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The porous medium equation as a finite-speed approximation to a Hamilton-Jacobi equation

by

D. G. ARONSON [1], [2]
School of Mathematics, University of Minnesota,
Minneapolis, MN 55455

and

J. L. VAZQUEZ [1] [3]
División de Matemáticas, Universidad Autónoma de Madrid,
28049 Madrid, Spain

ABSTRACT. — It is known that solutions of the porous medium equation $u_t = (u^m)_{xx}$ converge to solutions of the heat equation $u_t = u_{xx}$ as $m \downarrow 1$ if the initial datum $u(x, 0)$ is kept fixed. For porous medium flow u represents a suitably scaled density and $v = mu^{m-1}/(m-1)$ represents the pressure. We prove that v converges to a solution of the Hamilton-Jacobi equation $v_t = (v_x)^2$ as $m \downarrow 1$ if $v(x, 0)$ is fixed. Moreover, if $v(x, 0)$ has compact support the interface for the porous medium equation tends to the interface for the latter equation. The limit $m \uparrow 1$ is also discussed. In this fast-diffusion case no interfaces appear.

Key-words: Porous medium flow, Hamilton-Jacobi equation, Finite speed of propagation, Viscosity solutions, Interfaces.

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(3) Long-term visitor at the Institute for Mathematics and its Applications, 1985.

RÉSUMÉ. — On sait que les solutions de l'équation des milieux poreux $u_t = (u^m)_{xx}$ convergent quand $m \downarrow 1$ vers des solutions de l'équation de la chaleur $u_t = u_{xx}$ si l'on fixe la donnée initiale $u(x, 0)$. Comme modèle de l'écoulement d'un gaz dans un milieu poreux u représente la densité et $v = mu^{m-1}/(m-1)$ est la pression. Nous démontrons que v converge quand $m \downarrow 1$ vers une solution de l'équation de Hamilton-Jacobi $v_t = (v_x)^2$ si $v(x, 0)$ reste fixée. Si en plus $v(x, 0)$ est à support compact alors l'interface de la solution de l'équation des milieux poreux tend vers celle correspondant à $v_t = (v_x)^2$. On discute aussi le cas de diffusion rapide $m \uparrow 1$ où il n'y a pas d'interfaces.

INTRODUCTION AND RESULTS

The density $u = u(x, t)$ of an ideal gas flowing isentropically through a one-dimensional $(x \in \mathbb{R})$ porous medium obeys the equation

$$u_t = (u^m)_{xx} \quad \text{in} \quad Q = \mathbb{R} \times \mathbb{R}^+ \tag{0.1}$$

where $m > 1$ is a constant. It is known, [BC1], that the solutions u to (0.1) depend continuously in the $C(\mathbb{R}^+ : L^1(\mathbb{R}))$-norm on both the initial datum $u(.,0) = u_0 \in L^1(\mathbb{R}))$ and on m. In particular if u_0 is kept fixed and $m \downarrow 1$, then $u = u(x, t; m)$ converges to a solution of the heat conduction equation

$$u_t = u_{xx} \tag{0.2}$$

with initial datum u_0. Thus, for m near 1, the porous medium equation can be regarded as a perturbation of the heat equation.

Despite the convergence of solutions of the nonlinear equation (0.1) to solutions of the linear equation (0.2), there is a marked difference in the behaviour of solutions of these equations stemming from the fact that (0.1) is of degenerate parabolic type. Perhaps the most striking consequence of that degeneracy is the finite speed of propagation of disturbances from rest for the porous medium equation as opposed to the infinite speed associated with the heat equation. Specifically, suppose that u_0 is a continuous, nonnegative and bounded real function such that $u_0 = 0$ on $\mathbb{R}^+$ and $u_0 \equiv 0$. Then there exists a unique continuous, nonnegative and bounded function $u(x, t)$ in Q which solves (0.1) in a generalized sense and is such that $u(.,0) = u_0$, cf. [OKC], [AB]. Moreover the support of $u(.,t)$ is bounded away from $x = \infty$ for every $t > 0$ and the finite function

$$\zeta(t) = \sup \{ x \in \mathbb{R} : u(x, t) > 0 \} \tag{0.3}$$

is continuous and nondecreasing in $\mathbb{R}^+$. The curve $x = \zeta(t)$ is called the (right-hand) *interface* of u. If the support of u_0 is not an interval of the form $(-\infty, a)$ other interfaces will exist, but their properties are essentially the same as those of the right-hand interface. The interfaces have been the object of intense study in recent years and their behaviour is now reasonably well understood. Detailed results and references can be found in [V3]. In particular the interface exists and has the properties described above as long as the initial datum is a nonnegative function in $L^1_{loc}(\mathbb{R})$ which grows at most like $0(|x|^{2/(m-1)})$ as $|x| \to \infty$, cf. [AC], [BCP], [DK].

The finite speed of propagation associated with the porous medium equation is, of course, reminiscent of *hyperbolic equations.* Our purpose in this paper is to investigate the precise nature of the relationship between (0.1) and the Hamilton-Jacobi equation

$$v_t = (v_x)^2 , \tag{0.4}$$

sometimes called the nonstationary eikonal equation. To obtain (0.4) as the limit of (0.1) we proceed as follows. In view of the application in mind it is natural to restrict attention to nonnegative solutions of (0.1). We can then replace the variable u by the corresponding scaled *pressure*

$$v = \frac{m}{m-1} u^{m-1} . \tag{0.5}$$

Formally v satisfies the equation

$$v_t = (m-1)vv_{xx} + (v_x)^2 . \tag{0.6}$$

The local velocity of the flow at any point $(x, t) \in Q$ is given by the function $-v_x$ (Darcy's law) and the interface is characterized by the relationship

$$\zeta'(t) = -v_x(\zeta(t), t) , \tag{0.7}$$

where $v_x(\zeta(t), t)$ means lim $v_x(x, t)$ as $x \uparrow \zeta(t)$ with $t > 0$, cf. [A2], [Kn] (1).

Equations (0.4) and (0.6) share the property of finite propagation speed. Thus we can view (0.6) as a finite-speed viscous approximation to (0.4). In fact (0.4) and (0.6) formally agree on the interfaces. This agreement has been rigorously established in [CF] where it is shown that $v_t - (v_x)^2 \to 0$ as $(x, t) \to (\zeta(t_0), t_0)$ with $t_0 > 0$ and $v(x, t) > 0$ (2).

On the other hand, equations (0.4) and (0.6) formally agree everywhere when $m \to 1$ and it is this limit which is our main concern here. Set

$$\varepsilon = m - 1 .$$

(1) If the time t^* at which the interface starts to move *(waiting time) is* positive then ζ may not be differentiable. In that case formula (0.7) holds at $t = t^*$ with $\zeta'(t)$ replaced by the right-hand derivative $D^+\zeta(t^*)$, cf. [CF], [ACK], [ACV].

(2) At $t_0 = t^*$ we also need $t \geq t_0$.

For every $\varepsilon > 0$ let $v_{\varepsilon 0}$ be a continuous, nonnegative and bounded real function and consider the initial-value problem

$$\begin{aligned} v_t &= \varepsilon v v_{xx} + (v_x)^2 \quad \text{in} \quad Q, \\ v(x, 0) &= v_{\varepsilon 0}(x) \quad \text{in } \mathbb{R}. \end{aligned} \tag{P_ε}$$

For $\tau > 0$ let $Q_\tau = \{ (x, t) \in Q : t > \tau \}$. The following is our main convergence result:

THEOREM 1. — A) *Suppose that for every small* $\varepsilon > 0$, $v_{0\varepsilon}$ *is a continuous real function such that*

$$0 \leqslant v_{\varepsilon 0} \leqslant N \tag{0.8}$$

for some constant $N > 0$ *and* $\{ v_{\varepsilon 0} \}$ *converges to a function* v_0 *uniformly on compact subsets of* $\mathbb{R}$ *as* $\varepsilon \to 0$. *Let* $v_\varepsilon = v_\varepsilon(x, t)$ *denote the solution to* (P_ε). *Then as* $\varepsilon \to 0$, *the family* $\{ v_\varepsilon \}$ *converges uniformly on compact subsets of* $\overline{Q}$ *to a function* $v \in C(\overline{Q})$ *such that*

i) $v \in \operatorname{Lip}(Q_\tau)$ *for every* $\tau > 0$ *and* $v_t = (v_x)^2$ *a. e. in* Q,
ii) $v(x, 0) = v_0(x)$ *for all* $x \in \mathbb{R}$.
iii) $v_{xx} \geqslant -1/2t$ *in* $\mathscr{D}'(Q)$.

Moreover $v_{\varepsilon x} \to v_x$ *in* $L^p_{loc}(Q)$ *for every* $1 \leqslant p < \infty$.

B) *The limit function* $v = v(x, t)$ *is uniquely characterized as a solution in* $C(\overline{Q})$ *of the initial-value problem*

$$\left\{ \begin{aligned} v_t &= (v_x)^2 \quad \text{in} \quad Q, \\ v(.,0) &= v_0 \quad \text{in} \quad \mathbb{R} \end{aligned} \right. \tag{P_0}$$

by the semiconvexity property iii).

The proof of Theorem 1 is given in section 2 after the required estimates for v_ε and its derivatives are derived in Section 1. Here we shall discuss some relevant aspects of our result. First, note that the convergence of v_ε to v is compatible with the statement that u_ε converges to a solution of the heat equation. This is possible since (0.5) implies that $u_\varepsilon \to 0$ uniformly in Q as $\varepsilon \to 0$ if v_ε is bounded.

The semiconvexity property iii), which selects the « correct » solution of problem (P_0), is an immediate consequence of the estimate

$$v_{xx} \geqslant -\frac{1}{(\varepsilon + 2)t}, \tag{0.9}$$

valid for all nonnegative solutions of (0.1) ([AB]). A similar estimate, $\Delta u \geqslant -k/t$, also holds in several space dimensions, the equation being then $u_t = \Delta(u^m)$ in $Q = \mathbb{R}^d \times \mathbb{R}^+$ with $d > 1$, and has played a key role in the general theory of the porous medium equation, cf. [AC], [BCP]. Theorem 1(B) sheds a new light on its significance, but it would be interesting to better understand its physical meaning (in this connection see (0.13) below).

