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Asymptotic behaviour of minimal graphs over exterior domains

by

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ABSTRACT. — It is proved that if u is a $C^2(\mathbb{R}^n \sim \Omega)$ solution of the minimal surface equation, if Ω is bounded, and if $n \leqq 7$, then $Du(x)$ has a limit (in $\mathbb{R}^n$) as $|x| \to \infty$. This extends a result of L. Bers for the case $n = 2$. The result here is actually obtained as a special application of a more general result valid for all n.

Key-words: Minimal surface, Tangent cone at ∞.

RÉSUMÉ. — On démontre que si u est une solution $C^2(\mathbb{R}^n \sim \Omega)$ de l'équation de la surface minimale, si Ω est borné $n \leqq 7$, alors $Du(x)$ a une limite (dans $\mathbb{R}^n$) telle que $|x| \to \infty$. Ceci étend un résultat de L. Bers dans le cas $n = 2$. Notre résultat est en fait un corollaire d'un résultat plus général, valable quel que soit n.

A well known result of L. Bers [BL] says that if u is a C^2 solution of the minimal surface equation over $\mathbb{R}^2 \sim \Omega$, where Ω is a bounded open subset of $\mathbb{R}^2$, then $Du(x)$ has a limit $a \in \mathbb{R}^2$ as $|x| \to \infty$. A more geometric proof, valid for a solution u of any equation of minimal surface type, was given in [SL5].

Here we want to show that Bers' original result is also valid in dimension n, $3 \leq n \leq 7$; specifically, we shall prove

THEOREM 1. — *If u is a C^2 solution of the minimal surface equation over $\mathbb{R}^n \sim \Omega$, Ω bounded open in $\mathbb{R}^n$, $3 \leq n \leq 7$, then $Du(x)$ is bounded and has a limit as $|x| \to \infty$.*

(Since $n \geq 3$ it in fact follows from this that there is a constant b such that

$$\lim_{\rho \uparrow \infty} | u - l |_{C^2(\mathbb{R}^n \sim B_\rho)} = 0,$$

where $l(x) = a . x + b$, $a = \lim_{|x| \to \infty} Du(x)$, and $B_\rho = \{ x \in \mathbb{R}^n : |x| < \rho \}$.)

Of course in case $\Omega = \phi$, the fact that Du is bounded implies that it is constant (so that u is linear + constant), because each partial derivative $D_l u$ satisfies a uniformly elliptic divergence-form equation. (Cf. [MJ] [BDM].) Thus Theorem 1 may be viewed as an extension of this « Bernstein » result (for $\Omega = \phi$, $3 \leq n \leq 7$), which was due originally to Bernstein, Fleming, De Giorgi, Almgren, and J. Simons (see [SJ]).

We actually here derive Theorem 1 as a special consequence of a more general result, valid in all dimensions $n \geq 3$. Specifically we shall prove (in § 2, 3 below):

THEOREM 2. — *If u is a C^2 solution of the minimal surface equation on $\mathbb{R}^n \sim \Omega$, Ω bounded, then either $Du(x)$ is bounded and has a limit as $|x| \to \infty$ or else all tangent cones of graph u at ∞ are cylinders of the form $C \times \mathbb{R}$, where C is an $(n-1)$-dimensional minimizing cone in $\mathbb{R}^n$ with $\partial C = 0$ and with $0 \in$ sing C. (In particular* spt C *is not a hyperplane in this latter case.)*

For the meaning of « tangent cone at ∞ », we refer to § 1 below. Of course here spt C is the support of C and sing C (the singular set of C) is the set of points $\xi \in$ spt C such that spt $C \cap B_\sigma(\xi)$ fails to be an embedded C^2 submanifold for each $\sigma > 0$. It will also be shown in § 2, 3 that C has the form $C = \partial [\![V]\!]$, with V an open conical domain in $\mathbb{R}^n$. (That is, V is open in $\mathbb{R}^n$ and $V = \{ \lambda y : y \in V \}$ for each $\lambda > 0$.)

Notice that Theorem 1 follows immediately from Theorem 2 because there are no $(n-1)$-dimensional minimizing cones C in $\mathbb{R}^n$ with $\partial C = 0$ and $0 \in$ sing C for $3 \leq n \leq 7$. (Indeed the regularity theory for minimizing currents guarantees that sing $T = \phi$ whenever T is an $(n-1)$-dimensional mass minimizing current with $\partial T = 0$ and $n \leq 7$; see e. g. [FH, 5.3.18] or [SL1, § 37].)

