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Large scale oscillatory behaviour in loaded asymmetric systems

by

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RÉSUMÉ. — On considère des équations du type $u''+g(u)=s(1+\varepsilon h(t))$, où $s\neq 0$ est une constante, ε un petit paramètre, $h(t)$ une fonction 2π-périodique, et où la fonction g vérifie $g'(-\infty)\neq g'(+\infty)$. Suivant les valeurs de $g'(+\infty)$ et $g'(-\infty)$, on montre l'existence d'un grand nombre de solutions 2π-périodiques d'amplitude voisine de s.

Mots clés : Nonlinear oscillations, jumping nonlinearity, periodic solutions, Hamiltonian systems.

ABSTRACT. — We consider equations $u''+g(u)=s(1+\varepsilon h(t))$, where $s\neq 0$ is a constant, ε a small parameter, $h(t)$ a 2π-periodic function, and $g'(-\infty)\neq g'(+\infty)$. According to the values of $g'(+\infty)$ and $g'(-\infty)$, we show that there exist many 2π-periodic solutions, the amplitude of which are close to s.

Classification A.M.S. : .

1. INTRODUCTION

The purpose of this paper is to study periodic solutions of asymmetric systems under an external force consisting of a large constant component plus a small oscillatory component. The type of system satisfies the equation

$$u'' + g(u) = F(t) \tag{1.1}$$

where in general F is a 2π-periodic function and g is asymptotically asymmetric, that is, satisfies $g'(-\infty) \neq g'(+\infty)$.

The simplest of these equations is the piecewise linear homogeneous problem.

$$u'' + bu^+ - au^- = F(t) \tag{1.2}$$

A physical realization of this system is given by a particle of mass one sandwiched between two springs, but attached to neither, and allowed to move only along a straight line. If the spring constant of the first spring is a and the second is b, then the restoring force due to the two spring would be $-bu^+ + au^-$. (Unilateral springs of this type are called "rest stops" in the engineering literature.) This paper is concerned with the two cases, $a>0$, $b>0$ and $a<0$, $b>0$. The second situation is more difficult to envisage, but can be pictured as follows. A particle of mass one is allowed to move on a curve given by $y=0$ for $x>0$ and the curve $x^2 + (y+a)^2 = a^2$ for $x<0$. Gravity acts in the negative y direction. A rest stop acts to the right of the origin pushing the particle to the left with force bx if $x>0$ and not affecting the particle if x is negative. The force due to gravity will be in the negative direction, proportional to $\sin \Theta$, where Θ is the angle subtended by the particle and the origin at $(0, -a)$. For small Θ, this is approximately the distance s along the curve from the origin. Thus, we expect the particle to satisfy, for small s, the equation

$$s'' + bs^+ - as^- = F(t)$$

where F is the forcing term and $b>0$, $a<0$.

Thus equation (1.2) has simple physical realizations in either of the two situation ($a>0$, $b>0$) and ($a<0$, $b>0$).

We shall be considering the equations (1.1) and (1.2) under the influence of a forcing term of the form $F(t) = s + \varepsilon h(t)$, namely a large constant term plus a small oscillatory term $h(t)$ of period 2π. We consider the existence of 2π-periodic solutions and we give substance to the following slightly vague principle: "the greater the asymmetry of the system, the greater the number of large-amplitude oscillatory 2π-periodic solutions".

As a measure of the asymmetry of the system, we use the interval (a, b) for (1, 2) [or for (1,1) the interval $(g'(-\infty), g'(+\infty))$]. We show that the number of periodic solutions is $2n$ where n is the number of eigenvalues j^2, $j>1$ in the interval (a, b). In the case $(a<0, b>0)$ all solutions are for s positive, although for $(a>0, b>0)$ some are for s positive and some for s negative. This is made precise in the main theorem of section 3.

In section 4, we show that at least in one simple case, our theorem is sharp.

We believe these results provide a new insight into "resonance". We have three essential ingredients, (a) a sufficiently large asymmetry in the system, (b) a large loading term and (c) a small oscillatory term. We show that in the absence of damping these three ingredients give rise to large oscillations which could not be predicted by the linear theory. Furthermore, the *magnitude* of the oscillation is that of the large load, not that of the small oscillatory term. One cannot help but be struck by the analogy to the problem of large oscillatory behaviour in suspension bridges, either under the influence of high winds (large constant terms plus small oscillatory periodic behaviour due to stall-flutter) or under the influence of soldiers marching (large constant force due to the weight of soldiers plus small periodic term due to their marching in step).

Indeed, consider the following idealization of a suspension bridge. We consider a beam of length L and a restoring force of the type bu^+. This latter force is to take account of the fact that a cable will tend to return to equilibrium if stretched, but will exert no restoring force if compressed. Consider a load which is of the form $\operatorname{Sin}\frac{\pi x}{L}(S+\varepsilon h(t))$ where S is a large constant, h is periodic and ε is small. (Thus the forcing term consists of a large uni-directional load with small oscillations.) Such a bridge will have obey the equation.

$$\begin{gathered} u_{tt}+ku_{xxxx}+bu^+=\operatorname{Sin}\frac{\pi x}{L}(S+\varepsilon h(t)) \\ u(0, t)=u(L, t)=0=u_{xx}(0, t)=u_{xx}(L, t) \end{gathered} \tag{1.3}$$

If we look for solutions of the form $y(t)\sin\frac{\pi x}{L}$, we find that y must satisfy

$$y''+\left(k\left(\frac{\pi}{L}\right)^4+b\right)y^+-k\left(\frac{\pi}{L}\right)^4y^-=S+\varepsilon h(t)$$

which is exactly of the type (1.2). Thus, large amplitude oscillations of (1.2) predict large oscillations in suspension bridges.

In a later paper, we investigate the stability properties of these solutions. In particular, we consider (1.3) with small damping, when it can be shown that large amplitude solutions exist. Preliminary computations have revealed that these solutions can be extremely stable.

This work arises from the earlier work of the authors plus D. Hart, on equation (1.1) with Dirichlet or Neumann conditions, where similar results were obtained (*see* [3], [5], [7]).

We wish to thank Ivar Ekeland for his helpful suggestions, which considerably shortened section 2.

2. PRELIMINARIES

In this section we make a geometric study of solutions of the differential equation

$$u''(t)+g(u(t))=\varepsilon h(t) \tag{2.1}$$

near a nonconstant periodic solution $u_0(t)$ of the unperturbed differential equation

$$u''(t)+g(u(t))=0 \tag{2.2}$$

where $u_0(t)$ has least period $T_0>0$. We assume that g is of class C^1 and that h is continuous and periodic with period kT_0 where k is a positive integer. Here ε is a small parameter.

The solution $u_0(t)$ is said to be nondegenerate, or $u_0(t)$ *has property* (ND), if every T_0-periodic solution of the second-order linear differential equation

$$y''(t)+g'(u_0(t))y(t)=0 \tag{2.3}$$

is of the form $cu'_0(t)$ for some number c. If $u_0(t)$ has property (ND) and $Z(t)$ is a solution of (2.3) which is not a multiple of $u'_0(t)$, then since $Z(t+T_0)$ is also a solution of (2.3), there exist constants C_1 and C_2 such that

$$Z(t+T_0) \equiv C_1 u'_0(t) + C_2 Z(t).$$

Since the Wronskian of u'_0 and Z is constant, $C_2 = 1$, and since Z is not T_0-periodic, $C_1 \neq 0$.

Since $Z(t+kT_0) \equiv Z(t) + kC_1 u'_0(t)$, it follows that *every* kT_0*-periodic solution of* (2.3) *must be of the* form $cu'_0(t)$, a fact which will be used below.

It was shown in [6] that if $g(\xi)\xi > 0$ for $\xi \neq 0$ and either g has *hardening characteristic* ($g'(\xi) > g(\xi)/\xi$ for $\xi \neq 0$) or *softening characteristic* ($g'(\xi) < g(\xi)/\xi$ for $\xi \neq 0$), then any nonconstant periodic solution of (2.2) has property (ND). In the next section we shall show that a for a certain class of *asymmetric* restoring terms g, similar to those considered in the previous section, nonconstant periodic solutions of (2.2) have property (ND). In [5] it was shown that if $g(\xi)\xi \neq 0$ for $\xi \neq 0$, g has hardening or softening characteristic, and $u_0(t)$ is a nonconstant T_0-periodic solutions of (2.2), then for any continuous T_0-periodic $h(t)$, for $|\varepsilon|$ sufficiently small, there exist at least two T_0-periodic solutions of (2.1) near translates of u_0. In this section, under the more general assumption that $u_0(t)$ has property (ND), we show that if $h(t)$ is kT_0-periodic then for $|\varepsilon|$ sufficiently small there exist at least two kT_0-periodic solutions of (2.1) near translates of u_0. As in [6] we exploit the fact that for any $t > 0$, the time t map of R^2 into R^2 associated with the first-order system corresponding to (2.1) is area preserving.

Regarding R^2 as the set of 2×1 column matrices we define

$$Y_0(t) = \mathrm{col}(u_0(t), u'_0(t)),$$
$$N(t) = \mathrm{col}(-u''_0(t), u'_0(t))$$

and let

$$C_0 = \{ Y_0(t) \mid 0 \leqq t < T_0 \}$$

Since $u_0(t)$ is nonconstant, $Y'_0(t) \neq \mathrm{col}(0,0)$ for all t. The vector $Y'_0(t)$ is tangent to C_0 at the point corresponding to $\mathrm{col}(u_0(t), u'_0(t))$ and $N(t)$ is normal to C_0 at the same point.

Using the inverse function theorem, one can show that if $s_0>0$ is sufficiently small, then the mapping

$$\operatorname{Col}(\tau, s) \rightarrow Y_0(\tau)+s N(\tau) \tag{2.4}$$

maps the strip

$$\underset{\sim}{S}=\{\operatorname{col}(\tau, s) \mid -\infty<\tau<\infty, |s|<s_0\}$$

onto open annular neighborhood of C_0, two points, (τ', s') and (τ'', s'') in $\underset{\sim}{S}$ have the same image under the mapping if and only if $s'=s''$ and $\tau'-\tau''=m T_0$ for some integer m, and the mapping restricted to a small neighborhood of a point in $\underset{\sim}{S}$ has a C^1 inverse (*see* [1], p. 350).

