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Linearization and normal form of the Navier-Stokes equations with potential forces (*)

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ABSTRACT. — We derive a normalization theory for the Navier-Stokes equations with potential (gradient) body forces by means of a global asymptotic expansion of a solution as time goes to infinity. The normal form is the linear Navier-Stokes system if the spectrum of the Stokes operator has no resonances. In the general case, the normal form is an equation in a suitable Fréchet space, whose nonlinear terms correspond to resonances. However, it can be solved by integrating successively an infinite sequence of linear nonhomogeneous differential equations. The normalization mapping is globally defined, analytic, one to one. We illustrate our theory by two simple examples. In particular we relate our normalization for the Burgers equation to the Hopf-Cole transform.

RÉSUMÉ. — On construit une forme normale pour les équations de Navier-Stokes soumises à des forces dérivant d'un potentiel, à l'aide d'un

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développement asymptotique global de la solution quand $t \to \infty$. La forme normale correspond au système linéaire de Navier-Stokes si le spectre de l'opérateur de Stokes n'a pas de résonance. Dans le cas général, la forme normale est une équation dans un espace de Fréchet convenable dont les termes non linéaires correspondent aux résonances. Cependant, on peut le résoudre en intégrant successivement une suite infinie d'équations différentielles linéaires non homogènes. L'opérateur de normalisation est défini globalement, de façon analytique et injective. Nous illustrons notre théorie par deux exemples, en particulier dans le cas de l'équation de Burgers, en reliant notre théorie à la transformée de Hopf-Cole.

Mots-clés : Navier-Stokes equations, Nonlinear spectral manifolds, Frechet space, Nonresonant spectrum, Parabolic nonlinear equations, Asymptotic expansion, Burgers equations. Normal forms.

INTRODUCTION

Let us consider the Navier-Stokes equations

$$(0.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u + \nabla p = 0 \\ \nabla . u = 0 \\ u(x,0) = u_0(x) \end{cases}$$

in $\Omega \times]0, \infty[$, where Ω is a smooth bounded open set in $\mathbb{R}^n$, $n = 2, 3$, or the cube $]0, L[^n$.

We supplement (0.1) with the boundary condition

$$(0.2) \qquad u\,|_{\partial\Omega} = 0\,,$$

or with the spatial periodic condition

$$(0.3) \qquad u(x + Le_j) = u(x) \qquad \forall x \in \mathbb{R}^n, \qquad 1 \leq j \leq n$$

where (e_j), $1 \leq j \leq n$ is the canonical basis in $\mathbb{R}^n$.

In our previous works [*7*] [*8*] we investigated the asymptotic behavior of regular (we shall give a precise meaning of this concept later on) solutions u of (0.1), (0.2) (resp. (0.1), (0.3)) as $t \to +\infty$. Essentially, $u(t)$ decays exactly as $e^{-\nu\Lambda(u_0)t}$, where $\Lambda(u_0) \in \{\Lambda_1, \Lambda_2, \ldots\}$, the set of eigenvalues of the Stokes operator. Moreover, it was shown that there exists in the space $\mathscr{R}$ of initial data u_0 a flag of smooth analytic manifolds M_k, $k = 1, 2, \ldots$ such that $\Lambda(u_0) = \Lambda_k$, the k^{th} distinct eigenvalue of Stokes operator, if and only if $u_0 \in M_{k-1} \backslash M_k$, where $M_0 = \mathscr{R}$; the *spectral manifolds* M_k

have finite codimension $m_1 + m_2 + \ldots + m_k$ (where m_j = multiplicity of Λ_j), and in the periodic case (0.3) are « genuinely » nonlinear.

In the paper [*7*] we also gave the first steps of an asymptotic expansion of $u(t)$ as $t \to +\infty$ (See [*7*], Theorem 2).

The aim of the present paper, which develops our Note [*9*], is to achieve this asymptotic expansion and to derive some consequences of its global properties. In particular, if the spectrum of the Stokes operator is *nonresonant*, we shall construct a nonlinear functional transformation which linearizes the Navier-Stokes equations and the nonlinear spectral manifolds M_k.

The precise sense of linearization will be given later on in this paper. Roughly speaking, the value of the solution u at time t is obtained by applying the *linear* semi-group associated to the Stokes operator (suitably extended) to the nonlinear transform $U(u_0)$ of the initial data u_0. The (analytic and one to one) mapping U associates to u_0 the generating part of the asymptotic expansion of u. Of course U is not given in a closed form and there is no serious hope that one could define it so, as for example the Cole-Hopf transform [*14*] [*24*] which reduces the viscous one dimensional Burgers equation to the linear heat equation. However our linearization has its own theoretical interest; moreover it is obtained in a totally different fashion than previous theories of « linearization » of nonlinear differential operators: inverse scattering theory developed e. g. for one dimensional nonlinear equations such as KdV, Schrödinger [*11*] [*19*]; the classical theory of normal forms for ordinary nonlinear differential equations (cf. for instance the description of the work of Poincaré and Dulac in [*1*]); the very interesting papers of Nikolenko [*16*] [*17*] [*26*] [*27*] and Zehnder [*25*] on the extension of Siegel's Theorem to a class of nonlinear-Schrödinger equation satisfying a diophantine condition; the work of M. S. Berger et al. on « diagonalization » of nonlinear ordinary differential operators [*2*].

If the spectrum of the Stokes operator is resonant, we also give the asymptotic expansion of the solution as $t \to \infty$; the coefficients are now polynomials in t. The equivalent of the linearization map U is a normalization mapping W which reduces the Navier-Stokes equations to a *normal form* where the nonlinear terms correspond to resonances. In the nonresonant case the normal form is of course the linear Navier-Stokes equations; although in general the normal equation is nonlinear, it can always be solved by integrating successively an infinite sequence of *linear* nonhomogeneous differential equations. The normalization map is also built from the generating part of the asymptotic expansion. As U, it has nice properties: it is analytic, one to one, its derivative at 0 is the identity. Moreover, our normalization mapping can be explicitely expressed from any analytic mapping which admits the identity as derivative at 0 and which linearizes the Navier-Stokes equations. In the nonresonant case, any such mapping reduces to our normalization. To our knowledge this kind of normalization of a nonlinear parabolic equation is new. N. V. Nikolenko gives in [*18*] a derivation (along the lines of the Poincaré-Dulac theory) of the truncated normal form for a class of perturbed KdV-Burgers equations.

Our methods and results extend to general (eventually abstract) parabolic nonlinear equations of the Navier-Stokes type (e. g. to the class of abstract parabolic equations studied by J. M. Ghidaglia in [*12*]). Also it is likely that they can be extended to equations having general polynomial nonlinearities, provided there exists a relationship between the linear and the nonlinear part which insures that some non empty open set of initial data leads to regular solutions decaying exponentially to zero when $t \to \infty$.

As an illustration, we consider two simple examples. We prove that the analogue of our normalization mapping for the Burgers equation can be expressed in term of the Cole-Hopf transform, and is therefore defined in a closed form. A similar result is stated for the Minea system.

