric }2\times 2\text{ matrix}\right\}.$$

Since $|D^2v|$ is also bounded, we are in the two dimensional case, v is C^2 at the origin. (cf. [9, Theorem 17.2]).

Thus u is C^2 at the origin, and u is a quadratic polynomial. □

LEMMA 2.2. – *Let u be a $C^{1,1}$ viscosity solution of* (1.1) *in* $\mathbb{R}^3$, *and* $|D^2u|\leqslant K$. *Then u is a quadratic polynomial.*

Proof. – Without loss of generality, we assume $u(0)=0$, $\nabla u(0)=0$. We “blow down” u at ∞.

Set

$$u_k(x)=\frac{u(kx)}{k^2},\quad k=1,2,3,\ldots.$$

We see that

$$\|u_k\|_{C^{1,1}(B_R)}\leqslant C(K,R),$$

then there exists a subsequence, still denoted by $\{u_k\}$ and a function $u_R\in C^{1,1}(B_R)$ such that $u_k\to u_R$ in $C^{1,\alpha}(B_R)$ as $k\to\infty$, and $|D^2u_R|\leqslant K$. By the fact that the family of viscosity solution is closed under C^0 uniform limit, we know that u_R is also a viscosity solution of $F(D^2u)=c$ in B_R. By the $W^{2,\delta}$ estimate (cf. [2, Proposition 7.4]), we have

$$\left\|D^2u_k-D^2u_R\right\|_{L^\delta(B_{R/2})}\leqslant C(K,R)\|u_k-u_R\|_{L^\infty(B_R)}\to 0\quad\text{as } k\to\infty.$$

Note that $|D^2u_k|,|D^2u_R|\leqslant K$, then for $p>0$, we have

$$\left\|D^2u_k-D^2u_R\right\|_{L^p(B_{R/2})}\to 0\quad\text{as } k\to\infty.$$

By the diagonalizing process, there exists another subsequence, again denoted by $\{u_k\}$ and $v\in C^{1,1}(\mathbb{R}^3)$ such that $u_k\to v$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3)$ as $k\to\infty$, $|D^2v|\leqslant K$, and v is viscosity solution of $F(D^2v)=c$ in $\mathbb{R}^3$.

Since u is viscosity solution of $F(D^2v)=c$ in $\mathbb{R}^3$, $(x,\nabla u(x))$ is a minimal surface in $\mathbb{R}^6$. Also note $D^2u_k\to D^2v$ in $W^{2,3}_{\mathrm{loc}}(\mathbb{R}^3)$, by the monotonicity formula (cf. [13, p. 84]) and Theorem 19.3 in [13], we conclude that $M_v=(x,\nabla v(x))$ is a cone. Since the tangent cone of M_v at each point away from the vertex is a 2-dimensional cone cross $\mathbb{R}^1$ (cf. [13, Lemma 35.5]), the 2-dimensional cone must be a linear space by the same arguments as in the end of the proof Lemma 2.1. Then we apply Allard's regularity result (cf. [13, Theorem 24.2]) to conclude that M_v is smooth away from the vertex. That is $v\in C^\infty(\mathbb{R}^3\backslash\{0\})$ and $v(tx)=t^2v(x)$. By Lemma 2.1, v is a quadratic polynomial, say, $v(x)=\frac{1}{2}x^tMx$. Again by Allard's regularity result (cf. [13, Theorem 24.2]), $u\equiv v$ is a quadratic polynomial. □

Recall that a locally integrable function u is in VMO(Ω) with modulus $\omega_u(R,\Omega)$ if

$$\omega_u(R,\Omega)=\sup_{\substack{x_0\in\Omega\\ 0<r\leqslant R}}\ \fint_{B_r(x_0)\cap\Omega}|u(x)-u_{x_{0,r}}|\to 0,\quad \text{as } R\to 0,$$

where $\fint_A u$ denotes the average of u over A and $u_{x_{0,r}}$ the average of u over $B_r(x_0)\cap\Omega$.