Let $u(t, \tau, s, \varepsilon)$ denote the solution of (2.1) such that

$$\operatorname{Col}(u(0, \tau, s, \varepsilon), u'(0, \tau, s, \varepsilon))=Y_0(\tau)+s N(\tau) \tag{2.5}$$

and let

$$Y(t, \tau, s, \varepsilon)=\operatorname{col}(u(t, \tau, s, \varepsilon), u'(t, \tau, s, \varepsilon)). \tag{2.6}$$

Since $u(t, \tau, 0, 0)=u_0(t+\tau)$, it follows that $Y(t, \tau, 0, 0)=Y_0(t+\tau)$.

From the fact that (2.4) defines a covering map (*see* [2]) and basic results concerning smooth dependence of solutions of differential equations on initial conditions and parameters we infer the existence of positive numbers ε_1 and $s_1<s_0$ and unique C^1 functions $\theta(t, \tau, s, \varepsilon)$ and $p(t, \tau, s, \varepsilon)$ defined for $|t|<2k T_0$, $-\infty<\tau<\infty$, $|s|<s_1$, and $|\varepsilon|<\varepsilon_{W1}$ such that

$$\theta(t, \tau+T_0, s, \varepsilon) \equiv \theta(t, \tau, s, \varepsilon)+T_0,\ p(t, \tau+T_0, s, \varepsilon) \equiv p(t, \tau, s, \varepsilon),$$
$$|p(t, \tau, s, \varepsilon)|<s_0,$$
$$\theta(0, \tau, s, \varepsilon)=\tau, \qquad p(0, \tau, s, \varepsilon)=s, \tag{2.7}$$
$$\theta(t, \tau, 0, 0)=t+\tau, \qquad p(t, \tau, 0, 0) \equiv 0, \tag{2.8}$$

and

$$Y(t, \tau, s, \varepsilon)=Y_0(\theta(t, \tau, s, \varepsilon))+p(t, \tau, s, \varepsilon) N(\theta(t, \tau, s, \varepsilon)) \tag{2.9}$$

LEMMA 2.1. — *Let $u_0(t)$ have property* (ND). *Then there exists $\varepsilon_0>0$ and s^*, $0<s^*<s_1$, and a continuous function $\bar{s}(\tau, \varepsilon)$ defined for $-\infty<\tau<\infty$ and $|\varepsilon|<\varepsilon_0$, such that $|\bar{s}(\tau, \varepsilon)|<s^*$, $\dfrac{\partial \bar{s}}{\partial \tau}$ is continuous, and*

$$\theta(k T_0, \tau, \bar{s}(\tau, \varepsilon), \varepsilon) \equiv \tau+k T_0. \tag{2.10}$$

Conversely, if $|\varepsilon| < \varepsilon_0$, $|s| < s^*$, and $\theta(k\,T_0, \tau, s, \varepsilon) = \tau + k\,T_0$, *then* $s = \bar{s}(\tau, \varepsilon)$.

Moreover, $\bar{s}(\tau + T_0, \varepsilon) \equiv \bar{s}(\tau, \varepsilon)$ and $\bar{s}(\tau, 0) \equiv 0$.

Remark. — Since $Y_0(\tau + k\,T_0) \equiv Y_0(\tau)$ and $N(\tau + k\,T_0) = N(\tau)$, it follows from (2.9) that

$$Y(k\,T_0, \tau, \bar{s}(\tau, \varepsilon), \varepsilon) = Y_0(\tau) + p(k\,T_0, \tau, \bar{s}(\tau, \varepsilon), \varepsilon)\,N(\tau).$$

Refering to (2.5) and (2.6) we see that geometrically this means that *on each sufficiently short normal to the curve* C_0 *there exists a unique point such that the solution of the system*

$$y_1' = y_2, \qquad y_2' = -g(y_1) + \varepsilon\,h(t) \tag{2.11}$$

which starts at *this point at time* $t = 0$ *returns to a point on the same normal after time* $k\,T_0$.

Proof of Lemma. — The proof is an application of compactness and the implicit function theorem.

Setting $\varepsilon = 0$ in (2.9), then differentiating with respect to s, then setting $s = 0$ and using (2.8), we find that for $|t| \leqq 2k\,T_0$, $-\infty < \tau < \infty$

$$\frac{\partial Y}{\partial s}(t, \tau, 0, 0) = \frac{\partial \theta}{\partial s}(t, \tau, 0, 0)\,Y_0'(t + \tau) + \frac{\partial p}{\partial s}(t, \tau, 0, 0)\,N(t + \tau) \tag{2.12}$$

We assert that

$$\frac{\partial \theta}{\partial s}(k\,T_0, \tau, 0, 0) \neq 0, \qquad -\infty < \tau < \infty. \tag{2.13}$$

To prove this, we suppose, on the contrary, that there exists τ_0 such that $\dfrac{\partial \theta}{\partial s}(k\,T_0, \tau_0, 0, 0) = 0$. As functions of t, the components of $Y(t, \tau, s, \varepsilon)$ satisfy the system (2.11). Setting $\mathrm{col}(y_1, y_2) = Y(t, \tau, s, 0)$ in (2.11), differentiating with respect to τ and s, then setting $\tau = \tau_0$, $s = 0$, we find that each of the vector functions

$$V(t) = \frac{\partial Y}{\partial \tau}(t, \tau_0, 0, 0), \qquad W(t) = \frac{\partial Y}{\partial s}(t, \tau_0, 0, 0)$$

is a solution of the homogeneous linear differential system

$$U'(t) = A(t)\,U(t) \tag{2.14}$$

where

$$A(t)=\begin{bmatrix} 0 & 1 \\ -g'(u_0(t+\tau_0)) & 0 \end{bmatrix}.$$

Since

$$Y(t, \tau, 0, 0)=\operatorname{col}(u_0(t+\tau_0),\ u_0'(t+\tau_0)),$$
$$V(t)=\operatorname{col}(u_0'(t+\tau_0),\ u_0''(t+\tau_0)),$$

so $V(t)$ is a $k T_0$-periodic solution of (2.14). From (2.5) and (2.6) we see that $V(0)=Y_0'(\tau_0)$ and $W(0)=N(\tau_0)$. Since $Y_0'(\tau_0)$ and $N(\tau_0)$ are nonzero and orthogonal, $V(t)$ and $W(t)$ are independent solutions of (2.14).

From the assumption that (2.13) is false at $\tau=\tau_0$ and from (2.12), we see that

$$W(k T_0)=b N(k T_0+\tau_0)=b N(\tau_0)=b W(0),$$

where $b=\frac{\partial p}{\partial s}(k T_0, \tau_0, 0, 0)$. Let $X(t)$ be the 2×2 matrix whose first and second columns are $V(t)$ and $W(t)$ respectively. Since the trace of $A(t)$ is identically zero, it follows from Liouville's theorem that the determinant of $X(t)$ is constant. Therefore, since $V(k T_0)=V(0)$ and the determinant of $X(0)$ is equal to the determinant of $X(k T_0)$, we must have $b=1$. Since this implies that $W(t)$ is $k T_0$-periodic it follows that all solutions of (2.14) are $k T_0$-periodic of equivalently all solutions of the second order differential equation

$$Z''(t)+g'(u_0(t+\tau_0))Z(t)=0 \tag{2.15}$$

are $k T_0$*-periodic. But since every solution* $y(t)$ of (2.3) is of the form $Z(t-\tau_0)$ where Z is a solution of (2.15), this contradicts the assumption that $u_0(t)$ has property (ND).

This contradiction proves the claim (2.13).

Let Z be the Banach space consisting of bounded real-valved T_0-periodic functions $z(\tau)$ defined for $-\infty<\tau<\infty$ with norm

$$|z|=\sup\{|z(\tau)|\ -\infty<\tau<\infty\}$$

and let U be the open subset of Z consisting of functions z such that $|z|<s_1$. Let $\psi: U\times(-\varepsilon_1, \varepsilon_1)\to Z$ be defined by

$$\psi(z, \varepsilon)(\tau)=\theta(k T_0, \tau, z(\tau), \varepsilon)-(\tau+k T_0).$$

We see that ψ has a continuous Frechet derivative with respect to the first variable given by

$$D_1 \psi(z, \varepsilon)(w)(\tau) = \frac{\partial \theta}{\partial s}(k T_0, \tau, z(\tau), \varepsilon) w(\tau).$$

If $\bar{0} \in Z$ denotes the function which is identically zero, then

$$D_1 \psi(\bar{0}, 0)(w)(\tau) = \frac{\partial \theta}{\partial s}(k T_0, \tau, 0, 0) w(\tau).$$

So, according to what has been shown above, $D_1 \psi(\bar{0}, 0)$ has a continuous inverse. Hence, by the implicit function theorem, there exist numbers $\varepsilon_0 > 0$ and $s^* > 0$ and a continuous mapping

$$\bar{z}: \quad (-\varepsilon_0, \varepsilon_0) \to U^* = \{ z \in Z \mid |z| < s^* \}$$

such that $\psi(\bar{z}(\varepsilon), \varepsilon) = \bar{0}$, $\bar{z}(0) = \bar{0}$, and if $\psi(z, \varepsilon) = \bar{0}$ with $|z| < S^*$ and $|\varepsilon| < \varepsilon_0$, then $z = \bar{z}(\varepsilon)$.

Setting $\bar{s}(\tau, \varepsilon) = \bar{z}(\varepsilon)(\tau)$, we have that (2.10) holds for all τ if $|\varepsilon| < \varepsilon_0$, $\bar{s}(\tau, 0) \equiv 0$, and $\bar{s}(\tau + T_0, \varepsilon) \equiv \bar{s}(\tau, \varepsilon)$. Conversely, suppose that for some τ_3, $\theta(k T_0, \tau_3, s_3, \varepsilon_3) = \tau_3 + k T_0$, where $|s_3| < s^*$ and $|\varepsilon_3| < \varepsilon_0$. Setting $z^*(\tau) = \bar{z}(\tau)$ for $\tau \neq \tau_3 + m T_0$, $m = 0, \pm 1, \pm 2, \ldots$, and $z^*(\tau_3) = s_3$, otherwise, we see that $z^* \in U^*$ and $\psi(Z^*, \varepsilon_3) = 0$. Hence $z^* = \bar{z}(\varepsilon_3)$ which means $s_3 = \bar{s}(\tau_3, \varepsilon_3)$.

By taking ε_0 and s^* smaller if necessary, we may assume that

$$\frac{\partial \theta}{\partial s}(k T_0, \tau, s, \varepsilon) \neq 0$$

for $-\infty < \tau < \infty$, $|s| < s^*$ and $|\varepsilon| < \varepsilon_0$.