The case of non zero, time independant, forces remain open. However it is likely that part of our results are still valid in the situation where the forces are small (then there exists a unique, stable, steady solution).

To conclude this introduction, we can say that we derived in this work a normalization theory for the Navier-Stokes equations with potential body forces by means of a global asymptotic expansion. In this situation, the dynamics is trivial and hence the normal form is not necessary to understand it! However, we think that our theory gives some new insights in the structure of the Navier-Stokes equations. Applications and the connection to the Poincaré-Dulac approach are presently investigated and will be the object of a subsequent paper.

The content will be as follows:

1. FUNCTIONAL SETTING. SOME PREVIOUS RESULTS

In this section we recall classical and useful facts on Navier-Stokes equations and state some previous results we shall need in the sequel.

Let Ω be either a bounded regular open set in $\mathbb{R}^n$, or the cube $]0, L[^n$, where $n = 2, 3$. In the first case, we set

$$(1.1) \qquad \mathscr{V} = \{ u \in C_0^\infty(\Omega)^n ; \operatorname{div} u = 0 \}$$

and in the second case, we set

$$(1.2) \qquad \mathscr{V} = \Big\{ u = \text{trigonometric polynomials with values in } \mathbb{R}^n, \operatorname{div} u = 0, \int_\Omega u dx = 0, u \text{ satisfy } (0.3) \Big\}.$$

In both cases, we introduce the classical spaces (cf. [21] [22])

$$\mathrm{H} = \text{closure of } \mathscr{V} \text{ in } \mathrm{L}^2(\Omega)^n$$
$$\mathrm{V} = \text{closure of } \mathscr{V} \text{ in } \mathrm{H}^1(\Omega)^n ,$$

where $\mathrm{H}^l(\Omega)$ $(l = 1, 2, \ldots)$ denotes the Sobolev spaces of u's in $\mathrm{L}^2(\Omega)$ such that $\mathrm{D}^\alpha u \in \mathrm{L}^2(\Omega)$, $|\alpha| \leq l$.

Also we set $\mathrm{E}^m(\Omega) = \mathrm{H} \cap \mathrm{H}^m(\Omega)^n$, $\mathrm{E}^\infty(\Omega) = \bigcap_{m \geq 0} \mathrm{E}^m(\Omega)$, $m \geq 0$. We shall use the notations:

$$|u|^2 = \int_\Omega |u(x)|^2 dx, \qquad \|v\|^2 = \int_\Omega |\nabla v(x)|^2 dx ,$$

$(u \in \mathrm{H}, v \in \mathrm{V})$ for norms in H (resp. V). The corresponding scalar products will be denoted by $(.,.)$, (resp. $((.,.))$). The norm in $\mathrm{H}^m(\Omega)$ is indicated as $\| \cdot \|_m$.

Let P be the orthogonal projection on H in $\mathrm{L}^2(\Omega)^n$, and let

$$\mathrm{A}u = - \mathrm{P}\Delta u, \qquad \mathrm{B}(v, w) = \mathrm{P}((v . \nabla) w) ,$$

defined for $u, v, w \in \mathrm{E}^2(\Omega) \cap \mathrm{V} = \mathrm{D}(\mathrm{A})$ and H-valued. We also set $b(u, v, w) = (\mathrm{B}(u, v), w)$, and recall that $b(u, v, w) = - b(u, w, v)$.

The operators A and B can be extended by continuity to linear (resp. bilinear) operators from V (resp. $\mathrm{V} \times \mathrm{V}$) into the dual $\mathrm{V}' \supset \mathrm{H} \supset \mathrm{V}$ of V. We shall use the classical estimates (cf. [21] [22])

$$(1.3) \qquad |\mathrm{B}(u, v)| \leq \mathrm{C}_1 \|u\| \, |\mathrm{A}v|, \qquad u \in \mathrm{V}, \qquad v \in \mathrm{D}(\mathrm{A})$$

$$(1.4) \qquad |\mathrm{B}(u, v)| \leq \mathrm{C}_2 |\mathrm{A}u| \, \|v\|, \qquad u \in \mathrm{D}(\mathrm{A}), \qquad v \in \mathrm{V}$$

$$(1.5) \qquad |\mathrm{B}(u, v)| \leq \mathrm{C}_3 \|u\| \, \|v\|^{1/2} |\mathrm{A}v|^{1/2}, \qquad u \in \mathrm{V}, \qquad v \in \mathrm{D}(\mathrm{A})$$

$$(1.6) \qquad |\mathrm{B}(u, v)| \leqq \mathrm{C}_4 |u|^{1/4} |\mathrm{A}u|^{3/4} \|v\|, \qquad u \in \mathrm{D}(\mathrm{A}), \qquad v \in \mathrm{V}$$

$$(1.7) \qquad \|\mathrm{B}(u, u)\|_{\mathrm{V}'} \leq \mathrm{C}_5 |u|^{1/2} \|u\|^{3/2}, \qquad u \in \mathrm{V}$$

$$(1.8) \qquad \|\mathrm{B}(u, v)\|_{m-1} \leq \mathrm{C}_6 \|u\|_m \|v\|_m, \quad \text{for} \quad u, v \in \mathrm{H}^m(\Omega)^n, \qquad m \geq 2 ;$$

in particular (1.8) implies that $\mathrm{B}(\mathrm{E}^\infty(\Omega), \mathrm{E}^\infty(\Omega)) \subset \mathrm{E}^\infty(\Omega)$. (Here and in the sequel, $\mathrm{C}_1, \mathrm{C}_2, \ldots$ will design positive constants with respect to the explicit variables of the formulas.)

The equations (0.1) completed with (0.2) or (0.3) are equivalent to

$$\frac{du}{dt} + \nu A u + B(u, u) = 0, \qquad u(0) = u_0 \in V. \tag{1.9}$$

Let us recall also that, by definition, a solution u of (1.8) is *regular* on some interval $I \subset [0, \infty)$ if $u|_I \in C(I; V)$.

If a solution u is regular on $I = [t_0, t_1]$, then $u|_I$ is uniquely determined by $u(t_0)$.

Let $\mathscr{R}$ be the set of initial data u_0 in V leading to regular solutions on $\mathbb{R}_+$. It is well known ([*21*]) that $\mathscr{R} = V$ if $n = 2$, and is an open set of V containing 0 if $n = 3$. For the nonlinear equations (0.1), we shall only consider non zero initial data in $\mathscr{R}$ (then by the backward uniqueness property, $u(t) \neq 0$ for all positive t).

In this case, one has in fact the smoothness property (cf. [*21*]).

$$u \in C^\infty([t_0, +\infty); E^\infty(\Omega) \cap V), \quad \forall t_0 > 0.$$

Finally, let us recall that A is in fact the Stokes operator: the equations

$$\begin{cases} -\nu \Delta u = f - \nabla p, & f \in L^2(\Omega)^n \\ \nabla . u = 0 \quad \text{in} \quad \Omega, \end{cases}$$

completed with the boundary conditions (0.2) or (0.3), are equivalent to

$$\nu A u = Pf \quad \text{in} \quad H.$$

We shall denote by $0 < \lambda_1 \leq \lambda_2, \ldots$, the increasing sequence of the eigenvalues of A, and by $0 < \Lambda_1 < \Lambda_2 < \ldots$ the sequence of *distinct* eigenvalues of A, Λ_k having multiplicity m_k.

