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Solutions with minimal period for Hamiltonian systems in a potential well

by

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ABSTRACT. — Let $U \in C^2(\Omega)$, where Ω is a bounded set in $\mathbb{R}^N$. Suppose that $U(x)$ tends to $+\infty$ as x tends to $\partial\Omega$. Our main results concern the existence of periodic solutions of $-\ddot{x}=U'(x)$ having a prescribed number T as minimal period. The results are also generalized to first order Hamiltonian systems.

RÉSUMÉ. — Soit $U \in C^2(\Omega)$, où Ω est un ouvert donné de $\mathbb{R}^N$. On suppose que $U(x) \to +\infty$ quand $x \to \partial\Omega$. On montre l'existence de solutions périodiques de $\ddot{x}+U'(x)=0$, de période minimale prescrite. On étend ces résultats aux systèmes hamiltoniens du premier ordre.

Key words : Hamiltonian systems, perioduc solutions, potential well.

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1. INTRODUCTION

Let Ω be a bounded, convex domain in $\mathbb{R}^N$ with boundary Γ and let $U \in C^2(\Omega, \mathbb{R})$ be such that $U(x) \to +\infty$ as $x \to \Gamma$. Denote by ∇U the gradient of U.

In this paper we study the existence of T-periodic solutions of

$$-\ddot{x} = \nabla U(x), \tag{1.1}$$

where x stands for the N-vector $(x_1, \ldots, x_N) \in \Omega$ and the period $T > 0$ is prescribed.

Our main result is that: if U is convex, then (1.1) has a solution having T has *minimal* period. See Theorem 2.1 for the precise statement. Extensions to general first order Hamiltonian Systems are also given, *see* paragraph 3.

The study of (1.1) for U constrained in a potential well was begun by Benci *in* [5]; without assuming convexity, Benci proves an existence result (with no minimality of period) for second order systems like (1.1). On the other hand, for systems where U is defined on all $\mathbb{R}^N$ (or even for general Hamiltonian Systems of that kind), together with existence of periodic solutions (*see*, for example, the survey paper [16] and references therein), many results concerning the minimality of the period are known, *see* [8], [2], [1], [11], [12] and the survey [13]. Here we fill the gap for systems constrained in a potential well.

The approach we use is completely different from the Benci's one, based on the study of critical points of a functional on an *open subset* of a Hilbert space. On the contrary, we use here the Clarke's Dual Action Principle ([6], [8]). In contrast to [5] and in spite that U is defined in the bounded well Ω, such a device allows us to work with a functional Φ defined on *all* a Banach space and the critical points of Φ are found by a straight application of the Mountain Pass theorem [3], [4]. The fact of having now a "Mountain Pass" critical point, permits to show, by an application of the theory developed by Ekeland and Hofer [11], that the corresponding solution of (1.1) is of *minimal* period. By the same method (with a suitable choice of the spaces) we can handle the case of first order Hamiltonian Systems, too.

We point out that our approach works also if one greatly relaxes the convexity assumptions (at the expense of the minimality of the period), *see* theorem 2.7, and indicates that the potential well is only a limiting case of the "superquadratic" Hamiltonian Systems (*see* [15], [9]).

Actually, one of the purposes of this paper was just to show that the usual critical point theory can be employed, in a very simple fashion, to study these classes of problems.

Besides the introduction, the paper consists of 2 more sections and an Appendix: in paragraph 2 we discuss second order systems; paragraph 3 contains extensions to some classes of Hamiltonian Systems. We point out that for first order systems even the existence result is new. Lastly, in the Appendix, we shortly indicate how some of the arguments of [11] can be adapted to be used in our framework.

2. SECOND ORDER SYSTEMS

Let $\Omega \subset \mathbb{R}^N$ be a bounded, convex domain with $0 \in \Omega$. We denote by Γ the boundary of Ω and, for $\varepsilon > 0$, by Γ_ε the set $\{x \in \Omega : \operatorname{dist}(x, \Gamma) < \varepsilon\}$. Here dist $(x, \Gamma) = \min \{|x-y| : y \in \Gamma\}$, $|\,.\,|$ denoting the Euclidean norm in $\mathbb{R}^N$ corresponding to the scalar product $(.\,,\,.)$.

Let $U : \Omega \to \mathbb{R}$ be given. We say that U satisfies *assumption (A)* if:

1° $U \in C^2(\Omega, \mathbb{R})$;

2° $U(0) = 0 = \min_{\Omega} U$;

3° U is such that:

$$U(x) \to +\infty \quad \text{as } x \to \Gamma \qquad \text{(uniformly)};$$

4° there exist $\varepsilon > 0$ and $\theta \in]0, \frac{1}{2}[$ such that:

$$U(x) \leqq \theta (x, \nabla U(x)), \qquad \forall x \in \Gamma_\varepsilon,$$

Let us point out that (A 4°) is the usual assumption of "superquadraticity" (near Γ). Moreover, if U is radial and convex, then 4° follows from 3°.

Our main result is:

THEOREM 2.1. — *Suppose* U *satisfies* (A) *and:*

(i) $\quad \exists k > 0 \quad s.t. \ (U''(x)y, y) \geqq k|y|^2, \qquad \forall x \in \Omega, \quad \forall y \in \mathbb{R}^N.$

Let ω_N *be the greatest eigenvalue of* $U''(0)$, *and* $T_0 := \frac{\sqrt{2}}{\sqrt{\omega_N}}$. *Then,* $\forall T$, $0<T<T_0$, (1.1) *has at least a* T*-periodic solution* $u \not\equiv 0$, *having* T *as minimal period.*

The proof will be carried out in several steps.

Step 1. – Use of the Dual Action Principle ([6], [8]) to transform (1.1) in a critical point problem for a functional Φ in a Banach space E.

Step 2. – Application of the Mountain Pass theorem to Φ.

Step 3. – Use Ekeland-Hofer's argument (*see* [11]) to show that the Mountain Pass solution has minimal period.

Step 1 (Dual Action Principle). – We begin introducing the Legendre transform U^* of U setting

$$U^*(y) = \sup_{x \in \Omega} \{ (x, y) - U(x) \}.$$

The properties of U^* are collected in the following lemma:

LEMMA 2.2. – $U^* \in C^2(\mathbb{R}^N; \mathbb{R})$ *and is strictly convex. Moreover there are constants* $c_1, c_2, c_3 > 0$ *such that* $\forall y \in \mathbb{R}^N$, $|y|$ *large, one has:*

$$c_1 |y| \leqq U^*(y) \leqq c_2 |y|; \tag{2.1}$$

$$|\nabla U^*(t)| \leqq c_3 \tag{2.2}$$

$$U^*(y) \geqq (1-\theta)\,(y, \nabla U^*(y)). \tag{2.3}$$

Proof. – $\forall y \in \mathbb{R}^N$ the function $f_y(x) := (x, y) - U(x)$, $x \in \Omega$, is strictly concave. Moreover from (A 3°) it follows that $f_y(x) \to -\infty$ as $x \to \Gamma$. Hence, U^* is well defined $\forall y \in \mathbb{R}^N$. Further, taking also into account assumption (i), it is a well known fact in convex analysis that U^* is C^2, strictly convex and that, letting $\xi \in \Omega$ be the (unique) solution of $\nabla U(\xi) = y$, one has:

$$U^*(y) = (\xi, y) - U(\xi) \qquad (y = \nabla U(\xi)), \tag{2.4}$$

as well as

$$\nabla U^*(y) = x \quad \text{iff} \quad \nabla U(x) = y. \tag{2.5}$$

From (2.4) and (A 4°) it follows, for $\xi \in \Gamma_\varepsilon$:

$$U^*(y) \geqq (1-\theta)(\xi, \nabla U(\xi)) \qquad (y = \nabla U(\xi)). \tag{2.6}$$

Inequality (2.6) and (2.5) imply (2.3).