Suppose that τ_2 is arbitrary and that $|\varepsilon_2| < \varepsilon_0$. From the two conditions

$$\theta(k T_0, \tau_2, \bar{s}(\tau_2, \varepsilon_2), \varepsilon_2) - (\tau_2 + k T_0) = 0,$$

$$\frac{\partial \theta}{\partial s}(k T_0, \tau_2, \bar{s}(\tau_2, \varepsilon_2), \varepsilon_2) \neq 0$$

and the classical implicit function theorem, we infer the existence of intervals I and J containing τ_2 and ε_2 respectively, with $J \subset (-\varepsilon_0, \varepsilon_0)$, and a function $\varphi: I \times J \to (-s^*, s^*)$ such that

$$\varphi(\tau_2, \varepsilon_2) = \bar{s}(\tau_2, \varepsilon_2), \qquad \theta(k T_0, ,\tau, \varphi(\tau, \varepsilon), \varepsilon) \equiv \tau + k T_0$$

for $\tau \in \mathrm{I}$, $\varepsilon \in \mathrm{J}$ and φ and $\dfrac{\partial \varphi}{\partial \tau}$ are continuous. By what we have shown above we must have $\varphi(\tau, \varepsilon)=\bar{s}(\tau, \varepsilon)$ for $(\tau, \varepsilon) \in \mathrm{I} \times \mathrm{J}$ and this establishes continuity of $\bar{s}$ and $\dfrac{\partial \bar{s}}{\partial \tau}$. This proves the lemma.

THEOREM 2.1. — *Let $u_0(t)$ be a nonconstant T_0-periodic solution of* (2.2) *which has* property (ND). *For $\alpha>0$ let $\mathrm{A}(\alpha)$ denote the annular region consisting of points of the form $\mathrm{Y}_0(\tau)+s\mathrm{N}(\tau)$ with $|s|<\alpha$, $\tau \in [0, \mathrm{T}_0]$. Given $\alpha>0$, there exists $\varepsilon^*>0$ such that for $|\varepsilon|<\varepsilon^*$ there exist at least two $k\mathrm{T}_0$-periodic solutions $u_j(t)$, $j=1, 2$ of* (2.2) *such that* $\mathrm{col}(u_j(0), u_j'(0)) \in \mathrm{A}(\alpha)$ *for $j=1, 2$.*

Proof. — Let s^*, ε_0 and $\bar{s}: \mathrm{R} \times (-\varepsilon_0, \varepsilon_0) \to (-s^*, s^*)$ be as in Lemma 2.1 and for $|\varepsilon|<\varepsilon_0$ let C_ε be the simple closed curve with representation

$$\tau \to \mathrm{Y}_0(\tau)+\bar{s}(\tau, \varepsilon)\mathrm{N}(\tau), \qquad 0 \leqq \tau < \mathrm{T}_0.$$

If $\Pi_\varepsilon: \mathrm{R}^2 \to \mathrm{R}^2$ denotes the time $k\mathrm{T}_0$ map defined by the system (2.11) and C_ε^* denotes the image of C_ε under Π_ε then, according to Lemma 2.1, C_ε^* has the representation

$$\tau \to \mathrm{Y}_0(\tau)+p(k\mathrm{T}_0, \tau, \bar{s}(\tau, \varepsilon), \varepsilon)\mathrm{N}(\tau), \quad 0 \leqq \tau < \mathrm{T}_0.$$

Suppose that $|\varepsilon|<\varepsilon_0$ and there exists a point q in $\mathrm{C}_\varepsilon \cap \mathrm{C}_\varepsilon^*$. Then

$$q=\mathrm{Y}_0(\tau_1)+\bar{s}(\tau_1, \varepsilon)\mathrm{N}(\tau_1) = \mathrm{Y}_0(\tau_2)+p(k\mathrm{T}_0, \tau_2, \bar{s}(\tau_2, \varepsilon), \varepsilon)\mathrm{N}(\tau_2).$$

Since

$$|\bar{s}(\tau_j, \varepsilon)|<s^*<s_1<s_0,$$

we have

$$|p(k\mathrm{T}_0, \tau_2, \bar{s}(\tau_2, \varepsilon), \varepsilon)|<s_0.$$

Therefore, since (2.4) is a covering map of the region $-\infty<\tau<\infty$, $|s|<s_0$ on $\mathrm{A}(s_0)$ we must have $\tau_2=\tau_1+m\mathrm{T}_0$ for some integer m. Therefore, since the functions $\mathrm{Y}(\tau)$, $\mathrm{N}(\tau)$, $p(t, \tau, s, \varepsilon)$, and $\bar{s}(\tau, \varepsilon)$ are T_0-periodic in τ, it follows that

$$q = Y_0(\tau_1) + p(k T_0, \tau_1, \bar{s}(\tau_1 \varepsilon), \varepsilon) N(\tau_1) = \Pi_\varepsilon(q).$$

Therefore, $Y(t, \tau_1, \bar{s}(\tau_1 \varepsilon), \varepsilon)$ is a $k T_0$-periodic solution of (2.11).

To prove the theorem, it is therefore sufficient to show that given $\alpha > 0$, C_ε and C_ε^* intersect in at least two points in $A(\alpha)$ if ε is sufficiently small.

Since the divergence of the time dependent vector field

$$\operatorname{col}(y_2, -g(y_1) + \varepsilon h(t))$$

is identically equal to zero, the time $k T_0$ map Π_ε associated with system (2.11) is area preserving; that is, if D is a measurable subset of the plane, then $\Pi_\varepsilon(D)$ has the same measure as D provided Π_ε is defined on D (*see* [4] for more details). If D_0 is the bounded region bounded by C_0 then Π_ε will be defined on some open set U containing $\bar{D}_0$, for $|\varepsilon|$ sufficiently small. Since $\Pi_0(C_0) = C_0$, both C_ε and C_ε^* are equal to C_0 when $\varepsilon = 0$. Let q_0 be a point in D_0 and let D_ε and D_ε^* be the bounded regions bounded by the simple closed curves C_ε and C_ε^* respectively. Since C_ε and C_ε^* depend continuously on ε, it follows (for example, by use of winding numbers) that $q_0 \in D_\varepsilon \cap D_\varepsilon^*$ for $|\varepsilon|$ sufficiently small. Therefore, by connectivity D_ε is in U and $\Pi_\varepsilon(D_\varepsilon) = D_\varepsilon^*$ for $|\varepsilon|$ sufficiently small. Since D_ε and D_ε^* have the same area this implies that $C_\varepsilon \cap C_\varepsilon^*$ contains at least two points for $|\varepsilon|$small. Since for given $\alpha > 0$, both C_ε and C_ε^* are in $A(\alpha)$ for $|\varepsilon|$ sufficiently small, this proves the theorem.

A number $\tau_0 \in [0, T_0]$ is said to be a *bifurcation value* [relative to $u_0(t)$ and (2.1)] if there exists a sequence $\{\varepsilon_n\}_1^\infty$ with $\varepsilon_n \neq 0$ for all and $\varepsilon_n \to 0$ as $n \to \infty$ and a corresponding sequence of $k T_0$-periodic functions $\{u_n(t)\}_1^\infty$ such that $u_n(t)$ is a solution of (2.1) when $\varepsilon = \varepsilon_n$ and $u_n(0) \to u_0(\tau_0)$, $u_n'(0) \to u_0'(\tau_0)$ as $n \to \infty$.

In addition to Theorem 2.1 we shall also need a result concerning bifurcation values which is known, although perhaps not stated in the following way:

THEOREM 2.2 (Loud). – *Assume that* $u_0(t)$ *has property* (ND) *and let*

$$F(\tau) = \int_0^{k T_0} u_0'(t+\tau) h(t) \, dt. \tag{2.16}$$

If

$$F(\tau_0) = 0, \qquad F'(\tau_0) \neq 0 \tag{2.17}$$

then for $|\varepsilon|$ *sufficiently small there exists a* $k T_0$*-periodic solution* $u(t, \varepsilon)$ *of* (2.1) such *that* $u(0, \varepsilon) \to u_0(\tau_0)$, $u'(0, \varepsilon) \to u_0'(\tau_0)$ as $\varepsilon \to 0$ and *there exists* *a neighborhood* U *of* $(u_0(\tau_0), u_0'(\tau_0))$, *depending on* ε *such that if* $\bar{u}(t)$ *is a* $k\,T_0$*-periodic solution of* (2.1) *with* $(\bar{u}(0), \bar{u}'(0)) \in U$ *then* $\bar{u}(t) = u(t, \varepsilon)$.

If τ_0 *is a bifurcation value then* $F(\tau_0) = 0$.

In [6] Loud, using implicit function techniques, showed that if $u^*(t)$ is a nonconstant T_0-periodic solution of (2.2) which has property (ND) and

$$\int_0^{k\,T_0} u^*(t)\,h(t)\,dt = 0, \qquad \int_0^{k\,T_0} u^{*\prime}(t)\,h(t)\,dt \neq 0$$

then for $|\varepsilon|$ small, there is a unique $k\,T_0$-periodic solution $u(t, \varepsilon)$ of (2.1) with $(u(0, \varepsilon), u'(0, \varepsilon))$ close to $(u^*(0), u^*(0))$. If u^* is the solution $u_0(t+\tau_0)$, then, from (2.16), we see that Loud's conditions reduce to the conditions (2.17).

If τ_0 is a bifurcation value relative to u_0 and (2.1) and the sequences $\{\varepsilon_n\}_1^\infty$ and $\{u_n(t)\}_1^\infty$ are as above then, by $k\,T_0$-periodicity of u_n, we have for each n

$$\int_0^{k\,T_0} u_n'(t)\,h(t)\,dt = \frac{1}{\varepsilon_n}\int_0^{k\,T_0} u_n'(t)\,[u_n''(t) + g(u_n(t))]\,dt$$
$$= \frac{1}{\varepsilon_n}\int_0^{k\,T_0} \frac{d}{dt}[u_n'(t)^2/2 + G(u_n(t))]\,dt = 0, \quad (2.18)$$

where G is an antiderivative of g. Since $u_n'(t) \to u_0'(t+\tau_0)$ uniformly on $[0, k\,T_0]$ as $n \to \infty$, letting $n \to \infty$ in (2.18) yields $F(\tau_0) = 0$. Therefore this condition is necessary.

Theorem 2.2 can also be derived as a special case of multiparameter bifurcation theory for second order periodic differential equations—*see* for example [8], Chapt. 8.