§ 1. PRELIMINARIES, TANGENT CONES AT ∞

In this section $n \geq 3$ is arbitrary and throughout we assume that u is a $C^2(\mathbb{R}^n \sim \Omega)$ solution of the minimal surface equation

$$(*) \qquad \sum_{i,j=1}^{n} \left(\delta_{ij} - (1 + |\operatorname{grad} u|^2)^{-1} (D_i u)(D_j u) \right) D_i D_j u = 0,$$

with Ω a bounded open subset of $\mathbb{R}^n$.

As a preliminary result, we establish the following lemma.

1.1. LEMMA. — *Either* $|Du|$ *is bounded on* $\mathbb{R}^n \sim \Omega$ *or else*

$$\lim_{j\to\infty} (\rho_j^{-1} \sup_{B^n_{\rho_j}\sim\Omega} |u|) = \infty$$

for each sequence $\{\rho_j\} \uparrow \infty$.

(Here, and subsequently, B^n_ρ *is the open ball of radius* ρ *and centre* 0 *in* $\mathbb{R}^n$.*)*

Proof. — Suppose there is a sequence $\{\rho_j\} \uparrow \infty$ with

$$\sup_{j\geq 1} (\rho_j^{-1} \sup_{B^n_{\rho_j}\sim\Omega} |u|) < \infty .$$

By the standard gradient estimates for solutions of the minimal surface equation (the version of [SL6; Theorem 1] is particularly convenient here, because $\sup_{\partial\Omega} |Du| < \infty$ by the assumption that u is C^2 on $\mathbb{R}^n \sim \Omega$), we have

$$\sup_{j\geq 1} \sup_{B^n_{\rho_j/2}\sim\Omega} |Du| < \infty ;$$

that is, $\sup |Du| < \infty$ as required. ■

Next we note that (since (*) asserts exactly that $G = \text{graph}\, u$ has zero mean curvature) we have the formula (see [SL2] or [MS] or [AW] for discussion)

$$1.2 \qquad \int_G \sum_{i=1}^{n+1} \nabla_i \phi^i d\mathcal{H}^n = 0 ,$$

where $\nabla_i = e_i . \nabla$, ∇ = gradient operator on $G, \phi^1, \dots, \phi^{n+1} \in C^1_c(\mathbb{R}^{n+1} \sim \overline{\Omega} \times \mathbb{R})$. Notice that if ν is the upward unit normal for G and if f is C^1 in some neighbourhood of G, then

$$1.3 \qquad \nabla_i f(x) = \sum_{j=1}^{n+1} (\delta_{ij} - \nu_i(x)\nu_j(x)) D_j f(x), \qquad x \in G,$$

where $D_j f = \partial f/\partial x^j$ are the usual partial derivatives of f taken in $\mathbb{R}^{n+1}$.

We also have the standard fact (see e. g. [SL2, § 3]) that

1.4   G *is mass minimizing in* $\mathbb{R}^{n+1} \sim (\overline{\Omega} \times \mathbb{R})$,

in the sense that if we equip G with a smooth orientation, so that it becomes a multiplicity 1 current, then

$$1.5 \qquad \underline{\underline{M}}(G \llcorner W) \leq \underline{\underline{M}}(T \llcorner W)$$

for any open $W \subset\subset \mathbb{R}^{n+1} \sim (\overline{\Omega} \times \mathbb{R})$ and for any integer multiplicity locally rectifiable current T in $\mathbb{R}^{n+1}$ with $(\partial T) \llcorner W = 0$ and $\text{spt}\,(T - G) \subset\subset W$.

Next we recall that (from 1.2—see e. g. [GT, Ch. 16] and note that the arguments easily modify to take account of the fact that we need $\operatorname{spt}\phi^i \cap (\overline{\Omega} \times \mathbb{R}) = \emptyset$ in 1.2) there are the volume bounds.

1.6 $$\mathcal{H}^n(G \cap B_\rho(y)) \le c\rho^n, \qquad 1 \le \rho < \infty,$$

for suitable constant c, where $B_\rho(y)$ is the ball of radius ρ and centre y in $\mathbb{R}^{n+1}$ (ρ, y arbitrary).