Let $\delta > 0$ be such that $\{x \in \mathbb{R}^N : |x| \leqq \delta\} \equiv B_\delta \subset \Omega$ and let $\bar{c} = \max_{B_\delta} U$. Then $U^*(y) \geqq \sup_{x \in B_\delta} \{(x, y) - \bar{c}\} = \delta |y| - \bar{c}$ and the left hand side inequality in (2.1) follows. Next, by (2.4) and since $U(\xi) \geqq 0$ and Ω is bounded, one has:

$$U^*(y) \leqq (\xi, y) \leqq |\xi| \cdot |y| \leqq c_2 |y|.$$

Lastly, (2.2) follows from (2.5) because Ω is bounded. ∎

Remark 2.3. – Clearly, from (2.3) it follows:

$$U^*(y) \geqq (1-\theta)(y, \nabla U^*(y)) - c, \qquad \forall y \in \mathbb{R}^N \tag{2.7}$$

for some constant c. ∎

We are now ready to state the Dual Action Principle for (1.1).

Let

$$E = \left\{ u \in L^1(0, T; \mathbb{R}^N) : \int_0^T u = 0 \right\}$$

with norm

$$\|u\|_1 = \int_0^T |u| \, dt.$$

For $u \in E$ we can define a linear selfadjoint operator L from E into E by setting:

$$Lu := v \quad \text{iff} \quad -\ddot{v} = u;$$

moreover, since $L(E) \subset W^{2,1}(0, T; \mathbb{R}^N)$, L is compact. It is easy to see that:

$$\|Lu\|_{L^\infty} \leqq \frac{T}{4} \|u\|_1. \tag{2.8}$$

Define the functional $\Phi : E \to \mathbb{R}$ setting:

$$\Phi(u) = \int_0^T \left[U^*(u) - \frac{1}{2}(u, Lu) \right] dt.$$

Remark that $\int_0^T U^*(u)$ makes sense because $U^*(u) \in L^1$, $\forall u \in E$ because of (2.1). Moreover, since $U^* \in C^1$, then Φ is Gateaux differentable on E (but, possibly, not C^1.

If $u \in E$ and $\Phi'(u) = 0$, then $\exists \xi \in \mathbb{R}^N : -Lu + \nabla U^*(u) = \xi$. Setting $x = \nabla U^*(u)$ one has: $x = Lu + \xi$, in particular $x \in W^{2,1}$. Now, if $u \in L^\infty$ (as it is always the case, see arguments of Lemma 2.4 below) then $\nabla U^*(u(t)) \in \tilde{\Omega} \subset\subset \Omega$ a.e., and, x being continuous, $x(t) \in \Omega$, $\forall t$. Finally, $-\ddot{x} = u =$ [by (2.5)] $= \nabla U(x)$. ■

Step 2 (Mountain Pass solution). – We shall apply the Mountain Pass theorem, as stated in [4], thm.5, p. 272, because Φ is not C^1.

LEMMA 2.4. – *Φ satisfies* (P.S.), *namely: if $\{u_n\} \subset E$ is such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ then, up to a subsequence, $u_n \to \bar{u}$ and $\Phi'(\bar{u}) = 0$.*

Proof. – One has:

$$\Phi'(u_n)[u_n] = \int_0^T [(u_n, \nabla U^*(u_n)) - (u_n, Lu_n)] \, dt.$$

Hence, by (2.7) and Hölder's inequality, we obtain:

$$\int_0^T (u_n, Lu_n) \leq \beta \int_0^T U^*(u_n) + \| \Phi'(u_n) \|_{L^\infty} \| u_n \|_1 + c, \left(\beta = \frac{1}{1-\theta} \right)$$

From $\Phi(u_n) \leq C$ and the preceding inequality it follows:

$$\int_0^T U^*(u_n) \leq C + \frac{1}{2} \int_0^T (u_n, Lu_n) \leq C' + \frac{\beta}{2} \int_0^T U^*(u_n) + \frac{1}{2} \| \Phi'(u_n) \|_{L^\infty} \| u_n \|_1.$$

Since $1 - \frac{\beta}{2} > 0$ and using (2.1) one finds c'', $c''' > 0$ such that:

$$\| u_n \|_1 \leq c'' + c''' \| \Phi'(u_n) \|_{L^\infty} \| u_n \|_1. \tag{2.9}$$

Using the fact that $\Phi'(u_n) \to 0$ in L^∞, (2.9) implies $\| u_n \|_1 \leq$ const. Then one has $Lu_n \to \bar{v}$ in C^0, up to a subsequence. On the other hand,

$\xi_n := \frac{1}{T}\int_0^T \nabla U^*(u_n)$ is bounded by (2.2); passing to a subsequence again, we have that $\xi_n \to \bar{\xi}$. Setting $z_n = L u_n + \xi_n + \Phi'(u_n)$, it follows that $z_n \to \bar{z} \equiv \bar{v} + \bar{\xi}$ in L^∞, with $\bar{z}$ continuous function. Since $z_n(t) = \nabla U^*(u_n(t)) \in \Omega$ a. e. (remark that $u_n \in L^1$ implies $|u_n(t)| < +\infty$ a. e.) we have that $\bar{z}(t) \in \bar{\Omega}$, $\forall t \in [0, T]$.

We now claim that, in fact,

$$\bar{z}(t) \in \Omega, \qquad \forall t \in [0, T]. \tag{$\bigstar$}$$

Suppose, first of all, that $\bar{z}(t) \in \partial\Gamma$, $\forall t \in [0, T]$. Then $\forall \varepsilon > 0$, $\exists \bar{n}$ s. t. $n \geqq \bar{n}$ implies $d(z_n(t), \Gamma) < \varepsilon$ a. e. Since $z_n(t) = \nabla U^*(u_n(t))$, we have that $u_n(t) = \nabla U(z_n(t))$ and hence from (A 3°-4°) it follows that $|u_n(t)| \geqq K_\varepsilon$ a. e. $\forall n \geqq \bar{n}$, with $K_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We deduce that $\int_0^T |u_n(t)| \geqq TK_\varepsilon$, in contradiction with the boundedness of $\|u_n\|_1$.