Remark. — Although it does not seem possible to derive Theorem 2.1 from Theorem 2.2, the generic case of Theorem 2.1 does follow from Theorem 2.2. In fact, the set of continuous, $k\,T_0$-periodic functions $h(t)$, for which the T_0-periodic C^1 function $F(\tau)$ in (2.16) has only simple zeros, is open and dense, with respect to the uniform topology. Since, by T_0-periodicity of $u_0(t)$ we have

$$\int_0^{T_0} F(\tau)\,d\tau = \int_0^{k\,T_0}\left(\int_0^{T_0} u_0'(t+\tau)\,d\tau\right)h(t)\,dt = 0$$

there exist numbers τ_1 and τ_2 with $0 \leqq \tau_1 < \tau_2 < T_0$ such that $F(\tau_1) = F(\tau_2) = 0$. Therefore, if F has only simple zeros, Theorem 2.2

implies that, for $|\varepsilon|$ sufficiently small, there are at least two $k\,T_0$-periodic solutions of (2.1) which are close to translates of $u_0(t)$. We are grateful to Ivar Ekeland for this observation.

Example. — Suppose that $u_0(t)$ is a nonconstant periodic solution of (2.1) with least period 2π. We show that there are exactly two bifurcation values τ_1 and τ_2 in $[0, 2\pi)$ relative to $u_0(t)$ and the differential equation

$$u''(t)+g(u(t))=\varepsilon \cos t. \tag{2.19}$$

Here, of course, $k=1$. Moreover, we show that $F'(\tau_j)\neq 0$, $j=1, 2$, where F is as in the previous lemma.

Suppose $u_0'(t_1)=0$. Then $u_0''(t_1)=-g(u_0(t_1)\neq 0$. Otherwise both $u_0(t)$ and the constant $C=u_0(t_1)$ would be solutions of 2.2 which have the same values and same derivatives at $t=t_1$, which contradicts the assumption that $u_0(t)$ is nonconstant. Therefore the zeros of $u_0'(t)$ are isolated. Let t_1 and t_2 be consecutive zeros of $u'(t)$ with $t_1<t_2$. Since $u_0(t)$ and $u_0(2t_2-t)$ are both solutions of (2.1) which have the same values and the same derivatives at $t=t_2$, $u_0(t)\equiv u_0(2t_2-t)$. Using this relation, it follows that if

$$u_1(t)\equiv u_0(t+2(t_2-t_1)),$$

then $u_1(t_1)=u_0(t_1)$ and

$$u_1'(t_1)=-u_0'(t_1)=0=u_0'(t_1)$$

and hence $u_0(t)$ is periodic with period $2(t_2-t_1)$. Since $u_0''(t_2)\neq 0$ and $u_0(t)=u_0(2t_2-t)$ we see that $u_0'(t)$ has opposite signs on the intervals (t_1, t_2) and $(t_2, 2t_2-t_1)$. Hence $2(t_2-t_1)$ is the least period of $u_0(t)$ so by assumption $2\pi=2(t_2-t_1)$. It follows that $u_0'(t+t_1)\neq 0$ for $t\in(0, \pi)$ or $t\in(\pi, 2\pi)$ and that $u_0'(t+t_1)$ changes sign at $t=\pi$. Hence

$$0\neq \int_0^{2\pi} u_0'(t+t_1)\sin t\,dt=\int_0^{2\pi} u_0'(t)\sin(t-t_1)\,dt=b\cos t_1-a\sin t_1$$

where

$$a=\int_0^{2\pi} u_0'(t)\cos t\,dt, \qquad b=\int_0^{2\pi} u_0'(t)\sin t\,dt.$$

It follows that $a^2+b^2\neq 0$.

Let δ be chosen so that if $r=\sqrt{a^2+b^2}$, then $a=r\cos\delta$, $b=r\sin\delta$. We have

$$F(\tau)=\int_0^{2\pi} u_0'(t+\tau)\cos t\,dt$$
$$=\int_0^{2\pi} u_0'(t)\cos(t-\tau)\,dt$$
$$=r\cos\tau\cos\delta+r\sin\tau\sin\delta=r\cos(\delta-\tau).$$

It follows that $F(\tau)$ has exactly two zeros on $(0, 2\pi]$, which are both simple. By Theorem 2.2, for $|\varepsilon|\neq 0$ and $|\varepsilon|$ small, (2.19) has exactly two 2π-periodic solutions near translates of $u_0(t)$.

This example will be used to establish sharpness of a result given below.

3. PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH ASYMMETRIC NONLINEARITIES

In this section we study the differential equation

$$y''(t)+f(y(t))=s(1+\varepsilon h(t)). \tag{3.1}$$

We assume that f is a C^1-function, the limits

$$\lim_{\xi\to-\infty} f'(\xi)=a,\qquad \lim_{\xi\to\infty} f'(\xi)=b \tag{3.2}$$

exist and are finite, and $a<b$. The function $h(t)$ is continuous and 2π-periodic and s and ε are constants with $|s|$ large and $|\varepsilon|$ small. Our goal is to give lower and upper bounds for the number of 2π-periodic solutions of (3.1) for suitably restricted s and ε in terms of the number of squares of integers in the interval (a, b). We consider in detail the case where $a>0$. We study (3.1) under the following assumptions.

A_1: *There exist integers p and q with $p\geqq 0$ such that*

$$p^2<a<(p+1)^2\leqq q^2<b<(q+1)^2$$

A_2: *The piece-wise linear homogeneous differential equation*

$$y''+by^+-ay^-=0 \tag{3.3}$$

has no nonconstant 2π-periodic solutions. (This is easily seen to be equivalent to the assumption that $1/\sqrt{a}+1/\sqrt{b}\neq 2/N$ for $N=1, 2, \ldots$)

We first study the autonomous differential equation

$$y''(t)+f(y(t))=s$$

in which we make the substitution $y(t)=su(t)$ where $s>0$. According to the assumptions (3.2) we may write

$$f(\xi)=b\,\xi^{+}-a\,\xi^{-}+f_0(\xi)$$

where $\lim_{|\xi|\to\infty} f_0(\xi)/\xi=0$. Since both ξ^{+} and ξ^{-} are positively homogeneous of degree one, $u(t)$ satisfies the differential equation

$$u''+bu^{+}-au^{-}+f_0(su)/s=1 \tag{3.4}$$

For large $s>0$, the function $b\,\xi^{+}-a\,\xi^{-}+f_0(s\,\xi)/s-1=0$ has a unique zero C_s and $C_s\to 1/b$ as $s\to\infty$. Since $b\,\xi^{+}-a\,\xi^{-}+f_0(s\,\xi)/\xi$ tends to $+\infty$ and $-\infty$ as ξ tends to $+\infty$ and $-\infty$ respectively, for s large and positive all solutions of (3.4) are periodic. Moreover, the trajectories of the corresponding system

$$u'=v, \qquad v'=-(bu^{+}-au^{-}+f_0(su)/s-1)$$

coincide with the level curves of the function

$$E(u,\,v)=v^2/2+U(u,\,s)$$

where

$$U'(\xi,\,s)=b\,\xi^{+}-a\,\xi^{-}+f_0(s\,\xi)/s-1$$

and

$$U(C_s,\,s)=0.$$

For $s>0$ and s large, C_s is the unique constant solution of (3.4).

LEMMA 3.1. — *Let $s^*>0$ be so large that* for $s\geqq s^*$ (3.4) *has the unique constant solution C_s and all solutions* of (3.4) *are periodic. Let $\hat{T}=\pi/\sqrt{b}+\pi/\sqrt{a}$ be the minimal period of the nonconstant solutions* of (3.3). Given *any number $\delta>0$ there exists a* number $r=r(\delta)$ *such that if $s\geqq s^*$ and $u(t)$ is a solution of* (3.4) *with $|u|_\infty\geqq r$, then the minimal period of u is greater than $\hat{T}-\delta$.*

Proof. — In the contrary case there exists a sequence of numbers $\{S_m\}_1^\infty$ with $S_m\geqq S^*$ for all m, and a corresponding sequence of functions

$\{u_m\}_1^\infty$ such that $u_m(t)$ is a solution of (3.3) when $S=S_m$, $u_m(t)$ is periodic with least period T_m which satisfies $0<T_m \leqq \hat{T}-\delta$, and $|u_m|\infty \to \infty$ as $m \to \infty$. If we set $Z_m(t)=u_m(t)/|u_m|_\infty$ then

$$Z''_m(t)+b\,Z_m(t)^+ - a\,Z_m(t)^- + \frac{f_0(s_m u_m(t))}{s_m|u_m|_\infty} = \frac{1}{|u_m|_\infty}$$

and $|Z_m|_\infty=1$. For all $m \geqq 1$. From the differential equation, we see that $|Z''_m|_\infty$ is bounded independently of m and hence $|Z'_m|_\infty$ is bounded independently of m. By Ascoli's lemma we may assume, without loss of generality, that $\lim_{m\to\infty} Z_m(t)=Z(t)$ and $\lim_{m\to\infty} Z'_m(t)=Z'(t)$ uniformly with respect to $t\in[-\hat{T}, \hat{T}]$, where $Z\in C^1[-\hat{T}, \hat{T}]$ and $|Z|_\infty=1$. From the differential equation satisfied by $Z_m(t)$ and the fact that $\lim_{|\xi|\to\infty} f_0(\xi)/\xi=0$, it follows that the sequence $\{Z''_m(t)\}_1^\infty$ converges uniformly on $[-\hat{T}, \hat{T}]$. Hence $Z\in C^2[-\hat{T}, \hat{T}]$ and

$$Z''+b\,Z^+ - a\,Z^- = 0.$$

Since $0<T_m \leqq \hat{T}-\delta$ we may assume without loss of generality that $\lim_{m\to\infty} T_m=T_0\in[0, \hat{T}-\delta]$. If $T_0=0$ then, since $Z_m(0)=Z_m(T_m)$ and $Z'_m(0)=Z'_m(T_m)$ it follows by uniform convergence and the mean value theorem that $Z'(0)=Z''(0)=0$. But this is impossible since $|Z|_\infty=1$, the zeros of a nontrivial solution of (3.3) are all simple, and $Z''(t)\neq 0$ if $Z(t)\neq 0$. Hence $0<T_0\leqq \hat{T}-\delta$. From the conditions $Z_m(0)=Z_m(T_m)$ and $Z'_m(0)=Z'_m(T_m)$ and uniform convergence it follows $Z(0)=Z(T_0)$ and $Z'(0)=Z'(T_0)$. Since $\hat{T}$ is the minimum period of $Z(t)$, this is a contradiction and the lemma is proved.

LEMMA 3.2. — *Given* $\delta>0$, *there exists* $S(\delta)$ *such that if* $s\geqq S(\delta)$, *then* (3.4) *has a nonconstant periodic solution with minimal period less* than $\dfrac{2\pi}{\sqrt{b}}+\delta$.