Let also $(w_m)_{m=1}^\infty$ be the orthonormal (in H or V) basis of the associated eigenvectors:

$$A w_m = \lambda_m w_m, \qquad m = 1, 2, \ldots$$

The orthogonal projection in H on the linear span of $w_1, \ldots, w_m$ will be denoted by P_m; R_j will stand for the orthogonal projection on the eigenspace of Λ_j:

$$R_j w = \sum_{\lambda_k = \Lambda_j} (w, w_k) w_k$$

Then, we have of course

$$R_j R_k = 0 \quad \text{if} \quad j \neq k \qquad \text{and} \qquad R_1 + R_2 + \ldots = I.$$

We shall also consider the Fréchet space containing H:

$$\mathscr{S}_A = R_1 H \oplus R_2 H \oplus \ldots,$$

equipped with the topology of convergence of components. The Stokes operator A and the semi-group e^{-At} generated by A extend in the obvious way to $\mathscr{S}_A$.

We now summarize the results of [*7*] [*8*] we shall need in the following (See also Guillopé [*13*], Ghidaglia [*12*] for related results.)

THEOREM 1. — ([*7*] [*8*]). Let $u_0 \in \mathcal{R}\backslash\{0\}$, and u the corresponding regular solution of (1.9).

1) The limit $\lim_{t\to\infty} \frac{\| u(t) \|^2}{| u(t) |^2} = \Lambda(u_0)$ exists and is one of the eigenvalues of the Stokes operator.

2) $\lim_{t\to\infty} e^{\nu\Lambda(u_0)t}u(t)$ exists in V and H (in particular) and is not zero.

3) There exist smooth connected analytic manifolds M_k, $k = 1, 2, \ldots$ in $\mathcal{R}$ *(the nonlinear spectral manifolds)* such that

i) $\mathcal{R} = M_0 \supset M_1 \supset M_2, \ldots$

ii) $\mathrm{Codim}_H M_k = \mathrm{Codim}_V M_k = m_1 + \ldots + m_k$.

iii) M_k is invariant by the nonlinear semi-group $(S(t))_{t\geq 0}$ on $\mathcal{R}$ generated by the Navier-Stokes equations, i. e. $S(t)M_k \subset M_k$ $\forall t \geq 0$.

iv) $u_0 \in M_k\backslash M_{k+1} \Leftrightarrow \Lambda(u_0) = \Lambda_{k+1}$, $k = 0, 1, 2, \ldots$

v) In the periodic case (0.3), the M_k's are « genuinely » nonlinear (that is to say they are not *linear* manifolds) but they contain an unbounded, infinite dimensional *linear* manifold.

2. ASYMPTOTIC EXPANSION (NONRESONANT CASE)

Before stating our results, we need to introduce a technical notion on the spectrum of A.

DEFINITION 1. — We call *resonance* in the spectrum of A, a relation of the type

(2.1) $a_1\Lambda_1 + a_2\Lambda_2 + \ldots + a_k\Lambda_k = \Lambda_{k+1}$, where $a_i \in \mathbb{N}$, $1 \leq i \leq k$.

If no resonance occurs in the spectrum of A, A will be called *nonresonant*.

For abstract operators of the type of A, the nonresonant case is generic for a natural topology (See [*10*]). On the other hand, the periodic boundary condition (0.3) always leads to resonance, since in this case, one has for instance $\Lambda_2 = 2\Lambda_1$ (cf. [*22*]).

For the Dirichlet boundary condition (0.2) we think that the nonresonant case is generic with respect to the domain Ω. Indeed this is true for self-adjoint scalar second order elliptic operators with Dirichlet boundary conditions (See [*10*]).

Finally, we shall denote

$$0 < \mu_1 = \Lambda_1 < \mu_2 < \ldots$$

the elements of the additive semi-group $\mathcal{S}$ generated by Λ_j, $j = 1, 2, \ldots$

In the present paragraph we shall deal only with the nonresonant case and we shall suppose that no resonance occurs in the spectrum of A.

Then we get the following asymptotic expansion for a regular solution of (1.8):

THEOREM 2. — For each $N \in \mathbb{N}$, the solution u of (1.9) admits the expansion in H:

$$(2.2) \quad u(t) = W_{\mu_1} e^{-\nu\mu_1 t} + W_{\mu_2} e^{-\nu\mu_2 t} + \ldots + W_{\mu_N} e^{-\nu\mu_N t} + v_N(t), \qquad \forall t > 0,$$

where $W_{\mu_j} = W_{\mu_j}(u_0) \in E^{\infty}(\Omega) \cap V$, $j = 1, \ldots, N$ and

$v_N \in C([0, \infty); V) \cap L^2_{loc}(0, \infty; D(A)) \cap C^{\infty}([t_0, \infty); E^{\infty}(\Omega) \cap V), \qquad \forall t_0 > 0.$

This expansion satisfies the following properties:

i) $\| v_N(t) \|_m = O(e^{-\nu(\mu_N + \varepsilon_N)t})$, for some $\varepsilon_N > 0$, $m = 0, 1, 2, \ldots$

ii) $R_j W_{\Lambda_j} = W_{\Lambda_j}$ for $\Lambda_j \leq \mu_N$

iii) For $\mu_j = \alpha_1 \Lambda_1 + \ldots + \alpha_{j-1}\Lambda_{j-1}$, with $\alpha_1 + \ldots + \alpha_{j-1} \geq 2$, W_{μ_j} is a function of $W_{\Lambda_1}, \ldots, W_{\Lambda_{j-1}}$ which is homogeneous of degree $\leq \alpha_1$ in W_{Λ_1}, of degree $\leqq \alpha_2$ in $W_{\Lambda_2}, \ldots$, of degree $\leqq \alpha_{j-1}$ in $W_{\Lambda_{j-1}}$.

More precisely, one has in this case:

$$(2.3) \qquad \nu(-\mu_j I + A) W_{\mu_j} + \sum_{\mu_i + \mu_k = \mu_j} B(W_{\mu_i}, W_{\mu_k}) = 0 . \qquad \blacksquare$$

REMARK 1. — The first $W_{\mu_j} \neq 0$ in (2.2) corresponds to $\mu_j = \Lambda(u_0)$.