Thus there exists $\bar{t} \in [0, T]$ s. t. $\bar{z}(\bar{t}) \in \Omega$. Since $\bar{z}(t)$ is continuous, $\exists t_1, t_2$ s. t. $\bar{z}(t) \in \Omega$, $\forall t \in [t_1, t_2]$. Since $z_n(t) \to \bar{z}(t)$ in L^∞, then $z_n(t) \in \tilde{\Omega} \subset\subset \Omega$ a. e. for $t \in [t_1, t_2]$ for n large ($\tilde{\Omega} \subset\subset \Omega$ means that $\tilde{\Omega}$ is a compact subset of Ω), and hence $u_n(t) = \nabla U(z_n(t)) \in L^\infty([t_1, t_2])$. Moreover, since ∇U is Lipschitz continuous in $\tilde{\Omega}$, then:

$$u_n(t) = \nabla U(z_n(t)) \to \bar{u}(t) = \nabla U(\bar{z}(t)) \qquad \text{in} \quad L^\infty([t_1, t_2]).$$

Setting $w_n = L u_n + \xi_n$, then, weakly, $u_n(t) = -\ddot{w}_n(t)$, and

$$-\ddot{w}_n(t) \to \nabla U(\bar{z}(t)) \qquad \text{in} \quad L^\infty([t_1, t_2])$$

$$w_n(t) \to \bar{z}(t) \qquad \text{in} \quad L^\infty([t_1, t_2])$$

It follows that, weakly:

$$-\ddot{\bar{z}}(t) = \nabla U(\bar{z}(t)) \qquad \text{in} \quad]t_1, t_2[.$$

From the usual regularity theorem, we then deduce that $\bar{z}(t)$ is a classical solution of $-\ddot{x} = \nabla U(x)$, $\forall t \in]t_1, t_2[$. In particular the conservation of energy holds, i. e.:

$$\frac{1}{2}|\dot{\bar{z}}(t)|^2+U(\bar{z}(t))=\bar{c}\equiv\frac{1}{2}|\dot{\bar{z}}(\bar{t})|^2+U(\bar{z}(\bar{t})),\qquad \forall t\in]t_1, t_2[.$$

Suppose now that $\bar{z}(t)\in\Omega$, $\forall t\in[t_1, t_3[$, $t_2<t_3\leqq T$ and that $\bar{z}(t_3)\in\Gamma$. Since, repeating the preceeding argument, one has that $U(\bar{z}(t))\leqq\bar{c}$, $\forall t\in]t_1, t_3-\varepsilon[$, $\forall\varepsilon>0$ small, it follows that $\bar{z}(t_3)\in\Omega$, contradiction which proves (★).

By the claim it follows that $\bar{z}(t)$ is a continuous function with $\bar{z}(t)\in\Omega$, $\forall t\in[0, T]$. This and $z_n\to\bar{z}$ in L^∞ imply, by the same arguments used in proving the claim, that $u_n\to\nabla U(\bar{z})$ in $L^\infty(0, T)$, and in particular in $L^1(0, T)$; since $\Phi'(\bar{u}) = -L\ \bar{u}+\nabla U^*\ (\bar{u})-\xi=-\bar{v}+\bar{z}=0$, then (P. S.) holds. Moreover it follows that every critical point of Φ is in L^∞, as claimed at the end of Step 1. ■

The behaviour of Φ at $u=0$ and infinity is described in the following lemma.

LEMMA 2.5. — 1° $\exists r, a>0$ *such that* $\Phi(u)\geqq a$, $\forall\|u\|_1=r$;
2° $\exists\bar{v}\in E$, $\|\bar{v}\|_1>r$ *such that* $\Phi(\bar{v})\leqq 0$.

Proof. — Let $\varepsilon>0$ be such that $T<\sqrt{2}/\sqrt{(\omega_N+\varepsilon)}$ and let $\delta>0$ be such that $U(x)\leqq\frac{1}{2}(\omega_N+\varepsilon)|x|^2$, $\forall|x|\leqq\delta$. We define $\rho: [0, \delta[\to\mathbb{R}^+$ to be a continuous, convex function such that:

$$\rho(s)=\frac{1}{2}(\omega_N+\varepsilon)s^2,\quad 0\leqq s\leqq\frac{\delta}{2};\qquad \rho(s)\geqq\frac{1}{2}(\omega_N+\varepsilon)s^2,\quad \frac{\delta}{2}\leqq s<\delta;$$
$$\rho(s)\to+\infty\quad\text{as } s\to\delta^-.$$

Then one has $U(x)\leqq\rho(|x|)$, $\forall|x|<\delta$. By duality, it follows:

$$U^*(y)\geqq\rho^*(|y|),\qquad \forall y\in\mathbb{R}^N, \tag{2.10}$$

where ρ^* is the Legendre Transform of ρ. From (2.10) and (2.8) we obtain

$$\Phi(u)\geqq\int_0^T\rho^*(|u|)-\frac{T}{4}\|u\|_1^2.$$

Remark that $\rho^*(|u|)\in L^1(0, T)$, $\forall u\in E$ because the properties of ρ imply that $\rho^*(\sigma)$ grows linearly at infinity (see similar properties valid for U^*). Using Jensen's integral inequality [14], p. 133, it follows:

$$\Phi(u) \geqq \mathrm{T}\,\rho^*\left(\frac{1}{\mathrm{T}}\|u\|_1\right)-\frac{\mathrm{T}}{4}\|u\|_1^2.$$

Now, for $\sigma \geqq 0$ and small enough, $\rho^*(\sigma)=\frac{1}{2}\frac{1}{(\omega_N+\varepsilon)}\sigma^2$, hence:

$$\Phi(u) \geqq \frac{1}{2(\omega_N+\varepsilon)}\frac{1}{\mathrm{T}}\|u\|_1^2-\frac{\mathrm{T}}{4}\|u\|_1^2. \tag{2.11}$$

provided $\|u\|_1$ is sufficiently small. Since $\mathrm{T}<\sqrt{2}/\sqrt{(\omega_N+\varepsilon)}$, 1° follows.

To prove 2°, we take $v=\lambda\bar{u}$, $\lambda>0$, for any fixed $\bar{u}\in \mathrm{E}$, $\bar{u}\neq 0$. Using (2.1) one has:

$$\Phi(\lambda\bar{u}) \leqq c_2\lambda\|\bar{u}\|_1-\frac{1}{2}\lambda^2\int_0^{\mathrm{T}}(\bar{u}, \mathrm{L}\bar{u});$$

taking λ large enough, 2° follows. ■

In view of lemmas 2.4 and 2.5 ans since Φ' is strong-to-weak* continuous, we can apply the Mountain Pass theorem [4], thm.5, p. 272 finding a critical point $u\in\mathrm{E}$ of Φ. Such an u gives rise to a T-periodic solution of (1.1), $\forall\,\mathrm{T}<\mathrm{T}_0$, according to the Dual Action Principle.