Recall also that one of the versions of the monotonicity formula can be written

1.7 $$n\,\mathcal{H}^n(G \cap B_\rho \sim B_\sigma) = \rho b_\rho - \sigma b_\sigma, \qquad R_0 \le \sigma < \rho < \infty,$$

where R_0 is large enough to ensure ∂G ($\equiv$ graph $(u \mid \partial\Omega)$) $\subset B_{R_0}$ (all balls have centre 0 unless explicitly indicated otherwise), and where

$$b_\rho = \frac{d}{d\rho}\int_{G\cap B_\rho} |\nabla r|^2 d\mathcal{H}^n = \int_{G\cap\partial B_\rho} |\nabla r|\, d\mathcal{H}^{n-1},$$

with $r(x) \equiv |x|$. (The last equality follows from the co-area formula.) The identity 1.7 follows from 1.2 simply by substituting $\phi^i(x) = \psi(r)x^i$ in 1.2, and then letting ψ approach the characteristic function of the interval (σ, ρ). Notice that 1.7 (with $\sigma = R_0$) can be written (since $|\nabla r|^2 = 1 - (x.\nu)^2/r^2$)

$$\frac{d}{d\rho}(\rho^{-n}\mathcal{H}^n(G \cap B_\rho \sim B_{R_0})) = \frac{d}{d\rho}\int_{G\cap B_\rho} \frac{(x.\nu)^2}{r^{n+2}}\, d\mathcal{H}^n - \rho^{-n-1}R_0 b_{R_0}$$

(in the sense of distributions) for $\rho > R_0$, so that by integration we have, for $\rho > R \ge R_0$,

1.8 $$\rho^{-n}\mathcal{H}^n(G \cap B_\rho \sim B_{R_0}) - R^{-n}\mathcal{H}^n(G \cap B_R \sim B_{R_0}) = \int_{G\cap B_\rho \sim B_R} \frac{(x.\nu)^2}{r^{n+2}}\, d\mathcal{H}^n + E(\rho, R),$$

where

$$|E(\rho, R)| \le cR^{-n}.$$

c independent of ρ, R. (Cf. the standard monotonicity identities of [AW], [MS], [SL1].)

Now for $\lambda > 0$ we let u_λ be the scaled function $u_\lambda(x) = \lambda u(\lambda^{-1}x)$, $x \in \mathbb{R}^n \sim \Omega_\lambda$, $\Omega_\lambda = \{\lambda y : y \in \Omega\}$, and let G_λ be the graph of u_λ, so that viewing G_λ (equipped with an appropriate orientation) as a current, we may write

$$G_\lambda = \partial[\![U_\lambda]\!] \llcorner (\mathbb{R}^{n+1} \sim (\Omega_\lambda \times \mathbb{R})),$$

where $U_\lambda = \{(x, y) : y > u_\lambda(x),\ x \in \mathbb{R}^n \sim \Omega_\lambda\}$, and where $[\![U_\lambda]\!]$ denotes the current obtained by integration of $(n+1)$-forms $\omega \in \mathcal{D}^{n+1}(\mathbb{R}^{n+1})$ over U_λ.

By virtue of 1.5, 1.6 we can conclude from standard compactness results (see e. g. [FH] or [SL1, Ch. 7]) that for any sequence $\{\lambda_j\} \downarrow 0$ there is a subsequence $\{\lambda_{j'}\}$ and a current $T = \partial[\![U]\!]$ such that

$$1.9 \qquad T = \lim G_{\lambda_{j'}}$$

in the weak sense of currents, and in the sense that $U_{\lambda_{j'}}$ converges to U in the $L^1_{loc}(\mathbb{R}^{n+1})$ sense,

$$1.10 \qquad \mathscr{H}^n \llcorner G_{\lambda_{j'}} \to \mathscr{H}^n \llcorner S$$

in the sense of Radon measures on $\mathbb{R}^{n+1}$, where

$$S = \{ x \in \mathbb{R}^{n+1} : \limsup_{\rho \downarrow 0} \rho^{-n} \underline{\underline{M}}(T \llcorner B_\rho(x)) > 0 \},$$

$$1.11 \qquad G_{\lambda_{j'}} \to \operatorname{spt} T$$

locally in the Hausdorff distance sense in $\mathbb{R}^{n+1} \sim (\{0\} \times \mathbb{R})$,

$$1.12 \qquad T \text{ is minimizing in } \mathbb{R}^{n+1} \sim (\{0\} \times \mathbb{R}),$$

$$1.13 \qquad \underline{\underline{M}}(T \llcorner B_\rho(y)) \le c\rho^n, \qquad \rho > 0, \qquad y \in \mathbb{R}^{n+1}$$

(notice that this includes $y \in \{0\} \times \mathbb{R}$).