Proof of Theorem 2. — We shall proceed by induction on N. Let us start with the first term in (2.2). Applying the projection R_1 to (1.9) we get:

$$(2.4) \qquad \frac{d}{dt} R_1 u(t) + \nu\Lambda_1 R_1 u(t) + R_1 B(u(t), u(t)) = 0 ,$$

and

$$(2.5) \qquad e^{\nu\Lambda_1 t} R_1 u(t) = \int_t^{\infty} e^{\nu\Lambda_1 \tau} R_1 B(u(\tau), u(\tau)) d\tau + W_{\mu_1}$$

where

$$(2.6) \qquad W_{\mu_1} = \lim_{t \to \infty} e^{\nu\Lambda_1 t} R_1 u(t) .$$

The integral in (2.5) converges since (cf. (1.7)):

$| R_1 B(u(\tau), u(\tau)) | \leq C_6 \| B(u(\tau), u(\tau)) \|_{V'} \leq C_7 | u(\tau) |^{1/2} \| u(\tau) \|^{3/2} \leq C_8 e^{-2\nu\Lambda_1 \tau}$.

On the other hand, we have:

$$(2.7) \qquad \frac{d}{dt}(I - R_1)u + \nu A(I - R_1)u + (I - R_1)B(u, u) = 0$$

and, taking the scalar product with u,

$$(2.8)\quad \frac{1}{2}\frac{d}{dt}|(\mathrm{I}-\mathrm{R}_1)u|^2 + \nu\|(\mathrm{I}-\mathrm{R}_1)u\|^2 \leq |(\mathrm{B}(u,u),(\mathrm{I}-\mathrm{R}_1)u)|$$
$$\leq \mathrm{C}_9\|(\mathrm{I}-\mathrm{R}_1)u\|\,\|u\|^{3/2}|u|^{1/2}\quad (\text{by } (1.7))$$
$$\leq \varepsilon\|(\mathrm{I}-\mathrm{R}_1)u\|^2 + \frac{\mathrm{C}_{10}}{\varepsilon}e^{-4\nu\Lambda_1 t}\quad \text{for}\quad \varepsilon > 0.$$

Using the obvious inequality

$$\Lambda_2|(\mathrm{I}-\mathrm{R}_1)u|^2 \leq \|(\mathrm{I}-\mathrm{R}_1)u\|^2,$$

(2.8) yields:

$$(2.9)\quad \frac{d}{dt}|(\mathrm{I}-\mathrm{R}_1)u(t)|^2 + 2\nu\Lambda_2(1-\varepsilon)|(\mathrm{I}-\mathrm{R}_1)u|^2 \leq \frac{\mathrm{C}_{10}}{\varepsilon}e^{-4\nu\Lambda_1 t}$$

Hence,

$$|(\mathrm{I}-\mathrm{R}_1)u(t)|^2 \leq e^{-2\nu\Lambda_2(1-\varepsilon)t}|(\mathrm{I}-\mathrm{R}_1)u_0|^2 + \frac{\mathrm{C}_{10}}{\varepsilon}e^{-2\nu\Lambda_2(1-\varepsilon)t}\int_0^t e^{2\nu(\Lambda_2(1-\varepsilon)-2\Lambda_1)\tau}d\tau$$

i. e.:

$$(2.10)\qquad |(\mathrm{I}-\mathrm{R}_1)u(t)|^2 \leq \mathrm{C}_{11}(e^{-2\nu\Lambda_2(1-\varepsilon)t} + e^{-4\nu\Lambda_1 t}).$$

Finally we obtain:

$$(2.11)\qquad |(\mathrm{I}-\mathrm{R}_1)u(t)| = 0(e^{-\nu(\Lambda_1+\varepsilon_1)t})$$

with some convenient $\varepsilon_1 > 0$ (1):

Let us now estimate

$$(2.12)\quad |\mathrm{R}_1u(t) - e^{-\nu\Lambda_1 t}\mathrm{W}_{\mu_1}| \leq e^{-\nu\Lambda_1 t}\int_t^\infty e^{\nu\Lambda_1\tau}|\mathrm{R}_1\mathrm{B}(u(\tau),u(\tau))|\,d\tau$$
$$\leq \mathrm{C}_{12}e^{-2\nu\Lambda_1 t},$$

by using (1.7) and the fact that all norms in $\mathrm{R}_1\mathrm{H}$ are equivalent.

We have just proven that

$$(2.13)\quad u(t) = \mathrm{R}_1u(t) + (\mathrm{I}-\mathrm{R}_1)u(t) = \mathrm{W}_{\mu_1}e^{-\nu\mu_1 t} + v_1(t),$$

where $\mu_1 = \Lambda_1$ and $|v_1(t)| = \mathrm{O}(e^{-\lambda(\mu_1+\varepsilon_1)t})$. Moreover, $\mathrm{R}_1\mathrm{W}_{\mu_1} = \mathrm{W}_{\mu_1}$.

The decay estimate of v_1 is in fact valid in stronger norms, as shows the

LEMMA 1. — One has

$$(2.14)\qquad \|v_1(t)\|_m = \mathrm{O}(e^{-\nu(\mu_1+\varepsilon_1)t})\qquad \forall m \geq 1.$$

(1) The numbers ε_i, $i = 1, 2, \ldots$ will denote various positive quantities, independent of t which appear in the exponential terms. The index i will refer to the step of the expansion.

Proof. — By (2.13) it suffices to show the corresponding assertion for $(I - R_1)u(t)$.

One easily obtains from (1.8)

$$\frac{1}{2}\frac{d}{dt}\| (I - R_1)u \|^2 + \nu | A(I - R_1)u |^2 \leq | B(u, u) | \; | A(I - R_1)u |$$
$$\leq C_{13} \| u \|^{3/2} | Au |^{1/2} | A(I - R_1)u |$$
$$\leq C_{14} e^{-2\nu\Lambda_1 t} | A(I - R_1)u(t) |$$

for t large enough. Hence, one gets (see the derivation of (2.9)):

$$\frac{d}{dt}\| (I - R_1)u \|^2 + 2\nu\Lambda_2(1 - \varepsilon) \| (I - R_1)u \|^2 \leq \frac{C_{15}}{\varepsilon} e^{-4\nu\Lambda_1 t}, \qquad \varepsilon > 0$$

and finally:

$$\| (I - R_1)u(t) \|^2 \leq C_{16}(e^{-2\nu\Lambda_2(1-\varepsilon)t} + e^{-4\nu\Lambda_1 t}), \quad \text{i. e.}$$
$$\| (I - R_1)u(t) \| = O(e^{-\nu(\Lambda_1 + \varepsilon_1)t}).$$

The assertion concerning the higher norms is slightly more involved. For this purpose, we set $u_j = \dfrac{d^j u}{dt^j}$. Then u_j is governed by the equation:

$$\frac{du_j}{dt} + \nu A u_j + \sum_{k=0}^{j} \binom{j}{k} B(u_k, u_{j-k}) = 0. \tag{2.15}$$

It is easily seen that for every j, and every m

$$\| u_j(t) \|_m = O(e^{-\nu\Lambda_1 t}). \tag{2.16}$$

More precisely, (2.16) for $j = 0$ and $m = 0, 1, 2$ is a classical fact ([*7*]). The case $m = 0, 1, j$ arbitrary, is derived for instance in [*13*]. The general case follows by induction, using the standard regularity results for the Stokes operator [*4*] [*20*] [*24*].