Step 3 (Minimality of the period). – To show that the solution of Mountain Pass found before has minimal period T, we follow the arguments of [11]. We report below the main steps in the proof, indicating the differences.

(i) Ekeland and Hofer deal with a functional like

$$\tilde{\Phi}(u)=\int_0^{\mathrm{T}}\left[\mathrm{H}^*(u)-\frac{1}{2}(u, \tilde{\mathrm{L}}u)\right]dt$$

defined on $\mathrm{E}_\alpha=\left\{u\in \mathrm{L}^\alpha: \int_0^{\mathrm{T}} u=0\right\}$, $1<\alpha<2$, where, roughly, H^* behaves like $|u|^\alpha$ and $\tilde{\mathrm{L}}$ is a compact, linear, self-adjoint operator in E, corresponding to the inverse of $v\to -\mathrm{J}\dot{v}$, J symplectic matrix (in fact they deal with first order Hamiltonian Systems, as we do in paragraph 3 below). Our functional Φ shares the properties of such a $\tilde{\Phi}$, our space being L^1.

(ii) In [11], the Mountain Pass theorem is applied to get a critical point of "Mountain Pass type", as we do.

(iii) Under some smoothness condition, such a critical point possesses a specific topological property. Unfortunaley, neither $\tilde{\Phi}$ in [11], nor our Φ

are C^2; to overcome this point Ekeland and Hofer use a finite-dimensional reduction. Here the fact that we are working in L^1 requires some technical modification: in the Appendix we esplicitely indicate how the major of these changes can be handled.

(iv) The preceding topological properties are expressed in terms of the Morse index, which has to be defined in an appropriate way [11], § II (still for the lack of smoothness). This leads to require, in [11], $H''(z)>0$, $\forall z\neq 0$. In our setting we can repeat the same procedure, provided we assume $U''(x)>0$, $\forall x$ [*see* (i) of theorem 2.1]. Actually, considering the hamiltonian $H(p, x)=\frac{1}{2}|p|^2+U(q)$, assumption (i) is nothing else but the hypothesis of positive defineteness of $H''(p, x)$ in [11]. We point out that this has as a consequence the fact that we find a solution with minimal period T, only for $T>0$ small enough. *See* also Remark 2.6 (i) below.

(v) Lastly, using the properties of the solution of "Mountain Pass type", jointly with the index theory for periodic solutions [10], Ekeland and Hofer show that such a critical point gives rise to a solution of minimal period T. This last step can be carried out in our setting without any modification. This completes the proof. ■

Remark 2.6. — (i) It is possible to obtain other results on the existence of solutions with minimal period for (1.1) using the method developed in [2] and [1], which requires a strenghtening of the convexity of U, but not $U''(0)>0$. This leads to conjecture that also in theorem 2.1 the hypothesis $U''(x)>0$, $\forall x$ could be relaxed to $U''(x)>0$, $\forall x\neq 0$.

(ii) $U\in C^2(\Omega, \mathbb{R})$ is not used in steps 1 and 2, but only in proving the minimality of the period. ■

We end this section showing how our approach, based on the Dual Action Principle, if employed to prove existence results only, permits to handle a wide class of non-convex potentials. Moreover, according to remark 2.6, we will drop hypothesis (i) and assume $U\in C^1(\Omega, \mathbb{R})$ only.

THEOREM 2.7. — *Suppose* U *satisfies assumption* (A) [*with* $U\in C^1$ *in* (A 1°)] *and:*

(*ii*) $\exists\, \delta,\ A>0: U(x)\leqq\frac{1}{2}A|x|^2,\ \forall |x|\leqq\delta;$

(iii) $\exists\, m>0$ *such that* $V(x):=\frac{1}{2}m|x|^2+U(x)$ *is convex.*

Then, $\forall T>0$, (1.1) *has at least a nontrivial* T-*periodic solution.*

Proof. — First of all we remark that V satisfies (A). In fact, 1°-2°-3° [with $V \in C^1$ in (A 1°)] are trivial. As for (A 4°), taking $\theta < \bar{\theta} < \frac{1}{2}$ one finds (for all $x \in \Gamma_\varepsilon$):

$$V(x) - \bar{\theta}(x, \nabla V(x)) \leqq (\theta - \bar{\theta})(x, \nabla U(x)) + \left(\frac{1}{2} - \bar{\theta}\right) m |x|^2.$$

Since $(x, \nabla U(x)) \geqq \frac{1}{\theta} U(x) \to +\infty$ as $x \to \Gamma$, Ω is bounded and $\theta - \bar{\theta} < 0$, it follows that $V(x) \leqq \bar{\theta}(x, \nabla V(x))$, $\forall x \in \Gamma_{\varepsilon'}$, for some $0 < \varepsilon' < \varepsilon$.

Next, we write (1.1) in the equivalent form:

$$-\ddot{x} + mx = \nabla V(x). \tag{2.12}$$

Let $X = L^1(0, T: \mathbb{R}^N)$ and set $X = \mathbb{R}^N \oplus E$, with E as in theorem 2.1. For $T > 0$ small enough $-\frac{d^2}{dt^2} + m$ is invertible in X, with inverse L_m, i.e. $L_m u = v$ iff $-\ddot{v} + mv = u$. It is easy to check that L_m is compact and:

$$\| L_m u \|_{L^\infty} \leqq \frac{T}{4 - m T^2} \| u \|_1, \qquad \forall u \in E. \tag{2.13}$$

As in theorem 2.1, we will find a solution of (2.12) [hence of (1.1)], by the Dual Action Principle, which now leads to look for critical points on X of:

$$\Phi(u) = \int_0^T \left[V^*(u) - \frac{1}{2}(u, L_m u) \right] dt.$$

Let us point out that $V \in C^1$ and strictly convex (which we can assume without loss of generality taking m possibly larger) suffices to have ($V^* \in C^1$ and) Φ Gateaux differentiable. By (ii) one has $V(x) \leqq \frac{1}{2}(m + A) |x|^2$, $\forall |x| \leqq \delta$, and using (2.13) one readily verifies, as for (2.11), that:

$$\Phi(u) \geqq \left(\frac{1}{2(A + m)} \frac{1}{T} - \frac{T}{4 - m T^2} \right) \| u \|_1^2, \qquad \forall u \in E, \quad \| u \|_1 \text{ small},$$

so that $\exists \alpha, r > 0$: $\Phi(u) \geqq \alpha$, $\forall u \in E$, $\| u \|_1 = r$, provided $T > 0$ is small enough. Next, $\forall \xi \in \mathbb{R}^N$ one has

$$\Phi(\xi) = \int_0^T V^*(\xi) - \frac{1}{2} \frac{|\xi|^2}{m} \leqq c |\xi| - \frac{1}{2m} |\xi|^2.$$

This fact and a slight modification of the arguments employed in lemma 2.5-2° leads to: $\exists R > r$ such that $\Phi(v) \leqq 0$, $\forall v \in \mathbb{R}^N \oplus \tilde{E}$, $\|v\|_1 \geqq R$, $\tilde{E}$ being any finite dimensional subspace of E.

