NONHOMOGENEOUS CAHN–HILLIARD FLUIDS

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ABSTRACT. – In this paper we are interested in the study of a model of nonhomogeneous diphasic incompressible flow. More precisely we consider a coupling of a Cahn–Hilliard and an incompressible Navier–Stokes equations where the densities of the phases are different.

For this general model we can only show the local existence of a unique very regular solution and the existence of weaker solutions is still an open problem. But, if we look at the behavior of the system when the densities tends to be equal (slightly nonhomogeneous case), we show the existence of a global weak solution and of a unique local strong solution (which is in fact global in 2D). Finally, an asymptotic stability result for the metastable states is shown in this slightly nonhomogeneous case.

RÉSUMÉ. – Dans cet article, nous nous intéressons à l'étude d'un modèle d'écoulement diphasique nonhomogène incompressible. Plus précisément, nous considérons un couplage entre une équation de Cahn–Hilliard et une équation de Navier–Stokes incompressible dans lesquelles les densités des deux phases sont différentes.

Pour ce modèle général, nous pouvons seulement prouver l'existence locale et l'unicité d'une solution très régulière, l'existence de solutions plus faibles restant un problème ouvert. En revanche, si nous considérons le comportement du système quand les densités des deux phases sont proches (cas faiblement nonhomogène), nous montrons l'existence de solutions faibles globales et l'existence et l'unicité de solutions fortes locales (en fait globales en dimension 2). Enfin, un résultat de stabilité asymptotique des états métastables est établi, toujours dans le cas faiblement nonhomogène.

1. Introduction

We are interested in the study of incompressible diphasic nonhomogeneous mixtures flows. We have proposed in [6] the derivation of a mathematical model for this kind of problem based on the coupling of a Cahn–Hilliard equation and a nonhomogeneous Navier–Stokes equation. The origine of this derivation lies on the works of numerous

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authors [9,11,14,23]. We obtain the following equations for the order parameter φ, the potential μ and the velocity v. If ρ_1^0, ρ_2^0 denote the densities of the two phases, the system reads

$$\frac{\partial \varphi}{\partial t} + v.\nabla\varphi - \operatorname{div}\left(\frac{1}{\rho_\varepsilon(\varphi)}\nabla\left(\frac{\mu}{\rho_\varepsilon(\varphi)}\right)\right) = 0, \tag{1.1}$$

$$\mu = -\alpha\Delta\varphi + F'(\varphi), \tag{1.2}$$

$$\rho_\varepsilon(\varphi)\left(\frac{\partial v}{\partial t} + v.\nabla v\right) - 2\operatorname{div}\big(\eta(\varphi)D(v)\big) + \nabla p$$

$$= \mu\nabla\varphi + \varepsilon\frac{1 - \varphi^2}{4}\nabla\left(\frac{\mu}{\rho_\varepsilon(\varphi)}\right) + \rho_\varepsilon(\varphi)g, \tag{1.3}$$

$$\operatorname{div}(v) = 0, \tag{1.4}$$

the normalized density being theoretically given by

$$\rho_\varepsilon(\varphi) = 1 + \varepsilon\frac{\varphi - 1}{2}, \tag{1.5}$$

with

$$\varepsilon = \frac{|\rho_1^0 - \rho_2^0|}{\max(\rho_1^0, \rho_2^0)},$$

representing the relative difference of the densities. Let us remark that we always have $\varepsilon \leqslant 1$. We recall the usual notation for the deformation tensor $D(u) = (\nabla u + \nabla u^t)/2$.

For this model, we are not able to prove in general (even if $\varepsilon = 0$) that the values of the order parameter remain in the physical-meaningful interval $[-1, 1]$. This implies that if we define ρ_ε with (1.5) we are not sure that the density remains always positive. That's the reason why, we introduce a slightly different definition for ρ_ε, namely it must be a function satisfying:

$$\rho_\varepsilon(1) = 1, \qquad |\rho_\varepsilon'|_\infty \leqslant \varepsilon$$

and

$$0 < \rho_1 \leqslant \rho_\varepsilon(\varphi) \leqslant \rho_2,$$

independtly of ε. One may keep in mind that ρ_ε is essentially given by (1.5) into the interval $[-1, 1]$.

In some particular cases, for example (see [5]) if $\varepsilon = 0$ and if we introduce a degenerate mobility (diffusion coefficient) in the model, then we can show that the values of the order parameter stay in the physical-meaningful interval $[-1, 1]$. Hence, we know *a posteriori* that the density is really given by (1.5). Such a qualitative result on the values of φ is also expected even in the case $\varepsilon > 0$, if one consider a logarithmic Cahn–Hilliard potential F of the form
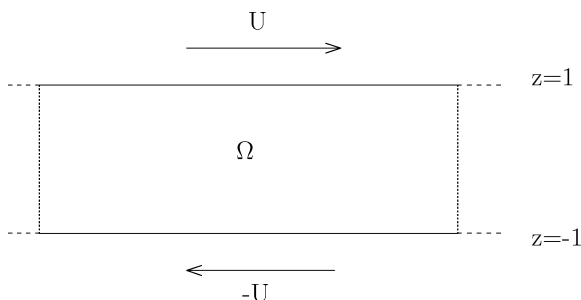
$$F(\varphi) = \theta\big(1 - \varphi^2\big) + (1 + \varphi)\log(1 + \varphi) + (1 - \varphi)\log(1 - \varphi),$$

which is a physical-relevant choice for F (see [14]).

From now on, we suppose that the dimension of the space is $d = 2$ or $d = 3$. Our following study takes place again [5] in the case of the channel under shear which corresponds to the physical experimental conditions, but our results are still true if we consider a bounded regular domain with homogeneous boundary conditions. Consequently, the previous system is provided with periodicity condition in the x, y-directions and on the other boundaries, with the conditions

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0, \tag{1.6}$$

$$v = U e_x \quad \text{on } \{z = 1\}, \qquad v = -U e_x \quad \text{on } \{z = -1\}. \tag{1.7}$$

We shown in [6] that the numerical simulations for this model give physical-relevant results. Moreover, the homogeneous case ($\varepsilon = 0$, $\rho_\varepsilon(\varphi) \equiv 1$) has been studied in [5], where it is shown the existence of weak solutions, the existence and uniqueness of strong solutions and an asymptotic stability result of the metastable states of the potential.

Our first objective is to study the existence of solutions to system (1.1)–(1.4), (1.6), (1.7). For any range of admissible values of ε, we can only show (Theorem 3.1) the existence and uniqueness of local very strong solutions. In this case, the existence of weak solutions is still an open question.

Nevertheless, in the slightly nonhomogeneous regime, that is to say if we suppose the smallness of the parameter ε, we can drasticaly improve the results in this direction. More precisely, we show (Theorem 3.2) that if ε is small enough, then there exists a global weak solution uniformly bounded in time in the appropriate spaces. Furthermore, this solution converges, up to an extraction of a subsequence, towards a weak solution of the homogeneous problem.

Moreover we show (Theorem 3.3), always under the condition that ε is small enough, the existence and uniqueness of strong solutions (global in 2D and local in 3D) for regular initial data.

Finally, we establish (Theorem 3.4) the same kind of asymptotic stability result than the one shown in [5], always in the slightly nonhomogeneous case. We point out that the asymptotic stability of the metastable stationnary states is shown even in 3D.

2. Notations and fundamental results

Functional spaces

Throughout this paper we denote by $|.|_p$ the usual norm on the space L^p, and by $\|.\|_s$ the usual norm on H^s. We let $\mathbb{L}^p = (L^p)^d$ and $\mathbb{H}^s = (H^s)^d$ the norms on these spaces being always denoted by $|.|_p$ and $\|.\|_s$.

We have to introduce the natural homogeneous boundary conditions associated to the problem (1.1)–(1.4), (1.6), (1.7). Namely, we introduce, if it makes sense, the conditions

φ is periodic in the x, y-directions and satisfies $\dfrac{\partial \varphi}{\partial \nu} = \dfrac{\partial \Delta \varphi}{\partial \nu} = 0$ on $\{z = \pm 1\}$.   (2.1)

u is periodic in the x, y-directions and satisfies $u = 0$ on $\{z = \pm 1\}$.   (2.2)

Then we define classicaly (see [5]) the spaces

$$\Phi_s = \{\varphi \in H^s, \text{ satisfying } (2.1)\},$$

$$V_s = \{u \in \mathbb{H}^s, \operatorname{div}(u) = 0, \text{ satisfying } (2.2)\}.$$

As usual, the space V_0 will be denoted by H, and the space V_1 by V. In the definition of H one must replace the boundary condition $u = 0$ by $u.\nu = 0$. Moreover, we will denote by $\mathcal{P}$ the orthogonal projector in $\mathbb{L}^2(\Omega)$ onto the space H.

Stokes operator

We recall (see [26]) that for any $u \in V_2$, there exists a unique $(Au, \pi) \in H \times (H^1/\mathbb{R})$ such that

$$Au = -\Delta u + \nabla \pi,$$

the operator $u \mapsto Au$ is a nonbounded operator in H of domain V_2 named the Stokes operator. Moreover, there exists $C_1, C_2, C_3 > 0$ such that for any $u \in V_2$ we have

$$\begin{cases} C_1 \|u\|_2 \leqslant |Au|_2 \leqslant C_2 \|u\|_2, \\ \|\pi\|_{H^1/\mathbb{R}} \leqslant C_3 \|u\|_2, \\ |\pi|_{L^2/\mathbb{R}} \leqslant C_3 \|u\|_1. \end{cases} \qquad (2.3)$$

Fundamental inequalities

We do not recall the classical Sobolev embeddings that we will use in this paper. We also refer to [3,18] for the different interpolation results we need in our estimates.

Let us recall the Poincaré's and Korn's inequalities: there exists $C_4, C_5 > 0$ such that for any $u \in V$, we have

$$C_4 \|u\|_1 \leqslant |\nabla u|_2 \leqslant C_5 |D(u)|_2. \qquad (2.4)$$

Furthermore, for any $f \in L^1$ we denote by $m(f) = \frac{1}{|\Omega|} \int_\Omega f$ its average. Then [25] there exists $C_6 > 0$ such that

$$\begin{cases} \|\varphi - m(\varphi)\|_1 \leqslant C_6 |\nabla \varphi|_2, & \forall \varphi \in \Phi_1, \\ \|\varphi - m(\varphi)\|_{s+2} \leqslant C_6 \|\Delta \varphi\|_s, & \forall s \geqslant 0, \ \forall \varphi \in \Phi_{s+2}. \end{cases} \tag{2.5}$$

As a consequence we will systematically use inequalities like

$$\|\nabla \varphi\|_1 = \left\| \nabla (\varphi - m(\varphi)) \right\|_1 \leqslant \|\varphi - m(\varphi)\|_2 \leqslant C_6 |\Delta \varphi|_2.$$

Finally we will use the two following Agmon's inequalities in dimension $d = 3$

$$\begin{cases} |f|_\infty \leqslant C \|f\|_1^{1/2} \|f\|_2^{1/2}, & \text{for any } f \in H^2, \\ |f|_\infty \leqslant C |f|_2^{1/2} \|f\|_3^{1/2}, & \text{for any } f \in H^3. \end{cases} \tag{2.6}$$

Stationary solutions

One can remark that, if we suppose that g is derived from a potential $\mathcal{G}$, that is to say $g = \nabla \mathcal{G}$, then we can construct a family of stationary solutions of (1.1)–(1.4)

$$\varphi^\infty = \omega, \qquad v_\infty^U = U z e_x, \tag{2.7}$$

where ω is a given constant. We will study the asymptotic stability of this solution in the Section 3.3, but introducing these solutions is necessary in order to state precisely the results we present here.

Mean conservation for the Cahn–Hilliard equation

We state here a fundamental property of the Cahn–Hilliard equation and more generally of Eq. (1.1) with the boundary conditions (1.6).

LEMMA 2.1. – *Any solution φ of (1.1) satisfying (1.6), with a velocity field $v(t) \in H$, satisfies*

$$\frac{\partial}{\partial t} m(\varphi) = 0,$$

which implies that

$$m(\varphi(t)) = m(\varphi_0), \text{ as long as } \varphi \text{ exists}.$$

The proof is straightforward by choosing the constant function 1 as a test function for (1.1). We will use this property systematically in the following.

General assumptions

To conclude with, we make precise here the assumptions we make in the whole paper. First, we assume that the external forces term g lies in $\mathbb{L}^2$ and is independent of the time. In some sections, we will suppose in addition that g is a gradient of a potential of H^1.

Furthermore, we assume that the viscosity η is a regular function (typically of C^1-class) which satisfies

$$0 < \eta_1 \leqslant \eta(x) \leqslant \eta_2, \quad \text{for any } x \in \mathbb{R}.$$

As far as the Cahn–Hilliard potential is concerned, we make the following assumptions (see [5])

$$F \text{ is of } C^2 \text{ class, and } F \geqslant 0, \tag{2.8}$$

$$\exists \, F_1, F_2 > 0 \text{ such that } |F'(x)| \leqslant F_1 |x|^p + F_2, \ |F''(x)| \leqslant F_1 |x|^{p-1} + F_2, \ \forall x \in \mathbb{R}, \tag{2.9}$$

where $1 \leqslant p \leqslant 3$ if $d = 3$ and $1 \leqslant p < +\infty$ if $d = 2$,

$$\forall \gamma \in \mathbb{R}, \ \exists \, F_3(\gamma) > 0, \ F_4(\gamma) \geqslant 0 \text{ such that,}$$

$$(x - \gamma) F'(x) \geqslant F_3(\gamma) F(x) - F_4(\gamma), \ \forall x \in \mathbb{R}, \tag{2.10}$$

$$\exists \, F_5 \geqslant 0 \text{ such that } F''(x) \geqslant -F_5, \ \forall x \in \mathbb{R}. \tag{2.11}$$

As a remark, we point out that the condition $F \geqslant 0$ is not restrictive because a physical-meaningful potential is always bounded from below and adding a constant to the potential F does not change the equations.

Those assumptions allows the choice of a classical Cahn–Hilliard potential: polynomial of second order with positive dominant coefficient (see [12,25]).

3. Slightly nonhomogeneous mixtures

In the case of general nonhomogeneous mixtures we can only show a result of local existence of strong solutions. In fact, the solutions we obtain are stronger than the one obtained in the sequel (Theorem 3.3). This is not surprising because, when we do not suppose that ε is small, our system of Eqs. (1.1)–(1.4) is very strongly non-linear in particular in the Cahn–Hilliard equation (1.1). We give without proof the following result

THEOREM 3.1. – *For any $U > 0$, $v_0 \in v_\infty^U + V$ and $\varphi_0 \in \Phi_4$, there exists a time $T > 0$ depending on U, $\|v_0\|_1$ and $\|\varphi_0\|_4$ such that for any $\varepsilon \leqslant 1$ there exists a unique strong solution $(\varphi_\varepsilon, v_\varepsilon)$ of the problem (1.1)–(1.4) on $[0, T[$, satisfying*

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;\Phi_4)} + \|\varphi_\varepsilon\|_{L^2(0,T;\Phi_6)} + \|v_\varepsilon - v_\infty^U\|_{L^\infty(0,T;V)} + \|v_\varepsilon - v_\infty^U\|_{L^2(0,T;V_2)} \leqslant C,$$

$$\left\| \frac{\partial \varphi_\varepsilon}{\partial t} \right\|_{L^2(0,T;\Phi_1)} + \left\| \frac{\partial v_\varepsilon}{\partial t} \right\|_{L^2(0,T;H)} \leqslant C,$$

where $C > 0$ is independent of ε.

Remark 3.1. – The proof consists essentially in using slightly differently the same estimates than in the proofs of Theorems 3.2 and 3.3.

In order to prove more significant results, we are interested, until the end of the paper, in the study of the system (1.1)–(1.4) when the parameter ε is small. That is to say that we suppose that the densities of the two phases are close enough. Under those conditions we can show the existence of global weak solutions and the existence and uniqueness of strong solutions (global in 2D and local in 3D).