We then deduce from (2.15):

$$\frac{1}{2}\frac{d}{dt} | (I - R_1)u_j |^2 + \nu \| (I - R_1)u_j \|^2 \leq \left| \left(\sum_{k=0}^{j} \binom{j}{k} B(u_k, u_{j-k}), \quad (I - R_1)u_j \right) \right|$$
$$\leq C_{17} \| (I - R_1)u_j \| \left[\sum_{k=0}^{j} \binom{j}{k} \| u_k \|^{3/2} | u_{j-k} |^{1/2} \right]$$
$$\leq \varepsilon \| (I - R_1)u_j \|^2 + \frac{C_{18}}{\varepsilon} e^{-4\nu\Lambda_1 t}, \qquad \varepsilon > 0.$$

Finally it comes (cf. derivation of (2.11)) that

$$|(\mathrm{I} - \mathrm{R}_1)u_j(t)| = \mathrm{O}(e^{-\nu(\Lambda_1+\varepsilon_1)t}), \qquad j = 1, 2, \ldots \tag{2.17}$$

From (2.17) it is now easy to obtain (2.14) recursively, using (2.15), (2.16), the estimates (1.3), (1.8) and the regularity results for the Stokes operator. □

This closes the first step in the expansion.

Before proceeding to the general induction argument, it is instructive to look for the next term of the expansion.

Let us first write the equation for v_1.

$$\frac{dv_1}{dt} + \nu \mathrm{A} v_1 + \mathrm{B}(v_1, v_1) + e^{-2\nu\Lambda_1 t}\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1) + e^{-\nu\Lambda_1 t}[\mathrm{B}(\mathrm{W}_1, v_1) + \mathrm{B}(v_1, \mathrm{W}_1)] = 0 \tag{2.18}$$

where we set $\mathrm{W}_1 = \mathrm{W}_{\Lambda_1} = \mathrm{W}_{\mu_1}$.

We shall distinguish two cases

i) We suppose first of all that $\mu_2 = 2\Lambda_1$, i. e. $2\Lambda_1 < \Lambda_2$. From (2.18) and Lemma 1, above, we deduce

$$\frac{dv_1}{dt} + \nu \mathrm{A} v_1 + e^{-2\nu\Lambda_1 t}\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1) = h(t), \tag{2.19}$$

where $h(t)$ is smooth on $[\delta, +\infty)$ $\forall \delta > 0$ and satisfies

$$\| h(t) \|_m = \mathrm{O}(e^{-\nu(2\Lambda_1+\varepsilon_2)t}) \qquad \text{for all} \quad m \geq 0. \tag{2.19'}$$

Consequently,

$$e^{\nu\Lambda_1 t}\mathrm{R}_1 v_1(t) = \int_t^\infty e^{-\nu\Lambda_1\tau}\mathrm{R}_1\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1)d\tau + \int_t^\infty \mathrm{O}(e^{-\nu\Lambda_1(1+\varepsilon_2)\tau})d\tau$$
$$= \frac{1}{\nu\Lambda_1} e^{-\nu\Lambda_1 t}\mathrm{R}_1\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1) + \mathrm{O}(e^{-\nu\Lambda_1(1+\varepsilon_2)t}).$$

Hence,

$$\mathrm{R}_1 v_1(t) = \frac{1}{\nu\Lambda_1} e^{-2\nu\Lambda_1 t}\mathrm{R}_1\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1) + \mathrm{O}(e^{-2\nu\Lambda_1(1+\varepsilon_2)t}). \tag{2.20}$$

On the other hand, applying the projection R_k to (2.19) for $k \geq 2$ yields:

$$\frac{d}{dt}\mathrm{R}_k v_1(t) + \nu\Lambda_k \mathrm{R}_k v_1(t) = -e^{-2\nu\Lambda_1 k}\mathrm{R}_k\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1) + \mathrm{R}_k h(t) \tag{2.21}$$

and

$$\frac{d}{dt} e^{2\nu\Lambda_1 t}\mathrm{R}_k v_1(t) + \nu(\Lambda_k - 2\Lambda_1)e^{2\Lambda_1 \nu t}\mathrm{R}_k v_1(t) = -\mathrm{R}_k\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1) + e^{2\nu\Lambda_1 t}\mathrm{R}_k h(t)$$

$$e^{2\nu\Lambda_1 t}\mathrm{R}_k v_1(t) = e^{-\nu(\Lambda_k - 2\Lambda_1)t}\mathrm{R}_k v_1(0) + \int_0^t e^{-\nu(\Lambda_k - 2\Lambda_1)(t-\tau)}e^{2\nu\Lambda_1\tau}\mathrm{R}_k h(\tau)d\tau + \frac{1}{\nu(\Lambda_k - 2\Lambda_1)}[e^{-\nu(\Lambda_k - 2\Lambda_1)t} - 1]\mathrm{R}_k\mathrm{B}(\mathrm{W}_1, \mathrm{W}_1) \tag{2.22}$$

This shows that, as $t \to +\infty$,

$$R_k v_1(t) = -\frac{R_k B(W_1, W_1)}{\nu(\Lambda_k - 2\Lambda_1)} e^{-2\nu\Lambda_1 t} + R_k b_k(t), \tag{2.23}$$

$$\left| \sum_{k \geq 2} R_k b_k(t) \right| = O(e^{-\nu(2\Lambda_1 + \varepsilon_2)t}), \tag{2.23'}$$

In fact, one has

$$b_k(t) = e^{-\nu\Lambda_k t} v_1(0) + \int_0^t e^{-\nu\Lambda_k(t-\tau)} h(\tau) d\tau + \frac{1}{\nu(\Lambda_k - 2\Lambda_1)} e^{-\nu\Lambda_k t} B(W_1, W_1) \tag{2.23''}$$

The treatment of the contribution of the first and third terms in (2.23″) is straightforward, while for the second term we have to use the fact that the R_k's are mutually orthogonal. Now, we set $2\Lambda_1 = \mu_2$ and

$$S_{\mu_2}(W_1, W_1) = -\frac{1}{\nu} \sum_{\Lambda_k \geq \Lambda_2} \frac{R_k B(W_1, W_1)}{(\Lambda_k - 2\Lambda_1)}. \tag{2.24}$$

This series converges in H and in D(A). We then define $W_{\mu_2} \in D(A)$ as

$$W_{\mu_2} = \frac{1}{\nu\Lambda_1} R_1 B(W_1, W_1) + S_{\mu_2}(W_1, W_1), \tag{2.25}$$

and set

$$u(t) = W_{\mu_1} e^{-\nu\mu_1 t} + W_{\mu_2} e^{-\nu\mu_2 t} + v_2(t). \tag{2.26}$$

Let us now check point *iii*) in Theorem 2. W_{μ_2} is clearly quadratic in W_{μ_1}. Moreover,

$$\nu(-2\Lambda_1 I + A)W_{\mu_2} + B(W_1, W_1)$$
$$= -2\Lambda_1 \left[\frac{1}{\Lambda_1} R_1 B(W_1, W_1) - \sum_{\Lambda_k \geq \Lambda_2} \frac{R_k B(W_1, W_1)}{\Lambda_k - 2\Lambda_1} \right] + R_1 B(W_1, W_1)$$
$$- \sum_{\Lambda_k \geq \Lambda_2} \frac{\Lambda_k R_k B(W_1, W_1)}{\Lambda_k - 2\Lambda_1} + B(W_1, W_1)$$
$$- \sum_{k=1}^{\infty} R_k B(W_1, W_1) + B(W_1, W_1) = 0$$

This proves in particular that $W_{\mu_2} \in E^\infty(\Omega)$.
For the remainder term v_2 in (2.26) one has the decay estimate

$$\| v_2(t) \|_m = O(e^{-\nu(\mu_2 + \varepsilon_2)t}).$$

We omit the proof since it is a particular case of Lemma 3 below
ii) We now assume that $\mu_2 = \Lambda_2$, i. e. $2\Lambda_1 > \Lambda_2$.