Then we can apply the linking theorem 6, p. 274 of [4], finding a nontrivial critical point of Φ on X, provided $T > 0$ is small enough, say $T < \tilde{T}$. This critical point gives rise to a T-periodic solution of (1.1), $\forall\, 0 < T < \tilde{T}$. If $T \geqq \tilde{T}$ it suffices to take an integer k such that $T/k < \tilde{T}$ and apply the above result. ■

Remark 2.8. – If U is defined on all $\mathbb{R}^N$ and behaves like $|x|^\beta$, $\beta > 2$ for x large, then $U^* : \mathbb{R}^N \to \mathbb{R}$ is like $|y|^\alpha$ for y large, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, hence $1 < \alpha < 2$. Then the case discussed in theorem 2.7 can be regarded as a limiting case of the above one. ■

3. HAMILTONIAN SYSTEMS

In this section we discuss first order Hamiltonian Systems. Let:

$$C = \{ (p, q) \in \mathbb{R}^N \times \mathbb{R}^N : q \in \Omega \},$$

Ω being, as in paragraph 2, a bounded, convex domain in $\mathbb{R}^N$. We still set $\Gamma = \partial\Omega$.

We say that $H : C \to \mathbb{R}$ satisfies *Assumption B* if:

1° $H \in C^2(C; \mathbb{R})$;

2° $H(0, 0) = 0 = \min_C H$;

3° There exist $U_i : \Omega \to \mathbb{R}\ (i = 1,2)$, $\varepsilon > 0$, $k > 0$ and $\theta \in \left]0, \frac{1}{2}\right[$ such that:

(i) U_1, U_2 are convex and satisfy (A 1°-2°-3°);

(ii) $U_i(q) \leqq \theta(q, \nabla U_i(q))$, $\forall q \in \Gamma_\varepsilon$;

(iii) $U_1(q) \leqq U_2(q) \leqq U_1(q) + k$;

(iv) $\begin{cases} a_1 |p|^{1/\theta} + U_1(q) \leqq H(p, q) \leqq a_2 |p|^{1/\theta} + U_2(q), \\ \qquad \forall (p, q) \in C,\ a_1, a_2 > 0; \end{cases}$

4° $\exists K \subset\subset C$ such that $H(p, q) \leqq \theta[(p, H_p) + (q, H_q)]$, $\forall (p, q) \in C - K$.

Here H_p, H_q denote the partial derivatives $\partial H/\partial p$, $\partial H/\partial q$.

First of all, we deduce from (B) the following lemma concerning the behaviour of H_p, H_q.

LEMMA 3.1. — *Suppose* H *satisfies* (B) *and is convex. Then* $\exists c_1, c_2>0$ *s. t.*

$$|H_p(p, q)| \leqq c_1 |p|^{(1/\theta)-1} + c_2, \qquad \forall (p, q) \in C. \tag{3.1}$$

Moreover, for all compact subset $\Omega_1 \subset \Omega$, $\exists c_3, c_4>0$ *s. t.*

$$|H_q(p, q)| \leqq c_3 |p|^{1/\theta} + c_4, \qquad \forall p \in \mathbb{R}^N, \quad \forall q \in \Omega_1. \tag{3.2}$$

Proof. — Since H is convex, $\forall x, p \in \mathbb{R}^N$, $\forall q \in \Omega$

$$H(p, q) + (H_p(p, q), x-p) \leqq H(x, q)$$

and, using [B 3° (iii)-(iv)], one has:

$$(H_p(p, q), x-p) \leqq a_2 |x|^{1/\theta} - a_1 |p|^{1/\theta} + k.$$

Taking the supremum for $|x-p|=|p|$, it follows:

$$|H_p(p, q)| \,.\, |p| \leqq a_2 |2p|^{1/\theta} - a_1 |p|^{1/\theta} + k.$$

Then (3.1) follows directly.

Let $\Omega_1 \subset\subset \Omega$ and set $\bar{\varepsilon} = d(\Omega_1, \Gamma)$. As before, $\forall q \in \Omega_1$, $y \in \Omega$,

$$(H_q(p, q), y-q) \leqq (a_2 - a_1) |p|^{1/\theta} + U_2(y) - U_1(q).$$

Since $U_1(q) \geqq 0$, and taking the supremum for $|y-q| = \bar{\varepsilon}/2$, it follows

$$|H_q(p, q)| \bar{\varepsilon}/2 \leqq (a_2 - a_1) |p|^{1/\theta} + \sup_{|y-q|=\bar{\varepsilon}/2} U_2(q).$$

Since $\{y : |y-q| = \bar{\varepsilon}/2\} \subset \Omega - \Gamma_{\bar{\varepsilon}/2}$, $\forall q \in \Omega_1$, one has:

$$\sup_{|y-q|=\bar{\varepsilon}/2} U_2(y) \leqq c_5$$

and (3.2) holds. ■

We look for T-periodic solutions of:

$$\begin{aligned} \dot{p} &= -H_q(p, q) \\ \dot{q} &= H_p(p, q) \end{aligned} \tag{3.3}$$

We will show:

THEOREM 3.2. — *Suppose* H *satisfies assumption* (B) *and let* $H''(p, q) > 0$, $\forall (p, q) \in C - \{(0, 0)\}$:

(*a*) *if* $U_2''(0)$ *is positive definite, then* $\exists T_0>0$ *such that* $\forall 0<T<T_0$ (3.1) *has a* T-*periodic solution with minimal period* T;

(*b*) *if* $U_2''(0)=0$, *then* $\forall T>0$ (3.1) *has a periodic solution with minimal period* T.

The argument is similar to that of theorem 2.1, but requires some modifications because now H has a different behaviour in p and q.

We set

$$H^*(x, y)=\sup_{(p, q)\in C}\{(x, p)+(y, q)-H(p, q)\}$$

The properties of H^* are collected in the following lemma:

LEMMA 3.3. – (i) $H^*\in C^1(\mathbb{R}^{2N}; \mathbb{R})\cap C^2(\mathbb{R}^{2N}-\{0\}; \mathbb{R})$ *and strictly vonvex;*

(ii) $H^*(w)\geqq(1-\theta)(w, \nabla H^*(w))$, $w=(x, y)\in\mathbb{R}^{2N}$, $|w|$ *large enough;*

(iii) $a_2^*|x|^\beta+U_2^*(y)\leqq H^*(x, y)\leqq a_1^*|x|^\beta+U_1^*(y)$, $\forall(x, y)\in\mathbb{R}^{2N}$, *where* $\beta=\dfrac{1}{1-\theta}$ *is the conjugate exponent of* $1/\theta$, $a_i^*>0$ *and* U_i^* *are the Legendre transforms of* U_i $(i=1, 2)$, *hence satisfy the properties listed in lemma* 2.2:

(iv) $|H_y^*(x, y)|\leqq c_6$;

(v) $|H_x^*(x, y)|\leqq c_7|x|^{\beta-1}+c_8$

(*here and in the following* c_6, c_7, ... *denote positive constants*).