3.1. Weak solutions

In this subsection we are concerned with the proof of the following result.

THEOREM 3.2. – *Let* $U > 0$, $v_0^\varepsilon \in v_\infty^U + H$, $\varphi_0^\varepsilon \in \Phi_3$, *such that* $m(\varphi_0^\varepsilon)$ *is independent of* ε. *We suppose that there exists* C_0 *independent of* ε *satisfying*

$$\|\varphi_0^\varepsilon\|_1 + |v_0^\varepsilon|_2 + \varepsilon^{1/2}\|\varphi_0^\varepsilon\|_2 + \varepsilon^{3/4}\|\varphi_0^\varepsilon\|_3 \leqslant C_0.$$

There exists ε_0 *depending only on* C_0, U *and* F *such that for any* $\varepsilon < \varepsilon_0$ *there exists a weak solution* $(\varphi_\varepsilon, v_\varepsilon)$ *of* (1.1)–(1.4) *on* $\mathbb{R}^+$ *for the initial data* $(\varphi_0^\varepsilon, v_0^\varepsilon)$, *satisfying*

$$\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^+;\Phi_1)} + \varepsilon^{1/2}\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^+;\Phi_2)} + \varepsilon^{3/4}\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^+;\Phi_3)} + \|v_\varepsilon - v_\infty^U\|_{L^\infty(\mathbb{R}^+;H)} \leqslant C, \tag{3.1}$$

$$\|\varphi_\varepsilon\|_{L^2(t_0,t_0+\tau;\Phi_3)} + \varepsilon^{1/2}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;L^2)} + \varepsilon^{3/4}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;H^1)}$$
$$+ \|\mu_\varepsilon\|_{L^2(t_0,t_0+\tau;\Phi_1)} + \|v_\varepsilon - v_\infty^U\|_{L^2(t_0,t_0+\tau;V)} \leqslant C(\tau), \quad \text{for any } t_0 \geqslant 0, \ \tau > 0, \tag{3.2}$$

$$\left\|\mathcal{P}\frac{\partial\rho_\varepsilon v_\varepsilon}{\partial t}\right\|_{L^2(0,T;V'_{d/2})} \leqslant M(T), \tag{3.3}$$

where C, $C(\tau)$, $M(T)$ *are independent of* ε *and* t_0.
    Moreover, if

$$\varphi_0^\varepsilon \to \varphi_0 \quad \text{and} \quad v_0^\varepsilon \to v_0, \quad \text{when } \varepsilon \to 0,$$

then, up to an extraction of a subsequence, $(v_\varepsilon, \varphi_\varepsilon)$ *converges towards a solution of the homogeneous limit system* ($\varepsilon = 0$).

Remark 3.2. – The following proof is given in the case $d = 3$. The estimates in the 2D case are made in the same way but are in fact much easier to derive. As an exception, we point out the difference between the 2D and 3D case for inequalities (3.28) and (3.29).

Proof. – In the following, it is convenient to drop the superscript ε for φ_0^ε, v_0^ε and the subscript ε for φ_ε, v_ε and ρ_ε, but one may keep in mind that any quantity which is estimated may depend on ε.
    We will only give the formal derivation of the energy estimates (3.1)–(3.3). The complete proof can be performed through an approximation process (a Galerkin method, for example, [5]) and we will make precise at the end of the proof, the way we obtain the compactness necessary to take the limit in the approximated solutions.
    From now on, we are mainly concerned with the proof of estimates (3.1), (3.2).

Step 1. Following [20], for $\lambda > 0$ we introduce a vector field v_λ as in [5] depending only in z and satisfying:

$$\operatorname{div}(v_\lambda) = 0,$$
$$v_\lambda . \nabla v_\lambda = 0,$$
$$|v_\lambda|_\infty = 1,$$
$$|v_\lambda|_4 \leqslant C\lambda,$$
$$|\nabla v_\lambda|_2^2 \leqslant C\frac{1}{\lambda^4}(1 + \lambda^8), \tag{3.4}$$

and

$$v_\lambda = e_x \quad \text{on } \{z = 1\}, \qquad v_\lambda = -e_x \quad \text{on } \{z = -1\}.$$

Now we let $v = u + U v_\lambda$ so that u satisfies homogeneous boundary conditions (2.2) and the equations

$$\frac{\partial \varphi}{\partial t} + u . \nabla \varphi - \operatorname{div}\left(\frac{1}{\rho}\nabla\left(\frac{\mu}{\rho}\right)\right) = -U v_\lambda . \nabla \varphi, \tag{3.5}$$

$$\mu = -\alpha \Delta \varphi + F'(\varphi), \tag{3.6}$$

$$\rho\left(\frac{\partial u}{\partial t} + u . \nabla u\right) - 2\operatorname{div}\bigl(\eta(\varphi)D(u)\bigr) + \nabla p$$
$$= -U\rho v_\lambda . \nabla u - U\rho u . \nabla v_\lambda + 2U \operatorname{div}\bigl(\eta(\varphi)D(v_\lambda)\bigr)$$
$$+ \mu \nabla \varphi + \varepsilon \frac{1 - \varphi^2}{4}\nabla\left(\frac{\mu}{\rho}\right) + \rho g, \tag{3.7}$$

$$\operatorname{div}(u) = 0. \tag{3.8}$$

Just as in the remaining of this paper, we have denoted by ρ the density $\rho(\varphi)$.

Step 2. We first try to get classical energy estimates for these equations. We take the inner product of (3.5) in L^2 with μ and of (3.7) in $\mathbb{L}^2$ with u, and we get

$$\frac{d}{dt}\left(\frac{\alpha}{2}|\nabla\varphi|_2^2 + \frac{1}{2}|\sqrt{\rho}u|_2^2 + \int_\Omega F(\varphi)\right) + 2\int_\Omega \eta(\varphi)|D(u)|^2 + \int_\Omega \frac{1}{\rho^2}|\nabla\mu|^2$$
$$= \int_\Omega \mu\frac{\rho'(\varphi)}{\rho^3(\varphi)}\nabla\varphi . \nabla\mu + \frac{\varepsilon}{4}\int_\Omega (1 - \varphi^2)\nabla\left(\frac{\mu}{\rho}\right).u + \int_\Omega \rho g.u$$
$$+ \frac{1}{2}\int_\Omega |u|^2\left(\frac{\partial \rho}{\partial t} + u . \nabla \rho\right) - U\int_\Omega (v_\lambda . \nabla\varphi)\mu$$
$$- U\int_\Omega \rho(u . \nabla v_\lambda)u - U\int_\Omega (v_\lambda . \nabla u)u + 2U\int_\Omega \eta(\varphi)D(v_\lambda) : D(u). \tag{3.9}$$

Remark 3.3. – One can easily see that this last estimate is useless if we do not have an estimate for $\partial\varphi/\partial t$. In fact, this point is the main difference in comparison with the classical nonhomogeneous fluids model [19,24]: as our model takes into account

exchange phenomena at the interface, the density does not satisfy the local conservation equation

$$\frac{\partial \rho}{\partial t} + v.\nabla \rho = 0,$$

so that we have

$$\int_\Omega \left(\rho \left(\frac{\partial u}{\partial t} + u.\nabla u \right) \right).u = \frac{1}{2} \frac{\partial}{\partial t} \left(\int_\Omega \rho |u|^2 \right) - \frac{1}{2} \int_\Omega |u|^2 \left(\frac{\partial \rho}{\partial t} + u.\nabla \rho \right),$$

where the last term does not vanish.

Nevertheless, we have global conservation of the order parameter (and of the density if it is defined by (1.5)) in the sense that (Lemma 2.1)

$$\frac{\partial}{\partial t} \left(\int_\Omega \varphi \right) = 0.$$

That's the reason why we have to multiply (3.5) by $\partial \varphi / \partial t$ to get after integration

$$\left| \frac{\partial \varphi}{\partial t} \right|_2^2 + \int_\Omega (u.\nabla \varphi) \frac{\partial \varphi}{\partial t} + U \int_\Omega (v_\lambda.\nabla \varphi) \frac{\partial \varphi}{\partial t}$$

$$= - \int_\Omega \frac{1}{\rho} \nabla \left(\frac{\mu}{\rho} \right).\nabla \frac{\partial \varphi}{\partial t}$$

$$= -\alpha \int_\Omega \frac{1}{\rho^2} \Delta \varphi \Delta \frac{\partial \varphi}{\partial t} - \alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^3(\varphi)} \Delta \varphi \nabla \varphi.\nabla \frac{\partial \varphi}{\partial t} - \int_\Omega \frac{1}{\rho} \nabla \left(\frac{F'(\varphi)}{\rho(\varphi)} \right).\nabla \frac{\partial \varphi}{\partial t}$$

$$= - \frac{d}{dt} \left(\frac{\alpha}{2} \int_\Omega \frac{1}{\rho^2} |\Delta \varphi|^2 \right) + \alpha \int_\Omega |\Delta \varphi|^2 \frac{\rho'(\varphi)}{\rho^3(\varphi)} \frac{\partial \varphi}{\partial t}$$

$$- \alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^3(\varphi)} \Delta \varphi \nabla \varphi.\nabla \frac{\partial \varphi}{\partial t} - \int_\Omega \frac{1}{\rho} \nabla \left(\frac{F'(\varphi)}{\rho(\varphi)} \right).\nabla \frac{\partial \varphi}{\partial t}. \qquad (3.10)$$

We see another time that this last estimate requires to have some information concerning $\nabla \frac{\partial \varphi}{\partial t}$. In this direction, we multiply (3.5) by $\frac{\partial}{\partial t}(\frac{\mu}{\rho})$ and integrate on Ω to have

$$\frac{1}{2} \frac{d}{dt} \left(\int_\Omega \frac{1}{\rho} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|^2 \right) + \int_\Omega u.\nabla \varphi \frac{\partial}{\partial t} \left(\frac{\mu}{\rho} \right) + U \int_\Omega (v_\lambda.\nabla \varphi) \frac{\partial}{\partial t} \left(\frac{\mu}{\rho} \right)$$

$$= - \int_\Omega \frac{\partial \varphi}{\partial t} \frac{\partial}{\partial t} \left(\frac{\mu}{\rho} \right) - \int_\Omega \frac{\rho'(\varphi)}{\rho^2(\varphi)} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|^2 \frac{\partial \varphi}{\partial t}$$

$$= \alpha \int_\Omega \frac{\partial \varphi}{\partial t} \frac{\partial}{\partial t} \left(\frac{\Delta \varphi}{\rho} \right) - \int_\Omega \frac{\partial \varphi}{\partial t} \frac{\partial}{\partial t} \left(\frac{F'(\varphi)}{\rho(\varphi)} \right) - \int_\Omega \frac{\rho'(\varphi)}{\rho^2(\varphi)} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|^2 \frac{\partial \varphi}{\partial t}$$

$$= \alpha \int_\Omega \frac{1}{\rho} \frac{\partial \varphi}{\partial t} \frac{\partial \Delta \varphi}{\partial t} - \alpha \int_\Omega \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\rho'(\varphi)}{\rho^2(\varphi)} \Delta \varphi$$

$$- \int_\Omega \left| \frac{\partial \varphi}{\partial t} \right|^2 \left(\frac{F''(\varphi)}{\rho(\varphi)} - \frac{F'(\varphi)\rho'(\varphi)}{\rho^2(\varphi)} \right) - \int_\Omega \frac{\rho'(\varphi)}{\rho^2(\varphi)} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|^2 \frac{\partial \varphi}{\partial t}$$

$$= -\alpha \int_\Omega \frac{1}{\rho} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2 - \alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^2(\varphi)} \frac{\partial \varphi}{\partial t} \nabla \frac{\partial \varphi}{\partial t} . \nabla \varphi - \alpha \int_\Omega \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\rho'(\varphi)}{\rho^2(\varphi)} \Delta \varphi$$

$$- \int_\Omega \left| \frac{\partial \varphi}{\partial t} \right|^2 \left(\frac{F''(\varphi)}{\rho(\varphi)} - \frac{F'(\varphi)\rho'(\varphi)}{\rho^2(\varphi)} \right) - \int_\Omega \frac{\rho'(\varphi)}{\rho^2(\varphi)} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|^2 \frac{\partial \varphi}{\partial t}. \qquad (3.11)$$

We use assumptions (2.8)–(2.11) on the function F, to deduce (see [5])

$$C\, F_4(m(\varphi_0)) + \frac{1}{\rho_2^2} |\nabla \mu|_2^2 \geqslant \frac{1}{2\rho_2^2} |\nabla \mu|_2^2 + C |\nabla \varphi|_2^2 + C |\Delta \varphi|_2^2 + C F_3(m(\varphi_0)) \left(\int_\Omega F(\varphi) \right), \qquad (3.12)$$

and

$$|m(\mu)| = \left| -\alpha \int_\Omega \Delta \varphi + \int_\Omega F'(\varphi) \right| \leqslant C \left(1 + |\varphi|_p^p \right) \leqslant C \left(1 + |\nabla \varphi|_2^p \right), \qquad (3.13)$$

because in any cases $H^1 \subset L^p$. Finally we have

$$\frac{1}{4} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|_2^2 \leqslant \frac{1}{4\rho_2^2} |\nabla \mu|_2^2 + C\varepsilon^2 |\mu \nabla \varphi|_2^2$$

$$\leqslant \frac{1}{4\rho_2^2} |\nabla \mu|_2^2 + C\varepsilon^2 |\nabla \mu|_2^2 |\Delta \varphi|_2^2 + C\varepsilon^2 m(\mu)^2 |\nabla \varphi|_2^2$$

$$\leqslant \frac{1}{4\rho_2^2} |\nabla \mu|_2^2 + C\varepsilon^2 |\nabla \mu|_2^2 |\Delta \varphi|_2^2 + C\varepsilon^2 \left(1 + |\nabla \varphi|_2^{2p} \right) |\nabla \varphi|_2^2. \qquad (3.14)$$