From (2.19), one obtains, taking first the projection R_2 and multiplying by $e^{\nu\Lambda_2 t}$:

$$(2.27) \qquad \frac{d}{dt}(e^{\nu\Lambda_2 t}R_2 v_1(t)) = e^{-\nu(2\Lambda_1-\Lambda_2)t}R_2 B(W_1, W_1) + O(e^{-\nu((2\Lambda_1-\Lambda_2)+\varepsilon_2)t})$$

which proves that $\lim_{t\to\infty} e^{\nu\Lambda_2 t}R_2 v_1(t)$ exists. We shall call it W_{μ_2}. In fact, using (2.18) it is easily seen that

$$(2.28) \quad W_{\mu_2} = R_2 u_0 - \int_0^\infty e^{\nu\Lambda_2 t}R_2\{B(v_1, v_1) + e^{-2\nu\Lambda_1 t}B(W_1, W_1) + e^{-\nu\Lambda_1 t}(B(W_1, v_1) + B(v_1, W_1))\}\,dt.$$

Thus from (2.27) we finally infer:

$$R_2 v_1(t) = e^{-\nu\mu_2 t}W_{\mu_2} + O(e^{-2\nu\Lambda_1 t})$$

and

$$(2.29) \qquad u(t) = W_{\mu_1}e^{-\nu\mu_1 t} + W_{\mu_2}e^{-\nu\mu_2 t} + v_2(t).$$

It can be shown that $\| v_2(t)\|_m = O(e^{-\nu(\mu_2+\varepsilon_2)t}\forall m$, but we omit the proof since it is a particular case of Lemma 2 below.

After having given the first two steps of the expansion, we now proceed to the general induction argument.

Let us assume that we have the expansion

$$(2.30) \qquad u(t) = e^{-\nu\mu_1 t}W_{\mu_1} + \ldots + e^{-\nu\mu_N t}W_{\mu_N} + v_N(t), \qquad N \geq 2$$

where W_{μ_j}, $j = 1, \ldots, N$ and v_N satisfy the properties *i*), *ii*), *iii*) of Theorem 1 (The case $N = 1$ has been proved above).

Inserting the expansion (2.30) into (1.9), we obtain the following equation for v_N

$$(2.31) \quad \frac{dv_N}{dt} + \nu A v_N + B(v_N, v_N) + \sum_{j=1}^{N} e^{-\nu\mu_j t}[B(W_{\mu_j}, v_N) + B(v_N, W_{\mu_j})] + \nu \sum_{\substack{\mu_j\leq\mu_N \\ \mu_j\notin \mathrm{Sp}A}} (AW_{\mu_j} - \mu_j W_{\mu_j})e^{-\nu\mu_j t} + \sum_{i,k=1}^{N} e^{-\nu(\mu_i+\mu_k)t}B(W_{\mu_i}, W_{\mu_k}) = 0.$$

Let Λ_{k_N+1} be the first eigenvalue of A which is strictly greater than μ_N. We shall distinguish two cases:

CASE I. — For $j, k = 1, \ldots, N$, one has either

$$(2.32) \qquad \mu_j + \mu_k \leqq \mu_N \qquad \text{or} \qquad \mu_j + \mu_k > \Lambda_{k_N+1}.$$

Then, obviously, $\mu_{N+1} = \Lambda_{k_N+1}$. By property *iii*), (2.3) of Theorem 2, (2.31) can be simplified as follows:

$$(2.33)\quad \frac{dv_N}{dt} + \nu A v_N + B(v_N, v_N) + \sum_{j=1}^{N} e^{-\nu\mu_j t}[B(W_{\mu_j}, v_N) + B(v_N, W_{\mu_j})] + \sum_{\substack{\mu_1 \leq \mu_i, \mu_j \leq \mu_N \\ \mu_i + \mu_j > \Lambda_{k_N+1}}} e^{-\nu(\mu_i+\mu_j)t} B(W_{\mu_i}, W_{\mu_j}) = 0 .$$

We apply the projection R_{k_N+1} on (2.33) to get

$$(2.34)\quad \frac{d}{dt} R_{k_N+1} v_N + \nu \Lambda_{k_N+1} R_{k_N+1} v_N + R_{k_N+1}\{ \dots \} = 0 ,$$

where $\{ \dots \}$ denotes the remaining terms in (2.33). Finally, one obtains

$$(2.35)\quad \frac{d}{dt}(e^{\nu\Lambda_{k_N+1}t} R_{k_N+1} v_N(t)) = O(e^{-\nu\varepsilon_{N+1}t}) ,$$

which proves that

$$(2.36)\quad W_{\Lambda_{k_N+1}} \underset{\text{def}}{=} \lim_{t\to\infty} e^{\nu\Lambda_{k_N+1}t} R_{k_N+1} v_N(t) \text{ exists} .$$

Let us notice at this stage (this will be useful in the proof of the analyticity of our linearizing transformation that from (2.33) it follows at once an explicit expression for $W_{\Lambda_{k_N+1}}$:

$$(2.37)\quad W_{\Lambda_{k_N+1}} = R_{k_N+1} u_0 - \sum_{i=1}^{N} R_{k_N+1} W_{\mu_i} - \int_0^\infty e^{\nu\Lambda_{k_N+1}t} R_{k_N+1} B(v_N, v_N) dt - \int_0^\infty \Big\{ \sum_{j=1}^{N} e^{\nu(\Lambda_{k_N+1}-\mu_j)t} [R_{k_N+1}[B(W_{\mu_j}, v_N) + B(v_N, W_{\mu_j})]] + \sum_{\substack{\mu_1 \leq \mu_i, \mu_k \leq \mu_N \\ \mu_i + \mu_k > \Lambda_{k_N+1}}} e^{-\nu(\mu_i+\mu_k-\Lambda_{k_N+1})t} R_{k_N+1} B(W_{\mu_i}, W_{\mu_k}) \Big\} dt .$$

We have finally proved:

$$R_{k_N+1} v_N(t) = e^{-\nu\Lambda_{k_N+1}t} W_{\Lambda_{k_N+1}} + O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t})$$

and

$$(2.38)\quad u(t) = W_{\mu_1} e^{-\nu\mu_1 t} + \dots + W_{\Lambda_{k_N+1}} e^{-\nu\Lambda_{k_N+1}t} + v_{N+1}(t) ,$$

where $W_{\Lambda_{k_N+1}} \in R_{k_N+1}H$ and v_{N+1} is as smooth as u. In particular, $v_{N+1} \in C^\infty([t_0, \infty); E^\infty(\Omega) \cap V)$, $\forall t_0 > 0$.