Crandall and P. L. Lions, [CL] (see also [CEL]) have recently introduced the notion of *viscosity solution* to characterize the « good » solutions of Hamilton-Jacobi equations. To be specific, if the equation is

$$u_t + \mathrm{H}(\mathrm{D}u) = 0 \quad \text{in} \quad \mathrm{Q} = \mathbb{R}^d \times (0, \mathrm{T}) \tag{0.10}$$

with $\mathrm{H} : \mathbb{R}^d \to \mathbb{R}$ continuous and $\mathrm{D}u = (u_{x_1}, \ldots, u_{x_d})$, then a function $u \in \mathrm{C}(\mathrm{Q})$ is a viscosity solution of (0.10) if for every $\phi \in \mathrm{C}^1(\mathrm{Q})$ we have

$$\phi_t + \mathrm{H}(\mathrm{D}\phi) \leqslant 0 \tag{0.11 a}$$

at all local maxima of $u - \phi$ and

$$\phi_t + \mathrm{H}(\mathrm{D}\phi) \geqslant 0 \tag{0.11 b}$$

at all local minima of $u - \phi$. Our limit function $v(x, t)$ of Theorem 1 is a viscosity solution of $v_t = (v_x)^2$ because v_x is locally bounded and v_{xx} is locally bounded below in Q. It then follows from Theorem 10.2 of [Li].

It should also be noted that since our estimates break down at $t = 0$ the current uniqueness proofs for viscosity solutions do not apply. In fact our uniqueness result, Theorem 1(B), is an improvement of Lemma 2.1 of [B] to cover the case in which the bounds for the derivatives are not uniform in x, t. As explicit examples show, the estimates which we derive i. e. $v_x = 0(t^{-1/2})$, $v_{xx} \geqslant 0(1/t)$, are actually attained for general initial data.

The idea that solutions of $v_t = (v_x)^2$ must approximate to leading order solutions of $v_t = \varepsilon v v_{xx} + (v_x)^2$ for very small $\varepsilon > 0$ is used by Kath and Cohen in [KC] where they study shock formation at the waiting-time on the interfaces of solutions of (0.6) for ε small using singular perturbation methods.

Since the equation $v_t = (v_x)^2$ is invariant under translations, in particular in v, the restriction to positive bounded solutions is equivalent to working with just bounded solutions. However, setting the lower bound at $u = v = 0$ plays an important role in the convergence discussed above because of the degeneracy of the diffusion term $\varepsilon v v_{xx}$ when v vanishes.

We also prove convergence of the interfaces which appear when $v_{\varepsilon 0}$ and v_0 vanish in some interval. To make things simple suppose that $v_{\varepsilon 0} \equiv v_0$ and v_0 vanishes in $\mathbb{R}^+$, but $v_0 \not\equiv 0$. Let $\zeta_\varepsilon(t)$ denote the (right-hand) interface for the solution v_ε of (P_ε) and let $\zeta(t)$ denote the interface for the solution v to (P_0). Then $\zeta_\varepsilon(t)$ and $\zeta(t)$ are Lipschitz continuous, nondecreasing functions of t for $0 \leqslant t < \infty$ and we have the following convergence result.

THEOREM 2. — *As* $\varepsilon \downarrow 0$ *we have* $\zeta_\varepsilon(t) \to \zeta(t)$ *uniformly on* $[0, \mathrm{T}]$ *for every* $\mathrm{T} > 0$ *and* $\zeta'_\varepsilon(t) \to \zeta'(t)$ *a. e. and in* $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^+)$ *for every* $p \in [1, \infty)$.

Further properties of the convergence of the interfaces, as well as proof of these results are given in Section 3.

The convergence results of Theorems 1 and 2 cannot be substantially improved because of the lack of regularity of the solution v to (0.4). In fact v_x is in general discontinuous and so is $\zeta'(t)$, cf. [D] or [La]. Stronger convergence results are obtained in Section 4 under the further assumption that the initial data $v_{\varepsilon 0}$ are concave on their support, cf. Theorem 3. The result depends on the fact that, for concave data, v is a C^1 function on the set where it does not vanish and $\zeta \in C^1[0, \infty)$. We also prove that if $v_{0\varepsilon} \uparrow v_0$ as $\varepsilon \downarrow 0$ then v_ε also converges monotonically to v (Theorem 4).

As we noted above the local velocity of propagation of solutions of (0.1) is given by $w = -v_x$. As a consequence of Theorem 1 it follows that the family $w_\varepsilon = v_{\varepsilon x}$, $\varepsilon > 0$, converges in $L^p_{loc}(Q)$ to a solution w of the *conservation law*

$$w_t + (w^2)_x = 0 \quad \text{in} \quad \Delta'(Q) \tag{0.12}$$

and w satisfies the *entropy condition*

$$w_x \leqslant \frac{1}{2t}. \tag{0.13}$$

If in addition v_{0x} exists in a suitable sense then w is the unique distributional solution of (0.12) satisfying (0.13) and taking the initial value $w(x, 0) = -v_0'(x)$, cf. [0], [LP].

It is also of some interest to consider what happens if we take the limit $m \to 1$ for $m < 1$ in (0.6). For $m \in (0, 1)$ the initial value problem for (0.1) has a unique solution provided that $u(x, 0)$ is nonnegative and locally integrable ([AB], [HP]). Moreover $u \in C^\infty(Q)$ and is positive everywhere in Q so that, in particular, there are no interfaces. The analog of Theorem 1 with $m \uparrow 1$ is proved in Section 5. Note that in this case v, which is still defined by (0.5), is negative.

Before we turn to the proofs of our results we pause to describe an important conection between equations (0.1) or (0.6) and (0.4). Let $m > 1$. Recall that if u is a nonnegative solution of (0.1) then v satisfies

$$v_t = (m-1)vv_{xx} + (v_x)^2$$

and

$$v_{xx} \geqslant -\frac{1}{(m+1)t}. \tag{0.15}$$

It follows that

$$v_t + \frac{m-1}{(m+1)t} v \geqslant (v_x)^2 \tag{0.16}$$

Now define

$$V(x, \tau) = v(x, t)t^{\frac{m-1}{m+1}}, \tag{0.17 a}$$

where t and τ are related by

$$\tau = \frac{m+1}{2} t^{\frac{2}{m+1}}. \qquad (0.17\,b)$$

With this change of variables (0.16) and (0.15) become

$$V_\tau \geqslant (V_x)^2 \quad \text{and} \quad V_{xx} \geqslant -\frac{1}{2\tau}. \qquad (0.18)$$

Thus if u is a solution of porous medium equation (0.1) then the change of variables (0.5), (0.17) transforms it into a viscosity supersolution of the equation (0.4) (3).

The equalities in (0.18) hold for the *Barenblatt solutions*

$$\bar{u}(x, t) = t^{-\frac{1}{m+1}} \left(C - \frac{m-1}{2m(m+1)} \frac{x^2}{t^{2/(m+1)}} \right)_+^{\frac{1}{m-1}}, \qquad (0.19)$$

where $(.)_+$ means $\max(., 0)$. For each $C > 0$ this function is a solution of (0.1) with initial data which is a multiple of the Dirac measure concentrated at $x = 0$. If $\bar{v}$ is defined through (0.5) from $\bar{u}$ then

$$\bar{v}_{xx} = -\frac{1}{(m+1)t}$$

whenever $\bar{u} > 0$. The change of variables (0.17) gives

$$\bar{V}(x, \tau) = \left(K - \frac{x^2}{4\tau} \right)_+ \quad \text{with} \quad K = \frac{Cm}{m-1}. \qquad (0.20)$$

The functions (0.20) are the well-known bounded self-similar solutions to $V_\tau = (V_x)^2$. Note that the initial value if

$$V(x, 0) = \begin{cases} 0 & \text{if } x \neq 0, \\ K & \text{if } x = 0. \end{cases} \qquad (0.21)$$

This rather striking correspondence has some deep consequences. It is known that as $t \to \infty$ every solution of (0.1) with $u(x, 0) \in L^1(\mathbb{R})$, $u_0 \geqslant 0$ and $u_0 \not\equiv 0$, converges with the appropriate scaling to the Barenblatt solution $\bar{u}$ with the same mass, $\int u(x, t)dx = \int \bar{u}(x, t)dx = M$. Specifically

$$t^{1/(m+1)} \| u(., t) - \bar{u}(., t) \|_\infty \to 0 \quad \text{as} \quad t \to \infty,$$

(cf. [K] and also [V1], [V2] for further details). The self-similar solutions

(3) We say that v is a viscosity supersolution of (0.10) if the condition (0.11 b) is satisfied. For the proof that v satisfying (0.18) is a viscosity supersolution see [Li], Theorem 10.2.

(0.20) play the same role for bounded solutions for (0.4). Thus we see that *the asymptotic behaviour of these classes of solutions for equations (0.1) and (0.4) coincide under the transformation (0.5), (0.17).* This asymptotic similarity was first observed in [V2] where, in particular, it was shown that if $m > 1$ then the velocity $w = -v_x$ associated with the solutions of (0.1) behaves at $t \to \infty$ like the finite *N-waves* that are the typical profiles of solutions of first-order conservation laws if $m > 1$. Corresponding results hold for $m \leqslant 1$ as we show in Section 5.

The convergence results Theorem 1 and its analog for $m \leqslant 1$, Theorem 6, also hold for $x \in \mathbb{R}^d$ for any $d > 1$. While our discussion of the case $m \leqslant 1$ in Section 5 is valid in several space dimensions, the case $m > 1$ requires new estimates and will be studied in [LSV].

1. ESTIMATES

In this section we collect various results for the solutions of problem (P_ε) that are needed in the sequel. In particular we obtain several estimates, none of which is entirely new, but we shall give new proofs in some cases in order to get the precise dependence on ε.