Proof. — For brevity we write

$$p(\xi, s)=b\,\xi^+ - a\,\xi^- + f_0(s\,\xi)/s - 1$$

For large s, $p(\xi, s)=0$ has the unique solution $\xi=C_s$ where $C_s\to 1/b$ as $s\to +\infty$. Since

$$\frac{\partial p}{\partial \xi}(c_s, s) = b + f'_0(s C_s) \to b > 0$$

as $s \to +\infty$, $(C_s, 0)$ is a stable equilibrium point of the conservative system

$$u' = v, \qquad v' = -p(u, s).$$

Therefore, if $\operatorname{col}(u(t), v(t))$ is a solution of the system with $|u(0) - C_s|$ and $|v(t)|$ small, then $|u(t) - C_s|$ and $|v(t)|$ will remain small for all t. Let col $(u(t), v(t))$ be such a solution and let $w(t) = u(t) - C_s$. We have that $w(t)$ is periodic and satisfies

$$w''(t) + Q(t) w(t) = 0 \tag{3.5}$$

where

$$Q(t) = \frac{p(w(t) + C_s, s) - p(C_s, s)}{w(t)} = \int_0^1 \frac{\partial p}{\partial \xi}(\tau w(t) + C_s, s)\, d\tau.$$

From this it follows that the maximum of $|Q(t) - b|$ for $-\infty < t < \infty$ can be made arbitrarily small if $|w(0)|$ and $|w'(0)|$ are sufficiently small and s is sufficiently large. If $w(t)$ is a nontrivial solution of (3.5), then the Sturm comparison theorem implies that the distance between two consecutive zeros of $w(t)$ is between $\pi/\sqrt{\max Q(t)}$ and $\pi/\sqrt{\min Q(t)}$. Therefore, the minimal period of $w(t)$ is between $2\pi/\sqrt{\max Q(t)}$ and $2\pi/\sqrt{\min Q(t)}$. Therefore for any $\delta > 0$, (3.4) has a nonconstant solution with minimal period less than $2\pi/\sqrt{b} + \delta$. This proves the lemma.

LEMMA 3.3. — *Let $u_0(t)$ be a periodic solution of*

$$u'' + bu^+ - au^- = 1 \tag{3.6}$$

with least period $T > 2\pi/\sqrt{b}$ such *that* $u_0(0) = 1/b$, $u_0'(0) > 0$. *Let $\chi(u_0)$ denote the characteristic function of the set where u_0 is positive and $\chi(-u_0)$ denote the characteristic function of the set where u_0 is negative. If $v(t)$ is the solution of the differential equation*

$$y'' + [b\chi(u_0) + a\chi(-u_0)] y = 0 \tag{3.7}$$

such that $v(0) = 0$ and $v'(0) = 1$, then $v(T) < 0$.

Proof. — Setting $C = 1/b$ and $u_0(t) = w(t) + C$ we see that

$$w''(t) + b(w(t) + C)^+ - 1 - a(w(t) + C)^- = 0. \tag{3.8}$$

Since $T > 2\pi/\sqrt{b}$, w cannot satisfy $w'' + bw = 0$ and hence $X(-u_0) \not\equiv 0$. It is easy to see that (3.6) cannot have a negative solution. If $0 < t_1 < t_2$, and t_1 and t_2 are the first and second zeros of $w(t)$ after $t = 0$, then since

$$w'(t)^2 + [bu_0^+(t)^2 + au_0^-(t)^2]/2 - u_0(t) = \text{const.}$$

it follows that $u'(t_2) = u'(0)$ and hence $t_2 = T$.

We prove the assertion of the lemma by use of the Sturm comparison theorem. From (3.8) we see that

$$w''(t) + p(t)\,w(t) = 0$$

where

$$p(t) = \begin{cases} b & \text{if} \quad w(t) + C > 0 \\ \dfrac{a(w(t)+C) - 1}{w(t)} & \text{if} \quad w(t) + C \leqq 0. \end{cases}$$

On the interval $[0, t_1]$,

$$p(t) = b = [b\,\chi(u_0) + a\,\chi(-u_0)](t).$$

Therefore $v(t)$ and $w(t)$ are solutions of the same differential equation on this interval, and since $v(0) = w(0) = 0$, $v(t)$ is a multiple of $w(t)$ on $[0, t_1]$. Thus $v(t_1) = w(t_1) = 0$. On the interval (t_1, t_2),

$$p(t) = b = [b\,\chi(u_0) + a\,\chi(-u_0)](t) \qquad \text{if} \quad u_0(t) > 0.$$

If $u_0(t) < 0$, then

$$p(t) = a + (aC - 1)/w(t) > a = [b\,\chi(u_0) + a\,\chi(-u_0)](t)$$

Since $u_0(t)$ must be negative somewhere on (t_1, t_2), we see that

$$p(t) \geqq [b\,\chi(u_0) + a\,\chi(-u_0)](t)$$

and

$$p(t) \not\equiv [b\,\chi(u_0) + a\,\chi(-u_0)](t)$$

on (t_1, t_2). Hence, by the Sturm comparison theorem, it follows that $v(t)$ cannot vanish for $t_1 < t \leqq t_2$. Since $v(t_1) = 0$ and $v'(t_1) < 0$, $v(t_2) = v(T) < 0$. This proves the lemma.

LEMMA 3.4. — *Let* $\delta > 0$ *be chosen so that* $2\pi/\sqrt{b} + \delta < \hat{T} - \delta$ *where* $\hat{T}$ *is as in Lemma* 3.4. *There exists a number* $S^* = S^*(\delta) > 0$, *independent of* $T \in [2\pi/\sqrt{b} + \delta, \hat{T} - \delta]$, *such that for* $S \geqq S^*(\delta)$, (3.4) *has a unique constant* *solution* C_s, *all other solutions are periodic and if* $u(t)$ *is a solution with* $u(0)=C_s$, $u'(0)>0$, *and* $u(t)$ *has minimal period* T, *then the solution* $v(t)$ *of the linear differential equation*

$$Z''+f'(su(t))Z=0 \tag{3.9}$$

such that $v(0)=0$, $v'(0)=1$, *satisfies* $v(T)<0$.

Remark. — Since (3.4) can be written in the form

$$u''(t)+f(su(t))/s-1=0$$

and the assertion of the lemma implies that not all solutions of (3.9) can be T-periodic, the lemma implies that *such a solution* $u(t)$ *satisfying these conditions has property* (ND).

Proof of Lemma 3.4. — Assuming that the statement of the lemma is false, there exists a sequence of numbers $\{S_m\}_1^\infty$ such that $S_m \to \infty$ as $m \to \infty$ and a corresponding sequence of numbers $\{T_m\}_1^\infty$ such that $T_m \in [2\pi/\sqrt{b}+\delta, T-\delta]$ and when $s=S_{m'}$ (3.4) has a periodic solution $u_m(t)$ with minimal period T_m satisfying $u_m(0)=C_{s'}$ $u_m'(0)>0$, such that when $s=S_m$ and $u(t)=u_m(t)$, the solution $v_m(t)$ of (3.9) which satisfies the initial conditions $v_m(0)=0$, $v_m'(0)=1$, must satisfy $v_m(T_m)\geqq 0$.

According to Lemma 3.1, there exists a number $r>0$ such that $|u_m|_\infty \leqq r$ for all m. From the relation

$$u_m''(t)+bu_m(t)^+-au_m(t)^-+\frac{f(S_m u_m(t))}{S_m}=1$$

we see that $|u_m''|_\infty$ is bounded independently of m. Therefore by the same type of argument that was used in Lemma 3.1, we may assume that $u_m(t)\to u_0(t)$, $u_m'(t)\to u_0'(t)$ as $m\to\infty$ uniformly with respect to t $[-\hat{T},\hat{T}]$. Since $f_0(s\xi)/s\to 0$ as $\xi\to\infty$ uniformly with respect to ξ in bounded sets, it follows from the differnetial equation that the sequence $\{u_m''(t)\}_1^\infty$ converges uniformly on $[-\hat{T},\hat{T}]$. Hence $u_0\in C^2$, $u_0(0)=0$, and

$$u_0''(t)+bu_0(t)^+-au_0(t)^-=1.$$

We may also assume that $\lim_{m\to\infty} T_m=T_0$, where $2\pi/\sqrt{b}+\delta\leqq T_0\leqq\hat{T}-\delta$.

Since $u_m(T_m)=u_m(0)$, $u_m'(T_m)=u_m'(0)$ it follows that $u_0(T_0)=u_0(0)$, $u_0'(T_0)=u_0'(0)$ and hence, $u_0(t)$ is T_0-periodic.

We assert that $u_0(t)$ is nonconstant and T_0 is the minimal period of $u_0(t)$. Assuming first, on the contrary, that $u_0(t)$ is constant, we must have that $u_0(t) \equiv 1/b$. If C_m is the unique zero of $f(S_m \xi)/S_m - 1$ for m large, then $C_m \to 1/b$ as $m \to \infty$. Writing $u_m(t) = C_m + w_m(t)$, it follows that since $u_m(t) \to 1/b$ uniformly with respect to $t \in [-\hat{T}, \hat{T}]$, $w_m(t) \to 0$ uniformly with respect to $t \in [-\hat{T}, \hat{T}]$ as $m \to \infty$. Using the same argument given in the proof of Lemma 3.2, we find that $w_m''(t) + Q_m(t) w_m(t) = 0$ where $Q_m(t) \to b$ uniformly as $m \to \infty$. Hence, by the argument given in the proof of Lemma 3.2, the minimal period of $w_m(t)$ approaches $2\pi/\sqrt{b}$ as $m \to \infty$, contradicting the fact that for all $m \geqq 1$ the minimal period of $w_m(t)$ is $T_m \geqq 2\pi/\sqrt{b} + \delta$. Therefore $u_0(t)$ is nonconstant.

Next we assume, contrary to the claim, that the minimal period of $u_0(t)$ is L, where $0 < L < T_0$. Since the distance between two consecutive zeros of $u_0'(t)$ is one-half the period (see the example at end of the previous section), there exist numbers t_0 and t_1 in $(-\hat{T}, \hat{T})$ with $t_0 < t_1$ and $t_1 - t_0 = L/2$, such that $u_0'(t_0) = u_0'(t_1) = 0$. Let $\alpha > 0$ be chosen so that $L + 4\alpha < T_0$. Since $u_0(t)$ is nonconstant, $u_0''(t_k) \neq 0$, $k = 1,2$. Therefore, since $u_m'(t) \to u_0'(t)$ and $u_m''(t) \to u_0''(t)$ as $m \to \infty$ uniformly on $[-\hat{T}, T]$, it follows that for m sufficiently large, $u_m'(t)$ has a zero in $(t_0 - \alpha, t_0 + \alpha)$ and a zero in $(t_0 - \alpha, t_0 + \alpha)$ and these two zeros are distinct from each other. Since the distance between these zeros is at most $t_1 - t_0 + 2\alpha = L/2 + 2\alpha$, it follows that for m sufficiently large the minimal period of $u_m(t)$ is at most $L + 4\alpha$. Since the minimal period of $u_m(t)$ tends to T_0 as $m \to \infty$ this gives a contradiction. Therefore T_0 is the minimal period of $u_0(t)$.