Proof. – H^* is defined in all $\mathbb{R}^{2N}$ because H is convex, [B 3° (iv)] holds and U_1 satisfies (A).

(i) It follows from the usual regularity of the Legendre transform.

(ii) It follows by duality from (B 4°).

(iii) It follows by duality from [B 3° (iv)].

(iv) We recall that

$$(x, y)=(H_p(p, q), H_q(p, q)) \quad \text{iff} \quad (p, q)=(H_x^*(x, y), H_y^*(x, y))$$

Since $q\in\Omega$ and Ω is bounded, (iv) follows.

(v) As in lemma 3.1 (replacing H with H^*), one uses (iii) above to find:

$$(H_x^*(x, y), \xi-x)\leqq a_1^*|\xi|^\beta+U_1^*(y)-a_2^*|x|^\beta-U_2^*(y) \qquad (3.4)$$

It is easy to check that [B 3° (iii)] implies

$$U_1^*(y)-U_2^*(y)\leqq k.$$

Hence, from (3.4):

$$(H_x^*(x, y), \xi - x) \leqq a_1^* |\xi|^\beta - a_2^* |x|^\beta + k.$$

Taking the supremum for $|\xi - x| = |x|$, (v) follows as in lemma 3.1. ■

According to lemma 3.3 (iii) and (2.1), the behaviour at infinity of H^* is linear in y and like $|x|^\beta$ in x. This fact suggests the following choice of the space to work with: let $X = E_\beta \times E_1$, where

$$E_\kappa = \left\{ u \in L^\kappa(0, T; \mathbb{R}^N) : \int_0^T u = 0 \right\},$$

and define $\forall u \in E_\kappa (\kappa = 1, \beta)$ $\mathscr{L} u = z$ iff $\dfrac{dz}{dt} = u$. We remark that:

$$\mathscr{L}(E_\beta) \subset W^{1, \beta} \subset\subset C^\circ; \tag{3.5}$$

while

$$\mathscr{L}(E_1) \subset W^{1, 1} \subset\subset L^{1/\theta}. \tag{3.6}$$

For $(v, w) \in X$, we set

$$f(v, w) = \int_0^T \left[H^*(v, w) - \frac{1}{2}(w, \mathscr{L} v) + \frac{1}{2}(v, \mathscr{L} w) \right] dt$$

Let us remark that f is well defined because: $H^*(v, w) \in L^1$ by lemma 3.3 (iii) and (2.1); $\mathscr{L} v \in C^\circ$ and $\mathscr{L} w \in L^{1/\theta}$ in view of (3.5) and (3.6). Moreover from lemma 3.3 (iv)-(v) we infer that f is Gateaux differentiable on X and

$$D_v f(v, w)[\varphi] = \int_0^T [(H_x^*(v, w), \varphi) + (\varphi, \mathscr{L} w)] \, dt, \qquad \forall \varphi \in E_\beta$$

and

$$D_w f(v, w)[\psi] = \int_0^T [(H_y^*(v, w), \psi) - (\psi, \mathscr{L} v)] \, dt, \qquad \forall \psi \in E_1$$

If (v, w) is a critical point of f on X, $\exists (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

$$H_x^*(v, w) + \mathscr{L} w = \xi, \qquad H_y^*(v, w) - \mathscr{L} v = \eta \tag{3.7}$$

Setting $p = \xi - \mathscr{L} w$ and $q = \eta + \mathscr{L} v$, one deduces $H_x^*(v, w) = p$ and $H_y^*(v, w) = q$, namely, by duality:

$$v=H_p(p, q), \qquad w=H_q(p, q). \tag{3.8}$$

Moreover, from $p=\xi-\mathscr{L} w$, $q=\eta+\mathscr{L} v$ and the definition of $\mathscr{L}$, one has:

$$\dot{p}=-w, \qquad \dot{q}=v. \tag{3.9}$$

Actually $w \in L^\infty$ (see arguments of lemma 3.4 below) and we deduce, as for the Dual Action Principle in paragraph 2, that $(p(t), q(t)) \in C$ for all t and, from (3.8), (3.9), that is a T-periodic solution of (3.3).

According to the preeceding discussion, we will look for non-trivial critical points of f on X. For this we begin proving:

LEMMA 3.4. – *(P.S.) holds for f on* X.

Proof. – Let $f(v_n, w_n)$ be bounded and $f'(v_n, w_n):=(\varphi_n, \psi_n)$, with

$$\varphi_n \equiv H_x^*(v_n, w_n)+\mathscr{L} w_n-\xi_n \to 0 \quad \text{in } L^{1/\theta} \tag{3.10}$$

and

$$\psi_n \equiv H_y^*(v_n, w_n)-\eta_n \to 0 \quad \text{in } L^\infty. \tag{3.11}$$

By the same arguments used in lemma 2.4, one finds:

$$\left(1-\frac{\beta}{2}\right)\int_0^T H^*(v_n, w_n) \leqq c_9+c_{10}[\|\varphi_n\|_{1/\theta}\|v_n\|_\beta+\|\psi_n\|_{L^\infty}\|w_n\|_1].$$

From lemma 3.3 (iii) and (2.1), if follows:

$$c_{11}(\|v_n\|_\beta^\beta+\|w_n\|_1) \leqq c_9+c_{10}[\|\varphi_n\|_{1/\theta}\|v_n\|_\beta+\|\psi_n\|_{L^\infty}\|w_n\|_1].$$

and also here we find that $\|v_n\|_\beta+\|w_n\|_1$ is bounded. From (3.5)-(3.6) it follows (without relabeling)

$$\mathscr{L} v_n \to \bar{q} \quad \text{in } C^\circ,$$
$$\mathscr{L} w_n \to \bar{p} \quad \text{in } L^{1/\theta}.$$

Also:

$$\xi_n=\frac{1}{T}\int H_y^*(v_n, w_n) \leqq \text{const.}$$

by lemma 3.3 (iv) and

$$\eta_n = \frac{1}{T}\int H_x^*(v_n, w_n) \leqq \text{const.}$$

because $\|v_n\|_\beta \leqq$ const. and lemma 3.3 (v) holds. Therefore

$$\mathscr{L} w_n - \xi_n \to \bar{p} - \bar{\xi} \quad \text{in } L^{1/\theta}$$
$$\mathscr{L} v_n - \eta_n \to \bar{q} - \bar{\eta} \quad \text{in } C^0$$

and

$$p_n \equiv \varphi_n - \mathscr{L} w_n + \xi_n \to -\bar{p} + \bar{\xi} \quad \text{in } L^{1/\theta} \tag{3.12}$$
$$q_n \equiv \psi_n + \mathscr{L} v_n + \eta_n \to \bar{q} + \bar{\eta} \quad \text{in } L^\infty. \tag{3.13}$$