We consider a family of measurable, nonnegative and bounded initial functions $\{ v_{0\varepsilon} : \varepsilon > 0 \}$ defined on the real line. We further assume that they are bounded uniformly in ε, i. e., there exists a constant $N > 0$ such that

$$0 \leqslant v_{\varepsilon 0}(x) \leqslant N \tag{1.1}$$

for a. e. $x \in \mathbb{R}$ and every $\varepsilon > 0$. We define the corresponding initial densities by

$$u_{\varepsilon 0}(x) = \left(\frac{\varepsilon}{\varepsilon + 1} v_{\varepsilon 0}(x) \right)^{1/\varepsilon}.$$

For every $\varepsilon > 0$ there exists a unique bounded continuous function u_ε defined in Q which satisfies

$$\begin{aligned} u_t &= (u^{1+\varepsilon})_{xx} \quad \text{in} \quad \mathscr{D}'(\mathrm{Q}), \\ u(.,t) &\to u_{\varepsilon 0} \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}) \quad \text{as} \quad t \downarrow 0. \end{aligned}$$

This follows from the known existence and uniqueness theory for the porous medium equation, cf. [OKC] [AC] [BCP] [DK]. If we recover the pressure v_ε by inverting the change of variables, i. e.

$$v_\varepsilon(x,t) = \frac{\varepsilon + 1}{\varepsilon} u(x,t)^\varepsilon,$$

then v_ε is a solution of (P_ε):

$$v_t = \varepsilon v v_{xx} + (v_x)^2 \quad \text{in} \quad Q, \tag{1.2 a}$$

$$v(x, 0) = v_{\varepsilon 0}(x) \quad \text{in} \quad \mathbb{R}. \tag{1.2 b}$$

in the sense that v is continuous, nonnegative and bounded in Q, and satisfies

$$\iint_Q \{ \varepsilon v v_x \psi_x + (\varepsilon - 1)(v_x)^2 \psi - v \psi_t \} = \int_{\mathbb{R}} v_{0_\varepsilon}(x) \psi(x, 0) dx$$

for every test function $\psi \in C^\infty(\overline{Q})$, cf. [A2]. Moreover u_ε and v_ε are C^∞ functions on the open subset of Q where they are positive.

The solutions satisfy the maximum principle: if $v_\varepsilon^{(1)}$ and $v_\varepsilon^{(2)}$ are two solutions of (P_ε) with initial data ${v_{\varepsilon 0}}^{(1)}$ and ${v_{\varepsilon 0}}^{(2)}$ and ${v_{\varepsilon 0}}^{(1)} \geqslant {v_{\varepsilon 0}}^{(2)}$ a. e. then $v_\varepsilon^{(1)} \geqslant v_\varepsilon^{(2)}$ everywhere in Q, cf. [OKC] [BCP] [DK]. From this we obtain our first estimate.

LEMMA 1.1. — *For every $(x, t) \in Q$ we have*

$$0 \leqslant v_\varepsilon(x, t) \leqslant \| v_{\varepsilon 0} \|_{L^\infty(\mathbb{R})} \leqslant N. \tag{1.3}$$

The following remark also follows from the maximum principle. If $x_0 \in \mathbb{R}$ is such that ess lim inf $v_{0\varepsilon}(x)$ is positive as $x \to x_0$ then for every $t > 0$ $v_\varepsilon(x_0, t) > 0$. (To prove this compare with a small Barenblatt solution centered at $x = x_0$). Using this remark we conclude that if $v_{\varepsilon 0}(x)$ is C^∞ and positive everywhere then u_ε is C^∞ in Q, cf. [OKC]. Since the solutions u_ε depend continuously on the initial data $u_{\varepsilon 0}$, ([BC1] [BCP] [DK]), every solution can be suitably approximated by C^∞ solutions.

Our next estimates are taken from [AB]:

LEMMA 1.2. — i) *For every $t > 0$, $v_{\varepsilon xx}(., t)$ is a locally bounded measure on $\mathbb{R}$ and*

$$v_{\varepsilon xx}(., t) \geqslant -\frac{1}{(\varepsilon + 2)t} \quad \text{in} \quad \mathscr{D}'(\mathbb{R}). \tag{1.4}$$

ii)

$$v_{\varepsilon t}(., t) \geqslant -\frac{\varepsilon v(., t)}{(\varepsilon + 2)t} \quad \text{in} \quad \mathscr{D}'(\mathbb{R}). \tag{1.5}$$

We remark that both estimates are sharp as can be seen by checking them on the Barenblatt solutions.

It was proved in [A1] that the velocity $-v_{\varepsilon x}$ is bounded in $Q_\tau = \{ (x, t) \in Q : t > \tau \}$ for every $\tau > 0$. We give a new and simple proof of this bound using

Lemmas 1.1 and 1.2, which exhibits the explicit dependence of the bound on N, t and ε:

LEMMA 1.3. — *For every $t > 0$ the function $v_\varepsilon(.,t)$ is Lipschitz continuous and satisfies*

$$| v_{\varepsilon x}(x,t)|^2 \leqslant \frac{2}{(\varepsilon+2)t} \| v(.,t) \|_\infty \leqslant \frac{2N}{(\varepsilon+2)t} \tag{1.6}$$

a. e. in Q.

REMARK. — Again the bound is attained by the Barenblatt solution.

Proof. — The argument proceeds at fixed time $t > 0$ by applying the estimates $0 \leqslant v_\varepsilon \leqslant N$ and $v_{\varepsilon xx} \geqslant -((\varepsilon+2)t)^{-1}$ to the function $x \to v(x,t)$.

For $t > 0$ and $y \in \mathbb{R}$ we define

$$\phi(x) = v(x+y,t) + \frac{x^2}{2(\varepsilon+2)t}.$$

Then ϕ is continuous, nonnegative and convex ($\phi'' \geqslant 0$ in $\mathscr{D}'(\mathbb{R})$). Therefore for every $h > 0$ we have

$$\phi(x \pm h) \geqslant \phi(x \pm h) - \phi(x) \geqslant \pm \phi'(x)h$$

Assume that $\phi'(x) \neq 0$. In the above inequality take the sign which gives $|\phi'(x)|h$ in the right-hand member to get

$$|\phi'(x)| \leqslant \frac{1}{h} \| \phi \|_{L^\infty(x-h,x+h)}.$$

Letting $x = 0$ and taking into account the definition of ϕ this means that for every $y \in \mathbb{R}$

$$| v_{\varepsilon x}(y,t)| \leqslant \frac{1}{h} \left(\| v_\varepsilon(.,t) \|_{L^\infty(\mathbb{R})} + \frac{1}{2(\varepsilon+2)t} h^2 \right).$$

The result now follows by choosing h so as to minimize the right-hand member of the inequality, i. e.

$$h^2 = 2(\varepsilon+2)t \| v_\varepsilon(.,t) \|_{L^\infty(\mathbb{R})}. \qquad \#$$

In view of Lemma 1.3 the family $\{ v_\varepsilon \}$ is uniformly Lipschitz continuous with respect to x in Q_τ for $\tau > 0$ with Lipschitz constant $L_\tau = (2N/(\varepsilon+2)\tau)^{1/2}$ uniform in ε. Using the results of [G] the solutions are then Hölder-continuous in Q_τ with respect to the variable t with exponent 1/2. We give a simple direct proof of this fact that shows the dependence on ε.

LEMMA 1.4. — *The family $\{ v_\varepsilon \}$ is uniformly Hölder-continuous with* *respect to t in* Q_τ, $\tau > 0$, *with exponent 1/2. More precisely, for every* $x \in \mathbb{R}$ *and* $t_1 \geqslant t_0 \geqslant \tau > 0$ *we have*

$$| v_\varepsilon(x, t_1) - v_\varepsilon(x_0, t_0) | \leqslant A\varepsilon^{1/2}N^{1/2}L_\tau(t_1 - t_0)^{1/2} + BL_\tau^2(t_1 - t_0) \quad (1.7)$$

for some constants A, B > 0 *independent of* ε, N, L_τ.

Proof. — We may assume that $v_\varepsilon \in C^\infty(Q)$. For convenience we temporarily drop the subscript ε. If x, t_1, t_0 and τ are as above, it follows from $| v_x | \leqslant L_\tau$ in Q_τ that for every $y \in \mathbb{R}$ such that $| y - x | \leqslant \lambda$ we have

$$| v(y, t_i) - v(x, t_i) | \leqslant L \, | y - x | \leqslant L\lambda, \quad (1.8)$$

where $L = L_\tau$. We want to estimate $h = v(x, t_1) - v(x, t_0)$ in terms of $\delta = t_1 - t_0$. Suppose, for example, that $h > 0$. Then take $\lambda < h/(2L)$ and set $I = [x - \lambda, x + \lambda]$ and $S = I \times [t_0, t_1]$. Integrating the equation $v_t = \varepsilon(vv_x)_x + (1 - \varepsilon)v_x^2$ in S we obtain

$$\int_I (v(y, t_1) - v(y, t_0))dy = \int_I dy \int_{I_0}^{t_1} v_t(y, t)dt =$$
$$= \varepsilon \int_{t_0}^{t_1} \{ (vv_x)(x + \lambda, t) - (vv_x)(x - \lambda, t) \} \, dt + (1 - \varepsilon) \iint_S (v_x)^2(y, t)dydt.$$

In view of (1.8) we have

$$v(y, t_1) - v(y, t_0) \geqslant v(x, t_1) - v(x, t_0) - 2L\lambda = h - 2\lambda L.$$

Therefore, using Lemmas 1 and 2, we find

$$2(h - 2\lambda L)\lambda \leqslant 2\varepsilon N\delta L + 2(1 - \varepsilon)L^2\lambda\delta.$$

Now if we set $\lambda = \dfrac{h}{4L}$ we get

$$h^2 \leqslant 8\varepsilon NL^2\delta + 2(1 - \varepsilon)L^2h\delta,$$

from which it follows that

$$h \leqslant 2(1 - \varepsilon)L^2\delta + (8\varepsilon)^{1/2}N^{1/2}L\delta^{1/2}, \quad (1.9)$$

and the assertion is proved. #

In Lemma 1.3 we derived a bound for $v_{\varepsilon x}$ which does not depend on smoothness of the initial data but which is only useful for t bounded away from zero. We shall also need a bound which is valid down to $t = 0$. For this we employ the Bernstein method as in reference [A1].

LEMMA 1.5. — *Suppose that* $v'_{\varepsilon 0}$ *is bounded in some interval* $(a, b) \in \mathbb{R}$. *For any* $T \in \mathbb{R}^+$ *and* $\delta \in \left(0, \dfrac{b-a}{2}\right)$ *let* $R = (a, b) \times (0, T]$ *and* $\mathbb{R}^* = (a + \delta,$

$b-\delta)\times(0,\mathrm{T}]$. *There exists a constant* $\mathrm{C}>0$ *independent of* a, b, ε, δ, T *and* $v_{\varepsilon 0}$ *such that*

$$|v_{\varepsilon x}(x,t)| \leqslant 2\,\|v'_{\varepsilon 0}\|_{\mathrm{L}^\infty(a,b)} + \mathrm{C}\delta^{-1}\,\|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R})} \tag{1.10}$$

for all $(x,t)\in\overline{\mathbb{R}}^*$.