PROPOSITION 2.3. – *Let u be a $C^{1,1}$ viscosity solution of* (1.1) *in $B_1\subset\mathbb{R}^3$ and $|D^2u|\leqslant K$. Then $D^2u\in VMO(B_{1/2})$ and the VMO modulus of D^2u, $\omega_{D^2u}(r)\leqslant\omega(r)$, where ω only depends on K and $\omega(r)\to 0$ as $r\to 0^+$.*

Proof. – Suppose the conclusion of the proposition is not true. Then there exists $\varepsilon_0\geqslant 0$, $r_k\to 0$, $x_k\in B_{1/2}$, and a family of $C^{1,1}$ viscosity solutions of $F(D^2u)=c$, $\{u_k\}$, $|D^2u_k|\leqslant K$, such that

$$\fint_{B_{r_k}}\left|D^2u_k-\left(D^2u_k\right)_{x_k,r_k}\right|\geqslant\varepsilon_0.$$

We "blow up" $\{u_k\}$, set

$$v_k(y)=\frac{u_k(x_k+r_ky)-\nabla u_k(x_k)\cdot r_ky-u_k(x_k)}{r_k^2}\quad\text{for } |y|\leqslant\frac{1}{r_k},$$

we see that $F(D^2v_k)=c$ in the viscosity sense and $\|v_k\|_{C^{1,1}(B_R)}\leqslant C(K,R)$. Then there exists a subsequence, still denoted by $\{v_k\}$ and a function $v_R\in C^{1,1}(B_R)$ such that $v_k\to v_R$ in $C^{1,\alpha}(B_R)$ as $k\to\infty$, and $|D^2v_R|\leqslant K$. By the fact that the family of viscosity solution is closed under C^0 uniform limit, we know that v_R is also a viscosity solution of $F(D^2v_R)=c$ in B_R. By the $W^{2,\delta}$ estimate (cf. [2, Proposition 7.4]), we have

$$\left\|D^2v_k-D^2v_R\right\|_{L^\delta(B_{R/2})}\leqslant C(K,R)\|v_k-v_R\|_{L^\infty(B_R)}\to 0\quad\text{as } k\to\infty.$$

Note that $|D^2v_k|,|D^2v_R|\leqslant K$, then for $p>0$, we have

$$\left\|D^2v_k-D^2v_R\right\|_{L^p(B_{R/2})}\to 0\quad\text{as } k\to\infty.$$

By the diagonalizing process, there exists another subsequence, again denoted by $\{v_k\}$, and $v\in C^{1,1}(\mathbb{R}^3)$ such that $v_k\to v$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3)$ as $k\to\infty$, $|D^2v|\leqslant K$, and v is viscosity solution of $F(D^2v)=c$ in $\mathbb{R}^3$. By Lemma 2.2, we know v is a quadratic polynomial. Hence

$$0=\fint_{B_1}\left|D^2v-\left(D^2v\right)_{0,1}\right|=\lim_{k\to\infty}\fint_{B_1}\left|D^2v_k-\left(D^2v_k\right)_{0,1}\right|$$
$$=\lim_{k\to\infty}\fint_{B_{r_k}}\left|D^2u_k-\left(D^2u_k\right)_{x_k,r_k}\right|\geqslant\varepsilon_0.$$

This is a contradiction. □

3. $C^{2,\alpha}$ estimates

Once the VMO modulus of Hessian D^2u is available, we can get the Hölder estimate of the Hessian of the solutions to general uniformly elliptic equation $F(D^2u)=0$ with ∇F being continuous. In the following Lemma 3.1 and Propositions 3.2, 3.3, we confine ourselves to the special Lagrangian equation in any dimension

$$F\left(D^2u\right)=\arctan\lambda_1+\cdots+\arctan\lambda_n=c, \tag{3.1}$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of the Hessian D^2u. In the end of section, relying on Proposition 2.3 which is true in three dimensional case, we give the proof of Theorem 1.1.

First we need the following modifying lemma in the proof of Proposition 3.2.

LEMMA 3.1. – *Let u be a $C^{1,1}$ viscosity solution of* (3.1). *Then for all quadratic polynomial P satisfying $|D^2P|\leqslant\|D^2u\|_{L^\infty}=K$, we can modify it to $\overline{P}=P+\frac{1}{2}sx^2$ so that*

$$F\left(D^2\overline{P}\right)=c,$$
$$|s|\leqslant C(n,K)\|u-P\|_{L^\infty(B_1)},$$
$$\|u-\overline{P}\|_{L^\infty(B_1)}\leqslant C(n,K)\|u-P\|_{L^\infty(B_1)}.$$

Proof. – Note that $F(D^2P)-F(D^2u)=F(D^2P)-c$, applying the Alexandrov–Bakelman–Pucci maximum principle, we have

$$\left|F\left(D^2P\right)-c\right|\leqslant C(n,K)\|u-P\|_{L^\infty(B_1)}.$$

Since $F(M)$ is elliptic for $|M|\leqslant K$ with ellipticity $\lambda(K)$, there exists s with

$$|s|\leqslant C(n,K)\|u-P\|_{L^\infty(B_1)}$$

so that

$$F\left(D^2P+sI\right)=c.$$

Now let $\overline{P}=P+\frac{1}{2}sx^2$, we arrive at the conclusion of the above lemma. □

The next proposition shows once D^2u concentrates on a point, then u gets close to a polynomial.