From (3.10), (3.11) we deduce:

$$v_n = H_p(p_n, q_n)$$
$$w_n = H_q(p_n, q_n).$$

Assuming $\bar{q}(t) + \bar{\eta} \in \Omega_1 \subset\subset \Omega$, $\forall t \in [0, T]$, it is easy to complete the proof. In fact (3.1) [resp. (3.2)] and (3.12) [resp. (3.13)] enables to use the generalized Lebesgue Dominated Convergence Theorem (as stated for example, in [17], Theorem 3.9) and get

$$v_n \to H_p(-\bar{p} + \bar{\xi}, \bar{q} + \bar{\eta}) \quad \text{in } L^\beta$$

respectively

$$w_n \to H_q(-\bar{p} + \bar{\xi}, \bar{q} + \bar{\eta}) \quad \text{in } L^1.$$

Lastly, to show that $\bar{q}(t) + \bar{\eta} \in \Omega_1 \subset\subset \Omega$ one argues as in lemma 2.4, proving first of all that $\{\bar{q}(t) + \bar{\eta}\}_{t \in [0, T]} \not\subset \Gamma$ so that $\bar{q}(t) + \bar{\eta} \in \Omega$ for $t \in [t_1, t_2] \subset [0, T]$. Then, by the same limiting procedures used above, one shows again that $(-\bar{p} + \bar{\xi}, \bar{q} + \bar{\eta})$ is a solution of (3.3) in $[t_1, t_2]$, and hence cannot reach Γ because $H(-\bar{p}(t) + \bar{\xi}, \bar{q}(t) + \bar{\eta}) = \text{const}$. This completes the proof. ■

Proof of theorem 3.2. — In order to apply the Mountain Pass theorem to f on X, it remains to study the behaviour of f at $v = w = 0$. For this we use lemma 3.3 (iii) and find:

$$f(v, w) \geqq c_{12} \|v\|_\beta^\beta + \int_0^T U_2^*(w) - c_{13}[\|v\|_\beta^2 + \|w\|_1^2].$$

Consider first case (*a*). Arguing as in lemma 2.5 we conclude that (0, 0) is a strict local minimum for f on X, provided $T > 0$ is small enough. In case (*b*) $U_2(q) \leqq \frac{1}{2} a |q|^2$ near $q=0$ with $a > 0$ arbitrary, and (0, 0) is a strict local minimum for every $T > 0$.

Applying the Mountain Pass Theorem we find a critical point $(v, w) \neq (0, 0)$ of f which corresponds to a T-periodic solution of (3.3). Lastly, the minimality of the period T follows as in step 3 of the proof of theorem 2.1. This time our setting is even closer to that of [11], because we are dealing now with first order systems and, thanks to the hypothesis $H''(p, q) > 0$, $\forall (p, q) \neq (0, 0)$, step 3 (iv) in the proof of theorem 2.1 is just the same as in [11]. ■

We end the paper with some remarks.

Remark 3.5. — By the same arguments, many situations as the preeceding ones can easily be handled. For example, one could study second order systems as (1.1) when $U: \mathscr{F} \to \mathbb{R}$, $\mathscr{F} = \{(x_1, x_2) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : x_1 \in \Omega\}$, $\Omega \subset \mathbb{R}^k$ bounded and convex, and U has a superquadratic behaviour in x_2 and $U \to +\infty$ as $x_1 \to \Gamma$.

Remark 3.6. — We remark that also here, as in paragraph 2, it would be possible to state existence results for (3.3) relaxing the convexity assumption at the expenses of the minimality of the period.

Remark 3.7. — We are not able to handle the case of a Hamiltonian H defined in a *bounded* convex domain $D \subset \mathbb{R}^{2N}$, with $H \to +\infty$ as $z \to \partial D$ (which, however, from the physical point of view, seems to be less significant than the one studied in theorem 3.2). In fact, in such a case, we should work in $L^1 \times L^1$ and, due to a lack of compactness, we do not know whether (P.S.) holds true.

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APPENDIX

First we indicate how the proof of lemma 7 in [11] can be carried out in our setting.

Recall that such a lemma deals with a finite-dimensional reduction used to overcome the lack of smoothness. We keep most of the notation of [11], which is assumed familiar to the reader.

The argument is by contradiction: in our setting, we have to suppose:

$$\exists \xi_n \in \mathbb{R}^N, \quad \exists v_n \in Z = \left\{ u \in L^1(a, b; \mathbb{R}^N) : \int_a^b u = 0 \right\},$$

$v_n \neq 0$ and $\gamma_n \to 0$ such that:

$$\left| \xi_n - \int_a^b \bar{u} \right| \leqq \varepsilon; \tag{A 1}$$

$$\| v_n \|_1 \leqq \gamma_n; \tag{A 1'}$$

$$\Phi(r(\xi_n) + v_n) = \inf \{ \Phi(r(\xi_n) + v) : v \in Z, \| v \|_1 \leqq \gamma_n \}. \tag{A 1''}$$

where $[a, b] \subset [0, T]$, $\Phi(u)$ stands for

$$\Phi_{a, b}(u) := \int_a^b \left[U^*(u) - \frac{1}{2}(u, L u) \right] dt$$

and $r = r_{a, b}$ is a C^1 map from $\mathbb{R}^N$ to $C([a, b], \mathbb{R}^N)$ defined in lemma 6 of [11]. We have to show that $v_n \to 0$ uniformly in $[a, b]$. The difference with [11] is that $\| \cdot \|_1$ is subdifferentiable (it is convex), but is not differentiable (not even Gateaux differentiable), so that the optimality condition will be written, according to the Lagrange multiplier rule [7], Thm. 6.1.1, as inclusions involving subdifferentials: $\exists \eta_n \in \mathbb{R}^N$ and $\lambda_n \geqq 0$ such that:

$$\left. \begin{array}{c} 0 \in \Phi'(r(\xi_n) + v_n) + \lambda_n \partial \| \cdot \|_1 (v_n) + \eta_n \\ \lambda_n (\| v_n \|_1 - \gamma_n) = 0 \end{array} \right\} \tag{A 2}$$

Recall that

$$\partial \| \cdot \|_1 (v_n) = \left\{ p \in L^\infty(a, b) : \int_a^b (p, v_n) = \| v_n \|_1, \| p \|_{L^\infty} = 1 \right\}$$

([4], 4.6.10). By the definition of $r(\xi_n)$ one has also $\Phi'(r(\xi_n))=\eta'_n\in\mathbb{R}^N$. Hence:

$$\left.\begin{array}{c}\Phi'(r(\xi_n)+v_n)-\Phi'(r(\xi_n))-\eta''_n\in-\lambda_n\,\partial\,\|\,.\,\|_1(v_n),\\ \eta''_n\in\mathbb{R}^N.\end{array}\right\}\tag{A 3}$$

From (A 1) it follows that $\xi_n\to\xi$ (up to subsequences). Then $r(\xi_n)\to r(\xi)$ in $C([a,b])$, and from (A 1') $r(\xi_n)+v_n\to r(\xi)$ in L^1. Since Φ is C^1 one has: $\Phi'(r(\xi_n)+v_n)-\Phi'(r(\xi_n))\to 0$ in L^∞. Remark that (A 3) means that $\exists\, p_n\in\partial\,\|\,.\,\|_1(v_n)$ such that:

$$\Phi'(r(\xi_n)+v_n)-\Phi'(r(\xi_n))-\eta''_n=-\lambda_n p_n.$$

Multiplying by $v_n/\|v_n\|_1$ and integrating, this leads to:

$$-\lambda_n=\int_a^b\left(\frac{v_n}{\|v_n\|_1},\,\Phi'(r(\xi_n)+v_n)-\Phi'(r(\xi_n))\right)dt;$$

then it follows: $\lambda_n\to 0$ and $|\eta''_n|\to 0$.