Proof. — We may assume that v_0 is positive and smooth in R. Set

$$\phi(r) = \mathrm{M}r(4-r)/3$$

where $\mathrm{M} = \|v_\varepsilon\|_{\mathrm{L}^\infty(\mathbb{R})}$ and define w by $v_\varepsilon = \phi(w)$. Note that $w\in[0,1]$. Let $p = w_x$ and $z = \zeta^2 p^2$, where $\zeta = \zeta(x)$ is a $\mathrm{C}^\infty([a,b])$ function with values in $[0,1]$ which vanishes in a neighborhood of $x=a$ and $x=b$. Then, as in [A1], at points of R where z has a relative maximum we have

$$\begin{aligned} &-\{m\phi'' + (m-1)\phi(\phi''/\phi')'\}\,p^4\zeta^2 \\ &\leqslant -\left\{(m+1)\phi' + 2(m-1)\phi\,\frac{\phi''}{\phi'}\right\}p^3\zeta\zeta' \\ &+\{2(m-1)\phi\,|\zeta'|^2 - (m-1)\phi\zeta\zeta''\}\,p^2. \end{aligned} \tag{1.11}$$

Now let $\bar\zeta = \bar\zeta(x)$ be a $\mathrm{C}^\infty(\mathbb{R})$ function with $\bar\zeta(0)=1$ and $\bar\zeta = 0$ for all $|x|\geqslant 1$ such that $\bar\zeta\in[0,1]$, $|\bar\zeta'|\leqslant 2$, and $|\bar\zeta''|\leqslant 4$ in $\mathbb{R}$. For an arbitrary fixed $\bar x\in[a+\delta,b-\delta]$ set $\zeta(x) = \bar\zeta\left(\frac{x-\bar x}{\delta}\right)$. Then substituting in (1.11) and taking into account the estimates $\frac{2\mathrm{M}}{3}\leqslant\phi'\leqslant\frac{4\mathrm{M}}{3}$, $\phi'' = -\frac{2\mathrm{M}}{3}$, $\left|\frac{\phi''}{\phi'}\right|\leqslant 1$ and $\left(\frac{\phi''}{\phi'}\right)'\leqslant 0$ we obtain

$$\zeta^2p^4 \leqslant \mathrm{C}_1\delta^{-2}p^2 + \mathrm{C}_2\delta^{-1}\zeta\,|p|^3,$$

where the constants C_1 and C_2 are independent of a, b, ε, δ, M and T. Since $2\mathrm{C}_2\delta^{-1}\zeta p \leqslant \zeta^2p^2 + \mathrm{C}_2^2\delta^{-2}$ we conclude that

$$\max z \leqslant (2\mathrm{C}_1 + \mathrm{C}_2^2)\delta^{-2} \equiv \mathrm{C}_3^2\delta^{-2}. \tag{1.12}$$

Suppose now that

$$\|v_{\varepsilon x}\|_{\mathrm{L}^\infty(\mathbb{R}^*)} > 2\,\|v'_{\varepsilon 0}\|_{\mathrm{L}^\infty(a,b)}.$$

Let $(\tilde x,\tilde t)\in\overline{\mathbb{R}}^*$ be a point where the maximum value of $|v_{\varepsilon x}|$ is achieved. Clearly $\tilde t>0$. Moreover, since $v_{\varepsilon x} = \phi'(w)w_x$ we have

$$\begin{aligned} \|w_x\|_{\mathrm{L}^\infty(\mathbb{R}^*)} \geqslant |w_x(\tilde x,\tilde t)| &> \frac{2}{|\phi'(w)|}\,\|v'_{\varepsilon 0}\|_{\mathrm{L}^\infty(a,b)} \\ &\geqslant \frac{3}{2\mathrm{M}}\,\|v'_{\varepsilon 0}\|_{\mathrm{L}^\infty(a,b)} \geqslant \|w_x(.,0)\|_{\mathrm{L}^\infty(a,b)}. \end{aligned}$$

Therefore if $(\bar{x}, \bar{t}) \in \mathbb{R}^*$ is a point at which $|w_x|$ attains its maximum value then $t > 0$. Set

$$z(x, t) = w_x^2(x, t)\zeta\left(\frac{x-\bar{x}}{\delta}\right)$$

where ζ is the function described in the last paragraph. Since

$$\| z \|_{L^\infty(R)} \geqslant z(\bar{x}, \bar{t}) = |w_x|^2(\bar{x}, \bar{t}) = \| w_x \|^2_{L^\infty(R^*)} > \| w_x(.,0) \|^2_{L^\infty(a,b)} \geqslant \| z(.,0) \|_{L^\infty(a,b)}$$

it follows that for any point $(x, t) \in R$ where z achieves its maximum we must have $t > 0$. Thus we can apply (1.12) to conclude that

$$w_x^2(\bar{x}, t) = z(x, t) \leqslant \| z \|_{L^\infty(R)} \leqslant C_3^2\delta^{-2}.$$

Therefore

$$|v_x(x, t)| = |\phi'(w)w_x| \leqslant \frac{4}{3}MC_3\delta^{-1}$$

and (1.10) holds with $C = \frac{4}{3}C_3$.

REMARK. — In case $(a, b) = \mathbb{R}$, the maximum principle holds for $v_{\varepsilon x}$ and we have

$$|v_x(x, t)| \leqslant \| v'_{\varepsilon 0} \|_{L^\infty(\mathbb{R})}. \tag{1.13}$$

This can be proved by a slight alteration of the above argument. A proof for more general equations of the form $u_t = \phi(u)_{xx}$, with ϕ a continuous increasing real function, can be found in [V2] along with a discussion of the appropriate concept of velocity and its behaviour.

By essentially the same argument used to prove Lemma 1.4 we can derive the following consequences of Lemma 1.5.

COROLLARY 1.1. — *Under the hypothesis of Lemma 5, v_ε is Hölder continuous with respect to t in $\mathbb{R}^*$ with exponent 1/2 and Hölder constant independent of* ε.

2. PROOF OF THEOREM 1

2.1. Passage to the limit $\varepsilon \downarrow 0$.

In view of Lemma 1 the family $\{ v_\varepsilon \}$ of solutions in Theorem 1 is uniformly bounded in Q. Moreover, by Lemmas 1.3 and 1.4, $\{ v_\varepsilon \}$ is equicontinuous in Q_τ for any $\tau > 0$. Therefore there exists a sequence $\varepsilon_n \downarrow 0$ such that $v_n \equiv v_{\varepsilon_n} \to v \in C(Q)$ uniformly on Q_τ for every $\tau > 0$. It is clear from Lemmas 1 and 2 that

$$0 \leqslant v \leqslant \| v_0 \|_\infty \quad \text{in} \quad Q, \tag{2.1}$$

$$v_{xx} \geqslant -\frac{1}{2t} \quad \text{in} \quad \mathscr{D}'(Q) \tag{2.2}$$

and

$$v_t \geqslant 0 \quad \text{in} \quad \mathscr{D}'(Q).$$

By Lemma 3, v is uniformly Lipschitz continuous with respect to x in Q_τ for any $\tau > 0$. Letting $\varepsilon \to 0$ in estimate (1.7) it follows that v is also uniformly Lipschitz continuous with respect to t in Q_τ for any $\tau > 0$. Moreover, the sequence $\{ v_{nx} \}$ is relatively compact in $L^p_{loc}(Q)$. This is a consequence of the following compactness result, which is a variation of Lemma 10.1 of [Li]

LEMMA 2.1. — *Let* $\{ V_n \} \geqslant 1$ *be a sequence in* $C(Q)$ *such that on every compact subset* $Q' \subset\subset Q$ *we have*

i) $\{ V_n \}$ *converges uniformly,*

ii) $\{ V_{nx} \}$ *is bounded in* $L^\infty(Q')$,

iii) *There exists a constant* $C = C(Q') > 0$ *such that for every* $n \geqslant 1$

$$V_{nxx} \geqslant - C(Q') \quad \text{in} \quad \mathscr{D}'(Q'). \tag{2.3}$$

Then $\{ V_{nx} \}$ *is relatively compact in* $L^p_{loc}(Q)$ *for every* $p \in [1, \infty)$.

The main idea of the proof is that an estimate of the type (2.3) implies a bound for V_{nxx} in $\mathscr{M}_b(Q')$, the space of bounded measures on Q', cf. also [Li], Lemma 3.1.

In view of the relative compactness of $\{ v_{xx} \}$ we have, after passing to subsequence if necessary,

$$v_{nx} \to v_x \quad \text{in} \quad L^p_{loc}(Q) \quad \text{and} \quad \text{a. e.} \tag{2.4}$$

Since the v_n satisfy

$$\int_Q (\varepsilon_n v_n v_{nx} \psi_x + (\varepsilon_n - 1)(v_{nx})^2 \psi - v_n \psi_t) = 0 \tag{2.5}$$

for all functions $\psi \in C^1_0(Q)$, letting $n \to \infty$ we obtain

$$\int (v_x)^2 \psi + v \psi_t = 0 .$$

Thus v satisfies (0.4) $v_t = (v_x)^2$ in $\mathscr{D}'(Q)$ and almost everywhere. Since it also satisfies (2.2), it is a viscosity solution of (0.4), cf. [Li], Theorem 10.2.

We now consider the convergence at $t = 0$. Assume to begin with that $v_{0\varepsilon} = v_0 \in C^1(\mathbb{R})$ and $\| v_0' \|_\infty = L$. By Lemmas 1.4 and 1.5 and the subsequent Remark, for any $\delta > 0$ we can find a $\tau = \tau(N, L)$ such that if $\varepsilon \in (0, 1)$ we have

$$| v_\varepsilon(x, t) - v_0(x) | \leqslant \delta/2 \tag{2.6}$$

for $0 < t \leqslant \tau$ and $x \in \mathbb{R}$. Letting $\varepsilon \to 0$ we get

$$| v(x, t) - v_0(x) | \leqslant \delta/2 , \tag{2.7}$$

hence $|v_\varepsilon(x,t) - v(x,t)| \leqslant \delta$ and the convergence is uniform in x near $t = 0$. Thus in this case $v \in W^{1,\infty}(Q)$.