The asymptotic behavior of v_{N+1} is precised by the

LEMMA 2. — For every $m \geqq 0$, one has

$$\| v_{N+1}(t) \|_m = O(e^{-\nu(\Lambda_{k_{N+1}} + \varepsilon_{N+1})t}). \tag{2.39}$$

Proof. — We first establish that

$$|(R_1 + \dots + R_{k_N+1})v_{N+1}(t)| = O(e^{-\nu(\Lambda_{k_{N+1}} + \varepsilon_{N+1})t}). \tag{2.40}$$

We recall that there $\mu_{N+1} = \Lambda_{k_N+1}$.

Since $v_N = v_{N+1} = W_{\mu_{N+1}} e^{-\nu\mu_{N+1}t}$, one gets

$$R_{k_N+1}v_{N+1}(t) = R_{k_N+1}v_N(t) - W_{\mu_{N+1}}e^{-\nu\mu_{N+1}t},$$

and by (2.35), we have

$$| W_{\mu_{N+1}} - e^{\nu\mu_{N+1}} R_{k_N+1}v_N(t)| = \left| \int_t^\infty e^{-\nu\varepsilon_{N+1}\tau} d\tau \right| = O(e^{-\nu\varepsilon_{N+1}t}),$$

which proves that

$$| R_{k_N+1}v_{N+1}(t) | = O(e^{-\nu(\Lambda_{k_{N+1}} + \varepsilon_{N+1})t}).$$

On the other hand, for any Λ_j, $1 \leq j \leq k_N$, it follows from (2.33):

$$\frac{d}{dt} R_j v_N(t) + \nu\Lambda_j R_j v_N(t) = O(e^{-\nu(\mu_{N+1} + \varepsilon_{N+1})t})$$

and

$$\frac{d}{dt}(e^{\nu\Lambda_j t} R_j v_N(t) = O(e^{-\nu(\mu_{N+1} - \Lambda_j + \varepsilon_{N+1})t}).$$

This equality yields, since $| v_N(t) | = O(e^{-\nu(\Lambda_{k_N} + \varepsilon_N)t})$:

$$e^{\nu\Lambda_j t} R_j v_N(t) = \int_t^\infty O(e^{-\nu(\mu_{N+1} - \Lambda_j + \varepsilon_{N+1})\tau}) d\tau$$

i. e.

$$R_j v_N(t) = R_j v_{N+1}(t) = O(e^{-\nu(\mu_{N+1} + \varepsilon_{N+1})t}),$$

and (2.40) is proved.

Let now $Q = I - (R_1 + \dots + R_{k_N+1})$, so that $Qv_{N+1} = Qv_N$. By (2.33) one has

$$\frac{d}{dt} Qv_N + \nu AQv_N = b_N,$$

where $\| b_N(t) \|_m = O(e^{-\nu(\Lambda_{k_{N+1}} + \varepsilon_{N+1})t})$ for every $m \geqslant 0$. Therefore for any $\varepsilon > 0$ small enough we have:

$$\frac{1}{2}\frac{d}{dt} | Qv_N(t) |^2 + \nu(1 - \varepsilon) \| Qv_N(t) \|^2 = O(e^{-2\nu(\Lambda_{k_{N+1}} + \varepsilon_{N+1})t})$$

and

$$\frac{d}{dt}|Qv_N(t)|^2 + 2\nu\Lambda_{k_N+2}(1-\varepsilon)|Qv_N(t)|^2 = O(e^{-2\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}).$$

Finally one gets:

$$(2.41)\quad |Qv_{N+1}(t)|^2 = |Qv_N(t)|^2 = e^{-2\Lambda_{k_N+2}(1-\varepsilon)t}|Qv_N(0)|^2 + \int_0^t e^{-2\nu\Lambda_{k_N+2}(t-\tau)}O(e^{-2\nu(\mu_{N+1}+\varepsilon_{N+1})\tau})d\tau = O(e^{-2\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}),$$

if we chose *a priori* $\varepsilon > 0$ small enough in order that $\Lambda_{k_N+2}(1-\varepsilon) > \Lambda_{k_N+1}$.

It remains to deal with the higher Sobolev norms. To do that we set $v_N^{(j)} = \dfrac{d^j v_N}{dt^j}$. This function obeys the equation:

$$(2.42)\quad \frac{dv_N^{(j)}}{dt} + \nu A v_N^{(j)} + \sum_{k=0}^{j}\binom{j}{k}B(v_N^{(k)}, v_N^{(j-k)}) + \sum_{i=1}^{N}\sum_{k=0}^{j}\binom{j}{k}(-\nu\mu_i)^k e^{-\nu\mu_i t}[B(W_{\mu_i}, v_N^{(j-k)}) + B(v_N^{(j-k)}, W_{\mu_i})] + \sum_{\substack{\mu_i+\mu_k>\Lambda_{k_N+1}\\ \mu_i,\mu_k\leqq\mu_N}}[-\nu(\mu_i+\mu_k)]^j e^{-\nu(\mu_i+\mu_k)t}B(W_{\mu_i}, W_{\mu_k}) = 0.$$

By induction on j we obtain from (2.37) (2.42) (1.8) and the decay estimate on $v_N(t)$ that

$$(2.43)\quad \|v_N^{(j)}(t)\|_m = O(e^{-\nu(\mu_N+\varepsilon_N)t}) \qquad \forall m \geqq 0.$$

We now apply the projection Q to equation (2.42) and we get (using (2.43) and (1.7))

$$(2.44)\quad \frac{1}{2}\frac{d}{dt}|Qv_N^{(j)}|^2 + \nu\|Qv_n^{(j)}\|^2 \leqq C_{19}e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t} \times \|Qv_N^{(j)}\| \leqq \varepsilon\|Qv_N^{(j)}\|^2 + \frac{C_{20}}{\varepsilon}e^{-2\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}, \qquad \varepsilon > 0$$

and we deduce from (2.44) that

$$(2.45)\quad |Qv_N^{(j)}(t)| = O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}), \qquad j = 1, 2, \ldots.$$

From (2.45) and (2.42) one obtains inductively, using (2.31) (2.43), the estimates (1.3) (1.8), and the regularity results on the Stokes operator, that

$$\| Qv_N(t) \|_m = O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}) \qquad (\forall m \geqq 1).$$

This together with (2.40) yields (2.39). □

CASE II. — There exists $\mu_i \leqq \mu_N$, $\mu_j \leqq \mu_N$ such that

$$\mu_N < \mu_i + \mu_j < \Lambda_{k_N+1}. \tag{2.46}$$

We then set

$$\mu_{N+1} = \text{Inf}\,\{ \mu_i + \mu_j ;\ \mu_i, \mu_j \text{ satisfying } (2.46) \}.$$