Summing (3.9), $\varepsilon \times$ (3.10), $\varepsilon^{3/2} \times$ (3.11), (3.12) and (3.14) we finally get the energy estimate we need

$$\frac{d}{dt} \left(\frac{\alpha}{2} |\nabla \varphi|_2^2 + \frac{1}{2} |\sqrt{\rho} u|_2^2 + \int_\Omega F(\varphi) + \varepsilon \frac{\alpha}{2} \left| \frac{1}{\rho} \Delta \varphi \right|_2^2 + \varepsilon^{3/2} \left| \frac{1}{\sqrt{\rho}} \nabla \left(\frac{\mu}{\rho} \right) \right|_2^2 \right)$$

$$+ 2 \int_\Omega \eta(\varphi) |D(u)|^2 + \frac{1}{4\rho_2^2} |\nabla \mu|^2 + \frac{1}{4} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|_2^2 + C |\nabla \varphi|_2^2$$

$$+ C |\Delta \varphi|_2^2 + C \left(\int_\Omega F(\varphi) \right) + \varepsilon \left| \frac{\partial \varphi}{\partial t} \right|_2^2 + \varepsilon^{3/2} \alpha \int_\Omega \frac{1}{\rho} \left| \nabla \frac{\partial \varphi}{\partial t} \right|^2$$

$$\leqslant C\, F_4(m(\varphi_0)) + C\varepsilon^2 |\nabla \mu|_2^2 |\Delta \varphi|_2^2 + C\varepsilon^2 m(\mu)^2 |\nabla \varphi|_2^2$$

$$+ \frac{\varepsilon}{4} \int_\Omega (1 - \varphi^2) \nabla \left(\frac{\mu}{\rho} \right) . u + \int_\Omega \frac{\rho'(\varphi)}{\rho^3(\varphi)} \mu \nabla \varphi . \nabla \mu + \int_\Omega \rho g . u$$

$$- \varepsilon\alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^3(\varphi)} \Delta\varphi \nabla\varphi . \nabla \frac{\partial\varphi}{\partial t} + \frac{1}{2} \int_\Omega \rho'(\varphi)|u|^2 u.\nabla\varphi + \frac{1}{2} \int_\Omega \rho'(\varphi)|u|^2 \frac{\partial\varphi}{\partial t}$$

$$- \varepsilon \int_\Omega u.\nabla\varphi \frac{\partial\varphi}{\partial t} + \varepsilon\alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^3(\varphi)} |\Delta\varphi|^2 \frac{\partial\varphi}{\partial t} - \varepsilon \int_\Omega \frac{1}{\rho} \nabla\left(\frac{F'(\varphi)}{\rho(\varphi)} \right).\nabla\frac{\partial\varphi}{\partial t}$$

$$- \varepsilon^{3/2} \int_\Omega u.\nabla\varphi \frac{\partial}{\partial t}\left(\frac{\mu}{\rho} \right) - \varepsilon^{3/2}\alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^2(\varphi)} \frac{\partial\varphi}{\partial t} \nabla\frac{\partial\varphi}{\partial t}.\nabla\varphi - \varepsilon^{3/2}\alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^2(\varphi)} \left| \frac{\partial\varphi}{\partial t} \right|^2 \Delta\varphi$$

$$- \varepsilon^{3/2} \int_\Omega \left| \frac{\partial\varphi}{\partial t} \right|^2 \left(\frac{F''(\varphi)}{\rho(\varphi)} - \frac{F'(\varphi)\rho'(\varphi)}{\rho^2(\varphi)} \right) - \varepsilon^{3/2} \int_\Omega \frac{\rho'(\varphi)}{\rho^3(\varphi)} \left| \nabla\left(\frac{\mu}{\rho} \right) \right|^2 \frac{\partial\varphi}{\partial t}$$

$$- U \int_\Omega (v_\lambda.\nabla\varphi)\mu - U \int_\Omega \rho(u.\nabla v_\lambda)u - U \int_\Omega (v_\lambda.\nabla u)u + 2U \int_\Omega \eta(\varphi)D(v_\lambda):D(u)$$

$$- \varepsilon U \int_\Omega (v_\lambda.\nabla\varphi)\frac{\partial\varphi}{\partial t} - \varepsilon^{3/2}U \int_\Omega (v_\lambda.\nabla\varphi)\frac{\partial}{\partial t}\left(\frac{\mu}{\rho} \right). \tag{3.15}$$

Step 3. We introduce the following functionals

$$y_\varepsilon(t) = \frac{\alpha}{2}|\nabla\varphi|_2^2 + \frac{1}{2}|\sqrt{\rho}u|_2^2 + \int_\Omega F(\varphi) + \varepsilon\frac{\alpha}{2}\left| \frac{1}{\rho}\Delta\varphi \right|_2^2 + \varepsilon^{3/2}\left| \frac{1}{\sqrt{\rho}}\nabla\left(\frac{\mu}{\rho} \right) \right|_2^2,$$

$$z_\varepsilon(t) = \eta_1|\nabla u|_2^2 + \frac{1}{4\rho_2^2}|\nabla\mu|_2^2 + \frac{1}{4}\left| \nabla\left(\frac{\mu}{\rho} \right) \right|_2^2 + C|\nabla\varphi|_2^2 + C|\Delta\varphi|_2^2 + C\left(\int_\Omega F(\varphi) \right)$$

$$+ \varepsilon\left| \frac{\partial\varphi}{\partial t} \right|_2^2 + \varepsilon^{3/2}\frac{\alpha}{\rho_2}\left| \nabla\frac{\partial\varphi}{\partial t} \right|_2^2.$$

Then we can prove the following result.

LEMMA 3.1. – *There exists $\beta, C > 0$ such that for any $0 < \varepsilon \leqslant 1$, we have*

$$\varepsilon^{3/2}|\nabla\Delta\varphi|_2^2 \leqslant C(y_\varepsilon + y_\varepsilon^\beta), \tag{3.16}$$

$$\varepsilon^{3/4}|\Delta\varphi|_2^2 \leqslant C(y_\varepsilon + y_\varepsilon^\beta), \tag{3.17}$$

and

$$y_\varepsilon \leqslant Cz_\varepsilon. \tag{3.18}$$

Proof. – We have from Eq. (3.6)

$$\Delta\varphi = \frac{1}{\alpha}(-\mu + F'(\varphi)),$$

and so

$$\nabla\Delta\varphi = -\frac{1}{\alpha}\rho\nabla\left(\frac{\mu}{\rho} \right) - \frac{1}{\alpha}\frac{\mu}{\rho}\nabla\rho + \frac{F''(\varphi)}{\alpha}\nabla\varphi,$$

from which we deduce

$$|\nabla\Delta\varphi|_2^2 \leqslant C\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2 + C\varepsilon^2\left|\frac{\mu}{\rho}\nabla\varphi\right|_2^2 + C|\nabla\varphi|_2^2 + C\int_\Omega |\varphi|^{2p-2}|\nabla\varphi|^2$$

$$\leqslant C\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2 + C\varepsilon^2\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2|\Delta\varphi|_2^2 + C|\nabla\varphi|_2^2 + C|\varphi|_{3p-3}^{2p-2}|\nabla\varphi|_6^2$$

$$\leqslant C\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2 + C\varepsilon^2\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2|\Delta\varphi|_2^2 + C|\nabla\varphi|_2^2 + C|\nabla\varphi|_2^{2p-2}|\Delta\varphi|_2^2.$$

Finally we have

$$\varepsilon^{3/2}|\nabla\Delta\varphi|_2^2 \leqslant C\left(y_\varepsilon + \varepsilon y_\varepsilon^2 + \varepsilon^{1/2}y_\varepsilon^p\right),$$

so that we get (3.16) with $\beta = \max(2, p) > 1$. In order to show (3.17) we integrate by parts and we use (3.16),

$$\varepsilon^{3/4}|\Delta\varphi|_2^2 \leqslant \varepsilon^{3/4}\int_\Omega |\nabla\varphi|\,|\nabla\Delta\varphi| \leqslant \varepsilon^{3/4}|\nabla\varphi|_2\,|\nabla\Delta\varphi|_2 \leqslant \frac{1}{2}|\nabla\varphi|_2^2 + \frac{1}{2}\varepsilon^{3/2}|\nabla\Delta\varphi|_2^2$$

$$\leqslant C\left(y_\varepsilon + y_\varepsilon^{(\beta+1)/2}\right) \leqslant C\left(y_\varepsilon + y_\varepsilon^\beta\right).$$

The third point is clear from the definitions of y_ε and z_ε.   □

Remark 3.4. –
– The first point of the lemma give us a control on the H^3-norm of φ in terms of y_ε which is not a priori obvious.
– The second point will be very useful in the sequel. Indeed, if we look at the definition of y_ε we have

$$\varepsilon^{3/2}|\Delta\varphi|_2^2 \leqslant Cy_\varepsilon,$$

whereas (3.17) let us estimate $|\Delta\varphi|_2^2$ with a smaller power of ε (namely $\varepsilon^{3/4}$) under the condition that we allow the presence of powers of y_ε greater than 1.
– The third point of the lemma is the key-point of the end of the proof, when an ordinary differential equation argument is used to conclude. Estimates (3.12)–(3.14) are just derived in order for this control of y_ε by z_ε to be true.

Let us go back to the proof of Theorem 3.2. We obtain from (3.15) the differential inequality

$$\frac{d}{dt}y_\varepsilon + z_\varepsilon \leqslant C\,F_4\big(m(\varphi_0)\big) + C\varepsilon y_\varepsilon z_\varepsilon + C\varepsilon^2\big(1 + y_\varepsilon^p\big)z_\varepsilon + I_1 + \cdots + I_{20}, \qquad (3.19)$$

where $I_1, \ldots, I_{20}$ denote the twenty integrals of the right-hand side of (3.15). From now on, we wish to estimate each of these terms in function of ε, y_ε and z_ε.

Using (3.8) and the boundary conditions on u, the first term reads after integration by parts

$$|I_1| \leqslant \frac{\varepsilon}{2}\left|\int_\Omega \frac{\varphi}{\rho(\varphi)}\mu\nabla\varphi.u\right| = \frac{\varepsilon}{2}\left|\int_\Omega \frac{\varphi}{\rho(\varphi)}(\mu - m(\mu))\nabla\varphi.u\right|$$

$$\leqslant C\varepsilon\big(|\varphi|_6 + |m(\varphi_0)|\big)|\nabla\varphi|_2|\mu - m(\mu)|_6|u|_6$$

$$\leqslant C\varepsilon\big(|\nabla\mu|_2^2 + |\nabla u|_2^2\big)\big(|\nabla\varphi|_2^2 + |\nabla\varphi|_2\big) \leqslant C\varepsilon\big(y_\varepsilon + y_\varepsilon^{1/2}\big)z_\varepsilon. \qquad (3.20)$$

We notice that $|\rho'(\varphi)| \leqslant \varepsilon$ to obtain, with the Sobolev embedding $H^{1/2} \subset L^3$ (in dimension $d \leqslant 3$),

$$|I_2| \leqslant \frac{\varepsilon}{\rho_1^2}\big(|\mu - m(\mu)|_6 + |m(\mu)|\big)|\nabla\varphi|_3|\nabla\mu|_2 \leqslant C\varepsilon|\nabla\mu|_2\big(1 + |\nabla\mu|_2 + |\nabla\varphi|_2^p\big)|\nabla\varphi|_3$$

$$\leqslant C\varepsilon^{3/4}|\nabla\mu|_2\big(1 + |\nabla\mu|_2 + |\nabla\varphi|_2^p\big)\big(|\nabla\varphi|_2^{1/2}\big)\big(\varepsilon^{1/4}|\Delta\varphi|_2^{1/2}\big)$$

$$\leqslant C\varepsilon^{3/4}y_\varepsilon^{1/2}z_\varepsilon + C\varepsilon z_\varepsilon. \tag{3.21}$$

The third term is obviously estimated as follows using (3.18) and Young's inequality

$$|I_3| \leqslant \rho_2|g|_2|u|_2 \leqslant Cy_\varepsilon^{1/2} \leqslant C + \frac{1}{8}z_\varepsilon. \tag{3.22}$$

Using the previous lemma and particularly (3.16) we obtain

$$|I_4| \leqslant C\varepsilon^2\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2|\nabla\varphi|_4|\Delta\varphi|_4 \leqslant C\varepsilon^2\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2|\Delta\varphi|_2|\nabla\Delta\varphi|_2$$

$$\leqslant C\varepsilon^{1/2}\bigg(\varepsilon^{3/4}\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2|\Delta\varphi|_2\bigg)\big(\varepsilon^{3/4}|\nabla\Delta\varphi|_2\big) \leqslant C\varepsilon^{1/2}z_\varepsilon^{1/2}\big(y_\varepsilon + y_\varepsilon^\beta\big)z_\varepsilon. \tag{3.23}$$

Using another time that $|\rho'(\varphi)| \leqslant \varepsilon$ we get

$$|I_5| \leqslant \frac{\varepsilon}{2}\int_\Omega |u|^3|\nabla\varphi| \leqslant C\varepsilon|u|_{18/5}^3|\nabla\varphi|_6$$

$$\leqslant C\varepsilon\|u\|_{2/3}^3\|\nabla\varphi\|_1 \leqslant C\varepsilon|u|_2|\nabla u|_2^2|\Delta\varphi|_2$$

$$\leqslant C\sqrt{\varepsilon}|\nabla u|_2^2\big(|u|_2^2 + \varepsilon|\Delta\varphi|_2^2\big) \leqslant C\sqrt{\varepsilon}y_\varepsilon z_\varepsilon. \tag{3.24}$$

The sixth and seventh terms are estimated as follows

$$|I_6| \leqslant C\varepsilon\int_\Omega |u|^2\bigg|\frac{\partial\varphi}{\partial t}\bigg| \leqslant C\varepsilon|u|_3^2\bigg|\frac{\partial\varphi}{\partial t}\bigg|_3$$

$$\leqslant C\varepsilon|u|_2|\nabla u|_2\bigg|\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2}\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2} \leqslant C\varepsilon^{3/8}|u|_2\bigg(|\nabla u|_2\varepsilon^{1/4}\bigg|\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2}\varepsilon^{3/8}\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2}\bigg)$$

$$\leqslant \varepsilon^{3/8}|u|_2\bigg(|\nabla u|_2^2 + \varepsilon\bigg|\frac{\partial\varphi}{\partial t}\bigg|_2^2 + \varepsilon^{3/2}\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2^2\bigg) \leqslant C\varepsilon^{3/8}y_\varepsilon^{1/2}z_\varepsilon, \tag{3.25}$$

$$|I_7| \leqslant \varepsilon|u|_4|\nabla\varphi|_2\bigg|\frac{\partial\varphi}{\partial t}\bigg|_4 \leqslant C\varepsilon|\nabla u|_2|\nabla\varphi|_2\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2$$

$$\leqslant C\varepsilon^{1/4}|\nabla\varphi|_2\bigg(|\nabla u|_2^2 + \varepsilon^{3/2}\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2^2\bigg) \leqslant C\varepsilon^{1/4}y_\varepsilon^{1/2}z_\varepsilon. \tag{3.26}$$

Using (3.16), we get

$$|I_8| \leqslant \frac{\alpha\varepsilon^2}{2\rho_1^2}|\Delta\varphi|_3^2\bigg|\frac{\partial\varphi}{\partial t}\bigg|_3 \leqslant C\varepsilon^2|\Delta\varphi|_2|\nabla\Delta\varphi|_2\bigg|\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2}\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2}$$

$$\leqslant C\varepsilon^{5/8}\big(\varepsilon^{3/4}|\nabla\Delta\varphi|_2\big)\bigg(\varepsilon^{1/4}\bigg|\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2}\bigg)\bigg(\varepsilon^{3/8}\bigg|\nabla\frac{\partial\varphi}{\partial t}\bigg|_2^{1/2}\bigg)|\Delta\varphi|_2$$

$$\leqslant C\varepsilon^{5/8}\big(y_\varepsilon^{1/2} + y_\varepsilon^{\beta/2}\big)z_\varepsilon. \tag{3.27}$$

In order to estimate the next term, we must derive estimates for $|\nabla F(\varphi)|_2$ and $|\nabla F'(\varphi)|_2$. More precisely, thanks to the assumption (2.9) on F we have

- If $d = 2$, or $d = 3$ and $p \leqslant 2$, thanks to the embedding $H^1 \subset L^{3p}$

$$
\begin{aligned}
\left|\nabla F(\varphi)\right|_2^2 &= \int_\Omega \left|F'(\varphi)\right|^2 |\nabla\varphi|^2 \leqslant C|\nabla\varphi|_2^2 + C \int_\Omega \left|\varphi - m(\varphi_0)\right|^{2p} |\nabla\varphi|^2 \\
&\leqslant C|\nabla\varphi|_2^2 + C|\nabla\varphi|_2^{2p}|\Delta\varphi|_2^2.
\end{aligned} \tag{3.28}
$$

- If $d = 3$ and $2 < p \leqslant 3$, thanks to the embedding $H^{3/2-1/p} \subset L^{3p}$

$$
\left|\nabla F(\varphi)\right|_2^2 \leqslant |\nabla\varphi|_2^2 + C|\nabla\varphi|_2^{p+2}|\Delta\varphi|_2^p. \tag{3.29}
$$

In the same way, in both dimensions $d = 2$ and $d = 3$, we have thanks to the embedding $H^1 \subset L^{3p-3}$

$$
\begin{aligned}
\left|\nabla F'(\varphi)\right|_2^2 &= \int_\Omega \left|F''(\varphi)\right|^2 |\nabla\varphi|^2 \leqslant C|\nabla\varphi|_2^2 + C \int_\Omega \left|\varphi - m(\varphi_0)\right|^{2p-2} |\nabla\varphi|^2 \\
&\leqslant C|\nabla\varphi|_2^2 + C|\nabla\varphi|_2^{2p-2}|\Delta\varphi|_2^2.
\end{aligned} \tag{3.30}
$$