For general $v_{0\varepsilon}$ we use a barrier argument, taking advantage of the above result. Given $I = (a, b)$, take $\delta > 0$ and construct functions $\overline{v}_0, \underline{v}_0 \in C^1(\mathbb{R})$ with bounded derivative such that for a certain ε_0

$$0 < \underline{v}_0(x) \leqslant v_{\varepsilon 0}(x) \leqslant \overline{v}_0(x)\,, \tag{2.7}$$

$$\overline{v}_0(x) - \underline{v}_0(x) < \delta/2, \quad \text{on} \quad I\,, \tag{2.8}$$

if $x \in I$ and $0 < \varepsilon < \varepsilon_0$. Denote by $\overline{v}_\varepsilon(x,t)$, $\underline{v}_\varepsilon(x,t)$ the solutions to (P_ε) with initial data $\overline{v}_0, \underline{v}_0$ resp. and by $\overline{v}(x,t)$, $\underline{v}(x,t)$ their limits as $\varepsilon \to 0$. Using (2.6), (2.7) on $\overline{v}_0$ and $\underline{v}_0$, there exists a time $\tau > 0$ such that if $x \in I$ and $0 < t \leqslant \tau$ then

$$v_\varepsilon(x,t) \leqslant \overline{v}_\varepsilon(x,t) \leqslant \overline{v}_0(x) + \delta/2 \leqslant v_0(x) + \delta$$

and

$$v(x,t) \geqslant \underline{v}(x,t) \geqslant \underline{v}_0(x) - \delta/2 \geqslant v_0(x) - \delta\,.$$

Therefore $v_\varepsilon(x,t) - v(x,t) \leqslant 2\delta$. In a similar way we obtain $v_\varepsilon(x,t) - v(x,t) \geqslant 2\delta$. It follows that $v_\varepsilon \to v$ uniformly on compact subsets of $\overline{Q}$. We conclude that $v \in C(\overline{Q})$ and $v(x, 0) = v_0(x)$.

2.2. **Uniqueness.**

We have just shown the existence of a sequence $\{ v_{\varepsilon_n} \}$ from the family $\{ v_\varepsilon \}$ which converges to a function $v \in C(Q)$ with the properties *i*), *ii*), *iii*) of Theorem 1(A). We shall now show that the whole family $\{ v_\varepsilon \}$ converges to v as $\varepsilon \downarrow 0$ by showing the uniqueness of the solution of problem (P_0) in that class of functions. In fact, part (B) of Theorem 1 follows from the following result which extends a result of Benton [B].

PROPOSITION 2.1. — *For $i = 1, 2$ let $v_i \in C(\overline{Q})$ be solutions of $v_t = (v_x)^2$ in $\mathscr{D}'(Q)$ which satisfy in the sense of distributions in* Q

i) $$v_{ixx} \geqslant -A/t$$

and

ii) $$|v_{ix}| \leqslant B/t^{1/2}$$

for some constants A, B > 0. *For arbitrary $a, b \in \mathbb{R}$ with $a < b$, let*

$$D = \{ (x,t) \in \mathbb{R}^2 : a + 4Bt^{1/2} \leqslant x \leqslant b - 4Bt^{1/2}, 0 \leqslant t \leqslant T \}\,,$$

where $T = \{ (b-a)/8B \}^2$ *and let* $D_t = \{ x \in \mathbb{R} : (x,t) \in D \}$. *Then for every* $\mu > 0$ *the function*

$$\Phi_\mu(t) = t^{-2A} \int_{D_t} \{ (v_2 - v_1)^+ \}^\mu dx \tag{2.8}$$

is nonincreasing on $[0, T]$. *Moreover for every* $t \in [0, T]$

$$\max_{x \in D_t} \{ v_2(x, t) - v_1(x, t) \}^+ \leqslant \max_{a \leqslant x \leqslant b} \{ v_2(x, 0) - v_1(x, 0) \}^+ . \qquad (2.9)$$

Remarks. — 1) As we have shown in the proof of Lemma 3, condition *ii*) is a consequence of *i*) and bound for the v_i. Indeed if $0 \leqslant v_i \leqslant N$ then $B \leqslant \sqrt{2NA}$. Thus the essential assumption is the semiconvexity assumption *i*).

2) The conditions *i*) and *ii*) need only hold in $D \subset [a, b] \times [0, T]$.

Proof. — We begin by showing that if Φ_μ is nonincreasing for every $\mu \in \mathbb{R}^+$ then (2.9) holds. For any t and τ with $0 < \tau < t \leqslant T$, $\Phi_\mu(t) \leqslant \Phi_\mu(\tau)$ implies

$$\left[\int_{D_t} \{ (v_2 - v_1)^+ \}^\mu dx \right]^{1/\mu} \leqslant \left(\frac{t}{\tau} \right)^{2A/\mu} \left[\int_{D_\tau} \{ (v_2 - v_1)^+ \}^\mu dx \right]^{1/\mu} .$$

Thus, letting $\mu \uparrow \infty$ we obtain

$$\max_{D_t} (v_2 - v_1)^+ \leqslant \max_{D_\tau} (v_2 - v_1)^+$$

and (2.9) follows by letting $\tau \downarrow 0$.

For arbitrary field $\mu \in \mathbb{R}^+$ set $W = \{ (v_2 - v_1)^+ \}^\mu$. Then

$$W_t = \mathscr{A} W_x \quad \text{in} \quad \mathscr{D}'(Q) \qquad (2.10)$$

where $\mathscr{A} = v_{1x} + v_{2x}$ satisfies

$$| \mathscr{A} | \leqslant 2Bt^{-1/2}.$$

Assume temporarily that the $v_i \in C^2(Q)$ and write (2.10) in the form

$$W_t = (\mathscr{A} W)_x - \mathscr{A}_x W . \qquad (2.11)$$

For $\sigma, \tau \in (0, T)$ with $\sigma < \tau$ integrate over $D_\sigma^\tau \equiv \{ (x, t) \in D : \sigma \leqslant t \leqslant \tau \}$ to obtain

$$\int_{D_\tau} W dx = \int_{D_\sigma} W dx - \iint_{D_\sigma^\tau} \mathscr{A}_x W dx dt + \int_\sigma^\tau W(\mathscr{A} - g') \Big]_{x = b - g} dt$$
$$- \int_\sigma^\tau W(\mathscr{A} + g') \Big]_{x = a + g} dt ,$$

where $g = g(t) = 4Bt^{1/2}$. The third and fourth integrals on the right hand side are nonpositive since $| \mathscr{A} | \leqslant 2Bt^{-1/2} = g'$. On the other hand, by the semiconvexity condition *i*), $- \mathscr{A}_x \leqslant 2A/t$. Therefore if we set

$$f(t) = \int_{D_t} W dx ,$$

then

$$f(\tau) \leqslant f(\sigma) + 2A \int_\sigma^\tau \frac{1}{t} f(t) dt . \qquad (2.12)$$

We conclude that

$$f(\tau) \leqslant \left(\frac{\tau}{\sigma}\right)^{2A} f(\sigma)$$

so that $\Phi_\mu(t) = t^{-2A} f(t)$ is nonincreasing on $(0, T)$. By continuity, the same holds on $[0, T]$.

If the $v_i \notin C^2(Q)$ we approximate $\mathscr{A} = v_{1x} + v_{2x}$ by a sequence of $C^2(Q)$-functions $\mathscr{A}^n$ such that $|\mathscr{A}^n| \leqslant 2Bt^{-1/2}$, $\mathscr{A}^n_x \geqslant -2At^{-1}$ and $\mathscr{A}^n \to \mathscr{A}$ in $L^1_{loc}(Q)$. Write (2.10) in the form

$$W_t = (\mathscr{A}^n W)_x - \mathscr{A}^n_x W + (\mathscr{A} - \mathscr{A}^n) W_x .$$

Since

$$\left| \iint_{D^\tau_\sigma} (\mathscr{A} - \mathscr{A}^n) W_x dx dt \right| \leqslant (2B\sigma^{-1/2}) C \| \mathscr{A} - \mathscr{A}^n \|_{L^1(D^\tau_\sigma)} ,$$

arguing as above we find

$$f(\tau) \leqslant f(\sigma) + 2A \int_\sigma^\tau \frac{1}{t} f(t) dt + (2B\sigma^{-1/2}) C \| \mathscr{A} - \mathscr{A}^n \|_{L^1(D^\tau_\sigma)} .$$

from which we derive (2.12) by letting $n \to \infty$.

3. CONVERGENCE OF THE INTERFACE

In this section we consider problem (P_ε) with a fixed initial datum v_0 which we assume to be bounded, nonnegative and continuous. In addition, we assume that v_0 vanishes on $\mathbb{R}^+$. Without loss of generality, we may assume that $0 = \sup \{ x : v_0(x) > 0 \}$. The right-hand interface of v_ε is then $x = \zeta_\varepsilon(t)$, where

$$\zeta_\varepsilon(t) = \sup \{ x \in \mathbb{R} : v_\varepsilon(x, t) > 0 \}, \quad 0 \leqslant t < \infty . \tag{3.1a}$$

It is known that ζ_ε is a continuous, nondecreasing function in $[0, \infty)$. We shall prove that as $\varepsilon \downarrow 0$, $\zeta_\varepsilon(t)$ converges to the right-hand interface $\zeta(t)$ of the solution v of (0.4) obtained as a limit of v_ε, i. e., to

$$\zeta(t) = \sup \{ x \in \mathbb{R} : v(x, t) > 0 \}, \quad 0 \leqslant t < \infty . \tag{3.1b}$$

Before stating our main result we summarize the relevant properties of $\zeta_\varepsilon(t)$ and $\zeta(t)$. For every $\varepsilon > 0$ and $\tau > 0$, $\zeta_\varepsilon \in C[0, \infty) \cap \mathrm{Lip}\,[\tau, \infty)$ and there exists a waiting time $\tau^*_\varepsilon \in [0, \infty)$ such that $\zeta_\varepsilon(t) = 0$ for $0 \leqslant t \leqslant t^*_\varepsilon$, and $\zeta_\varepsilon \in C^1(t^*_\varepsilon, \infty)$ with $\zeta'_\varepsilon(t) > 0$ if $t > t^*_\varepsilon$. It is shown in [V3] that

$$T_\varepsilon/(B_\varepsilon)^\varepsilon \leqslant t^*_\varepsilon \leqslant \mu_\varepsilon T_\varepsilon/(B_\varepsilon)^\varepsilon , \tag{3.2}$$

where

$$B_\varepsilon = B_\varepsilon(u_0) = \sup_{x<0}\left\{ |x|^{-\frac{\varepsilon+2}{\varepsilon}} \int_x^0 u_0(\xi)d\xi \right\},$$

$$T_\varepsilon = \frac{1}{2(\varepsilon+1)}\left(\frac{\varepsilon}{\varepsilon+2}\right)^{\varepsilon+1},$$

and μ_ε is a constant such that $\mu_\varepsilon > 1$, $\mu_\varepsilon \to 1$ as $\varepsilon \downarrow 0$. Note that $B_\varepsilon < \infty$ is the necessary and sufficient condition to have positive t_ε^*. The equation (0.7), $\zeta_\varepsilon'(t) = -v_{\varepsilon,x}(\zeta_\varepsilon(t), t)$, is satisfied on the interface if $t \neq t_\varepsilon^*$, [CF]. Moreover we have ([V1])

$$\zeta_\varepsilon'' + \frac{\varepsilon+1}{(\varepsilon+2)t}\zeta_\varepsilon' \geqslant 0 \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^+). \tag{3.3}$$