Since $n \ge 3$ it is easy to check that 1.12 and 1.13 imply

$$1.14 \qquad T \text{ is minimizing in } \mathbb{R}^{n+1}.$$

It is also standard that then (since $T = \partial[\![U]\!]$ implies that T has multiplicity 1 $\mathscr{H}^n$—a. e. in S (S as in 1.10), and since the density function of a minimizing current is upper semi-continuous—see e. g. [FH] or [SL1, Ch. 7]) $\liminf_{\rho \downarrow 0} \omega_n^{-1} \rho^{-n} \underline{\underline{M}}(T \llcorner B_\rho(y)) \ge 1$ at *each* point of spt T, and hence

$$1.15 \qquad S = \operatorname{spt} T \quad (S \text{ as in } 1.10)$$

and we can (and shall) take U to be open with

$$1.16 \qquad \operatorname{spt} T = \partial U .$$

From the De Giorgi regularity theorem (see e. g. [SL1, § 24] or [G]) we have furthermore that for each $y \in \operatorname{spt} T$ with $\lim_{\rho \downarrow 0} \omega_n^{-1} \rho^{-n} \underline{\underline{M}}(T \llcorner B_\rho(y)) = 1$ there is $\sigma > 0$ such that

1.17 $\operatorname{spt} T \cap B_\sigma(y) (= \partial U \cap B_\sigma(y))$ *is an embedded* C^∞ *submanifold of* $\mathbb{R}^{n+1}$.

This guarantees in particular that the points of sing T (i. e. the points $y \in \operatorname{spt} T$ such that 1.17 fails for each $\sigma > 0$) form a closed set of $\mathscr{H}^n$-measure zero.

Finally we note that T is a *cone*; that is, if η is any homothety $x \mapsto \lambda x$ ($\lambda > 0$ fixed), then $\eta_\# T = T$. Indeed using 1.8, 1.10, 1.15 it is easy to see that

$$\rho^{-n} \mu(B_\rho) = \sigma^{-n} \mu(B_\sigma), \qquad 0 < \sigma < \rho < \infty,$$

where $\mu = \mathscr{H}^n \llcorner \operatorname{spt} T$, and then (since T is minimizing) that an identity like 1.8 holds for T with $R_0 = 0$ and $E \equiv 0$, thus giving

$$\int_{\operatorname{spt} T} (\nu . x)^2 d\mu(x) = 0 ,$$

where ν is the unit normal of ∂U (which is well defined on $\operatorname{reg} T \equiv \operatorname{spt} T \sim \operatorname{sing} T$). The fact that $\eta_{\#} T = T$ for any homothety η now readily follows from this and the homotopy formula for currents. (See for example [SL1, Ch. 7] or [G] for similar arguments.)

Subsequently, any T obtained as described above will be called a *tangent cone for graph u at infinity*. In case $|Du|$ is bounded we can prove that there is a unique such T and it is a hyperplane. In fact we have the following result:

1.18. LEMMA. — *If* Du *is bounded* (see Lemma 1.1) *then it has a limit at infinity.*

Proof. — Since $|Du|$ is bounded, every tangent cone T of graph u at ∞ (obtained as above) is the graph of a Lipschitz weak solution of the minimal surface equation. From standard elliptic regularity theory, such solutions are smooth (see e. g. [GT, Ch. 13]). Hence since the graphs of these solutions are cones, they must all be linear functions.

It follows that for any given $\varepsilon > 0$ there is $R(\varepsilon) \geq 1$ such that if $R \geq R(\varepsilon)$ then there is a linear function l (possibly depending on R) such that

$$\sup_{B^n_{2R} \sim B^n_{R/4}} | u - l | \leq \varepsilon R .$$

Combining this with the Schauder theory [(GT, Ch. 6]), applied to $u - l$ (which we may do since $u - l$ satisfies a linear elliptic equation with coefficients having finite C^1 norm—this follows from the fact that the C^2 norm of u is finite), we deduce

$$\sup_{B^n_R \sim B^n_{R/2}} | D^2 u | = \sup_{B^n_R \sim B^n_{R/2}} | D^2(u - l) | \leq c\varepsilon/R .$$