We deduce from these estimates that

$$
\begin{aligned}
|I_9| &\leqslant \varepsilon \left|\nabla\left(\frac{F'(\varphi)}{\rho(\varphi)}\right)\right|_2 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 \leqslant C\varepsilon|\nabla F'(\varphi)|_2 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 + C\varepsilon^2|\nabla F(\varphi)|_2 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 \\
&\leqslant C\varepsilon\left(|\nabla\varphi|_2 + |\nabla\varphi|_2^{p-1}|\Delta\varphi|_2\right)\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 + C\varepsilon^2\left(|\nabla\varphi|_2 + |\nabla\varphi|_2^p|\Delta\varphi|_2\right)\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 \\
&\leqslant C\varepsilon^{1/4}\left(1 + y_\varepsilon^{(p-1)/2}\right)z_\varepsilon + C\varepsilon^{5/4}\left(1 + y_\varepsilon^{p/2}\right)z_\varepsilon \leqslant C\varepsilon^{1/4}\left(1 + y_\varepsilon^{p/2}\right)z_\varepsilon. \tag{3.31}
\end{aligned}
$$

We can now write

$$
|I_{10}| \leqslant \varepsilon^{3/2}\left|\int_\Omega u.\nabla\varphi\left(\frac{F''(\varphi)}{\rho(\varphi)} - \frac{F'(\varphi)\rho'(\varphi)}{\rho^2(\varphi)}\right)\frac{\partial\varphi}{\partial t}\right| + \alpha\varepsilon^{3/2}\left|\int_\Omega u.\nabla\varphi\frac{\partial}{\partial t}\left(\frac{\Delta\varphi}{\rho(\varphi)}\right)\right|.
$$

We estimate separately the two terms A and B of this last inequality, using (3.17) and (3.28)–(3.30). The following computations are made in the case $d = 3$ and $2 < p \leqslant 3$, but one clearly have the same kind of estimates in the other case

$$
\begin{aligned}
|A| &\leqslant \varepsilon^{3/2}|u|_3|\nabla F'(\varphi)|_2 \left|\frac{\partial\varphi}{\partial t}\right|_6 + \varepsilon^{5/2}|u|_3|\nabla F(\varphi)|_2 \left|\frac{\partial\varphi}{\partial t}\right|_6 \\
&\leqslant C|u|_2^{1/2}|\nabla u|_2^{1/2}\left(\varepsilon^{3/2}|\nabla F'(\varphi)|_2 + \varepsilon^{5/2}|\nabla F(\varphi)|_2\right)\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 \\
&\leqslant C|u|_2^{1/2}\left(\varepsilon^{3/4}\left(|\nabla\varphi|_2^{1/2} + |\nabla\varphi|_2^{p-1}|\Delta\varphi|_2^{1/2}\right)\right. \\
&\quad + \varepsilon^{7/4}\left(|\nabla\varphi|_2^{1/2} + |\nabla\varphi|_2^{p/2+1}|\Delta\varphi|_2^{(p-1)/2}\right)\left)\left(\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2|\nabla u|_2^{1/2}|\Delta\varphi|_2^{1/2}\right)\right. \\
&\leqslant C\varepsilon^{3/8}y_\varepsilon^{1/4}\left(y_\varepsilon^{1/4} + y_\varepsilon^{(p+1+\beta)/2}\right)z_\varepsilon,
\end{aligned}
$$

$$|B| \leqslant \varepsilon^{3/2}\alpha \left| \int_\Omega \frac{1}{\rho} u.\nabla\varphi \frac{\partial\Delta\varphi}{\partial t} \right| + \varepsilon^{3/2}\alpha \left| \int_\Omega u.\nabla\varphi\Delta\varphi \frac{\rho'(\varphi)}{\rho^2(\varphi)} \frac{\partial\varphi}{\partial t} \right|$$

$$\leqslant \varepsilon^{3/2}\alpha \left| \int_\Omega \nabla\left(\frac{1}{\rho}u.\nabla\varphi\right)\nabla\frac{\partial\varphi}{\partial t} \right| + \varepsilon^{3/2}\alpha \left| \int_\Omega u.\nabla\varphi\Delta\varphi \frac{\rho'(\varphi)}{\rho^2(\varphi)} \frac{\partial\varphi}{\partial t} \right|$$

$$\leqslant \varepsilon^{3/2}\alpha \left| \int_\Omega (u.\nabla\varphi)\frac{\rho'(\varphi)}{\rho^2(\varphi)}\nabla\varphi.\nabla\frac{\partial\varphi}{\partial t} \right| + \varepsilon^{3/2}\alpha \left| \int_\Omega \frac{1}{\rho}\nabla(u.\nabla\varphi).\nabla\frac{\partial\varphi}{\partial t} \right|$$

$$+ \varepsilon^{3/2}\alpha \left| \int_\Omega u.\nabla\varphi\Delta\varphi \frac{\rho'(\varphi)}{\rho^2(\varphi)} \frac{\partial\varphi}{\partial t} \right|$$

$$\leqslant C\varepsilon^{5/2}|\nabla\varphi|_6^2 |u|_6 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 + C\varepsilon^{3/2}|\nabla(u.\nabla\varphi)|_2 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2$$

$$+ C\varepsilon^{5/2}|u|_6|\nabla\varphi|_6|\Delta\varphi|_2 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2$$

$$\leqslant C\varepsilon^{3/4}\left(\varepsilon|\Delta\varphi|_2^2\right)\left(|\nabla u|_2\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2\right) + C\varepsilon^{3/2}|\nabla(u.\nabla\varphi)|_2 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2.$$

Moreover, with Agmon's inequalities (2.6) and the Sobolev embedding $H^{3/4} \subset L^4$ in both dimensions 2 and 3, we have

$$\left|\nabla(u.\nabla\varphi)\right|_2 \leqslant |\nabla u|_2|\nabla\varphi|_\infty + |u.D^2\varphi|_2 \leqslant |\nabla u|_2|\Delta\varphi|_2^{1/2}|\nabla\Delta\varphi|_2^{1/2} + |u|_4|D^2\varphi|_4$$

$$\leqslant |\nabla u|_2|\Delta\varphi|_2^{1/2}|\nabla\Delta\varphi|_2^{1/2} + |u|_2^{1/4}|\nabla u|_2^{3/4}|\Delta\varphi|_2^{1/4}|\nabla\Delta\varphi|_2^{3/4},$$

so that finally, we get

$$|B| \leqslant C\varepsilon^{3/4}y_\varepsilon z_\varepsilon + C\varepsilon^{3/16}\left(\varepsilon^{3/16}|\Delta\varphi|_2^{1/2}\varepsilon^{3/8}|\nabla\Delta\varphi|_2^{1/2}\right)\left(|\nabla u|_2\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2\right)$$

$$+ C\varepsilon^{3/16}\left(|u|_2^{1/4}\varepsilon^{9/16}|\nabla\Delta\varphi|_2^{3/4}\right)\left(|\nabla u|_2^{3/4}|\Delta\varphi|_2^{1/4}\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2\right)$$

$$\leqslant C\varepsilon^{3/4}y_\varepsilon z_\varepsilon + \varepsilon^{3/16}\left(y_\varepsilon + y_\varepsilon^{\beta/2}\right)z_\varepsilon.$$

As a consequence, we get the following estimate for the tenth term

$$|I_{10}| \leqslant C\varepsilon^{3/16}\left(y_\varepsilon^{1/2} + y_\varepsilon^{(p+1+\beta)/2}\right)z_\varepsilon. \tag{3.32}$$

The estimates for the two next terms are straightforward

$$|I_{11}| \leqslant C\varepsilon^{5/2}\left|\frac{\partial\varphi}{\partial t}\right|_6\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2|\nabla\varphi|_3 \leqslant C\varepsilon^{3/4}\left(|\nabla\varphi|_2^{1/2}\varepsilon^{1/4}|\Delta\varphi|_2^{1/2}\right)\left(\varepsilon^{3/2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2\right)$$

$$\leqslant C\varepsilon^{3/4}y_\varepsilon^{1/2}z_\varepsilon, \tag{3.33}$$

and

$$|I_{12}| \leqslant C\varepsilon^{5/2}\left|\frac{\partial\varphi}{\partial t}\right|_3^2|\Delta\varphi|_3 \leqslant C\varepsilon^{1/8}\left(\varepsilon^{3/8}|\Delta\varphi|_2\varepsilon^{3/4}|\nabla\Delta\varphi|_2\right)\left(\varepsilon^{1/2}\left|\frac{\partial\varphi}{\partial t}\right|_2\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2\right)$$

$$\leqslant C\varepsilon^{1/8}\left(y_\varepsilon + y_\varepsilon^\beta\right)z_\varepsilon. \tag{3.34}$$

Using assumption (2.9) on F and the Sobolev embeddings, we derive the following estimate

$$
\begin{aligned}
|I_{13}| &\leqslant C\varepsilon^{3/2} \int_\Omega \left|\frac{\partial\varphi}{\partial t}\right|^2 |F''(\varphi)| + C\varepsilon^{5/2} \int_\Omega \left|\frac{\partial\varphi}{\partial t}\right|^2 |F'(\varphi)| \\
&\leqslant C\varepsilon^{3/2} \left|\frac{\partial\varphi}{\partial t}\right|_2^2 + C\varepsilon^{3/2} \int_\Omega \left|\frac{\partial\varphi}{\partial t}\right|^2 |\varphi|^{p-1} + C\varepsilon^{5/2} \int_\Omega \left|\frac{\partial\varphi}{\partial t}\right|^2 |\varphi|^p \\
&\leqslant C\varepsilon^{3/2} \left|\frac{\partial\varphi}{\partial t}\right|_2^2 + C\varepsilon^{3/2} \left|\frac{\partial\varphi}{\partial t}\right|_3^2 |\varphi|_{3p-3}^{p-1} + C\varepsilon^{5/2} \left|\frac{\partial\varphi}{\partial t}\right|_4^2 |\varphi|_{2p}^p \\
&\leqslant C\varepsilon^{3/2} \left|\frac{\partial\varphi}{\partial t}\right|_2^2 + C\varepsilon^{3/2} \left|\frac{\partial\varphi}{\partial t}\right|_2 \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 |\nabla\varphi|_2^{p-1} + C\varepsilon^{5/2} \left|\frac{\partial\varphi}{\partial t}\right|_2^{1/2} \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{3/2} |\nabla\varphi|_2^p \\
&\leqslant C\varepsilon^{1/2} z_\varepsilon + C\varepsilon^{1/4} z_\varepsilon y_\varepsilon^{(p-1)/2} + C\varepsilon^{9/8} z_\varepsilon y_\varepsilon^{p/2}. \quad (3.35)
\end{aligned}
$$

From the Cahn–Hilliard equation (3.5) we deduce

$$
\Delta\left(\frac{\mu}{\rho}\right) = \rho\left(\frac{\partial\varphi}{\partial t} + u.\nabla\varphi + Uv_\lambda.\nabla\varphi + \frac{\rho'(\varphi)}{\rho^2(\varphi)}\nabla\varphi.\nabla\left(\frac{\mu}{\rho}\right)\right),
$$

so that,

$$
\left|\Delta\left(\frac{\mu}{\rho}\right)\right|_2^2 \leqslant C\left(\left|\frac{\partial\varphi}{\partial t}\right|_2^2 + |u|_3^2 |\nabla\varphi|_\infty^2 + U^2 |\nabla\varphi|_2^2 + \varepsilon^2 \left|\nabla\left(\frac{\mu}{\rho}\right)\right|_3^2 |\nabla\varphi|_6^2\right),
$$

and then with the embedding $H^{1/2} \subset L^3$ we get

$$
\left|\Delta\left(\frac{\mu}{\rho}\right)\right|_2 \leqslant C\left(\left|\frac{\partial\varphi}{\partial t}\right|_2 + |u|_2.|\nabla\Delta\varphi|_2 + U|\nabla\varphi|_2 + \varepsilon^2 \left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2 |\Delta\varphi|_2^2\right). \quad (3.36)
$$

Finally, we obtain that the next term is estimated as follows

$$
\begin{aligned}
|I_{14}| &\leqslant C\varepsilon^{5/2} \left|\nabla\left(\frac{\mu}{\rho}\right)\right|_3^2 \left|\frac{\partial\varphi}{\partial t}\right|_3 \leqslant C\varepsilon^{5/2} \left|\left(\nabla\frac{\mu}{\rho}\right)\right|_2 \left|\Delta\left(\frac{\mu}{\rho}\right)\right|_2 \left|\frac{\partial\varphi}{\partial t}\right|_2^{1/2} \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{1/2} \\
&\leqslant C\varepsilon^{5/2} \left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2 \left(\left|\frac{\partial\varphi}{\partial t}\right|_2 + |u|_2 |\nabla\Delta\varphi|_2 + U|\nabla\varphi|_2\right. \\
&\quad \left. + \varepsilon^2 \left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2 |\Delta\varphi|_2^2\right) \left|\frac{\partial\varphi}{\partial t}\right|_2^{1/2} \left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{1/2} \\
&\leqslant C\varepsilon^{5/8}\left(\varepsilon^{3/4}\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2\right)\left(\varepsilon^{3/4}\left|\frac{\partial\varphi}{\partial t}\right|_2^{3/2}\right)\left(\varepsilon^{3/8}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\right) \\
&\quad + C\varepsilon^{5/8}|u|_2\left(\varepsilon^{3/4}|\nabla\Delta\varphi|_2\right)\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2\left(\varepsilon^{1/4}\left|\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\right)\left(\varepsilon^{3/8}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\right) \\
&\quad + C\varepsilon^{15/8}|\nabla\varphi|_2\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2\left(\varepsilon^{1/4}\left|\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\right)\left(\varepsilon^{3/8}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\right)
\end{aligned}
$$

$$+ C\varepsilon^{19/8}\left(\varepsilon^{3/4}\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2\right)\left(\varepsilon^{3/4}|\Delta\varphi|_2^2\right)\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2\left(\varepsilon^{1/4}\left|\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\right)$$

$$\times\left(\varepsilon^{3/8}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\right)$$

$$\leqslant C\varepsilon^{5/8}y_\varepsilon^{1/2}z_\varepsilon + C\varepsilon^{5/8}\left(y_\varepsilon + y_\varepsilon^{(\beta+1)/2}\right)z_\varepsilon + C\varepsilon^{15/8}y_\varepsilon^{1/2}z_\varepsilon$$

$$+ C\varepsilon^{19/8}\left(y_\varepsilon^{3/2} + y_\varepsilon^{\beta+1/2}\right)z_\varepsilon. \tag{3.37}$$

The next five integrals can be easily estimated as follows

$$|I_{15}| = \left|\int_\Omega Uv_\lambda.\nabla\varphi(\mu - m(\mu))\right|$$

$$\leqslant U|v_\lambda|_4|\nabla\varphi|_2|\mu - m(\mu)|_4 \leqslant U\lambda|\nabla\varphi|_2|\nabla\mu|_2 \leqslant U\lambda z_\varepsilon, \tag{3.38}$$

$$|I_{16}| \leqslant \left|U\int_\Omega \rho'(\varphi)(u.\nabla\varphi)(u.v_\lambda)\right| + \left|U\int_\Omega \rho(u.\nabla)uv_\lambda\right|$$

$$\leqslant \varepsilon U|\nabla\varphi|_2|\nabla u|_2^2|v_\lambda|_\infty + U|v_\lambda|_4|\nabla u|_2^2 \leqslant \varepsilon y_\varepsilon^{1/2}z_\varepsilon + \lambda U z_\varepsilon, \tag{3.39}$$

$$|I_{17}| \leqslant U|v_\lambda|_4|\nabla u|_2^2 \leqslant \lambda U z_\varepsilon, \tag{3.40}$$

$$|I_{18}| \leqslant CU|D(u)|_2|D(v_\lambda)|_2 \leqslant \frac{1}{8}z_\varepsilon + CU^2|D(v_\lambda)|_2^2, \tag{3.41}$$

$$|I_{19}| \leqslant \varepsilon^{1/4}U|v_\lambda|_4|\nabla\varphi|_2\left(\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2\right) \leqslant \varepsilon^{1/4}U\lambda z_\varepsilon. \tag{3.42}$$