(3.3) means that the function $\zeta'(t)t^{\frac{\varepsilon+1}{\varepsilon+2}}$ is nondecreasing in $[0, \infty)$. It follows from (0.7) and (1.6) that

$$\zeta_\varepsilon'' \geqslant -\left(\frac{\varepsilon+1}{\varepsilon+2}\right)\left(\frac{2}{\varepsilon+2}\right)^{1/2} N^{1/2}t^{-3/2}. \tag{3.4}$$

As for $\zeta(t)$ it is well-known that $\zeta \in C[0, \infty) \cap \operatorname{Lip}[\tau, \infty)$ for every $\tau > 0$, that (0.7) holds, and that ζ' is continuous except for an at most countable number of times t_i at which the one-sided derivatives $D^+\zeta(t_i)$ and $D^-\zeta(t_i)$ exist and satisfy $D^+\zeta(t_i) > D^-\zeta(t_i)$, ([D]). These properties of ζ also follow from our results below.

The following is an expanded version of Theorem 2.

THEOREM 2'. — A) *As $\varepsilon \downarrow 0$ the family $\{\zeta_\varepsilon(t)\}$ converges uniformly on compact subsets of $[0, \infty)$ to the interface $\zeta(t)$ of the function $v = \lim v_\varepsilon$.*

B) *Moreover $\zeta_\varepsilon' \to \zeta'$ in $L^p_{\text{loc}}(\mathbb{R}^+)$ for every $1 \leqslant p < \infty$ and ζ satisfies*

$$\zeta''(t) + \frac{1}{2t}\zeta'(t) \geqslant 0 \quad \textit{in} \quad \mathscr{D}'(\mathbb{R}^+). \tag{3.3'}$$

C) *The waiting time t^* of ζ_ε converges to the waiting time t^* of ζ given by*

$$t^* = 1/4B,$$

where

$$B = \sup_{x<0}\{|x|^{-2}v_0(x)\}.$$

Proof. — A) Let $N = \| v_0 \|_{L^\infty(\mathbb{R})}$. We claim that

$$0 \leqslant \zeta_\varepsilon(t) \leqslant 2\sqrt{Nt}. \tag{3.5}$$

To prove this we compare v_ε with the function

$$\bar{v}(x,t) = \begin{cases} \mathrm{N} & \text{in} \quad \mathbb{R}^- \times \mathbb{R}^+, \\ \mathrm{N} - x^2/4t & \text{in} \quad [0, 2\sqrt{\mathrm{N}t}) \times \mathbb{R}^+, \\ 0 & \text{in} \quad [2\sqrt{\mathrm{N}t}, \infty) \times \mathbb{R}^+. \end{cases}$$

Observe that $\bar{v}_t = (\bar{v}_x)^2$ in Q with initial values

$$v(x,0) = \begin{cases} \mathrm{N} & \text{for} \quad x \in \mathbb{R}^-, \\ 0 & \text{for} \quad x \in \mathbb{R}^+. \end{cases}$$

Moreover, in the support $\bar{\Omega}$ of $\bar{v}$ we have

$$\bar{v}_{xx} \leqslant 0 \quad \text{in} \quad \mathscr{D}'(\bar{\Omega}). \tag{3.6 a}$$

and in $\{ (x,t) \in \mathrm{Q} : 0 < x < 2\sqrt{\mathrm{N}t} \}$ we have precisely

$$\bar{v}_{xx} = -\frac{1}{2t}. \tag{3.6 b}$$

Therefore in $\bar{\Omega}$, $\bar{v}$ satisfies

$$\bar{v}_t - \varepsilon \bar{v} \bar{v}_{xx} - (\bar{v}_x)^2 = -\varepsilon \bar{v} \bar{v}_{xx} \geqslant 0$$

while for every $\varepsilon > 0$

$$v_{\varepsilon t} - \varepsilon v_\varepsilon v_{\varepsilon xx} - \varepsilon (v_x)^2 = 0 \tag{3.7 b}$$

in Q. Since $\bar{v}(x,0) \geqslant v_0(x)$ in $\mathbb{R}$ it follows from the maximum principle that $\bar{v} \geqslant v_\varepsilon$ in Q and this implies (3.5).

Since the solutions are not smooth at the interfaces the maximum principle cannot be applied directly and an auxiliary argument is necessary. Fix $\varepsilon > 0$ and let $w(x,t) = \bar{v}(x - \delta, t)$ with $\delta > 0$. We shall prove that for every t $\zeta_\varepsilon(t) < \bar{\zeta}(t) + \delta$ and $v \leqslant w$ in Q. Let $t_1 = \inf \{ t > 0 : \zeta_\varepsilon(t) \geqslant \bar{\zeta}(t) + \delta \}$. Clearly $0 < t_1 \leqslant \infty$. We consider now the region $\mathrm{S} = \{ (x,t) \in \Omega_\varepsilon : 0 < t \leqslant t_1 \}$. Since $\bar{\Omega} \supset \mathrm{S}$ it follows from the standard maximum principle that $v_\varepsilon \leqslant w$ in S. Therefore if $t_1 < \infty$ we have $v(.,t_1) \leqslant w(.,t_1)$ in $(-\infty, \zeta(t_1))$ and hence in $\mathbb{R}$. On the other hand at the point (x_1, t_1) with $x_1 = \zeta_\varepsilon(t_1) = \zeta(t_1) + \delta$ we have

i) $v_\varepsilon(x_1, t_1) = w(x_1, t_1)$,

ii) $v_{\varepsilon x}(x_1 -, t_1) = -\zeta'_\varepsilon(t_1) \leqslant -\bar{\zeta}'(t_1) = w_x(x_1 -, t_1)$

and

iii) $v_{\varepsilon xx}(x, t_1) = -\dfrac{1}{(\varepsilon + 2)t_1} > w_{xx}(x, t_1) = -\dfrac{1}{2t_1}$ if $0 < x < x_1$.

From *i*), *ii*), *iii*) it follows that $v_\varepsilon(.,t_1) > w(.,t_1)$ for $0 < x < x_1$, in contradiction to the above result. Therefore $t_1 = \infty$ and $v_\varepsilon \leqslant w$ in Q. Finally, let $\delta \downarrow 0$ to obtain $v_\varepsilon \leqslant \bar{v}$.

In view of (3.5), (0.7) and Lemma 1.3 the family $\{ \zeta_\varepsilon \}$ is uniformly bounded and equicontinuous on any compact subset on $\mathbb{R}^+$. Thus there exists a function $Z \in C(\mathbb{R}^+)$ and a sequence $\{ \varepsilon_n \}$ such that $\varepsilon_n \downarrow 0$ and $\zeta_n \equiv \zeta_{\varepsilon n} \to Z$ locally uniformly in $\mathbb{R}^+$. It is easily seen that Z satisfies (3.3′) and is locally Lipschitz continuous and nondecreasing. Moreover, (3.5) implies that $Z(0) = 0$ and Z is Hölder continuous with exponent 1/2 at $t = 0$. Therefore the convergence of ζ_n to Z is uniform on $[0, T]$ for every $T > 0$. We shall now show that Z is the right hand interface for v so that, in particular, the whole family ζ_ε converges to ζ.

Suppose that $(\bar{x}, \bar{t}) \in Q$ is such that $Z(\bar{t}) < \bar{x}$. Then $\zeta_n(\bar{t}) < \bar{x}$ for all sufficiently large n. It follows that $v_n(\bar{x}, \bar{t}) = 0$ for all sufficiently large n. Therefore $v(\bar{x}, \bar{t}) = \lim v_n(\bar{x}, \bar{t}) = 0$ so that

$$Z(\bar{t}) \geqslant \sup \{ x \in \mathbb{R} : v(x, \bar{t}) > 0 \} = \zeta(\bar{t}).$$

Since $\zeta_\varepsilon \geqslant 0$, it follows that $Z \geqslant 0$. If for some $t > 0$ we have $Z(\bar{t}) = 0$ then $Z = 0$ on $[0, \bar{t}]$ and Z is the interface for v on $[0, \bar{t}]$. Suppose now $Z(\bar{t}) > 0$ and consider an $\bar{x}$ such that

$$\frac{1}{3} Z(\bar{t}) < \bar{x} < Z(\bar{t}). \tag{3.8}$$

For sufficiently large n,

$$\bar{x} < \zeta_n(\bar{t}) = \int_0^{\bar{t}} \zeta_n'(t)dt < (\varepsilon_n + 2)\bar{t}\zeta_n'(\bar{t})$$

because of the nondecreasing nature of $\zeta_n'(t)t^{\frac{\varepsilon+1}{\varepsilon+2}}$. Therefore, in view of (0.7),

$$v_{nx}(\zeta_n(\bar{t}), \bar{t}) \leqslant - \frac{\bar{x}}{(\varepsilon_n + 2)t}.$$

By Taylor's theorem and Lemma 1.2 we have, with $\zeta_n = \zeta_n(t)$,

$$v_n(\bar{x}, \bar{t}) = v_n(\zeta_n, \bar{t}) + (\bar{x} - \zeta_n)v_{nx}(\zeta_n, \bar{t}) + \frac{1}{2}(\bar{x} - \zeta_n)^2 v_{nxx}(\cdot, t)$$

$$\geqslant \frac{(\zeta_n - x)}{(\varepsilon_n + 2)t} \left\{ \bar{x} - \frac{1}{2}(\zeta_n - \bar{x}) \right\} > 0.$$

Thus, if $\bar{x}$ satisfies (3.8) we let $n \to \infty$ to obtain,

$$v(\bar{x}, \bar{t}) \geqslant \frac{Z - \bar{x}}{2\bar{t}} \left\{ \bar{x} - \frac{1}{2}(Z - \bar{x}) \right\} > 0.$$

We conclude that $Z(\bar{t}) \leqslant \zeta(\bar{t})$.