PROPOSITION 3.2. – *Assume that u is a $C^{1,1}$ viscosity solution of* (3.1) *in $B_1 \subset \mathbb{R}^n$, $|D^2u| \leqslant K$ in B_1, and $D^2u \in VMO(B_{3/4})$ with VMO modulus $\omega(r)$. Then for any $\varepsilon > 0$, there exist $\eta = \eta(n, K, \omega, \varepsilon)$ and a quadratic polynomial P so that*

$$\left|\frac{1}{\eta^2}u(\eta x) - P(x)\right| \leqslant \varepsilon \quad \textit{for } x \in B_1,$$
$$F\left(D^2P\right) = c.$$

Proof. – Take ρ, $r > 0$ to be chosen later. Set

$$w_r(x) = \frac{1}{r^2}u(rx),$$
$$s_r = \operatorname{Tr} \fint_{B_1} D^2 w_r = \operatorname{Tr} \fint_{B_r} D^2 u,$$

then $|s_r| \leqslant nK$.

Solve

$$\begin{cases} \triangle v(x) = s_r & \text{in } B_1, \\ v(x) = w_r & \text{on } \partial B_1. \end{cases}$$

By the Alexandrov–Bakelman–Pucci maximum principle and the assumption $|D^2u| \leqslant K$, we have

$$\begin{aligned} \|w_r - v\|_{L^\infty(B_1)} &\leqslant C(n)\|\triangle w_r - \triangle v\|_{L^n(B_1)} \\ &\leqslant C(n)\left[\int_{B_1} \left|D^2u(rx) - \left(D^2u\right)_{0,r}\right|^n\right]^{1/n} \\ &\leqslant C(n, K)\left[\int_{B_1} \left|D^2u(rx) - \left(D^2u\right)_{0,r}\right|\right]^{1/n} \\ &= C(n, K)\left[\fint_{B_r} \left|D^2u(x) - \left(D^2u\right)_{0,r}\right|\right]^{1/n} \\ &\leqslant C(n, K)\omega^{1/n}(r). \end{aligned}$$

Also

$$\begin{aligned} \left\|D^3v\right\|_{L^\infty(B_{1/2})} &\leqslant C(n) \sup_{x \in \partial B_1} \left|w_r(x) - w_r(0) - \langle \nabla w_r(0), x\rangle - \frac{1}{2}s_r|x|^2\right| \\ &\leqslant C(n, K). \end{aligned}$$

If we take the quadratic part $\overline{P}$ of v at the origin, then

$$|w_r(x) - \overline{P}(x)| \leqslant C(n, K)\omega^{1/n}(r) + C(n, K)|x|^3.$$

Now let $x = \rho y$, $\overline{\overline{P}}(y) = \frac{1}{\rho^2}\overline{P}(\rho y)$. For $|y| \leqslant 1$, we have

$$\left|\frac{1}{\rho^2}w_r(\rho y) - \overline{\overline{P}}(y)\right| \leqslant C(n, K)\left[\frac{\omega^{1/n}(r)}{\rho^2} + \rho\right].$$

Since $\frac{1}{\rho^2} w_r(\rho y)$ satisfies $F(D^2u) = c$, by the modifying Lemma 3.1, we perturb $\overline{\overline{P}}(y)$ to another quadratic polynomial $P(y)$ so that

$$F\left(D^2 P\right) = c$$

with

$$\left|\frac{1}{\rho^2} w_r(\rho y) - P(y)\right| \leqslant C(n, K)\left[\frac{\omega^{1/n}(r)}{\rho^2} + \rho\right].$$

Finally we choose ρ, then r depending on $n, K, \omega, \varepsilon$ so that

$$\left|\frac{1}{\eta^2} u(\eta y) - P(y)\right| \leqslant \varepsilon,$$

where $\eta = \eta(n, K, \omega, \varepsilon) = \rho r$. $\square$

Finally, Proposition 3.3 indicates the inductive process by which, once u is close to a polynomial, it becomes $C^{2,\alpha}$.