Finally, using (3.28)–(3.30) and Agmon's inequalities (2.6), we can conclude our estimates with the last term in the following way if we suppose that $d = 2$ or $d = 3$ and $p \leqslant 2$

$$|I_{20}| \leqslant \varepsilon^{3/2}U\left|\int_\Omega v_\lambda.\nabla\varphi\left(\frac{F'}{\rho}\right)'(\varphi)\frac{\partial\varphi}{\partial t}\right| + \varepsilon^{3/2}U\left|\int_\Omega \frac{1}{\rho}\nabla(v_\lambda.\nabla\varphi).\nabla\frac{\partial\varphi}{\partial t}\right|$$

$$+ \varepsilon^{3/2}U\left|\int_\Omega v_\lambda.\nabla\varphi\frac{\rho'(\varphi)}{\rho^2(\varphi)}\frac{\partial\varphi}{\partial t}\Delta\varphi\right|$$

$$\leqslant C\varepsilon^{3/2}|v_\lambda|_\infty|\nabla F'(\varphi)|_2\left|\frac{\partial\varphi}{\partial t}\right|_2 + C\varepsilon^{5/2}|v_\lambda|_\infty|\nabla F(\varphi)|_2\left|\frac{\partial\varphi}{\partial t}\right|_2$$

$$+ C\varepsilon^{3/2}|\nabla v_\lambda|_2|\nabla\varphi|_\infty\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2$$

$$+ C\varepsilon^{3/2}|v_\lambda|_\infty|\Delta\varphi|_2\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2 + C\varepsilon^{5/2}|v_\lambda|_\infty|\Delta\varphi|_2^2\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2$$

$$\leqslant C\varepsilon\left(1 + |\nabla\varphi|_2^{p-1}\right)\left(|\Delta\varphi|_2\varepsilon^{1/2}\left|\frac{\partial\varphi}{\partial t}\right|_2\right) + C\varepsilon^2\left(1 + |\nabla\varphi|_2^p\right)\left(|\Delta\varphi|_2\varepsilon^{1/2}\left|\frac{\partial\varphi}{\partial t}\right|_2\right)$$

$$+ C\varepsilon^{3/2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2|\Delta\varphi|_2^{1/2}|\nabla\Delta\varphi|_2^{1/2} + C\varepsilon^{3/4}\left(|\Delta\varphi|_2\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2\right)$$

$$+ C\varepsilon^{5/4}\big(\varepsilon^{1/2}|\Delta\varphi|_2\big)\left(|\Delta\varphi|_2\varepsilon^{3/4}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2\right)$$
$$\leqslant C\varepsilon\big(1 + y_\varepsilon^{(p-1)/2}\big)z_\varepsilon + C\varepsilon^2\big(1 + y_\varepsilon^{p/2}\big)z_\varepsilon + C\varepsilon^{3/8}\big(1 + y_\varepsilon^{(\beta-1)/2}\big)z_\varepsilon$$
$$+ C\varepsilon^{3/4}z_\varepsilon + C\varepsilon^{5/4}y_\varepsilon^{1/2}z_\varepsilon. \tag{3.43}$$

In the case $d = 3$ and $2 < p \leqslant 3$, we use (3.29) instead of (3.28) and we easily get the same kind of estimate.

Step 4. With (3.20)–(3.43), our energy estimate (3.19) reads

$$\frac{\partial}{\partial t}y_\varepsilon + \frac{1}{4}z_\varepsilon \leqslant C + C\varepsilon^{1/2}z_\varepsilon + C\lambda U z_\varepsilon + C\varepsilon^{1/8}\big(1 + y_\varepsilon^q\big)z_\varepsilon, \tag{3.44}$$

where $q > 0$ depends only on p and β.

From now on, we impose $\varepsilon < \varepsilon_1 = 1/(16C)^2$ and we choose λ such that $C\lambda U < 1/16$, so that we have

$$\frac{\partial}{\partial t}y_\varepsilon + \frac{1}{8}z_\varepsilon \leqslant C + C\varepsilon^{1/8}\big(1 + y_\varepsilon^q\big)z_\varepsilon. \tag{3.45}$$

Let $M_\varepsilon > 0$ be the solution of

$$C\varepsilon^{1/8}\big(1 + M_\varepsilon^q\big) = \frac{1}{16}.$$

One can easily see that

$$M_\varepsilon \to +\infty, \quad \text{when } \varepsilon \to 0.$$

Moreover, thanks to the assumption on the initial data we have for a constant $K > 0$,

$$y_\varepsilon(0) \leqslant K(C_0 + U)$$

so that there exists $\varepsilon_0 < \varepsilon_1$ such that if $\varepsilon < \varepsilon_0$ we have

$$y_\varepsilon(0) \leqslant \frac{1}{2}M_\varepsilon.$$

Hence, if we choose now $\varepsilon < \varepsilon_0$, there exists a maximal time $T^* \in]0, +\infty]$ such that $y_\varepsilon(t) < M_\varepsilon$ for any $t \in [0, T^*[$. Thanks to this property and the definition of M_ε we infer from (3.45) that for any $t \in [0, T^*[$ we have

$$\frac{d}{dt}y_\varepsilon + \frac{1}{16}z_\varepsilon \leqslant C,$$

and so using (3.18) we deduce that on $[0, T^*[$ we have

$$\frac{d}{dt}y_\varepsilon + C'y_\varepsilon \leqslant C.$$

We easily get from this inequality the estimate

$$y_\varepsilon(t) \leqslant y_\varepsilon(0)e^{-C't} + \frac{C}{C'} \leqslant K(C_0 + U) + \frac{C}{C'},$$

and finally if ε is small enough so that $\frac{1}{2}M_\varepsilon > K(C_0 + U) + \frac{C}{C'}$ we have for any $t \in [0, T^*[$ the inequality

$$y_\varepsilon(t) \leqslant \frac{1}{2}M_\varepsilon.$$

If T^* is finite, this is in contradiction with the maximality of T^*. This implies that necesarily we have

$$T^* = +\infty,$$

that is to say that we have global and uniform in time estimates

$$\sup_{t \in \mathbb{R}^+} y_\varepsilon(t) \leqslant K(C_0 + U) + \frac{C}{C'},$$

$$\sup_{t \in \mathbb{R}^+} \int_t^{t+\tau} z_\varepsilon(s)\, ds \leqslant 16\left(K(C_0 + U) + \frac{C}{C'} + C\tau\right),$$

which implies estimates (3.1) and (3.2).

Step 5. As it is classical (see [5]), we only give now the sketch of the proof of (3.3) from (3.1)–(3.2) in the case $U = 0$ (for simplicity).

Denoting by Ω_T the set $]0, T[\times \Omega$, we take a test function $w \in L^2(0, T; V_{d/2})$ for the nonhomogeneous Navier–Stokes equation to get

$$\int_{\Omega_T} \mathcal{P}\left(\frac{\partial \rho v}{\partial t}\right).w = \int_{\Omega_T} \frac{\partial \rho v}{\partial t}.w = \int_{\Omega_T} \frac{\partial \rho}{\partial t} v.w - \int_{\Omega_T} \rho v.\nabla v.w - \int_{\Omega_T} 2\eta(\varphi) D(v) : D(w)$$

$$+ \int_{\Omega_T} \mu \nabla \varphi.w - \varepsilon \int_{\Omega_T} \frac{\varphi^2}{4} \nabla\left(\frac{\mu}{\rho}\right).w + \int_{\Omega_T} \rho g.w. \tag{3.46}$$

Using the fact that ρ and $1/\rho$ are uniformly bounded independtly of ε, and the estimate

$$\left| \int_{\Omega_T} \frac{\partial \rho}{\partial t} v.w \right| \leqslant \varepsilon \int_{\Omega_T} \left|\frac{\partial \varphi}{\partial t}\right| |v||w| \leqslant \varepsilon \left\|\frac{\partial \varphi}{\partial t}\right\|_{L^2(0,T;L^6(\Omega))} \|v\|_{L^\infty(0,T;H)} \|w\|_{L^2(0,T;\mathbb{L}^3(\Omega))}$$

$$\leqslant M(T) \|w\|_{L^2(0,T;V_{d/2})},$$

we see that finally

$$\left| \int_{\Omega_T} \mathcal{P}\left(\frac{\partial \rho v}{\partial t}\right).w \right| \leqslant M(T) \|w\|_{L^2(0,T;V_{d/2})}.$$

Indeed, the other terms in (3.46) are estimated classicaly (see [5]), the choice of the space $V_{d/2}$ being issued from the nonlinear term. Then, by a duality argument, estimate (3.3) is established.

Step 6. Passing to the limit in the equations satisfied by the approximated solutions (φ_n, ρ_n, v_n) is classical (see [5]) at the condition that we have some compactness on the velocity field (v_n). This is obtained here just like in [19].

More precisely, the compactness on φ_n and ρ_n is a straightforward consequence of (3.1) and (3.2), whereas we get from (3.2) and (3.3) that

$$\mathcal{P}(\rho_n v_n) \text{ is bounded in } L^2(0, T; V),$$

$$\frac{\partial}{\partial t}(\mathcal{P}(\rho_n v_n)) \text{ is bounded in } L^2(0, T; V'_{d/2}),$$

so that a classical compactness lemma [24] implies that

$$\mathcal{P}(\rho_n v_n) \text{ is compact in } L^2(0, T; H).$$

Furthermore, we have from (3.2) the weak convergence of $(\sqrt{\rho_n} v_n)$ in $L^2(\Omega_T)$ and

$$\int_{\Omega_T} |\sqrt{\rho_n} v_n|^2 = \int_{\Omega_T} \rho_n |v_n|^2 = \int_{\Omega_T} \mathcal{P}(\rho_n v_n).v_n$$

$$\rightarrow \int_{\Omega_T} \mathcal{P}(\rho v).v = \int_{\Omega_T} \rho |v|^2 = \int_{\Omega_T} |\sqrt{\rho} v|^2,$$

so that finally, we have proved the strong convergence of $\sqrt{\rho_n} v_n$ towards $\sqrt{\rho} v$ in $L^2(\Omega_T)$. This strong convergence allows us to pass to the limit in the nonlinear terms of the Navier–Stokes equation.

Hence, the proof of the existence of the solutions is complete.

Step 7. As far as the limit $\varepsilon \to 0$ is concerned, the key-point is that, we have enough compactness on the velocity v_ε thanks to estimate (3.1), (3.2) and (3.3) which are uniform in ε.

Moreover, thanks to (3.1) and (3.2), and the fact that $|\rho'_\varepsilon|_\infty \leqslant \varepsilon$ we see that

$$\rho_\varepsilon \to 1 \quad \text{in } L^\infty(\mathbb{R}^+; H^1) \text{ strong,}$$

$$\frac{\partial \rho_\varepsilon}{\partial t} \to 0, \quad \text{in } L^2(\Omega_T) \text{ strong for any } T > 0,$$

which allow us to perform the limit in the term

$$\rho_\varepsilon \frac{\partial v_\varepsilon}{\partial t}$$

in the Navier–Stokes equation. $\square$

3.2. Strong solutions

In this section, we have to suppose that $|\rho''_\varepsilon|_\infty \leqslant \varepsilon$. This assumption is clearly reasonable because we recall that ρ_ε is essentially linear (see (1.5)) in the physical-meaningful interval $[-1, 1]$.

THEOREM 3.3. – *Let $U > 0$, $v_0 \in v_\infty^U + V$, $\varphi_0 \in \Phi_3$ satisfying the boundary conditions. There exists $\varepsilon_0 > 0$ depending on U, $\|v_0\|_1$, $\|\varphi_0\|_3$ and F, such that if $\varepsilon < \varepsilon_0$ there exists a unique strong solution $(\varphi_\varepsilon, v_\varepsilon)$ of the problem for the initial data (φ_0, v_0).*
– If $d = 2$, this solution is global and satisfies

$$\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^+;\Phi_3)} + \|v_\varepsilon - v_\infty^U\|_{L^\infty(\mathbb{R}^+;V)} \leqslant C,$$

$$\|\varphi_\varepsilon\|_{L^2(t_0,t_0+\tau;\Phi_4)} + \|v_\varepsilon - v_\infty^U\|_{L^2(t_0,t_0+\tau;V_2)}$$

$$+ \left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;\Phi_1)} + \left\|\frac{\partial v_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;H)} \leqslant C(\tau), \quad \textit{for any } t_0, \tau > 0,$$

where C and $C(\tau)$ are independent of ε.

– *If $d = 3$, the solution is local and satisfies locally the same regularity results than for the 2D case.*

Proof. – In the following we concentrate our efforts on the existence part of the theorem. Indeed, the proof of the uniqueness is straightforward (see [5]) using the energy estimates that we obtain for these strong solutions.

Step 1. From Theorem 3.2, we obtain the following estimates for the weak solutions with initial data (φ_0, v_0) independent of ε:

$$\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^+;\Phi_1)} + \varepsilon^{1/2}\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^+;\Phi_2)} + \varepsilon^{3/4}\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^+;\Phi_3)}$$
$$+ \|v_\varepsilon - v_\infty^U\|_{L^\infty(\mathbb{R}^+;H)} \leqslant C, \tag{3.47}$$

$$\|\varphi_\varepsilon\|_{L^2(t_0,t_0+\tau;\Phi_3)} + \varepsilon^{1/2}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;L^2)} + \varepsilon^{3/4}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;H^1)}$$
$$+ \|\mu_\varepsilon\|_{L^2(t_0,t_0+\tau;\Phi_1)} + \|v_\varepsilon - v_\infty^U\|_{L^2(t_0,t_0+\tau;V)} \leqslant C(\tau), \quad \text{for any } t_0, \tau > 0. \tag{3.48}$$

Step 2. Using the fact that $\|\varphi_0\|_3$ and $\|v_0\|_1$ are independent of ε, we can derive additional energy estimates. We recall that we have set $v = u + Uv_\lambda$, and that in fact we study (3.5)–(3.8) and not (1.1)–(1.4).