According to the assumptions (3.2), if $u_0(t) \neq 0$, then

$$\lim_{m \to \infty} f'(S_m u_m(t)) = [b\chi(u_0) + a\chi(-u_0)](t)$$

and the convegence is uniform on any closed interval which does not contain a zero of $u_0(t)$.

For each $m \geqq 1$, let v_m be the solution of

$$y'' + f'(S_m u_m(t))\, y = 0$$

which satisfies the initial conditions $v_m(0) = 0$, $v_m'(0) = 1$. Since $f'(S_m u_m(t))$ is bounded independently of m, it is not difficult to show (for example, by considering the corresponding system and applying Gronwall's lemma) that the sequences $\{v_m(t)\}_1^\infty$ and $\{v_m'(t)\}_1^\infty$ are uniformly bounded on $[-\hat{T}, \hat{T}]$. Therefore, from the differential equation satisfied by v_m, we see that the sequence $\{v_m''(t)\}_1^\infty$ is also uniformly bounded on $[-\hat{T}, \hat{T}]$. Applying Ascoli's lemma, we may assume that

$$\lim_{m \to \infty} v_m(t) = v(t), \qquad \lim_{m \to \infty} v'_m(t) = v'(t).$$

Hence, by the form of the differential equations satisfied by v_m and what was shown in the previous paragraph, it follows that $v(t)$ has a piecewise continuous second derivative whose discontinuities can only occur at the zeros of $u_0(t)$. Moreover

$$v''(t) + [b\chi(u_0) + a\chi(-u_0)](t)\, v(t) = 0$$
$$v(0) = 0, \qquad v'(0) = 1.$$

According to what was assumed at the beginning of the proof, $v_m(T_m) \geqq 0$, and hence by uniform convergence, $v(T_0) = \lim_{m \to \infty} v_m(T_m) \geqq 0$. But $T_0 \geqq 2\pi/b + \delta$, so we have a contradiction to Lemma 3.3. This proves the assertion of Lemma 3.4.

LEMMA 3.5. — *Let* $\delta > 0$ *and* $S^*(\delta)$ *be as in Lemma* 3.4. *Assume that* $S^*(\delta)$ *is also so large that* $s \geqq S^*(\delta)$ *implies that* (3.4) *has nonconstant solutions with periods less than* $2\pi/\sqrt{b} + \delta$ (see Lemma 3.2). *Let* $s \geqq S^*(\delta)$ *and let* $u(t, r)$ *denote the solution of* (3.4) *such that* $u(0, r) = C_s$ *and* $u'(0, r) = r$ *where* C_s *is the unique constant solution of* (3.4). *If* $T(r)$ *denotes the minimal period of* $u(t, r)$, *then* $T(r)$ *is of class* C^1 *in* r *for* $0 < r < \infty$ *and there exist numbers* r_1 *and* r_2 *with* $0 < r_1 < r_2$ *such that* $2\pi/\sqrt{b} + \delta \leqq T(r) \leqq \hat{T} - \delta$ *if and only if* $r_1 \leqq r \leqq r_2$. Moreover, $T'(r) > 0$ for $r_1 \leqq r \leqq r_2$.

Proof. — Let $s \geqq S(\delta)$ be fixed, and let $r > 0$. If $t = T(r)$ is the second solution of $u(t, r) = C_s$ after $t = 0$, then, since (3.4) is conservative, $T(r)$ is the period of $u(t, r)$. Since

$$\frac{\partial u}{\partial t}(T(r), r) = u'(T(r), r) = r > 0,$$

it follows from the implicit function theorem that $T(r)$ is a C^1 function of r for $r > 0$. If

$$v(t, r) = \frac{\partial}{\partial r} u(t, r),$$

then

$$v'' + f'(su(t, r))\, v = 0$$
$$v(0, r) = 0, \qquad v'(0, r) = 1$$

Therefore, by Lemma 3.4, if $2\pi/\sqrt{b} + \delta \leqq T(r) \leqq \hat{T} - \delta$, then $v(T(r), r) < 0$. Assuming that $r > 0$, $T(r) \in [2\pi/\sqrt{b} + \delta, \hat{T} - \delta]$, and differentiating the identity $u(T(r), r) = 0$ we have

$$0 = u'(T(r), r)\, T'(r) + v(T(r), r)$$

and hence

$$T'(r) = -v(T(r), r)/r > 0.$$

Since for $r > 0$, either $T(r) = 2\pi/\sqrt{b} + \delta$ or $T(r) = \hat{T} - \delta$ implies that $T'(r) > 0$, it is clear that either of these equations has at most one solution for $0 < r < \infty$.

According to the way $S^*(\delta)$ was chosen, if r is sufficiently small and positive, then $T(r) < 2\pi/\sqrt{b} + \delta$. If $U(\xi, s)$ is the function that was defined before Lemma 3.1, then since

$$\frac{\partial U}{\partial \xi}(\xi, s)/\xi \to b > 0 \quad as \ \xi \to \infty,$$

and since

$$\frac{\partial U}{\partial \xi}(\xi, s)/\xi \to a > 0 \quad \text{as } \xi \to -\infty,$$
$$U(\xi, s) \to \infty \quad \text{as } |\xi| \to \infty.$$

Therefore, since

$$u'(t, r)^2/2 + U(u(t, r), s) = \text{const.} = r^2/2$$

it follows that

$$\max_t |u(t, r)| \to \infty \quad \text{as } r \to \infty.$$

Therefore, by Lemma 3.1, $T(r) > \hat{T} - \delta$, for r sufficiently large and positive. Combining this with what has already been established above, we infer the existence of numbers $r_1>0$ and $r_2>r_1$ such that

$$T(r_1)=2\pi/\sqrt{b}+\delta, \qquad T(r_2)=\hat{T}-\delta,$$

$$T'(r)>0 \quad \text{for} \quad r_1 \leqq r \leqq r_2,$$

$$T(r)<2\pi/\sqrt{b}+\delta \quad \text{for} \quad 0<r<r_1,$$

and

$$T(r)>\hat{T}-\delta \quad \text{for} \quad r>r_2.$$

This proves the lemma.

We can now prove the first half of the main result of this section.

3.1 A THEOREM *Assume* (3.2) *holds with* $0<a<b$ *and that assumptions* (A_1) *and* (A_2) *are satisfied. Let m be the number of integers k such that*

$$1/\sqrt{b}<1/k<1/2(1/\sqrt{b}+1/\sqrt{a}). \tag{3.10}$$

For $s>0$ *sufficiently large there are exactly m orbits of the autonomous system*

$$y'=Z, \qquad Z'=-f(y)+s \tag{3.11}$$

which correspond to periodic solutions having 2π *as a pariod. The minimal periods of these solutions are fo the form* $2\pi/k$ *where k is an integer satisfying* (3.10). *If* $h(t)$ *is continuous and* 2π*-periodic, then for* $s>0$ *sufficiently large and* $|\varepsilon|$ *sufficiently small,* (3.1) *has at least* $2m+1$ 2π*-periodic solutions.*

Proof. — For $s>0$ we have $y''+f(y)=s$ if and only if $u=y/s$ is a solution of (3.4). Thus, we may consider the autonomous system corresponding to (3.4) in the proof of the first part of the theorem. Let $\delta>0$ be chosen so that if k is an integer which satisfies (3.10), then

$$2\pi/\sqrt{b}+\delta<2\pi/k<\hat{T}-\delta \tag{3.12}$$

Assuming there is an integer k satisfying these inequalities, we fix it in the following paragraph.

Let $S^*(\delta)$ be as in Lemma 3.5 and let $s \geqq S^*(\delta)$. Since (3.4) has the unique constant solution C_s, and all other solutions are periodic, each of the orbits of the corresponding first order system, aside from the equilibrium point $(C_s, 0)$, must contain a point of the form (C_s, r) where $r>0$. That is, using the notation of Lemma 3.5, each such orbit can be represented by $\operatorname{col}(u(t,r), u'(t,r))$ where $r>0$ and $0 \leqq t \leqq T(r)$. According to Lemma 3.5, the equation $T(r)=2\pi/k$ has exactly one solution for $r>0$.

Thus we have shown that for $s \geqq S^*(\delta)$, there is exactly one orbit of the system (3.11) corresponding to a periodic solution with minimal period $2\pi/k$ if the integer k satisfies (3.10). To complete the proof of the first part we need only show that there are no other orbits corresponding to a 2π-periodic solution for $s>0$ sufficiently large. Because of assumptions (A_1) and (A_2), there exists a number $\alpha>0$ such that if $r>0$ is an integer which does not satisfy the inequalities (3.10), then either $2\pi/r<2\pi/\sqrt{b}-\alpha$ or $2\pi/r>\hat{T}+\alpha$. To complete the proof of the first part, it is only necessary to show that for $s>0$ and sufficiently large, (3.4) cannot have a nonconstant solution with period less than $2\pi/\sqrt{b}-\alpha$ or greater than $\hat{T}+\alpha$. Assuming the contrary, there exists a sequence of numbers $\{S_n\}$ with $\lim_{n\to\infty} S_n=\infty$ and a corresponding sequence of functions $\{u_n(t)\}_{n=1}^{\infty}$ such that $u_n(t)$ is a nonconstant solution of (3.4) when $s=S_{n'}$ and periodic with minimal period not in $[2\pi/\sqrt{b}-\alpha, \hat{T}+\alpha]$.

Suppose first, that some subsequence of the numerical sequence $\{|u_n|_\infty\}_1^\infty$ is bounded. The argument of the proof of Lemma 3.4 shows that we may then assume that $u_n(t)\to u_0(t)$ and $u_n'(r)\to u_0'(t)$ as $n\to\infty$ uniformly on any compact interval where $u_0(t)$ is a solution of (3.6). If $u_0(t)$ is a constant, the proof of Lemma 3.4 shows that the period of $u_n(t)$ tends to $2\pi/\sqrt{b}$ which is a contradiction. If $y_0(t)$ is nonconstant but does not change sign then $u_0(t)$ is nonnegative and it is easy to see that its period is $2\pi/\sqrt{b}$. In this case, the period of $u_n(t)$ tends to $2\pi/\sqrt{b}$ which is again a contradiction.