PROPOSITION 3.3. – *There exist $\mu \in (0,1)$, m depending on n, $K = \|D^2 u\|_{L^\infty(B_1)}$ and α so that, if $\|u - P\|_{L^\infty(B_1)} \leqslant \mu^{2+\alpha+m}$, $F(D^2 P) = c$, and $F(D^2 u) = c$ almost everywhere in $B_1 \subset \mathbb{R}^n$, where F is as in* (3.1). *Then we have a family of polynomials $P_k = a^k + \langle b^k, x\rangle + \frac{1}{2} x^t C^k x$, $k = 1, 2, \ldots,$ satisfying*

(i) $\|u - P_k\|_{L^\infty(B_{\mu^k})} \leqslant \mu^{k(2+\alpha)+m}$,
(ii) $|a^k - a^{k+1}|$, $\mu^k |b^k - b^{k+1}|$, $\mu^{2k}|C^k - C^{k+1}| \leqslant C(n, K)\ \mu^{k(2+\alpha)+m}$,
(iii) $F(C^k) = c$.

Proof. – Let $P_1 = P$, we prove this proposition by induction. Set

$$w(x) = \frac{(u - P_k)(\mu^k x)}{\mu^{k(2+\alpha)+m}} \quad \text{for } x \in B_1,$$

then $|w(x)| \leqslant 1$ and for $F^k(M) = \frac{1}{\mu^{k\alpha+m}} F(\mu^{k\alpha+m} M + C^k)$

$$F^k\left(D^2 w\right) = \frac{1}{\mu^{k\alpha+m}} F\left(D^2 u(\mu^k x)\right) = \frac{c}{\mu^{k\alpha+m}}$$

almost everywhere in B_1. Let v be the solution of

$$\begin{cases} \sum_{i,j=1}^n F_{ij}^k(0) D_{ij} v = 0 & \text{in } B_{3/4}, \\ v = w & \text{on } \partial B_{3/4}, \end{cases}$$

where $F_{ij} = \partial F(M)/\partial m_{ij}$, $M = (m_{ij})$. We see

$$\begin{aligned} F^k\left(D^2 v\right) &= F^k(0) + \sum_{i,j=1}^n F_{ij}^k(0) D_{ij} v + \mathrm{O}\left(\left\|\nabla^2 F\right\|\left\|D^2 v\right\|^2 \mu^{k\alpha+m}\right) \\ &= \frac{c}{\mu^{k\alpha+m}} + \mathrm{O}\left(\left\|\nabla^2 F\right\|\left\|D^2 v\right\|^2 \mu^{k\alpha+m}\right). \end{aligned}$$

Using the interior Hölder estimate on w with $\beta = \beta(n, K)$, for example, Proposition 4.10 in [2],

$$\|w\|_{C^\beta(\overline{B}_{3/4})} \leqslant C(n, K).$$

By the global Hölder estimate on v,

$$\|v\|_{C^{\beta/2}(\overline{B}_{3/4})} \leqslant C(n, K)\|w\|_{C^\beta(\partial B_{3/4})} \leqslant C(n, K).$$

Applying the interior estimate on v, we have for $\delta > 0$ to be chosen later

$$\|v\|_{C^3(B_{1/2})} \leqslant C(n, K),$$
$$\left\|D^2 v\right\|_{L^\infty(B_{3/4-\delta})} \leqslant C(n, K)\delta^{-2}.$$

By the Alexandrov–Bakelman–Pucci maximum principle, we have

$$\begin{aligned}\|w - v\|_{L^\infty(B_{3/4-\delta})} &\leqslant \sup_{\partial B_{3/4-\delta}} |w - v| + C(n, K)\left\|F^k\left(D^2 w\right) - F^k\left(D^2 v\right)\right\|_{L^\infty(B_{3/4-\delta})}\\ &\leqslant C(n, K)\left[\left(\delta^\beta + \delta^{\beta/2}\right) + \left\|\nabla^2 F\right\|\mu^{k\alpha+m}\delta^{-4}\right]\\ &\leqslant C(n, K)\left[\left(\delta^\beta + \delta^{\beta/2}\right) + \mu^{k\alpha+m}\delta^{-4}\right].\end{aligned}$$

Now take $\overline{P}$ to be the quadratic part of v at the origin, we have

$$\|w - \overline{P}\|_{L^\infty(B_\mu)} \leqslant C(n, K)\left[\mu^3 + \left(\delta^\beta + \delta^{\beta/2}\right) + \mu^{k\alpha+m}\delta^{-4}\right].$$

Since $F^k(D^2 w) = c/\mu^{k\alpha+m}$, by the modifying Lemma 3.1 properly scaled, we perturb $\overline{P}$ to another quadratic $\overline{\overline{P}}$ so that $F^k(D^2\overline{\overline{P}}) = c/\mu^{k\alpha+m}$ which is $F(D^2\overline{\overline{P}}) = c$, and

$$\left\|w - \overline{\overline{P}}\right\|_{L^\infty(B_\mu)} \leqslant C(n, K)\left[\mu^3 + \left(\delta^\beta + \delta^{\beta/2}\right) + \mu^{k\alpha+m}\delta^{-4}\right].$$