Hence, by integration along paths in $B^n_R \sim B^n_{R/2}$, we have

(1) $$| Du(x) - Du(y) | \leq c\varepsilon \qquad \forall x, y \in B^n_R \sim B^n_{R/2} .$$

On the other hand each component $\phi = D_j u$ of Du satisfies an equation of the form

(2) $$D_i(a_{ik} D_k \phi) = 0 ,$$

where

$$a_{ik} = (1 + | Du |^2)^{-\frac{1}{2}}(\delta_{ik} - (1 + | Du |^2)^{-1}(D_i u)(D_k u))$$

(as we see by differentiating the divergence form version

$$\sum_{i=1}^{n} D_i(D_i u/\sqrt{1+|Du|^2}) = 0$$

of the minimal surface equation). In particular $D_j u$ satisfies a maximum/minimum principle on bounded domains in $\mathbb{R}^n \sim \bar{\Omega}$; then in view of the arbitrariness of ε in (1), it follows that $\lim_{j\to\infty} D_j u$ exists. ■

§ 2. TANGENT CYLINDERS AT ∞

In this section we show that, unless Du is bounded as $|x| \to \infty$, every tangent cone T of graph u at ∞ (obtained as described in § 1) is a vertical cylinder:

2.1 $$T = C \times \mathbb{R},$$

where $C = \partial[\![V]\!]$, V open in $\mathbb{R}^n$ with $\partial V = \text{spt } C$, and where C is minimizing in $\mathbb{R}^n$.

In case $\Omega = \emptyset$ this was already known (a proof appears in [MM] for example). The extension here to case $\Omega \neq \emptyset$ is given mainly for the reader's convenience, since no really new ideas are involved. Note however that the fact (observed by Fleming in case $\Omega = \emptyset$) that sing $C \neq \emptyset$ is not so easy to prove in case $\Omega \neq \emptyset$; this will be done in § 3.

We first note that (in the notation of § 1) by 1.9 and 1.15

2.2 $$\lim_{j'\to\infty} \int_{G_{\lambda_{j'}}} \nu_{j'} . \phi \, d\mathcal{H}^n = \int_{\text{spt } T} \nu . \phi \, d\mathcal{H}^n$$

for each fixed $\phi = (\phi^1, \ldots, \phi^{n+1}) \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, where ν_j is the upward pointing unit normal for G_{λ_j} and where ν is the outward pointing normal of ∂U at regular points of $\partial U = \text{spt } \mathbf{T}$. (U as in 1.16.) Thus in particular (since $\nu_j . e_{n+1} > 0$ on G_{λ_j}) we have

2.3 $$\nu . e_{n+1} \geq 0 \quad \text{on} \quad \text{reg } T.$$

We already remarked in § 1 that $\mathcal{H}^n(\text{sing } T) = 0$. We also need to recall the further regularity theory for minimizing currents $T = \partial[\![U]\!]$:

2.4 $$\begin{cases} \text{sing } T = \phi, & 3 \leq n \leq 6 \\ \text{sing } \mathbf{T} \text{ is discrete}, & n = 7 \\ \mathcal{H}^{n-7+\alpha}(\text{sing } T) = 0 & \forall \alpha > 0 \quad \text{in} \quad \text{case } n \geq 8 \end{cases}$$

(See e. g. [G] or [SL1, Ch. 7]), so that, since T is a cone, in particular

2.5
$$\begin{cases} \text{T is a hyperplane,} & n \leq 6 \\ \text{sing T} \subset \{0\}, & n = 7. \end{cases}$$

Notice that in case $n = 7$ we then also trivially have that reg T is connected; otherwise reg T $\cap$ S^n would contain smooth compact *disjoint* embedded minimal surfaces Σ_1, Σ_2, and we could rotate Σ_1 until it touched Σ_2, thus contradicting the Hopf maximum principle. (Actually reg T is connected for *all n*, by a result Bombieri and Giusti [BG].)

Next we claim, under the present assumption that $|Du|$ is not bounded, that

2.6
$$L \subset \text{spt T},$$

where L is a vertical ray from 0: either $L = \{\lambda e_{n+1} : \lambda > 0\}$ or $\{\lambda e_{n+1} : \lambda < 0\}$. To see this we note that if we let $u_j = u_{\lambda_{j'}}$, then for each fixed $\sigma > 0$,

2.7
$$\lim_{j \to \infty} \sup_{|x| = \sigma} |u_j| = \infty,$$

because otherwise Lemma 1.1 tells us that $|Du|$ is bounded on $\mathbb{R}^n \sim \Omega$, contrary to hypothesis. Thus 2.7 is established, and 2.6 clearly follows from this due to 1.11 and the fact that $\lim_{j \to \infty} \inf_{|x| > j} |u| < \infty$ by a standard barrier argument involving the catenoid.