Therefore, $u_0(t)$ must assume both positive and negative values. Writing $u_0(t)=C+w(t)$, where $C=1/b$, and referring to the proof Lemma 3.3, we see that $w''(t)+p(t)\,w(t)=0$ where $p(t)$ is as before. If $t_0<t_1<t_2$ are three consecutive zeros of w such that $w(t)>0$ on (t_0, t_1), and $w(t)<0$ on (t_1, t_2), then since $p(t)=b$ on (t_0, t_1) and $a\leqq p(t)\leqq b$ on (t_1, t_2) and each inequality is strict somewhere on the interval, by the Sturm comparison theorem we have $t_1-t_0=\pi/\sqrt{b}$ and $\pi/\sqrt{b}\leqq t_2-t_1\leqq\pi/\sqrt{a}$. Hence, the period of $u_0(t)$ is between $2\pi\sqrt{b}$ and $\pi/\sqrt{b}+\pi/\sqrt{a}=\hat{T}$. Since the period of $u_n(t)$ approaches that of $u_0(t)$ as $n\to\infty$ we agian have a contradiction.

We are left with the case where $|u_n|_\infty\to\infty$ as $n\to\infty$. In this case, the proof of Lemma 3.1 shows that if $Z_n(t)=u_n(t)/|u_n|_\infty$ then it may be assumed that $Z_n(t)\to Z(t)$, $Z_n'(t)\to Z'(t)$ uniformly as $n\to\infty$, where $Z''+b\,Z^+-a\,Z^-=0$. Since $|Z|_\infty=1$, the period of Z is $\hat{T}$. Since the period of u_n approaches that of Z as $n\to\infty$ we again have a contradiction. Therefore

(3.4) has no 2π-periodic solutions, other than those with minimal period $2\pi/k$ for some k satisfying (3.10)b for s sufficiently large.

To prove the second part of the theorem we fix $s \geqq S^*(\delta)$. Again, $\delta > 0$ is chosen so that any integer k satisfying (3.10) also satisfies (3.12). According to Lemma 3.5, if k is such an integer, then there exists a unique number $r_k > 0$ that $T(r_k) = 2\pi/k$. By the remark following the statement of Lemma 3.4, the $2\pi/k$-periodic solution $u(t, r_k)$ of the differential equation $u'' + f(su)/s - 1 = 0$ defined in Lemma 3.5, has property (ND). Therefore, by Theorem 2.1 and 2π-periodicity of $h(t)$, if A_k is an arbitrary open neighborhood of the curve

$$\{ \operatorname{col}(u(t, r_k), u'(t, r_k)) \mid 0 \leqq t \leqq T(r_k) \},$$

then for $|\varepsilon|$ sufficiently small there exist two distinct 2π-periodic solutions of the perturbed differential

$$u'' + f(su)/s - 1 = \varepsilon h(t) \tag{3.13}$$

with starting points $\operatorname{col}(u(0), u'(0))$ contained in A_k. By choosing the A_k, as k ranges over all integers satisfying (3.10), to be disjoint, we see that for $|\varepsilon|$ sufficiently small there exist at least $2m$ distinct 2π-periodic solutions of (3.13).

The existence of another 2π-periodic solution can be obtained by a more standard perturbation argument. If C_s is the unique constant solution of (3.4), then since $C_s \to 1/b$ as $s \to \infty$, $f'(sC_s) \to b$ as $s \to \infty$. By assumption (A_1), we may choose $s \geqq S^*(\delta)$ so large that $f'(sC_s)$ is positive and not the square of an integer. Let X be the Banach space of C^2 2π-periodic functions with norm $|u| = |u|_\infty + |u'|_\infty + |u''|_\infty$ and Y the Banach space of continuous 2π-periodic functions with norm $|h|_\infty$. Let F be the C^1 mapping from X to Y defined by

$$F(x)(t) = x''(t) + f(sx(t))/s - 1$$

Let x_0 be the constant function $x_0(t) \equiv C_s$ where s is chosen so $s \geqq S^*(\delta)$ and $f'(sC_s) \neq n^2$, $n = 0, 1, 2, \ldots$ We have that $F(x_0) \equiv 0$ and that $F'(x_0)$ is the continuous linear mapping given by

$$F'(x_0)(w)(t) = w''(t) + f'(sC_s)w(t).$$

Because of the condition on $f''(sC_s)$, the linear mapping $F'(x_0) : X \to Y$ is one-to-one and onto. Therefore, by the inverse function theorem, for $|\varepsilon|$

sufficiently small, there exists u_ε in X such that $F(u)=\varepsilon h$ and $u_\varepsilon \to C_s$ as $\varepsilon \to 0$ in C^2.

The neighborhoods A_k for k satisfying (3.10) are disjoint from C_s as well from each other. Hence for $|\varepsilon|$ sufficiently small, (3.13) has at least $2m+1$ distinct 2π-periodic solutions, since $y(t)$ is a solution of (3.1) if and only $u(t)=y(t)/s$ is a solution of (3.4). This proves the Theorem.

The second part of the main theorem is proved using a series of lemmas concerning periodic solutions of the autonomous differential equation

$$u''+bu^+-au^-+f_0(\tau s)/\tau=-1 \tag{3.14}$$

where $T>0$ is a large parameter. The proofs of Lemmas 3.6, 3.7, 3.8, 3.9, and 3.10 stated below parallel those of Lemma 3.1, 3.2, 3.3, 3.4, and 3.5 respectively and are therefore omitted except for part of the proof of lemma 3.8.

LEMMA 3.6. — *There exists a number* $\tau^*>0$ *such that* for $\tau \geqq \tau^*$, (3.14) *has a unique constant solution* d_τ *such that* $d_\tau \to -1/a$ as $\tau \to \infty$ *and all other solutions are periodic. Given any number* $\delta>0$ *there exists* $r_1=r_1(\delta)$ *such that* if $\tau \geqq \tau_*$ *and* $u(t)$ *is a solution of* (3.14) *with* $|u|_\infty \geqq r_1$, *then the minimal period* of u *is less than* $\hat{T}+\delta$.

LEMMA 3.7. — *Given* $\delta>0$, there exists $S_1(\tau)$ *such that if* $s \geqq S_1(T)$, *then* (3.14) *has a nonconstant periodic solution with minimal period greater than* $2\pi/\sqrt{a}-\delta$.

LEMMA 3.8. — *Let* $u_0(t)$ *be a periodic solution of*

$$u''+bu^+-au^-=-1 \tag{3.15}$$

with least period $T<2\pi/\sqrt{a}$ *such that* $u_0(0)=-1/a$ and $u_0'(0)<0$. *If* $v(t)$ *is the solution of the differential equation*

$$y''+[b\chi(u_0)+a\chi(-u_0)]y=0 \tag{3.16}$$

such that $v(0)=0$, $v'(0)=-1$, then $v(T)<0$.

Sketch of proof. — If we set $u_0(t)=-1/a+w(t)$ and let t_1 and t_2 be the first and second zeros of $w(t)=0$, then $t_1=\pi/\sqrt{a}$ and $w(t)$ and $v(t)$ are multiples of one another on this interval. Using the Sturm comparison theorem as in the proof of Lemma 3.3, we can show that $v(t)$ must have a zero on the open interval (t_1, t_2). To show that $v(t)$ cannot have two zeros on the half-open interval $(t_1, t_2]$ we note that $Z(t)=w'(t)=u'(t)$ is a solution of (3.16) such that $Z'(0)<0$. Moreover, Z has exactly two zeros on $(0, t_2)$ since $t_2=T$ is the period of $u(t)$. If $v(t)$ had two zeros on $(t_1, t_2]$ it would have at least four zeros on $[0, t_2]$, so, by the Sturm separation theorem, $u'(t)=w'(t)$ would have at least three zeros on $(0, t_2)$ which is a contradiction. Hence $v(t)<0$.

LEMMA 3.9. — *Let* $\delta>0$ *be chosen so that* $\hat{T}+\delta<2\pi/\sqrt{a}-\delta$. *There exists a number* $S_1^*=S_1^*(\tau)$ *independent of* $T\in[\hat{T}+\delta, 2\pi/\sqrt{a}-\delta]$ *such that for* $\tau\geqq S_1^*(\delta)$, (3.14) *has a unique constant solution* d_s, *all other solutions are periodic and if* $u(t)$ *is a solution with* $u(0)=d_s$, $u'(0)<0$ *and* $u(t)$ *has minimal period* T, *then the solution* $v(t)$ *of the linear differential equation*

$$Z''(t)+f'(\tau u(t))Z(t)=0 \tag{3.16}$$

such that $v(0)$, $v'(0)=-1$, *satisfies* $v(T)<0$.

Remark. — *The lemma shows that such a solution* $u(t)$ *of* (3.14) satisfies condition (ND).

LEMMA 3.10. — *Let* $\delta>0$ *and* $S_1^*(\delta)$ *be as in Lemma* 3.9. *Assume that* $S_1^*(\delta)$ *is also so large that* (3.14) *has nonconstant solutions with periods greater than* $2\pi/\sqrt{a}-\delta$ for $\tau\geqq S_1^*(\delta)$. *Let* $\tau\geqq S_1^*(\delta)$ *and let* $u_1(t, r)$ *denote the solution of* (3.14) *such that* $u_1(0, r)=d_s$ and $u_1'(0, r)=r$. *If* $T_1(r)$ denotes the minimal period of $u_1(t, r)$, *then* $T_1(r)$ *is of class* C^1 *in r for* $-\infty<r<0$ *and there exists numbers* r_3 *and* r_4 *with* $-\infty<r_3<r_4<0$ *such that* $\hat{T}+\delta\leqq T_1(r)\leqq 2\pi/\sqrt{a}-\delta$ *if and only if* $r_3\leqq r\leqq r_4$. *Moreover* $T_1'(r)>0$ *on this interval.*

Sketch of proof. — If $T_1(r)$ is between $\hat{T}+\delta$ and $2\pi/\sqrt{a}-\delta$, then the solution $v(t)$ of (3.16) which satisfies $v(0)=0$, $v'(0)=-1$ is $-\dfrac{\partial u_1}{\partial r}(t, r)$, assuming $u(t)=u_1(t, r)$. According to Lemma 3.9, $v(T)<0$, so if $r<0$, $\dfrac{\partial u_1}{\partial r}(T_1(r), r)>0$. Assuming that $r<0$ and $\hat{T}+\delta\leqq T(r)\leqq 2\pi/\sqrt{a}-\delta$, we obtain after differentiating the identity

$$u_1(T_1(r), r)=0, \qquad r\,T_1'(r)=-\frac{\partial u_1}{\partial r}(T_1(r), r)<0.$$

Therefore $T_1'(r)>0$.