Since $\Phi'(u)=\nabla U^*(u)-Lu$, (A 3) becames

$$\nabla U^*(r(\xi_n))+Lv_n+\eta''_n\in\nabla U^*(r(\xi_n)+v_n)+\lambda_n\,\partial\,\|\,.\,\|_1(v_n).\tag{A 4}$$

Let

$$f_n(t):=\nabla U^*(r(\xi_n)(t))+Lv_n(t)+\eta''_n.\tag{A 5}$$

Remark taht $f_n(.)$ is continuous and $f_n\to\nabla U^*(r(\xi))$ uniformly in $[a,b]$, because $\eta''_n\to 0$, $v_n\to 0$ in L^1 and L is continuous from L^1 to $W^{2,1}\subset C^0$, $\xi_n\to\xi$ and r as well as ∇U^* are continuous (1).

Set, for $(t,y)\in[a,b]\times\mathbb{R}^N$:

$$U^*_n(t,y):=U^*(r(\xi_n)(t)+y)+\lambda_n|y|.$$

U^*_n is strictly convex in y because so is U^* and one has:

$$\partial_y U^*_n(t,y)=\nabla U^*(r(\xi_n)(t)+y)+\lambda_n\,\partial\,|\,.\,|(y).$$

(1) In the case of theorem 3.2, the convergence of one of the components of f_n would be in L^∞. However, $\nabla H^*(r(\xi))$ is still a continuous function, and the arguments can be carried over with minor changes only.

It is easy to verify that $p_n \in \partial \|.\|_1 (v_n)$ implies $p_n(t) \in \partial |.| (v_n(t))$, a. e. in $[a, b]$. Hence from (A 4) it follows:

$$f_n(t) \in \partial_y U_n^*(t, v_n(t)) \quad \text{a. e. in } [a, b].$$

By the Legendre reciprocity formula ([4], p. 203) one has:

$$v_n(t) \in \partial U_n(t, f_n(t)) \tag{A 6}$$

where $U_n(t, x) = \max_{y \in \mathbb{R}^N} \{(x, y) - U_n^*(t, y)\}$ indicates the Fenchel transform of U_n^*.

For $\lambda_n = 0$ $U_n^*(t, y)$ becames $U^*(r(\xi_n)(t) + y)$ and for its Fenchel transform $U_n^0(t, x)$ one has:

$$U_n^0(t, x) = U(x) - (x, r(\xi_n)(t)).$$

Moreover, since $U_n^*(t, y) \geqq U^*(r(\xi_n)(t) + y)$, it follows:

$$U_n(t, x) \leqq U_n^0(t, x).$$

Next, $U_n(t, .)$ is C^1 because U^* is strictly convex. Then ∂U_n is singleton and (A 6) turns out to be:

$$v_n(t) = \nabla U_n(t, f_n(t)). \tag{A 7}$$

From now on the proof proceeds as in [11] and leads to show that $v_n \to 0$ uniformly in $[a, b]$ since f_n converges uniformly to $\nabla U^*(r(\bar{\xi})) \equiv \bar{f}$, and obviously $\bar{f}(t) \in \Omega$, $\forall t \in [a, b]$. This shows that lemma 7 of [11] holds.

All the remaining arguments of [11] concerning the finite dimensional reduction are local in nature and work in our setting as well. ■

REFERENCES

[1] A. Ambrosetti, Nonlinear Oscillatiations with Minimal Period, *Proceed. Symp. Pure Math.*, Vol. **44**, 1985 pp. 29-35.

[2] A. Ambrosetti and G. Mancini, Solutions of Minimal Period for a Class of Convex Hamiltonian Systems, *Math. Ann.*, Vol. **255**, 1981, pp. 405-421.

[3] A. Ambrosetti and P. Rabinowitz, Dual Variational Methods in Critical Point Theory and Applications, *J. Funct. Anal.*, Vol. **14**, 1973, pp. 349-381.

[4] J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New York, 1984.

[5] V. Benci, Normal Modes of a Lagrangian System Constrained in a Potential Well, *Ann. I.H.P. "Analyse non lineare"*, Vol. **1**, 1984, pp. 379-400.

[6] F. CLARKE, Periodic Solutions of Hamiltonian Inclusions, *J. Diff. Eq.*, Vol. **40**, 1981, pp. 1-6.

[7] F. CLARKE, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[8] F. CLARKE and I. EKELAND, Hamiltonian Trajectories having Prescribed Minimal Period, *Comm. Pure and Appl. Math.*, Vol. **33**, 1980, pp. 103-116.

[9] I. EKELAND, Periodic Solutions to Hamiltonian Equations and a Theorem od P. Rabinowitz, *J. Diff. Eq.*, Vol. **34**, 1979, pp. 523-534.

[10] I. EKELAND, Une théorie de Morse pour les systèmes hamiltoniens convexes, *Ann. I.H.P.* "*Analyse non lineare*", Vol. **1**, 1984, pp. 19-78.

[11] I. EKELAND and H. HOFER, *Periodic Solutions with Prescribed Period for Convex Autonomous Hamiltonian Systems*, Inv. Math. **81** (1985), pp. 155-188).

[12] M. GIRARDI and M. MATZEU, *Periodic Solutions of Convex Hamiltonian Systems with a Quadratic Growth at the Origin and Superquadratic at Infinity*, preprint, Univ. degli Studi di Roma, Roma, 1985.

[13] M. GIRARDI and M. MATZEU, Some Results on Solutions of Minimal Period to Hamiltonian Systems, in *Nonlinear Oscillations for Conservative Systems*, A. AMBROSETTI Ed., Pitagora, Bologna, 1985, pp. 27-35.

[14] A. KUFNER, O. JOHN and S. FUCIK, *Function Spaces*, Academia, Prague, 1977.

[15] P. RABINOWITZ, Periodic Solutions of Hamiltonian Systems, *Comm. Pure and Appl. Math.*, Vol. **31**, 1978, pp. 157-184.

[16] P. RABINOWITZ, Periodic Solutions of Hamiltonian Systems: a Survey, *S.I.A.M. J. Math. Anal.*, Vol. **13**, 1982, pp. 343-352.

[17] J. J. BENEDETTO, *Real Variable and Integration*, B. G. Teubner, Stuttgart, 1976.

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