We first multiply (3.5) with $\Delta^2\varphi$ to obtain after integration by parts

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta\varphi|_2^2 = \int_\Omega \text{div}\left(\frac{1}{\rho}\nabla\left(\frac{\mu}{\rho}\right)\right)\Delta^2\varphi - \int_\Omega u.\nabla\varphi\Delta^2\varphi - U\int_\Omega v_\lambda.\nabla\varphi\Delta^2\varphi, \tag{3.49}$$

and one have

$$\text{div}\left(\frac{1}{\rho}\nabla\left(\frac{\mu}{\rho}\right)\right) = -\frac{\rho'(\varphi)}{\rho^2(\varphi)}\nabla\varphi.\nabla\left(\frac{\mu}{\rho}\right) + \frac{1}{\rho}\Delta\left(\frac{\mu}{\rho}\right)$$

$$= -\frac{\rho'(\varphi)}{\rho^2(\varphi)}\nabla\varphi.\nabla\left(\frac{\mu}{\rho}\right) + \frac{1}{\rho^2}\Delta\mu + 2\frac{(\rho'(\varphi))^2}{\rho^4(\varphi)}|\nabla\varphi|^2\mu - \frac{\rho''(\varphi)}{\rho^3(\varphi)}|\nabla\varphi|^2\mu$$

$$- 2\frac{\rho'(\varphi)}{\rho^3(\varphi)}\nabla\varphi.\nabla\mu - \frac{\rho'(\varphi)}{\rho^3(\varphi)}\mu\Delta\varphi,$$

with

$$\Delta\mu = -\alpha\Delta^2\varphi + \Delta F'(\varphi).$$

Finally, using the fact that $|\rho'|_\infty \leqslant \varepsilon$ and $|\rho''|_\infty \leqslant \varepsilon$, we obtain from the previous estimate

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta\varphi|_2^2 + \frac{\alpha}{\rho_2^2}|\Delta^2\varphi|_2^2 \leqslant \frac{\varepsilon}{\rho_1^2}\int_\Omega|\nabla\varphi|\left|\nabla\left(\frac{\mu}{\rho}\right)\right||\Delta^2\varphi| + \frac{1}{\rho_1^2}\int_\Omega|\Delta F'(\varphi)||\Delta^2\varphi|$$

$$+ \left(\frac{2\varepsilon^2}{\rho_1^4} + \frac{\varepsilon}{\rho_1^3} \right) \int_\Omega |\nabla\varphi|^2 |\mu| |\Delta^2\varphi|$$

$$+ 2\frac{\varepsilon}{\rho_1^3} \int_\Omega |\nabla\varphi| |\nabla\mu| |\Delta^2\varphi| + \frac{\varepsilon}{\rho_1^3} \int_\Omega |\Delta\varphi| |\mu| |\Delta^2\varphi|$$

$$+ \left| \int_\Omega u.\nabla\varphi\Delta^2\varphi \right| + U \left| \int_\Omega v_\lambda.\nabla\varphi\Delta^2\varphi \right|. \tag{3.50}$$

Then, if we denote by $J_1, \ldots, J_7$ the seven terms of the right-hand side of this inequality, we have

$$J_1 \leqslant C\varepsilon |\nabla\varphi|_\infty \left| \nabla\left(\frac{\mu}{\rho}\right) \right|_2 |\Delta^2\varphi|_2 \leqslant C\varepsilon |\nabla\Delta\varphi|_2 \left| \nabla\left(\frac{\mu}{\rho}\right) \right|_2 |\Delta^2\varphi|_2$$

$$\leqslant \frac{\alpha}{10\rho_1^2} |\Delta^2\varphi|_2^2 + C\varepsilon^2 |\nabla\Delta\varphi|_2^2 \left| \nabla\left(\frac{\mu}{\rho}\right) \right|_2^2. \tag{3.51}$$

The second term is estimated as in [5], to obtain for $q > 0$

$$J_2 \leqslant \frac{\alpha}{10\rho_1^2} |\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^{4/3}\left(1 + |\nabla\varphi|_2^q\right). \tag{3.52}$$

For the third term we get with (3.13)

$$J_3 \leqslant C\varepsilon |\nabla\varphi|_6^2 |\mu - m(\mu)|_6 |\Delta^2\varphi|_2 + C\varepsilon |m(\mu)| |\nabla\varphi|_4^2 |\Delta^2\varphi|_2$$

$$\leqslant \frac{\alpha}{10\rho_1^2} |\Delta^2\varphi|_2^2 + C\varepsilon^2 |\Delta\varphi|_2^4\left(1 + |\nabla\mu|_2^2 + |\nabla\varphi|_2^{2p}\right). \tag{3.53}$$

The next two terms are controlled as follows

$$J_4 \leqslant C\varepsilon |\nabla\varphi|_\infty |\nabla\mu|_2 |\Delta^2\varphi|_2 \leqslant \frac{\alpha}{10\rho_1^2} |\Delta^2\varphi|_2^2 + C\varepsilon^2 |\nabla\Delta\varphi|_2^2 |\nabla\mu|_2^2, \tag{3.54}$$

and

$$J_5 \leqslant C\varepsilon |\Delta\varphi|_3 |\mu - m(\mu)|_6 |\Delta^2\varphi|_2 + C\varepsilon |m(\mu)| |\Delta\varphi|_2 |\Delta^2\varphi|_2$$

$$\leqslant \frac{\alpha}{10\rho_1^2} |\Delta^2\varphi|_2^2 + C\varepsilon^2 |\nabla\mu|_2^2 |\nabla\Delta\varphi|_2^2 + C\varepsilon^2\left(1 + |\nabla\varphi|_2^{2p}\right)|\Delta\varphi|_2^2. \tag{3.55}$$

If we use the Agmon-like inequality

$$|\nabla\varphi|_\infty \leqslant C|\nabla\varphi|_2^{1/2} |\Delta^2\varphi|_2^{3/2},$$

we have

$$J_6 \leqslant |u|_2 |\nabla\varphi|_\infty |\Delta\varphi|_2 \leqslant C|u|_2 |\nabla\varphi|_2^{1/2} |\Delta^2\varphi|_2^{3/2}$$

$$\leqslant \frac{\alpha}{10\rho_1^2} |\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^2 |u|_2^4, \tag{3.56}$$

and

$$J_7 \leqslant \frac{\alpha}{10\rho_1^2} |\Delta^2\varphi|_2^2 + C|\nabla\varphi|_2^2. \tag{3.57}$$

Finally, the energy estimate reads

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta\varphi|_2^2 + \frac{\alpha}{10\rho_2^2}|\Delta^2\varphi|_2^2 \leqslant C\varepsilon^2|\nabla\Delta\varphi|_2^2\left(|\nabla\mu|_2^2 + \left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2\right) + C|\nabla\varphi|_2^{4/3}(1 + |\nabla\varphi|_2^q)$$
$$+ C\varepsilon^2(1 + |\nabla\varphi|_2^{2p})|\Delta\varphi|_2^2 + C|\nabla\varphi|_2^2|u|_2^4,$$

which can be also written using (3.47), (3.48)

$$\frac{1}{2}\frac{\partial}{\partial t}|\Delta\varphi|_2^2 + \frac{\alpha}{10\rho_2^2}|\Delta^2\varphi|_2^2 \leqslant f_\varepsilon(t), \tag{3.58}$$

where $f_\varepsilon(t)$ is bounded in $L^1(t_0, t_0 + \tau)$ uniformly in ε and $t_0 \geqslant 0$, for any $\tau > 0$. But we also know by (3.48) that $\Delta\varphi$ is bounded in $L^2(t_0, t_0 + \tau, L^2)$ independently in t_0 and ε. We can conclude by the uniform Gronwall lemma that

$$\|\varphi\|_{L^\infty(\mathbb{R}^+;\Phi_2)} \leqslant C, \qquad \|\varphi\|_{L^2(t_0,t_0+\tau;\Phi_4)} \leqslant C(\tau), \tag{3.59}$$

where $C, C(\tau)$ are independent of ε.

Step 3. We have now to obtain more regularity on the velocity. We multiply Navier–Stokes equation (3.7) by $\partial u/\partial t$ so that after integration we obtain

$$\rho_1\left|\frac{\partial u}{\partial t}\right|_2^2 + \frac{\partial}{\partial t}\left(\int_\Omega \eta(\varphi)|D(u)|^2\right)$$
$$\leqslant \left|\int_\Omega \rho(u.\nabla)u\frac{\partial u}{\partial t}\right| + \left|\int_\Omega \eta'(\varphi)\frac{\partial\varphi}{\partial t}|D(u)|^2\right|$$
$$+ U\left|\int_\Omega \rho(v_\lambda.\nabla)u\frac{\partial u}{\partial t}\right| + U\left|\int_\Omega \rho(u.\nabla)v_\lambda\frac{\partial u}{\partial t}\right| + 2U\left|\int_\Omega \text{div}\big(\eta(\varphi)D(v_\lambda)\big)\frac{\partial u}{\partial t}\right|$$
$$+ \left|\int_\Omega \mu\nabla\varphi.\frac{\partial u}{\partial t}\right| + \varepsilon\left|\int_\Omega \frac{1-\varphi^2}{4}\nabla\left(\frac{\mu}{\rho}\right).\frac{\partial u}{\partial t}\right| + \left|\int_\Omega \rho g.\frac{\partial u}{\partial t}\right|.$$

A parameter γ given being given (which will be fixed in the sequel), each term of this inequality can be easily estimated to give

$$\frac{\partial}{\partial t}\left(\int_\Omega \eta(\varphi)|D(u)|^2\right) + \frac{\rho_1}{10}\left|\frac{\partial u}{\partial t}\right|_2^2$$
$$\leqslant \gamma\frac{\eta_1}{20}|Au|_2^2 + \frac{\alpha}{20\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2 + \frac{1}{\gamma}I + C|\nabla u|_2^2 + C|\Delta v_\lambda|_2^2|\nabla u|_2^2$$
$$+ C|\nabla\Delta\varphi|_2^2|D(v_\lambda)|_2^2 + CU^2|\Delta v_\lambda|_2^2 + C(1 + \|\varphi\|_2^2)^2\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2 + C|g|_2^2, \tag{3.60}$$

where the term I is defined by

$$I = \begin{cases} C|u|_2^2|\nabla u|_2^4, & \text{if } d = 2, \\ C|\nabla u|_2^6, & \text{if } d = 3. \end{cases} \tag{3.61}$$

Estimate (3.60) must be supplied with a control on the $\mathbb{H}^2$-norm of u, that's the reason why we have to multiply the Navier–Stokes equation by the value of the Stokes operator $Au = -\Delta u + \nabla\pi$ to get

$$
-\int_\Omega \eta(\varphi)\Delta u.Au = 2\int_\Omega \eta'(\varphi)(D(u).\nabla\varphi)Au + 2U\int_\Omega \eta'(\varphi)(D(v_\lambda).\nabla\varphi).Au
$$

$$
+ U\int_\Omega \eta(\varphi)\Delta v_\lambda.Au - \int_\Omega \rho\frac{\partial u}{\partial t}.Au
$$

$$
- \int_\Omega \rho(u.\nabla)u.Au - U\int_\Omega \rho(v_\lambda.\nabla)u.Au - U\int_\Omega \rho(u.\nabla)v_\lambda.Au
$$

$$
+ \int_\Omega \mu\nabla\varphi.Au + \varepsilon\int_\Omega \frac{1-\varphi^2}{4}\nabla\left(\frac{\mu}{\rho}\right).Au + \int_\Omega \rho g.Au,
$$

and so

$$
\eta_1|Au|_2^2 \leqslant \left|\int_\Omega \eta(\varphi)\nabla\pi\, Au\right| + C\int_\Omega |D(u)||\nabla\varphi||Au|
$$

$$
+ CU\int_\Omega |D(v_\lambda)||\nabla\varphi||Au| + CU\int_\Omega |\Delta v_\lambda||Au| + C\int_\Omega \left|\frac{\partial u}{\partial t}\right||Au|
$$

$$
+ C\int_\Omega |u||\nabla u||Au| + CU\int_\Omega |v_\lambda||\nabla u||Au| + CU\int_\Omega |u||\nabla v_\lambda||Au|
$$

$$
+ \int_\Omega |\mu - m(\mu)||\nabla\varphi||Au| + \varepsilon\int_\Omega \left|\frac{1-\varphi^2}{4}\right|\left|\nabla\left(\frac{\mu}{\rho}\right)\right||Au| + C\int_\Omega |g||Au|.
$$

The first term of the right-hand side member of this inequality reads after integration par parts (remember that $\mathrm{div}(Au) = 0$ and $Au.\nu = 0$ on the boundary) and using (2.3)

$$
\left|\int_\Omega \eta(\varphi)\nabla\pi.Au\right| = \left|\int_\Omega \eta'(\varphi)\pi\nabla\varphi.Au\right|
$$

$$
\leqslant C|\pi|_2|\nabla\varphi|_\infty|Au|_2 \leqslant \frac{\eta_1}{10}|Au|_2^2 + C|\nabla\Delta\varphi|_2^2|\nabla u|_2^2,
$$

the other terms being easily estimated, one gets

$$
\frac{\eta_1}{10}|Au|_2^2 \leqslant C|\nabla\Delta\varphi|_2^2|\nabla u|_2^2 + CU^2|D(v_\lambda)|_2^2|\nabla\Delta\varphi|_2^2 + C\left|\frac{\partial u}{\partial t}\right|_2^2 + I
$$

$$
+ C|\nabla u|_2^2 + C|\Delta v_\lambda|_2^2|\nabla u|_2^2 + C|\nabla\mu|_2^2|\Delta\varphi|_2^2
$$

$$
+ C\varepsilon^2(1 + \|\varphi\|_2^2)\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2 + C|g|_2^2, \tag{3.62}
$$

the term I being always defined by (3.61).

If now we choose the value of the parameter γ (independently of ε) so that $\gamma C \leqslant \frac{\rho_1}{20}$, we get by summing (3.60) and $\gamma * (3.62)$ and using (3.59)

$$\frac{\partial}{\partial t}\left(\int_{\Omega}\eta(\varphi)|D(u)|^2\right)+\frac{\rho_1}{20}\left|\frac{\partial u}{\partial t}\right|_2^2+\gamma\frac{\eta_1}{20}|Au|_2^2$$

$$\leqslant\frac{\alpha}{20\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2+\left(\frac{1}{\gamma}+\gamma\right)I+C|\nabla u|_2^2+C|\Delta v_\lambda|_2^2|\nabla u|_2^2$$

$$+C|\nabla\Delta\varphi|_2^2|D(v_\lambda)|_2^2+CU^2|\Delta v_\lambda|_2^2+C\left(1+\|\varphi\|_2^2\right)^2\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2$$

$$+C\gamma|\nabla\Delta\varphi|_2^2|\nabla u|_2^2+C\gamma U^2|D(v_\lambda)|_2^2|\nabla\Delta\varphi|_2^2+C|g|_2^2,$$

$$\leqslant\frac{\alpha}{20\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2+\left(\frac{1}{\gamma}+\gamma\right)I+C+C|\nabla u|_2^2+C\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2. \tag{3.63}$$

Unfortunately, this estimate is not sufficient to conclude. Indeed, because of the first term in the right-hand side member of this inequality, we must now derive estimates on $\partial\varphi/\partial t$ in H^1 independently of ε.

Step 4. The Cahn–Hilliard equation (3.5) gives us

$$\left|\frac{\partial\varphi}{\partial t}\right|_2^2\leqslant|u|_2^2|\nabla\varphi|_\infty^2+U^2|\nabla\varphi|_2^2+\left|\mathrm{div}\left(\frac{1}{\rho}\nabla\left(\frac{\mu}{\rho}\right)\right)\right|_2^2,$$

and with the same computations than for the estimates (3.51)–(3.57), we deduce that

$$\left\|\frac{\partial\varphi}{\partial t}\right\|_{L^2(t_0,t_0+\tau;L^2)}\leqslant C(\tau), \tag{3.64}$$

independtly of t_0 and ε. This result is clearly stronger than (3.48).