We can now give the second half of the main result of this section.

THEOREM 3.1 B. — *Let the conditions of Theorem* 3.1 A *be satisfied. Let* m_1 *be the number of integers k such that*

$$1/2(1/\sqrt{b}+1/\sqrt{a})<1/k<1/\sqrt{a}. \tag{3.17}$$

For s sufficiently large and negative the autonomous system (3.11) *has exactly* m_1 orbits *which correspond to periodic solutions having* 2π *as a* *period. The minimal periods of these solutions are of the form* $2\pi/k$ *where* k *is an integer satisfying* (3.17).

If $h(t)$ *is continuous and* 2π*-periodic, then for* $s<0$ and $|s|$ *sufficiently large* and $|\varepsilon|$ *sufficiently small,* (3.1) *has at least* $(2m_1+1)\,2\pi$-periodic *solutions.*

The theorem is proved from Lemmas 3.6-3.10 in a way similar to the way that Theorem 3.1 A is proved from Lemmas 3.1-3.5. The important thing to note is that if $s<0$ and we let $\tau=-s$, then $y''+f(y)=s$, and only if $u=y/\tau$ satisfies

$$u''+f(\tau u)/\tau-1=0.$$

Combining Theorems 3.1 A and 3.1 B we obtain

THEOREM 3.2. – *If* (3.2) *holds and assumptions* (A_1) *and* (A_2) *hold then there exists an integer* r *with* $1\leqq r\leqq 2(q-p)+1$ *such that for* s *large and positive and* $|\varepsilon|$ *small,* (3.1) *has at least* r *solutions, and for* s *sufficiently large and negative and* $|\varepsilon|$ *small,* (3.1) *has at least* $2(q-p)+2-r$ *solutions.*

4. A CASE WHERE THEOREM 3.2 IS SHARP

Let a and b be chosen so that $0<a<1<b<4$ and let $f(\xi)$ be a C^1 function such that $f'(\xi)\to b$ as $\xi\to\infty$ and $f'(\xi)\to a$ as $\xi\to-\infty$. Let $h(t)$ be continuous and 2π-periodic. We claim that if $1/2(1/\sqrt{a}+1/\sqrt{b})<1$, then for s sufficiently large and positive and $|\varepsilon|\neq 0$ small,

$$y''+f(y)=s(1+\varepsilon\cos t) \tag{4.1}$$

has exactly one 2π-periodic solution while for s sufficiently large and negative and $|\varepsilon|\neq 0$ small there are exactly three solutions which are 2π-periodic. If $1<1/2(1/\sqrt{a}+1/\sqrt{b})$ then (4.1) has exactly three 2π-periodic solutions for s large and positive and $|\varepsilon|\neq 0$ and small and exactly one 2π-priodic solution for s large and negative and $|\varepsilon|\neq 0$ small.

Suppose then that $1/2(1/\sqrt{a}+1/\sqrt{b})\neq 1$. We claim that for s and ε in bounded intervals there exists a bound on $|u|_\infty$ if u is a 2π-periodic solution of (4.1). Indeed, if this were not the case, the argument used in the proof of Lemma 3.1 would give the existence of a 2π-periodic solution

Z of

$$Z'' + bZ^+ - aZ^- = 0$$

with $|Z|^\infty = 1$. The minimal period of Z would have to be $2\pi/k$ for some $k \geqq 1$ and since the minimal period is $\pi/\sqrt{a} + \pi/\sqrt{b} > \pi$, we have a contradiction.

Suppose $1/2(1/\sqrt{a} + 1/\sqrt{b}) > 1$. We suppose $s > 0$ and make the substitution $y = su$ in (4.1) obtaining

$$u'' + f(su)/s - 1 = \varepsilon \cos t. \tag{4.2}$$

Since $k = 1$ is the unique integer such that

$$1/\sqrt{b} < 1/k < 1/2(1/\sqrt{a} + 1/\sqrt{b}),$$

it follows from Theorem 3.1A that for s sufficiently large, there is a unique 2π-periodic solution u_0 of

$$u'' + f(su)/s - 1 = 0 \tag{4.3}$$

with $u_0(0) = C_s$ and $u_0'(0) > 0$, where C_s is the unique solution of $f(su)/s - 1 = 0$ and all solutions are periodic. Let *us* fix such an s. Let A and U be disjoint neighborhoods of the curve

$$C_0 = \{ \operatorname{col}(u_0(t), u_0'(t)) \mid 0 \leqq t \leqq 2\pi \}$$

and the point $\operatorname{col}(C_s, 0)$ respectively. Using Theorem 3.1A and the example given at the end of the second section, we infer that for $|\varepsilon| \neq 0$ and $|\varepsilon|$ small, there are exactly two 2π-periodic solutions of

$$u'' + f(su)/s - 1 = \varepsilon \cos t \tag{4.4}$$

with $\operatorname{col}(u(0), u'(0)) \in A$ and exactly one 2π-periodic solution with $\operatorname{sol}(u(0), u'(0)) \in U$. We claim that for $|\varepsilon| \neq 0$ and sufficiently small these are the only 2π-periodic solutions.

Suppose not. Then, there exists a sequence of numbers $\{\varepsilon_n\}_1^\infty$ such that $\varepsilon_n \neq 0$ for all n, $\lim_{n \to \infty} \varepsilon_n = 0$, and when $\varepsilon = \varepsilon_n$ (4.4) has a 2π-periodic solution $u_n(t)$ such that

$$\operatorname{col}(u_n(0), u_n'(0)) \in R^2 \setminus (A \cup U)$$

Since $|u_n|_\infty$ is bounded independently of n, it follows from (4.4) that $|u''_n|_\infty$ is bounded independetly of n. Thus, we may suppose that

$$\lim_{n \to \infty} u_n(t) = Z(t), \qquad \lim_{n \to \infty} u'_n(t) = Z'(t)$$

uniformly as $n \to \infty$, where Z is 2π-periodic, $Z \in C^2$ and

$$Z'' + f(sz)/s - 1 = 0.$$

Since $R^2 \setminus (A \cup U)$ is closed, we have $\mathrm{col}(Z(0), Z'(0)) \in R^2 \setminus (A \cup U)$ and this means that the system

$$u' = v, \qquad v' = -(f(su)/s - 1)$$

has a closed orbit other than C_0 which corresponds to a 2-periodic solution. This is a contradiction, and establishes the claim.

If $1 < 1/2(1/\sqrt{a} + 1/\sqrt{b})$ and s is sufficiently large and negative, then there are no orbits of the above system corresponding to nonconstant 2π-priodic solutions by Theorem 3.1 B. Therefore, the only 2π-periodic solution is the constant solution d_s. The same type of argument given above shows that for s large and negative and $|\varepsilon|$ small the only 2π-periodic solution is one with $\mathrm{col}(u(0), u'(0))$ near $\mathrm{col}(d_s, 0)$.

The case $1/2(1/\sqrt{a} + 1/\sqrt{b}) < 1$ is handled similarly.

5. THE CASE $a < 0$

We discuss briefly the case where $a < 0$ and there exists an integer $q \geqq 0$ such that

$$q^2 < b < (q+1)^2 \tag{5.1}$$

In this case assumption A_2 is always satisfied since (3.3) has no nonconstant periodic solutions.

We again substitute $y = su$ in $y'' + f(y) = s$ and consider the differential equation

$$u'' + f(su)/s - 1 = 0 \tag{5.2}$$

In [7] the phase portrait of the flow generated by the corresponding autonomous system

$$u' = v, \quad v' = -(f(su)/s - 1) \tag{5.3}$$

is analysed in detail and it is shown that, for s large and positive, the phase portrait is like that of the limiting system

$$u' = y, \qquad v' = -(bu^+ - au^- - 1).$$

For s large and positive, there are exactly two equilibium, points $(D_s, 0)$ and $(C_s, 0)$ such that $D_s \to 1/a < 0$ and $C_s \to 1/b$ as $s \to \infty$. The point $(C_s, 0)$ is a center and $(D_s, 0)$ is a saddle point. There is a homoclinic orbit such that solutions of (5.3) corresponding to the orbit tend to $(D_s, 0)$ as $t \to \pm\infty$. The homoclinic orbit together with the saddle form a simple closed curve which bounds a region which is the union of orbits corresponding to nonconstant periodic solutions of (5.3) and the center $(C_s, 0)$. As the orbits in this region approach the boundary of the region, the periods of the corresponding solutions tend to ∞ and, for s large and positive, the periods of the orbits near $(C_s, 0)$ are close to $2\pi/\sqrt{b}$. All other orbits are unbounded.

Using obvious modifications of the arguments used in the third section, one can show that for each integer k, with $0 < k \leqq q$, if $s > 0$ is sufficiently large, then there exists exactly one orbit of (5.3) corresponding to a solution with least period $2\pi/k$. Moreover, the corresponding solution of (5.2) has property (ND).

Let $h(t)$ be continuous and 2π-periodic. For s large and positive let Γ_k denote the unique orbit of (5.3) which corresponds to periodic solutions of (5.2) with least period $2\pi/k$. For $k = 1, \ldots, q$. The results of the second section imply that for $1 \leqq k \leqq q$ and $|\varepsilon|$ sufficiently small, there exist at least two 2π-periodic solutions of

$$u'' + f(su)/s = 1 + \varepsilon h(t) \tag{5.4}$$

$u_1(t)$ and $u_2(t)$ such that the point $(u_i(0), u_i'(0))$ tends to the orbit Γ_k as $\varepsilon \to 0$ for $i = 1, 2$.

The implicit function techniques used in the third section show that, for $|\varepsilon|$ small and for s large and positive there exist 2π-periodic solutions $u_1(t)$ and $u_2(t)$ of (5.4) with $(u_1(0), u_1'(0))$ close to $(C_s, 0)$ and $(u_2(0)), u_2'(0))$ close to $(D_s, 0)$.

Summarizing the above discussion we have

THEOREM 5.1. — *If* $a<0$ *and there exists an integer* $q \geqq 0$ *such that* (5.1) holds, *then for* s *sufficiently large and positive and* $|\varepsilon|$ *sufficiently small, there exist at least* $2(q+1)\,2\pi$*-periodic solutions of* (3.1).

As in the fourth section, it can be shown that the above result is sharp if $a<0$, $0<b<1$, and $h(t)=\cos t$.

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