B) In view of (0.7) and Lemma 3, the family $\{ \zeta_\varepsilon' \}$ is uniformly bounded in $[\tau, \infty)$ for any $\tau > 0$. Moreover, according to (3.4),

$$\zeta_\varepsilon'' \geqslant - \frac{N^{1/2}}{\tau^{3/2}}$$

in $[\tau, \infty)$ if $\varepsilon \leqslant 1$. It follows from Lemma 3.1 of [Li] that $\{\zeta_\varepsilon'\}$ is relatively compact in $L^p_{loc}(\mathbb{R}^+)$ for every $p \in [1, \infty)$. Thus, in particular, $\zeta_\varepsilon' \to \zeta'$ in $L^p_{loc}(\mathbb{R}^+)$ for every $p \in [1, \infty)$ and for almost every time $t > 0$.

C) By definition

$$\frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} = 2(\varepsilon + 1)\left(\frac{\varepsilon + 2}{\varepsilon}\right)^{\varepsilon+2} \left\{ \sup_{\mathbb{R}^-} |x|^{-\frac{\varepsilon+2}{\varepsilon}} \int_x^0 u_0 d\xi \right\}^\varepsilon$$
$$\cong \frac{2(\varepsilon + 2)^{\varepsilon+1}}{\varepsilon^\varepsilon} \sup_{\mathbb{R}^-} \left\{ |x|^{-2} \left(\frac{1}{|x|} \int_x^0 v_0^{\frac{1}{\varepsilon}} d\zeta \right)^\varepsilon \right\}.$$

If

$$B = \sup_{\mathbb{R}^-} \left\{ \frac{v_0(x)}{|x|^2} \right\}$$

then

$$\left(\frac{1}{|x|} \int_x^0 v_0^{\frac{1}{\varepsilon}} d\zeta \right)^\varepsilon \leqslant \frac{B|x|^2 \varepsilon^\varepsilon}{(2+\varepsilon)^\varepsilon}.$$

Therefore

$$\limsup_{\varepsilon \downarrow 0} \frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} \leqslant 4B.$$

On the other hand, for each $B_1 \in (0, B)$ there exist $x_1 = x_1(B_1) < 0$ and $\delta_1 = \delta_1(B_1) \in (0, -x_1)$ such that

$$v_0(x) \geqslant B_1 |x|^2 \quad \text{for} \quad |x - x_1| \leqslant \delta_1.$$

For $\delta \in (0, \delta_1]$ set $I_\delta = [x_1 - \delta, x_1 + \delta] \subset [x_1 - \delta, 0)$. If $x = x_1 - \delta$ then

$$|x|^{-2} \left(\frac{1}{|x|} \int_x^0 v_0^{\frac{1}{\varepsilon}} d\zeta \right)^\varepsilon \geqslant |x|^{-2} \left(\frac{1}{|x|} \int_{I_\delta} v_0^{\frac{1}{\varepsilon}} d\xi \right)^\varepsilon \geqslant \left(\frac{2\delta}{|x_1 - \delta|} \right)^\varepsilon B_1 \left| \frac{x_1 + \delta}{x_1 - \delta} \right|^2.$$

Hence

$$\frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} \geqslant \frac{2(\varepsilon + 2)^{\varepsilon+1}}{\varepsilon^\varepsilon} \left(\frac{2\delta}{|x_1 - \delta|} \right)^\varepsilon B_1 \left| \frac{x_1 + \delta}{x_1 - \delta} \right|^2,$$

which implies

$$\liminf_{\varepsilon \downarrow 0} \frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} \geqslant 4B_1 \left| \frac{x_1 + \delta}{x_1 - \delta} \right|^2.$$

Now let $\delta \downarrow 0$ and $B_1 \uparrow B$ to obtain

$$\liminf_{\varepsilon \downarrow 0} \frac{(B_\varepsilon)^\varepsilon}{T_\varepsilon} \geqslant 4B. \qquad \#$$

As a consequence of Theorem 2 we know that $\zeta'(t)t^{1/2}$ is nondecreasing and

$$\zeta_\varepsilon' \to \zeta' \tag{3.9}$$

almost everywhere in $\mathbb{R}^+$. We shall conclude this section by giving a more precise statement of (3.9). Note that it follows from the monotonicity of $\zeta'(t)t^{1/2}$, that $\zeta'(t)$ has at most jumping discontinuities with positive jumps.

THEOREM 3. — *For all $t > 0$ and every sequence $\varepsilon_m \downarrow 0$ such that $\zeta'_{\varepsilon_m}(t)$ converges we have*

$$\lim \zeta'_{\varepsilon_m}(t) \in [\zeta'(t-), \zeta'(t+)]. \tag{3.10}$$

In particular $\zeta'_\varepsilon \to \zeta'$ at every point where ζ' exists.

Proof. — Fix $t_0 \in \mathbb{R}^+$ and let $P = \zeta'(t_0+)$. We claim first that

$$\limsup_{\varepsilon\downarrow 0} \zeta'_\varepsilon(t_0) \leqslant P. \tag{3.11}$$

Suppose for contradiction that

$$\limsup_{\varepsilon\downarrow 0} \zeta'_\varepsilon(t_0) = P + 2\lambda$$

for some $\lambda > 0$. It follows from (3.3) that

$$\zeta'_\varepsilon(t) \geqslant \zeta'_\varepsilon(t_0)\left(\frac{t_0}{t}\right)^{\frac{\varepsilon+1}{\varepsilon+2}}.$$

Let $\{\varepsilon_m\}$ be a sequence such that $\zeta'_{\varepsilon_m}(t_0) \to P + 2\lambda$. Then for all sufficiently large m and $t > t_0$

$$\zeta'_{\varepsilon_m}(t) \geqslant (P + \lambda)\left(\frac{t_0}{t}\right)^{1/2}.$$

On the other hand, by the definition of P, there is a $\delta > 0$ such that $\zeta'(t) \leqslant \left(P + \frac{\lambda}{4}\right)$ for all t such that $0 < t - t_0 < \delta$. Therefore we have

$$\begin{aligned}\zeta_{\varepsilon_m}(t) \geqslant \zeta_{\varepsilon_m}(t_0) + 2(P+\lambda)t_0^{1/2}(t^{1/2} - t_0^{1/2}) &\to \zeta(t_0) + 2(P+\lambda)t^{1/2}(t^{1/2} - t_0^{1/2}) \\ &> \zeta(t_0) + \left(P + \frac{\lambda}{2}\right)(t - t_0) \geqslant \zeta(t) + \frac{\lambda}{4}(t - t_0)\end{aligned}$$

for all $t > t_0$ such that $2(P + \lambda)t_0^{1/2} > \left(P + \frac{\lambda}{4}\right)(t^{1/2} + t_0^{1/2})$. Since this contradicts the uniform convergence of $\zeta_{\varepsilon_m} \to \zeta$ we conclude that (3.11) holds. A similar argument with $t < t_0$ shows that

$$\liminf_{\varepsilon\downarrow 0} \zeta'_\varepsilon(t_0) \geqslant \zeta'(t_0-).$$

4. CONCAVE SOLUTIONS

If the initial data $v_{\varepsilon 0}$ are concave in their support then the limit function v_0 also has this property. Moreover, by the results of [GJ] and [BV] the corresponding solutions v_ε of (P_ε) an v of (P_0) are concave in their supports as functions of x for each fixed $t > 0$. In particular we have

$$0 \geqslant v_{\varepsilon xx}(.,t) \geqslant -\frac{1}{(2+\varepsilon)t}, \tag{4.1 a}$$

and

$$0 \geqslant v_{xx}(.,t) \geqslant -\frac{1}{2t} \tag{4.1 b}$$

for $t > 0$. The presence of upper estimates allows us to obtain the following convergence result:

THEOREM 4. — A) *Let $v_{\varepsilon 0}$, v_0, v_ε, v be as in Theorem 1 and assume in addition that $v_{\varepsilon 0}$ is concave on its support for each $\varepsilon > 0$. Then v_ε, $v_{\varepsilon x}$, $v_{\varepsilon t}$ converge uniformly to v, v_x, v_t resp. on compact subsets of the closure of*

$$\Omega = \{ (x,t) \in Q : v(x,t) > 0 \}.$$

B) *The interfaces $\zeta_\varepsilon(t)$ and $\zeta(t)$ corresponding to v_ε and v respectively are C^1 concave functions of t for $0 \leqslant t < \infty$ and $\zeta'_\varepsilon(t) \to \zeta'(t)$ uniformly in (τ, T) for every $\tau, T > 0$ with $\tau < T$.*

Proof. — A) To fix the ideas let us assume that each $v_{\varepsilon 0}$ vanishes outside a finite interval $I_\varepsilon = (a_\varepsilon, b_\varepsilon)$. Then the subset Ω_ε of Q where v_ε is positive has the form

$$\Omega_\varepsilon = \{ (x,t) : \zeta_\varepsilon^-(t) < x < \zeta_\varepsilon(t) \},$$

where $x = \zeta_\varepsilon^-(t)$ is the left-hand interface for v_ε.