Step 5. We come back to the inequality (3.11) which gives us

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\int_{\Omega}\frac{1}{\rho}\left|\nabla\left(\frac{\mu}{\rho}\right)\right|^2\right)+\frac{\alpha}{\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2$$

$$\leqslant C\varepsilon\int_{\Omega}\left|\frac{\partial\varphi}{\partial t}\right|\left|\nabla\frac{\partial\varphi}{\partial t}\right||\nabla\varphi|+C\varepsilon\int_{\Omega}|\Delta\varphi|\left|\frac{\partial\varphi}{\partial t}\right|^2+C\int_{\Omega}|F''(\varphi)|\left|\frac{\partial\varphi}{\partial t}\right|^2$$

$$+C\varepsilon\int_{\Omega}|F'(\varphi)|\left|\frac{\partial\varphi}{\partial t}\right|^2+C\varepsilon\int_{\Omega}\left|\nabla\left(\frac{\mu}{\rho}\right)\right|^2\left|\frac{\partial\varphi}{\partial t}\right|$$

$$+\left|\int_{\Omega}(u.\nabla\varphi)\frac{\partial}{\partial t}\left(\frac{\mu}{\rho}\right)\right|+U\left|\int_{\Omega}(v_\lambda.\nabla\varphi)\frac{\partial}{\partial t}\left(\frac{\mu}{\rho}\right)\right|. \tag{3.65}$$

We estimate the seven terms $K_1,\dots,K_7$ of this inequality in the following way

$$K_1\leqslant C\varepsilon\left|\frac{\partial\varphi}{\partial t}\right|_2\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2|\nabla\varphi|_\infty\leqslant\frac{\alpha}{10\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2+C\varepsilon^2|\nabla\Delta\varphi|_2^2\left|\frac{\partial\varphi}{\partial t}\right|_2^2, \tag{3.66}$$

$$K_2\leqslant C\varepsilon|\Delta\varphi|_3\left|\frac{\partial\varphi}{\partial t}\right|_3^2\leqslant C\varepsilon|\nabla\Delta\varphi|_2\left|\frac{\partial\varphi}{\partial t}\right|_2\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2$$

$$\leqslant \frac{\alpha}{10\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2 + C\varepsilon^2|\nabla\Delta\varphi|_2^2\left|\frac{\partial\varphi}{\partial t}\right|_2^2, \tag{3.67}$$

$$K_3 \leqslant C|F''(\varphi)|_\infty\left|\frac{\partial\varphi}{\partial t}\right|_2^2 \leqslant C\left(1+|\varphi|_\infty^{p-1}\right)\left|\frac{\partial\varphi}{\partial t}\right|_2^2 \leqslant C\left(1+\|\varphi\|_2^{p-1}\right)\left|\frac{\partial\varphi}{\partial t}\right|_2^2, \tag{3.68}$$

$$K_4 \leqslant C\varepsilon|F'(\varphi)|_\infty\left|\frac{\partial\varphi}{\partial t}\right|_2^2 \leqslant C\varepsilon\left(1+\|\varphi\|_2^p\right)\left|\frac{\partial\varphi}{\partial t}\right|_2^2, \tag{3.69}$$

$$K_5 \leqslant C\varepsilon\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_3^2\left|\frac{\partial\varphi}{\partial t}\right|_3 \leqslant C\varepsilon\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2\left|\Delta\left(\frac{\mu}{\rho}\right)\right|_2\left|\frac{\partial\varphi}{\partial t}\right|_2^{1/2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^{1/2}$$

$$\leqslant \frac{\alpha}{10\rho_1}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2 + C\left|\frac{\partial\varphi}{\partial t}\right|_2^2 + C\varepsilon^2\left|\Delta\left(\frac{\mu}{\rho}\right)\right|_2^2\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2. \tag{3.70}$$

The sixth term is estimated just like the term I_{10} (see (3.32)), in the proof of Theorem 3.2

$$K_6 \leqslant \frac{\alpha}{10\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2 + C|u|_2\left(|\nabla\varphi|_2 + |\nabla\varphi|_2^{q_1}|\Delta\varphi|_2^{q_2}\right)|\nabla u|_2$$

$$+ C|\Delta\varphi|_2^4|\nabla u|_2^2 + \left(|\nabla\Delta\varphi|_2|\Delta\varphi|_2 + |\Delta\varphi|_2^{1/2}|\nabla\Delta\varphi|_2^{1/2}\right)|\nabla u|_2^2, \tag{3.71}$$

and in the same way

$$K_7 \leqslant CU|\nabla\varphi|_2\left(|F''(\varphi)|_\infty + \varepsilon|F'(\varphi)|_\infty\right)\left|\frac{\partial\varphi}{\partial t}\right|_2 + CU^2\varepsilon|\Delta\varphi|_2^4|\nabla v_\lambda|_2^2$$

$$+ CU^2|\Delta\varphi|_2^2 + CU^2|\nabla v_\lambda|_2^2|\nabla\Delta\varphi|_2^2. \tag{3.72}$$

Thanks to (3.47) and (3.59), inequality (3.65) leads to

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega\frac{1}{\rho}\left|\nabla\left(\frac{\mu}{\rho}\right)\right|^2\right) + \frac{\alpha}{10\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2$$

$$\leqslant C + C\left|\frac{\partial\varphi}{\partial t}\right|_2^2 + C|\nabla\Delta\varphi|_2^2 + C|\nabla u|_2^2 + C|\nabla\Delta\varphi|_2^2|\nabla u|_2^2$$

$$+ C\varepsilon^2\left|\Delta\left(\frac{\mu}{\rho}\right)\right|_2^2\left|\nabla\left(\frac{\mu}{\rho}\right)\right|_2^2. \tag{3.73}$$

Step 6. We have now collected all the inequalities we need to conclude. Indeed, if we sum (3.63) and (3.73) we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega\frac{1}{\rho}\left|\nabla\left(\frac{\mu}{\rho}\right)\right|^2 + \int_\Omega\eta(\varphi)|D(u)|^2\right) + \frac{\alpha}{20\rho_2}\left|\nabla\frac{\partial\varphi}{\partial t}\right|_2^2 + \frac{\rho_1}{20}\left|\frac{\partial u}{\partial t}\right|_2^2 + \gamma\frac{\eta_1}{20}|Au|_2^2$$

$$\leqslant g_1(t) + g_2(t)\left(|\nabla u|_2^2 + |\nabla\mu|_2^2\right) + \left(\frac{1}{\gamma} + \gamma\right)I, \tag{3.74}$$

where $g_1(t)$ and $g_2(t)$ are two functions bounded in $L^1(t_0, t_0 + \tau)$ independently of ε and t_0 for any $\tau > 0$. More precisely, this last fact comes from (3.36) and (3.64).

The conclusion is now straightforward with (3.48).

- If the dimension is $d = 2$, the term I (see (3.61)) is of the form $g_3(t)|\nabla u|_2^2$ with $g_3(t)$ bounded in $L^1(t_0, t_0 + \tau)$ independtly of t_0 and ε. We obtain the desired regularity and globality from the uniform Gronwall lemma.
- If the dimension is $d = 3$, as for the simple Navier–Stokes equation, the term $I = C|\nabla u|_2^6$ limits the estimates to be local using the Gronwall lemma.

In each case, the estimates obtained are independent of ε and in particular, in the 3D case, the existence time of the solution is independent of ε. $\square$

3.3. Asymptotic behavior

We are interested here in proving a result of asymptotic stability for the metastable states, as in [5], but in the case of nonhomogeneous fluids. In this subsection we have to suppose that the external force term g is derived from a potentiel. That is to say we suppose that there exists $\mathcal{G} \in L^2$ such that

$$g = \nabla \mathcal{G}. \tag{3.75}$$

One can think of g as a gravity forces term.

THEOREM 3.4. – *Let $\omega \in \mathbb{R}$, and suppose that $F''(\omega) > 0$ (we say that ω is a metastable state of the potential F). Then there exists $\varepsilon_0 > 0$, satisfying*

$$\varepsilon_0 < \frac{F''(\omega)}{|F'(\omega)|} \rho_1,$$

and a $\delta_0 > 0$ such that for any $\delta < \delta_0$ and any data $U > 0$, $v_0 \in v_\infty^U + V$, $\varphi_0 \in \Phi_3$, with $m(\varphi_0) = \omega$, satisfying

$$U + \|v_0\|_1 + \|\varphi_0 - \omega\|_3 \leqslant \delta,$$

there exists for any $\varepsilon < \varepsilon_0$, a unique global strong solution $(\varphi_\varepsilon, v_\varepsilon)$ of problem (1.1)–(1.4) (even in 3D) with the initial data (φ_0, v_0). Moreover, if $\delta > 0$ is small enough, this solution satisifies

$$\varphi_\varepsilon(t) \to \omega, \ v_\varepsilon(t) \to v_\infty^U, \quad \text{when } t \to +\infty.$$

Proof. – Step 1. First of all, we have to change the Cahn–Hilliard potential we work with. More precisely, we introduce a function $R_\varepsilon(x)$ such that $R_\varepsilon(\omega) = 0$ and $R_\varepsilon' = \rho_\varepsilon$. Then we construct, near the point $x = \omega$ a function F_ω given, $\xi > 0$ small enough being fixed, by

$$F_\omega(x) = F(x) - \frac{F'(\omega)}{\rho_\varepsilon(\omega)} R_\varepsilon(x) - F(\omega), \quad \forall x \in [\omega - \xi, \omega + \xi].$$

One can easily show that we have

$$F_\omega(\omega) = 0, \quad F_\omega'(\omega) = 0, \quad F_\omega''(\omega) > 0,$$

this last condition being ensured because $F''(\omega) > 0$ and ε is chosen so that

$$\varepsilon < \frac{F''(\omega)}{|F'(\omega)|} \rho_1.$$

This function F_ω is strictly convex near ω, so that it is easy to extend F_ω to be defined and convex on $\mathbb{R}$, and to satisfy assumptions (2.8)–(2.11).

A very important point, is that in the assumption (2.10), we can take $F_4(\omega) = 0$ (because F_ω is convex, and $F_\omega(\omega) = 0$).

Step 2. We perform the same estimates than in the proof of Theorem 3.2 with this new potential F_ω, excepted for the term I_3 which can be written thanks to (3.75)

$$|I_3| = \left| \int_\Omega \rho \nabla \mathcal{G}.u \right| = \left| \int_\Omega \mathcal{G}\rho'(\varphi)\nabla(\varphi - m(\varphi)).u \right|$$
$$\leqslant \varepsilon |\mathcal{G}|_2 |\Delta\varphi|_2 |\nabla u|_2 \leqslant C\varepsilon z_\varepsilon. \tag{3.76}$$

Using this new estimate, inequality (3.44) becomes

$$\frac{\partial}{\partial t} y_\varepsilon + \frac{1}{4} z_\varepsilon \leqslant CU^2 |D(v_\lambda)|_2^2 + C\varepsilon^{1/2} z_\varepsilon + C\lambda U z_\varepsilon + C\varepsilon^{1/8}(1 + y_\varepsilon^q) z_\varepsilon. \tag{3.77}$$

Indeed, we have $F_4(\omega) = 0$ in (3.12) and (3.15), and the constant which appears in (3.22) is no more present in the new estimate of I_3 (3.76).

We recall that we can take

$$\lambda = \min\left(\frac{1}{16CU}, \sqrt[4]{U}\right),$$

so that with (3.4) we have

$$f_1(U) \equiv CU^2 |D(v_\lambda)|_2^2 \to 0, \quad \text{when } U \to 0.$$

From now on, f_i will always denote a real positive continous function satisfying $f_i(0) = 0$. We follow the proof as the one of Theorem 3.2 to get, if we suppose $\varepsilon < 1/(16C)^2$,

$$\frac{d}{dt} y_\varepsilon + \frac{1}{8} z_\varepsilon \leqslant f_1(U) + C\varepsilon^{1/8}(1 + y_\varepsilon^q) z_\varepsilon,$$

and finally introducing the same M_ε we have that, if ε is small enough so that $K\delta \leqslant \frac{1}{2} M_\varepsilon$ then we have for any time t the estimate

$$\frac{d}{dt} y_\varepsilon + \frac{1}{16} z_\varepsilon \leqslant f_1(U),$$

and so with (3.18)

$$\frac{d}{dt} y_\varepsilon + C' y_\varepsilon \leqslant f_1(U).$$

Finally, using the assumption on the data, we deduce from the Gronwall lemma

$$y_\varepsilon(t) \leqslant K\delta + \frac{f_1(U)}{C'} \equiv f_2(\delta),$$

where we used the fact that $U \leqslant \delta$. We deduce easily that

$$\|\varphi_\varepsilon - \omega\|_{L^\infty(\mathbb{R}^+;\Phi_1)} + \varepsilon^{1/2}\|\varphi_\varepsilon - \omega\|_{L^\infty(\mathbb{R}^+;\Phi_2)} + \varepsilon^{3/4}\|\varphi_\varepsilon - \omega\|_{L^\infty(\mathbb{R}^+;\Phi_3)}$$
$$+ \|v_\varepsilon - v_\infty^U\|_{L^\infty(\mathbb{R}^+;H)} \leqslant f_2(\delta), \tag{3.78}$$

$$\|\varphi_\varepsilon - \omega\|_{L^2(t_0,t_0+\tau;\Phi_3)} + \varepsilon^{1/2}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;L^2)} + \varepsilon^{3/4}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t_0,t_0+\tau;H^1)}$$
$$+ \|\mu_\varepsilon\|_{L^2(t_0,t_0+\tau;\Phi_1)} + \|v_\varepsilon - v_\infty^U\|_{L^2(t_0,t_0+\tau;V)}$$
$$\leqslant (1+\tau)f_2(\delta), \quad \text{for any } t_0, \tau > 0. \tag{3.79}$$

Step 3. We perform the same estimate than (3.50) to obtain

$$\frac{1}{2}\frac{d}{dt}|\Delta\varphi|_2^2 + \frac{\alpha}{10\rho_2^2}|\Delta^2\varphi|_2^2 \leqslant f_\varepsilon^\delta(t),$$

where f_ε^δ is estimated independently of ε in the following way using (3.78), (3.79):

$$\|f_\varepsilon^\delta\|_{L^1(t_0,t_0+\tau)} \leqslant (1+\tau)f_3(\delta).$$

We can deduce of this estimate, using the uniform Gronwall lemma that

$$\|\varphi - \omega\|_{L^\infty(\mathbb{R}^+;\Phi_2)} \leqslant f_4(\delta), \qquad \|\varphi - \omega\|_{L^2(t_0,t_0+\tau;\Phi_4)} \leqslant (1+\tau)f_4(\delta). \tag{3.80}$$

As H^2 is embedded in L^∞ (in both dimensions $d = 2$, $d = 3$), we deduce from (3.80) that

$$\|\varphi - \omega\|_{L^\infty(\mathbb{R}^+\times\Omega)} \leqslant f_4(\delta),$$

so that if we choose δ_0 small enough and $\delta < \delta_0$ we have

$$f_4(\delta) < \xi,$$

and so for almost every (t,x), $\varphi(t,x)$ lies in the interval $[\omega - \xi, \omega + \xi]$ where the potential is defined by

$$F_\omega(x) = F(x) - \frac{F'(\omega)}{\rho_\varepsilon(\omega)}R_\varepsilon(x) - F(\omega).$$

That is to say that φ is solution of

$$\frac{\partial\varphi}{\partial t} + v.\nabla\varphi - \text{div}\left(\frac{1}{\rho_\varepsilon(\varphi)}\nabla\left(\frac{-\alpha\Delta\varphi + F'_\omega(\varphi)}{\rho_\varepsilon(\varphi)}\right)\right) = 0,$$

$$\frac{\partial\varphi}{\partial t} + v.\nabla\varphi - \text{div}\left(\frac{1}{\rho_\varepsilon(\varphi)}\nabla\left(\frac{1}{\rho_\varepsilon(\varphi)}\left(-\alpha\Delta\varphi + F'(\varphi) - \frac{F'(\omega)}{\rho_\varepsilon(\omega)}\rho_\varepsilon(\varphi)\right)\right)\right) = 0,$$

$$\frac{\partial\varphi}{\partial t} + v.\nabla\varphi - \text{div}\left(\frac{1}{\rho_\varepsilon(\varphi)}\nabla\left(\frac{-\alpha\Delta\varphi + F'(\varphi)}{\rho_\varepsilon(\varphi)}\right)\right) = 0.$$

One recognize here the initial equations (1.1), (1.2) with the potential F, and so finally, the solution we construct here is a solution of our problem (1.1)–(1.4).

Step 4. As the existence of a global strong solution in 2D is given by the Theorem 3.3, we only have to show that the solution obtained is a strong solution in velocity, defined globally in the 3D case. We are going to use the fact that the initial data is chosen near a stationnary state.