Now we use the standard fact that $\Delta \nu . e_{n+1} + |A|^2 \nu . e_{n+1} = 0$ on reg T, where A is the second fundamental form of reg T, so that by 2.3

2.8
$$\Delta \nu . e_{n+1} \leq 0 \quad \text{on} \quad \text{reg T}.$$

In case $n \leq 7$ we can use connectedness of reg T, 2.3, 2.5, 2.8 and 2.6 (which guarantees that $\nu . e_{n+1} = 0$ at some points of reg T) to deduce by the Hopf maximum principle that $\nu . e_{n+1} \equiv 0$ on reg T. Hence, again using 2.5, we have 2.1 as required.

In case $n \geqslant 8$, the argument is only slightly more complicated: by [BG] and 2.8 we have

(*)
$$\inf_{\text{reg T} \cap B_{\rho/2}(y)} e_{n+1} . \nu \geq \rho^{-n} \int_{\text{reg T} \cap B_\rho(y)} e_{n+1} . \nu \, d\mathcal{H}^n$$

for any $\rho > 0$ and $y \in$ spt T. However we showed above that spt T contains a vertical $\frac{1}{2}$-line, and evidently $\inf_{\text{reg T} \cap B_\rho(y)} e_{n+1} . \nu = 0$ for any y in this $\frac{1}{2}$-line and any $\rho > 0$, thus by (*) $e_{n+1} . \nu \equiv 0$.

Thus we have established $\nu . e_{n+1} \equiv 0$ on reg T. Since $\partial T = 0$ it then easily follows (e. g. by using the homotopy formula for currents), that T is invariant under translations parallel to e_{n+1}. Thus (with U as in 1.16)

$U = V \times \mathbb{R}, T = C \times \mathbb{R}, C = \partial[\![V]\!]$ with V open in $\mathbb{R}^{n+1}$, V invariant under homotheties. Of course C is minimizing in $\mathbb{R}^n$, because T is minimizing in $\mathbb{R}^{n+1}$. This completes the proof of 2.1.

§ 3. PROOF OF THEOREM 2

In view of Lemmas 1.1, 1.17 and the fact 2.1, Theorem 2 of the introduction will be proved if we can establish that

3.1 $\quad \operatorname{sing} C \neq \emptyset$ (i. e. C *is not a hyperplane*)

for any C as in 2.1.

Suppose for contradiction that $C \times \mathbb{R}$, as in 2.1, is indeed a hyperplane H in $\mathbb{R}^{n+1}$. We first claim that in this case H is the unique tangent cone for graph u at ∞ and that in fact, if η is a unit normal for H, there exist $R_2 > R_1$ such that

3.2 $$G \sim B_{R_2} = \{ x + h(x)\eta : x \in H \sim B_{R_1} \} \sim B_{R_2},$$

with $h \in C^2(H \sim B_{R_1})$ satisfying

3.3 $$| h(x) | + | x | \, | Dh(x) | \leq c | x |^{1-\alpha}, \qquad x \in H \sim B_{R_1},$$

for some constant $\alpha > 0$. This is actually a special case of the general unique tangent cone result of [AA]. For a somewhat simpler proof, see [SL3, II, § 6].

Now suppose without loss of generality that $e_n = (0, 0, \ldots, 0, 1, 0)$ is normal to H (so that we can take $\eta = e_n$ in 3.2), and introduce new coordinates $(y^1, \ldots, y^{n+1})$ for $\mathbb{R}^{n+1}$ according to the transformation Q given by

$$\begin{cases} y' = x' \qquad (y' = (y^1, \ldots, y^{n-1}), \qquad x' = (x^1, \ldots, x^{n-1})) \\ y^n = x^{n+1} \\ y^{n+1} = x^n . \end{cases}$$