By Theorem 2 we know that $\zeta_\varepsilon(t) \to \zeta(t)$ uniformly in $[0, T]$ for any $T > 0$. In the same way $\zeta_\varepsilon^-(t)$ converges to the left-interface $\zeta^-(t)$ of the limit function v. It is clear that v is positive in the set

$$\Omega = \{ (x,t) \in Q : \zeta^-(t) < x < \zeta(t) \}.$$

In view of Lemma 1.3, $\{ v_{\varepsilon x} \}$ is bounded uniformly in Q_τ for $\tau > 0$. It follows from (4.1 *a*) that the family $\{ v_{\varepsilon x} \}$ is Lipschitz continuous with respect to the x-variable locally in Ω and uniformly in ε. Since $w_\varepsilon = v_{\varepsilon x}$ satisfies the equation

$$w_t = \varepsilon v_\varepsilon w_{xx} + (\varepsilon + 2) v_{\varepsilon x} w_x \tag{4.2}$$

in Ω_ε, it follows from [G] that the family $\{ w_\varepsilon \}$ is Hölder continuous in t with exponent 1/2 and the Hölder constant is locally bounded in Ω independent of ε. Therefore we conclude that

$$v_{\varepsilon x} \to v_x$$

uniformly on compact sets of Ω. Finally the convergence of $v_{\varepsilon t}$ follows from $v_{\varepsilon t} - (v_{\varepsilon x})^2 = \varepsilon v_\varepsilon v_{\varepsilon xx}$ since

$$0 \geqslant \varepsilon v_\varepsilon v_{\varepsilon xx} \geqslant \frac{\varepsilon N}{(\varepsilon + 2)t}. \tag{4.3}$$

B) The uniform convergence of ζ_ε' to ζ' follows from the estimate ([BV])

$$0 \geqslant \zeta_\varepsilon''(t) \geqslant \frac{1}{(\varepsilon + 2)t}\zeta'(t). \qquad \# \tag{4.4}$$

A second result that can be obtained with concave initial data concerns monotone convergence.

THEOREM 5. — *Let $v_{\varepsilon 0}$ and v_0 be as in Theorem 4 and assume, in addition, that $v_{\varepsilon 0}(x) \uparrow v_0(x)$ for every $x \in \mathbb{R}$ as $\varepsilon \downarrow 0$. Then $v_\varepsilon \uparrow v$.*

Proof. — We want to prove that, given $\varepsilon' > \varepsilon > 0$, we have $v_{\varepsilon'} \leqslant v_\varepsilon$ everywhere in Q. The idea of the argument is the following. Consider the equation

$$L_\varepsilon(w) = w_t - \varepsilon w w_{xx} - (w_x)^2 .$$

v_ε is a smooth solution of $L_\varepsilon w = 0$ in Ω_ε, whereas $v_{\varepsilon'}$ is a subsolution in $\Omega_{\varepsilon'}$ because $L_\varepsilon(v_{\varepsilon'}) = (\varepsilon' - \varepsilon)v_{\varepsilon'}v_{\varepsilon' xx} \leqslant 0$. Since $v_{\varepsilon 0} \geqslant v_{\varepsilon' 0}$ we can use the maximum principle to conclude that $v_{\varepsilon 0} \geqslant v_{\varepsilon' 0}$.

Since the domains Ω_ε and $\Omega_{\varepsilon'}$ of $v_\varepsilon, v_{\varepsilon'}$ are not necessarily the same there is a difficulty in applying the classical maximum principle which can be overcome as follows. Assume that the support of $v_{\varepsilon' 0}$ is bounded and that we replace v_ε by the solution $\bar{v}_\varepsilon$ of $L_\varepsilon w = 0$ with initial data $w(x, 0) = v_{\varepsilon 0} + \delta$ for some $\delta > 0$. Now since v_ε is positive and C^∞ everywhere in Q and $v_{\varepsilon'}(x, t) = 0$ for large $|x|$, we easily conclude that $v_{\varepsilon'} \leqslant v_\varepsilon$ in Q. The stated result follows by approximation since solutions of the porous medium equation depend continuously on the initial data. #

As an example consider the Barenblatt solution

$$V_\varepsilon(x, t) = \frac{(r_\varepsilon(t)^2 - x^2)_+}{2(\varepsilon + 2)(t + 1)} \tag{4.5}$$

where $r_\varepsilon(t) = K(1 + t)^{1/(\varepsilon + 2)}$ and K a positive constant. As $\varepsilon \to 0$ we have

$$V_\varepsilon(x, t) \uparrow V(x, t) = 1/4\left(K^2 - \frac{x^2}{t + 1}\right)_+ ,$$

while for the interfaces we get

$$r_\varepsilon(t) \uparrow r(t) = K(1 + t)^{1/2}.$$

5. THE CASE $m < 1$

We can also consider a limit process for the solutions of porous medium equation as $m \to 1$ with $m < 1$. As noted in the Introduction, if we look at the density (i. e., if the initial data $u_{\varepsilon 0}$ converge as $m \uparrow 1$) the solutions of $u_t = (u^m)_{xx}$ converge to a solution of the heat equation.

If we look at the variable v defined as before by

$$v = \frac{m}{m-1} u^{m-1} = -\frac{m}{1-m} u^{-(1-m)} \tag{5.1}$$

we see first that $u \geqslant 0$ implies $v \leqslant 0$ and also that $u \to 0$ implies $v \to -\infty$ while $u \to \infty$ implies $v \to 0$. Moreover if we put

$$\varepsilon = 1 - m$$

then v formally satisfies the equation

$$v_t = \varepsilon \,|\, v \,|\, v_{xx} + (v_x)^2 . \tag{5.2}$$

Now if the initial datum $v(x, 0)$ is kept fixed as $\varepsilon \to 0$ we want to prove that the solutions of (5.2) converge again to a solution of equation (0.4). Following the outline of the proof of Theorem 1 we obtain a solution v_ε of (P_ε), obtain estimates for v_ε and $v_{\varepsilon x}$ and finally pass to the limit $\varepsilon \to 0$.

To begin with, it is proved in [AB] that for every $0 < m < 1$ and non-negative $u_0 \in L^1(\mathbb{R})$ there exists a unique function $u \in C[0, \infty; L^1(\mathbb{R})) \cap C^\infty(Q)$ such that

$$\begin{aligned} &u_t,\ \Delta u^m \in L^1_{loc}(Q) , \\ &u_t = \Delta u^m \quad \text{in} \quad Q , \\ &u(.,0) = u_0 . \end{aligned} \tag{5.3}$$

Moreover u is positive everywhere in Q (so that there is no interface) and the following estimates hold ([AB], [BC])

$$\frac{(1-m)v}{(m+1)t} \leqslant v_t \leqslant -\frac{v}{t}, \tag{5.4}$$

and

$$v_{xx} \geqslant -\frac{1}{(m+1)t}, \tag{5.5}$$

where v is given by (5.1). Of course v satisfies equation (5.2) in Q. Using (5.4) and (5.5) we obtain directly from (5.2) the following interesting pointwise bound for $|v_x|$:

$$|v_x|^2 \leqslant \frac{2\,|v|}{(m+1)t}. \tag{5.6}$$

It is worth noting that, as compared with (1.2 *a*) and (1.6), the estimate for v_t is bilateral and the bound for $v_x(x, t)$ depends only on the value of v at (x, t).

Solutions with general initial data $u_0 \in L^1_{loc}(\mathbb{R})$ are constructed by [HP]. Moreover they prove that uniqueness in the appropriate class holds and that when $u_0 \geqslant 0$ the estimates noted above are valid for all the solutions.

Using these estimates and arguing as in Section 2 we can prove the following result. Consider a family of initial functions $v_{\varepsilon 0} \in C(\mathbb{R})$ that satisfy

$$0 > v_{\varepsilon 0}(x) \geqslant -N \tag{5.7}$$

for some $N > 0$ and every $x \in \mathbb{R}$, and let v_ε denote the solution of the problem

$$\begin{aligned} v_t &= -\varepsilon v v_{xx} + (v_x)^2 \quad \text{in} \quad Q, \\ v(x, 0) &= v_{0\varepsilon}(x) \quad \text{in} \quad \mathbb{R}. \end{aligned} \tag{P$_\varepsilon$}$$

We have

THEOREM 6. — *Assume that* as $\varepsilon \to 0$, $\{ v_{\varepsilon 0} \}$ *converges uniformly on compact subsets of* $\mathbb{R}$ *to a function* v_0. *Then* v_ε *converges uniformly on compact subsets of* Q *to the solution* v *of the problem*

$$\begin{aligned} v_t &= (v_x)^2 \quad \textit{in} \quad Q, \\ v(x, 0) &= v_0(x) \quad \textit{in} \quad \mathbb{R}, \end{aligned} \tag{P$_0$}$$

satisfying the condition $v_{xx} \geqslant -1/(2t)$ in $\mathscr{D}'(Q)$. *Moreover* $v_{\varepsilon x} \to v_x$ *in* $L^p_{\text{loc}}(Q)$ *for every* $p \in [1, \infty)$.

Note also that (5.4) implies in the limit the following estimate for negative solutions of (P_0)

$$0 \leqslant v_t \leqslant \frac{|v|}{t}. \tag{5.8}$$

Together with (5.7) this implies that solutions of $v_t = (v_x)^2$ approach the maximum value (here $v = 0$) as $t \to \infty$ and x is fixed with a rate at most $0(1/t)$. This is exactly the rate for the self-similar solutions (0.20).

Let us finally remark that the estimates (5.4)-(5.6) are true with suitable constants for the solutions of the corresponding d-dimensional problem

$$\begin{aligned} u_t &= \Delta u^m \quad \text{in} \quad Q = \mathbb{R}^d \times (0, \infty), \\ u(.,0) &= u_0 \in L^1_{\text{loc}}(\mathbb{R}^d), \end{aligned}$$

if $u_0 \geqslant 0$ and $d(1-m) < 2$. In particular the crucial estimate (5.6) becomes

$$|\nabla v|^2 \leqslant \frac{2}{2 - d(1-m)} \cdot \frac{|v|}{t} \tag{5.6'}$$

and the proof of Theorem 6 applies essentially unchanged in several dimensions.

Regarding the connection between (0.1) and (0.4), the transformation (0.17) is still valid if $0 < m < 1$ and transforms positive solutions of (0.1) into negative supersolutions of (0.4). In particular the Barenblatt solution

$$\bar{u}(x, t) = t^{-\frac{1}{m+1}} \left(C + \frac{1-m}{2m(m+1)} \frac{x^2}{t^{2/(m+1)}} \right)^{-\frac{1}{1-m}} \tag{5.7}$$

transforms into the solution

$$V(x, t) = -\left(K + \frac{x^2}{4\tau}\right)_+ \tag{5.8}$$

with $K = Cm/(1 - m) > 0$. The case $K = 0$ in (5.8) corresponds to the special solution of (0.1) given by

$$v(x, t) = -\frac{x^2}{2(m+1)t}. \tag{5.9}$$

As in the case $m > 1$, these particular solutions represent the asymptotic behaviour of a large class of solutions, cf. [V2].

For $m = 1$ the transformations (0.5), (0.17) should be replaced by

$$v(x, t) = \log(u(x, t)), \quad \tau = t, \quad \text{and} \quad V(x, \tau) = \log(u(x, t)t^{1/2}). \tag{5.10}$$

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