We finally choose μ, then δ and m, depending on n, $K = \|D^2 u\|_{L^\infty(B_1)}$, and α so that

$$\left\|w - \overline{\overline{P}}\right\|_{L^\infty(B_\mu)} \leqslant \mu^{2+\alpha}.$$

Rescaling back, we get

$$\left\|u - P_k - \mu^{k(2+\alpha)+m}\overline{\overline{P}}(\mu^{-k}x)\right\|_{L^\infty(B_{\mu^{k+1}})} \leqslant \mu^{(k+1)(2+\alpha)+m}.$$

Let $P_{k+1} = P_k + \mu^{k(2+\alpha)+m}\overline{\overline{P}}(\mu^{-k}x)$, we see (i), (ii), and (iii) hold. □

Proof of Theorem 1.1. – We apply Propositions 2.3, 3.2 to u and Proposition 3.3 to $u(\eta x)/\eta^2$. From (i) (ii) in Proposition 3.3, we see that the family of polynomials $\{P_k\}$ converges uniformly to a quadratic polynomial $Q(x)$ satisfying

$$\left|\frac{1}{\eta^2}u(\eta x) - Q(x)\right| \leqslant C(K)|x|^{2+\alpha}.$$

Let $y = \eta x$, for $|y| \leqslant \eta$ we get

$$\left| u(y) - \eta^2 Q\left(\frac{y}{\eta}\right) \right| \leqslant C(K) \frac{1}{\eta^\alpha} |y|^{2+\alpha}.$$

Similarly, one proves the above inequality at every point $x_0 \in B_{1/2}$, that is, for $|y - x_0| \leqslant \eta$

$$|u(y) - Q_{x_0}(y)| \leqslant C(K) \frac{1}{\eta^\alpha} |y - x_0|^{2+\alpha}.$$

Therefore

$$\|D^2 u\|_{C^\alpha(B_{1/2})} \leqslant C\left(\alpha, \|D^2 u\|_{L^\infty(B_1)}\right).$$

The proof of Theorem 1.1 is complete. $\square$

Acknowledgements

The author is grateful to Luis A. Caffarelli for suggestions. The author is partially supported by NSF Grant DMS 9970367.

REFERENCES

[1] Caffarelli L.A., Interior a priori estimates for solutions of fully nonlinear equations, Ann. Math. 130 (1989) 189–213.
[2] Caffarelli L.A., Cabré X., Fully Nonlinear Elliptic Equations, American Mathematical Society Colloquium Publications, Vol. 43, American Mathematical Society, Providence, RI, 1995.
[3] Caffarelli L.A., Crandall M.G., Kocan M., Świech A., On viscosity solutions of fully nonlinear equations with measurable ingredients, Comm. Pure Appl. Math. 49 (1996) 365–397.
[4] Caffarelli L.A., Nirenberg L., Spruck J., The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian, Acta Math. 155 (1985) 261–301.
[5] Caffarelli L.A., Yuan Y., A Priori estimates for solutions of fully nonlinear equations with convex level set, Indiana Univ. Math. J., to appear.
[6] Calabi E., Minimal immersions of surfaces in Euclidean spheres, J. Differential Geom. 1 (1967) 111–125.
[7] Chiarenza F., Frasca M., Longo P., $W^{2,p}$ solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc. 336 (1993) 841–853.
[8] Evans L.C., Classical solutions of fully nonlinear, convex, second-order elliptic equations, Comm. Pure Appl. Math. 35 (3) (1982) 333–363.
[9] Gilbarg D., Trudinger N.S., Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer-Verlag, 1983.
[10] Harvey R., Lawson H.B. Jr., Calibrated geometry, Acta Math. 148 (1982) 47–157.
[11] Huang Q.-B., On the regularity of solutions to fully nonlinear elliptic equations via Liouville property, Proc. Amer. Math. Soc., to appear.

[12] Krylov N.V., Boundedly nonhomogeneous elliptic and parabolic equations, Izv. Akad. Nauk SSSR Ser. Mat. 46 (3) (1982) in Russian; English translation in Math. USSR Izv. 20 (1983) 459–492, 487–523.
[13] Simon L., Lectures on Geometric Measure Theory, Proc. C. M. A., Austr. Nat. Univ., Vol. 3, 1983.