If we let

$$h(t) = \int_\Omega \frac{1}{\rho} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|^2 + \int_\Omega \eta(\varphi) |D(u)|^2,$$

estimate (3.74) reads in the 3D case

$$\frac{1}{2} \frac{d}{dt} h \leqslant g_1(t) + g_2(t) h(t) + C h(t)^3, \tag{3.81}$$

where g_1 and g_2 are bounded independtly of t_0 and ε in the following way

$$\int_{t_0}^{t_0+1} g_i(t)\, dt \leqslant f_5(\delta), \quad \text{for } i = 1, 2,$$

and thanks to (3.79) we also have

$$\int_{t_0}^{t_0+1} h(t)\, dt \leqslant f_5(\delta).$$

Moreover, thanks to the assumption on the data, there exists C' such that

$$h(0) < C' \delta^2.$$

Now let us introduce

$$f_6(\delta) = \left(\max \left(C' \delta^2, f_5(\delta) \right) + f_5(\delta) \right) e^{1 + f_5(\delta)}.$$

It is straightforward to show that

$$f_6(\delta) \to 0, \quad \text{when } \delta \to 0.$$

Let us now choose, δ_0 small enough so that $C f_6(\delta)^2 < 1$ for any $\delta < \delta_0$. In those conditions, one have

$$C h(0)^2 < 1.$$

If we denote by T the maximal time such that

$$C h(t)^2 < 1, \quad \forall t \in [0, T[,$$

we get from (3.81), for any $t \in [0, T[$,

$$\frac{d}{dt} h \leqslant g_1(t) + (1 + g_2(t)) h(t),$$

so that, using the uniform Gronwall lemma, we have for any $t \in [0, T[$

$$h(t) \leqslant f_6(\delta).$$

But, by the choice of δ_0, we have $C f_6(\delta)^2 < 1$, which implies that the existence time T of the solution is $+\infty$. Moreover, we deduce the following estimates on the solution (φ, v)

$$\|\varphi - m(\varphi)\|_{L^\infty(\mathbb{R}^+; \Phi_3)} + \|v - v_\infty^U\|_{L^\infty(\mathbb{R}^+; V)} \leqslant f_7(\delta), \tag{3.82}$$

$$\left\| \frac{\partial \varphi}{\partial t} \right\|_{L^2(t_0, t_0+\tau; \Phi_1)} + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(t_0, t_0+\tau; H)} + \|v\|_{L^2(t_0, t_0+\tau; V_2)} \leqslant f_7(\delta)(1 + \tau).$$

We have shown the global and uniform in time existence of strong solutions for the problem (1.1)–(1.4).

Step 5. In the previous step, we have obtained the stability of the stationary solution we are studying. In order to show the asymptotic stability of this solution, we have to study the convergence of the solution when t tends to $+\infty$.

First we study the convergence of the order parameter φ. If one takes the scalar product of (1.1) with $\varphi - \omega$ in L^2, one obtains using (1.4) and the boundary conditions on v,

$$\frac{d}{dt} |\varphi - \omega|_2^2 + \int_\Omega \frac{1}{\rho} \nabla\left(\frac{\mu}{\rho}\right).\nabla\varphi = 0,$$

which can be written, using the fact that $\mu = -\alpha \Delta\varphi + F_\omega'(\varphi)$ (or $F'(\varphi)$, it is strictly equivalent),

$$\frac{d}{dt} |\varphi - \omega|_2^2 - \alpha \int_\Omega \frac{1}{\rho} \nabla\left(\frac{\Delta\varphi}{\rho}\right).\nabla\varphi = -\int_\Omega \frac{1}{\rho} \nabla\left(\frac{F_\omega'(\varphi)}{\rho}\right).\nabla\varphi.$$

We get

$$\frac{d}{dt} |\varphi - \omega|_2^2 + \int_\Omega \frac{1}{\rho^2} |\Delta\varphi|^2$$

$$= \alpha \int_\Omega \frac{\rho'(\varphi)}{\rho^3(\varphi)} \Delta\varphi |\nabla\varphi|^2 - \int_\Omega \frac{1}{\rho^2} F_\omega''(\varphi) |\nabla\varphi|^2 + \int_\Omega \frac{F_\omega'(\varphi)\rho'(\varphi)}{\rho^3(\varphi)} |\nabla\varphi|^2,$$

and so, using the fact that F_ω is convex (by construction), and φ is bounded in L^∞, we deduce

$$\frac{d}{dt} |\varphi - \omega|_2^2 + \frac{1}{\rho_2^2} |\Delta\varphi|_2^2 \leqslant C\varepsilon |\Delta\varphi|_2^3 + C\varepsilon |\nabla\varphi|_2^2 \leqslant C\varepsilon C f_4(\delta) |\Delta\varphi|_2^2 + C f_4(\delta) |\nabla\varphi|_2^2,$$

because we have

$$\left|F'_\omega(\varphi)\right| = \left|F'_\omega(\varphi) - F'_\omega(\omega)\right| \leqslant \Big(\sup_{[\omega-\xi,\omega+\xi]} \left|F''_\omega\right| \Big) |\varphi - \omega|,$$

that is to say with (3.80)

$$\left|F'_\omega(\varphi)\right|_\infty \leqslant C f_4(\delta).$$

If δ is small enough such that $2C f_4(\delta) \leqslant 1/(2\rho_2^2)$, we deduce that we have

$$\frac{d}{dt}|\varphi - \omega|_2^2 + C|\Delta\varphi|_2^2 \leqslant 0,$$

which implies by (2.5) that

$$\frac{d}{dt}|\varphi - \omega|_2^2 + C|\varphi - \omega|_2^2 \leqslant 0,$$

and finally

$$|\varphi - \omega|_2^2 \leqslant |\varphi_0 - \omega|_2^2 \mathrm{e}^{-Ct},$$

which gives the desired convergence in Φ_0. The convergence in Φ_s for any $s < 2$ comes directly from the previous convergence and the fact that $(\varphi - \omega)$ is bounded uniformly in time in Φ_2.

Step 6. It remains to show the convergence of the velocity field. Let us write down the equation satisfied by $v - v_\infty^U$ in the following way (remember that $\Delta v_\infty^U = 0$ and that $g = \nabla \mathcal{G}$)

$$\rho(\omega)\left(\frac{\partial(v - v_\infty^U)}{\partial t} + (v - v_\infty^U).\nabla(v - v_\infty^U)\right) - 2\operatorname{div}\big(\eta(\varphi)D(v - v_\infty^U)\big) + \nabla p$$

$$= 2\operatorname{div}\big((\eta(\varphi) - \eta(\omega))D(v_\infty^U)\big) - \rho(\omega)\big(v_\infty^U.\nabla(v - v_\infty^U) + (v - v_\infty^U).\nabla v_\infty^U\big)$$

$$- (\rho(\varphi) - \rho(\omega))\left(\frac{\partial v}{\partial t} + v.\nabla v\right) + (\mu - m(\mu))\nabla\varphi$$

$$+ \varepsilon\left(\frac{1 - \varphi^2}{4} - \frac{1 - \omega^2}{4}\right)\nabla\left(\frac{\mu}{\rho}\right) + (\rho(\varphi) - \rho(\omega))g,$$

from which we deduce in a straightforward way

$$\frac{\rho(\omega)}{2}\frac{d}{dt}\big(|v - v_\infty^U|_2^2\big) + \eta_1\big|\nabla(v - v_\infty^U)\big|_2^2$$

$$\leqslant 2\int_\Omega |\eta(\varphi) - \eta(\omega)|\,|D(v_\infty^U)|\,|D(v - v_\infty^U)|$$

$$+ \rho(\omega)\int_\Omega |v_\infty^U|\,|v - v_\infty^U|\,|\nabla(v - v_\infty^U)| + \rho(\omega)\int_\Omega |v - v_\infty^U|^2\,|\nabla v_\infty^U|$$

$$+ \int_{\Omega} |\rho(\varphi) - \rho(\omega)| \left| \frac{\partial v}{\partial t} + v.\nabla v \right| |v - v_{\infty}^{U}|$$

$$+ \int_{\Omega} |\mu - m(\mu)| |\nabla \varphi| |v - v_{\infty}^{U}| + \varepsilon \int_{\Omega} \left| \frac{\varphi^2 - \omega^2}{4} \right| \left| \nabla \left(\frac{\mu}{\rho} \right) \right| |v - v_{\infty}^{U}|$$

$$+ \left| \int_{\Omega} (\rho(\varphi) - \rho(\omega)) g (v - v_{\infty}^{U}) \right|.$$

This last inequality leads to

$$\frac{\rho(\omega)}{2} \frac{\partial}{\partial t} (|v - v_{\infty}^{U}|_2^2) + \eta_1 |\nabla (v - v_{\infty}^{U})|_2^2$$

$$\leqslant 2\|\eta'\|_{\infty} |\varphi - \omega|_{\infty} |D(v_{\infty}^{U})|_2 |\nabla (v - v_{\infty}^{U})|_2 + C|v_{\infty}^{U}|_{\infty} |\nabla (v - v_{\infty}^{U})|_2^2$$

$$+ C|\nabla v_{\infty}^{U}|_{\infty} |v - v_{\infty}|_2^2 + C\varepsilon |\varphi - \omega|_{\infty} \left(\left| \frac{\partial v}{\partial t} \right|_2 + |\nabla v|_2 |Av|_2 \right) |v - v_{\infty}^{U}|_2$$

$$+ C|\nabla \mu| |\nabla \varphi|_4 |v - v_{\infty}^{U}|_2 + C\varepsilon |\varphi^2 - \omega^2|_{\infty} \left| \nabla \left(\frac{\mu}{\rho} \right) \right|_2 |v - v_{\infty}^{U}|_2$$

$$+ C\varepsilon |\varphi - \omega|_{\infty} |g|_2 |v - v_{\infty}^{U}|_2. \tag{3.83}$$

Using (2.3), (3.80), (3.82) and the fact that

$$|v_{\infty}^{U}|_{\infty} = |\nabla v_{\infty}^{U}|_{\infty} = U \leqslant \delta,$$

we obtain, for δ small enough, the estimate

$$\frac{\partial}{\partial t} |v - v_{\infty}^{U}|_2^2 + |v - v_{\infty}^{U}|_2^2 \leqslant g_1(t) + C' e^{-C_1 t} g_2(t),$$

where $g_1(t) \to 0$ when t goes to infinity, and g_2 is given by

$$g_2(t) = \left| \frac{\partial v}{\partial t} \right|_2^2 + |Av|_2^2 + |\nabla \mu|_2^2 + \left| \nabla \left(\frac{\mu}{\rho} \right) \right|_2^2,$$

so that we have thanks to (3.79) and (3.82)

$$\int_0^t g_2(s) \, ds \leqslant C(1 + t).$$

We can suppose that $C_1 < 1$, then we show that any function $y(t)$ satisfying

$$y'(t) + y(t) \leqslant g_1(t) + C e^{-C_1 t} g_2(t),$$

necesarily tends to zero when t goes to infinity. Indeed, we have

$$y(t) \leqslant e^{-t} y(0) + e^{-t} \int_0^t e^s g_1(s) \, ds + C e^{-t} \int_0^t e^{(1-C_1)s} g_2(s) \, ds.$$

In this inequality it is clear that the first term converges towards zero, but also the second term using the Cesaro theorem and the fact that $g_2(t) \to 0$. As far as the third term is concerned, we have

$$\mathrm{e}^{-t} \int_0^t \mathrm{e}^{(1-C_1)s} g_2(s)\, ds = \mathrm{e}^{-t} \mathrm{e}^{(1-C_1)t} \int_0^t g_2(s)\, ds \leqslant C \mathrm{e}^{-C_1 t}(1+t).$$

The conclusion is straightforward: the velocity field converges towards the stationnary velocity field in H and so in V_s for any $s < 1$ because we have shown that $(v - v_\infty^U)$ is bounded uniformly in time in the space V.   □

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REFERENCES

[1] Adams R.A., Sobolev Spaces, Pure and Applied Mathematics, Vol. 65, Academic Press, New York, 1975.

[2] Alikakos N., Bates P., Fusco G., Slow motion for the Cahn–Hilliard equation in one space dimension, J. Differential Equation 90 (1991) 81–135.

[3] Agmon S., Lectures on elliptic boundary value problems, Van Nostrand Mathematical Studies, Vol. 2, 1965.

[4] Bates P.W., Fife P.C., The dynamics of nucleation for the Cahn–Hilliard equation, SIAM J. Appl. Math. 53 (4) (1993) 990–1008.

[5] Boyer F., Mathematical study of multiphase flow under shear through order parameter formulation, Asymptotic Analysis 20 (2) (1999) 175–212.

[6] Boyer F., A theoretical and numerical model for the study of incompressible mixture flows, Computers and Fluids (2000), to appear.

[7] Colin A., Panizza P., Private communications.

[8] Carr J., Gurtin M., Slemrod M., Structured phase transitions on a finite interval, Arch. Rational Mech. Anal. 86 (1984) 317–351.

[9] Chella R., Vinals J., Mixing of a two-phase fluid by a cavity flow, Physical Review E 53 (1996) 3832.

[10] Debussche A., Dettori L., On the Cahn–Hilliard equation with a logarithmic free energy, Nonlinear Analysis 24 (10) (1995) 1491–1514.

[11] Doi M., Dynamics of domains and textures, in: Theoretical Challenges in the Dynamics of Complex Fluids, 1997, pp. 293–314.

[12] Elliott C.M., Garcke H., On the Cahn–Hilliard equation with degenerate mobility, SIAM J. Math. Anal. 27 (2) (1996) 404–423.

[13] Gilbarg D., Trudinger N.S., Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1977.

[14] Gunton J.D., San Miguel M., Sahni P.S., in: Domb, Lebowitz (Eds.), Phase Transitions and Critical Phenomena, Vol. 8, Academic, London, 1983.

[15] Gurtin M.E., Polignone D., Vinals J., Two-phase binary fluids and immiscible fluids described by an order parameter, Mathematical Models and Methods in Applied Sciences 6 (1996) 815.

[16] Horgan C.O., Korn's inequalities and their applications in continuum mechanics, SIAM Review 37 (4) (1995) 491–511.

[17] Jasnow D., Vinals J., Coarse-grained description of thermo-capillary flow, Phys. Fluids 8 (1996) 660.

[18] Lions J.L., Magenes E., Problèmes aux Limites non Homogènes et Applications, Dunod, 1968.

[19] Lions P.L., Mathematical Topics in Fluid Mechanics: Vol. 1, Incompressible Models, Oxford Science Publications, 1996.

[20] Miranville A., Upper bound on the dimension of the attractor for the shear-layer flow in space dimension 3, in: Dynamical Systems, Stockholm, 1992, World Sci. Publishing, River Edge, NJ, 1993, pp. 61–74.

[21] Miranville A., A model of Cahn–Hilliard equation based on a microforce balance, C. R. Acad. Sci. Paris Sér. I Math. 328 (12) (1999) 1247–1252.

[22] Nicolaenko B., Scheurer B., Temam R., Some global dynamical properties of a class of pattern formation equations, Comm. Partial Differential Equations 14 (2) (1989) 245–297.

[23] Onuki A., Phase transitions of fluids in shear flow, J. Phys.: Condens. Matter 9 (1997) 6119–6157.

[24] Simon J., Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (4) (1987) 65–96.

[25] Temam R., Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Science, Vol. 68, Springer-Verlag, 1997.

[26] Temam R., Navier–Stokes Equations, Studies in Mathematics and its Applications, Vol. 2, North-Holland, Amsterdam, 1977.

[27] Wei J., Winter M., Stationary solutions for the Cahn–Hilliard equation, Ann. Inst. Henri Poincaré: Analyse non linéaire 15 (4) (1998) 459–492.

[28] Wei J., Winter M., On the stationary Cahn–Hilliard equation: interior spike solutions, J. Differential Equations 148 (2) (1998) 231–267.

[29] Wei J., Winter M., On the stationary Cahn–Hilliard equation: bubble solutions, SIAM J. Math. Anal. 29 (6) (1998) 1492–1518.