Then for suitable compact K and suitable R we have

$$G \sim K = Q\ (\text{graph}\ h \,|\, \mathbb{R}^n \sim B_R),$$

so that we have a diffeomorphism ψ: $\mathbb{R}^n \sim \tilde{K} \to \mathbb{R}^n \sim B^n_R$,

$$x = (x^1, \ldots, x^n) \mapsto y = (y^1, \ldots, y^n),$$

defined by $y' = x'$, $y^n = u(x', x^n)$, where $\tilde{K} = \mathbb{R}^n \sim \pi(G \sim K)$, π the projection taking $z \in \mathbb{R}^{n+1}$ onto its first n-coordinates. The inverse is given by $x' = y'$, $x^n = h(y', y^n)$, so in particular we have $\dfrac{\partial u(x)}{\partial x^n} \dfrac{\partial h(y)}{\partial y^n} \equiv 1$ for $x \in \mathbb{R}^n \sim \tilde{K}$,

$y = \psi(x)$. Assuming without loss of generality that $u(te_n) \to \infty$ (rather than $-\infty$) as $t \to \infty$, we thus deduce

3.4 $$\frac{\partial h(y)}{\partial y^n} > 0, \qquad y \in \mathbb{R}^n \sim B^n_R.$$

Similarly if $G^- = \text{graph}(-u)$, then we have, for suitable compact K_1 and $R > 0$, that

3.5 $$G^- \sim K_1 = Q(\text{graph } h^-) \qquad \text{where} \qquad h^- \in C^2(H \sim B_R).$$

Notice that, for suitably large $\rho > R$ and any $c \in \mathbb{R}$,

$$\{(y', h(y', c)) : |(y', c)| > \rho \qquad \text{and} \qquad \{(y', h^-(y', c)) : |(y', c)| > \rho\}$$

coincide with $\{x : |(x', c)| > \rho, u(x) = c\}$ and $\{x : |(x', c)| > \rho, u(x) = -c\}$ respectively, so that

3.6 $$h(y', y^n) = h^-(y', -y^n), \qquad |y| > \rho.$$

Writing

$$w(y) = h(y) - h^-(y), \qquad |y| > \rho,$$

we see that (using 3.3, 3.6 and the fact that h, h^- satisfy the minimal surface equation)

3.7 $$\Delta w = \text{div}(A.Dw), \qquad |y| > \rho,$$

where the matrix A is smooth and

3.8 $$|A| + |DA|\,|y| \leq c|y|^{-\alpha}, \qquad |y| > \rho.$$

Also by 3.4, 3.6 we have

3.9 $$\partial w(y)/\partial y^n > 0, \qquad |y| > \rho$$

and

3.10 $$w(y', y^n) = -w(y', -y^n).$$

Now let $\{t_j\} \uparrow \infty$ be arbitrary, and define

$$w_j(y) = \frac{h(t_j y)}{h(t_j e_n)}.$$

Since $w(y) > 0$ for $y^n > 0$ (by 3.9, 3.10), in view of 3.7, 3.8, 3.9, 3.10 we can use Harnack's inequality and Schauder estimates in order to deduce that there is a subsequence $\{t_{j'}\}$ such that

$$w_{j'} \to w_* \quad \text{locally in} \quad C^1 \text{ on } \mathbb{R}^n \sim \{0\},$$

where w_* is harmonic on $\mathbb{R}^n \sim \{0\}$, $\partial w_*/\partial y^n \geq 0$, and $w_*(y', y^n) = -w_*(y', -y^n)$, $y \neq 0$. Thus w_* is bounded on $B_1(0) \sim \{0\}$, and hence the singularity at 0 is removeable; that is, w_* extends to a harmonic function on $\mathbb{R}^n$. But then

$\partial w_*/\partial y^n$ extends to be a non-negative harmonic function on all of $\mathbb{R}^n$ and hence must be constant by Liouville's theorem. Thus $w_*(y', y^n) \equiv cy^n$ for some constant c. Since $w_*(e_n) = 1$ (by construction) we then have $w_*(y', y^n) \equiv y^n$.

In view of the arbitrariness of the sequence $\{ t_j \}$ in the above argument, it follows that for each given $\varepsilon > 0$ there is a $T = T(\varepsilon) \geq \rho$ such that

$$w(2te_n) \geq 2(1 - \varepsilon)w(te_n)$$

for each $t \geq T$. Taking $t = 2^j T(\varepsilon)$ for ε small, and iterating, we then deduce that for any given $\beta \in (0, 1)$ there is $c = c(\beta)$ such that

$$w(te_n) \geq ct^{1-\beta}, \qquad t \geq T.$$

However, taking $\beta < \alpha$ (α as in 3.3), this contradicts 3.3. This completes the proof of Theorem 2.

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