From (2.31) and Property *iii*) of Theorem 2, we obtain

$$\begin{aligned} \frac{dv_N}{dt} + \nu A v_N + B(v_N, v_N) + \sum_{j=1}^{N} e^{-\nu\mu_j t}[B(W_{\mu_j}, v_N) + B(v_N, W_{\mu_j})] \\ + \sum_{\substack{\mu_i,\mu_j \leqslant \mu_N \\ \mu_i+\mu_j=\mu_{N+1}}} e^{-\nu\mu_{N+1}t} B(W_{\mu_i}, W_{\mu_j}) \\ + \sum_{\substack{\mu_i,\mu_j \leqslant \mu_N \\ \mu_i+\mu_j>\mu_{N+1}}} e^{-\nu(\mu_i+\mu_j)t} B(W_{\mu_i}, W_{\mu_j}) = 0. \end{aligned} \tag{2.47}$$

One gets by the induction hypothesis:

$$\frac{dv_N}{dt} + \nu A v_N + \sum_{\mu_i+\mu_j=\mu_{N+1}} e^{-\nu\mu_{N+1}t} B(W_{\mu_i}, W_{\mu_j}) = h_N(t), \tag{2.48}$$

where h_N is smooth and satisfies

$$\| h_N(t) \|_m = O(e^{-\nu(\mu_{N+1}+\varepsilon_{N+1})t}) \qquad \forall m \geq 0 \tag{2.49}$$

(Note that necessarily $\mu_N + \mu_j + \varepsilon_N > \mu_{N+1}$, $j = 1, \ldots, N$).

Proceeding as in the second step of the expansion (case (*i*)), we deduce from (2.49) that:

$$\begin{aligned} R_j v_N(t) = \frac{1}{\nu(\mu_{N+1} - \Lambda_j)} e^{-\nu\mu_{N+1}t} R_j \sum_{\mu_l+\mu_m=\mu_{N+1}} B(W_{\mu_l}, W_{\mu_m}) \\ + O(e^{-\nu(\mu_{N+1}+\varepsilon_{N+1})t}) \end{aligned} \tag{2.50}$$

for $j = 1, 2, \ldots, \Lambda_{k_N}$ ($\leq \mu_N$).

On the other hand, applying the R_k projection on (2.48) for $k \geq \Lambda_{k_N+1}$ leads to

$$\frac{d}{dt}(e^{\nu\mu_{N+1}t}R_k v_N(t)) + \nu(\Lambda_k - \mu_{N+1})e^{\nu\mu_{N+1}t}R_k v_N(t) = -R_k\left\{\sum_{\mu_i+\mu_j=\mu_{N+1}} B(W_{\mu_i}, W_{\mu_j})\right\} + e^{\nu\mu_{N+1}t}R_k h_N(t),$$

whence to

$$(2.51)\quad e^{\nu\mu_{N+1}t}R_k v_N(t) = e^{-\nu(\Lambda_k-\mu_{N+1})t}R_k v_N(0) + \int_0^t e^{-\nu(\Lambda_k-\mu_{N+1})(t-\tau)}e^{\nu\mu_{N+1}\tau}R_k h_N(\tau)d\tau + \frac{1}{\nu(\Lambda_k-\mu_{N+1})}[e^{-\nu(\Lambda_k-\mu_{N+1})t} - 1]R_k\left\{\sum_{\mu_i+\mu_j=\mu_{N+1}} B(W_{\mu_i}, W_{\mu_j})\right\}.$$

This shows that, as $t \to +\infty$,

$$(2.52)\quad R_k v_N(t) = -\frac{1}{\nu(\Lambda_k-\mu_{N+1})}R_k\left\{\sum_{\mu_i+\mu_j=\mu_{N+1}} B(W_{\mu_i}, W_{\mu_j})\right\}e^{-\nu\mu_{N+1}t} + R_k b_k(t),$$

where

$$(2.52)'\quad \left|\sum_{k\geq k_N+1} R_k b_k(t)\right| = O(e^{-\nu(\mu_{N+1}+\varepsilon_{N+1})t}).$$

The justification of (2.52)′ is similar to that of (2.23)′.

We set now

$$(2.53)\quad S_{\mu_{N+1}} = -\frac{1}{\nu}\sum_{\Lambda_k\geq\Lambda_{k_N+1}}\left\{\frac{1}{\Lambda_k-\mu_{N+1}}\sum_{\mu_i+\mu_j=\mu_{N+1}} R_k B(W_{\mu_i}, W_{\mu_j})\right\}$$

(Obviously this series converges in H and D(A)). Finally, we define $W_{\mu_{N+1}}$ by

$$W_{\mu_{N+1}} = \frac{1}{\nu}\sum_{j=1}^{k_N}\left\{\frac{1}{\mu_{N+1}-\Lambda_j}R_j\sum_{\mu_l+\mu_m=\mu_{N+1}} B(W_{\mu_l}, W_{\mu_m})\right\} + S_{\mu_{N+1}}$$

i. e.

$$(2.54)\quad W_{\mu_{N+1}} = \frac{1}{\nu}\sum_{j=1}^{\infty}\left\{\frac{1}{\mu_{N+1}-\Lambda_j}\sum_{\mu_l+\mu_m=\mu_{N+1}} R_j B(W_{\mu_l}, W_{\mu_m})\right\}.$$

It is now trivial to check (2.3) in Theorem 2. In fact, by (2.54) one gets

$$(2.55)\quad v(-\mu_{N+1}I+A)W_{\mu_{N+1}}=\sum_{j=1}^{\infty}\frac{-\mu_{N+1}}{\mu_{N+1}-\Lambda_j}\sum_{\mu_l+\mu_m=\mu_{N+1}}R_jB(W_{\mu_l},W_{\mu_m})$$
$$+\sum_{j=1}^{\infty}\frac{\Lambda_j}{\mu_{N+1}-\Lambda_j}\sum_{\mu_l+\mu_m=\mu_{N+1}}R_jB(W_{\mu_l},W_{\mu_m})$$
$$=-\sum_{\mu_l+\mu_m=\mu_{N+1}}B(W_{\mu_l},W_{\mu_m}).$$

It follows at once from (2.55) (and the induction hypothesis) that W_{μ_j} belongs to $E^\infty(\Omega)\cap V$ and satisfies *iii*) in Theorem 2.

We now set

$$(2.56)\qquad u(t)=e^{-v\mu_1 t}W_{\mu_1}+\dots+e^{-v\mu_{N+1}t}W_{\mu_{N+1}}+v_{N+1}(t)$$

Obviously v_{N+1} is as smooth as u. The following Lemma precises the decay of v_{N+1} as $t\to+\infty$.

LEMMA 3. — For every $m\geq 0$ one has

$$(2.57)\qquad \|v_{N+1}(t)\|_m=O(e^{-v(\mu_{N+1}+\varepsilon_{N+1})t}).$$

Proof. — We start with a finite dimensional part of v_{N+1}. Since $v_N=e^{-v\mu_{N+1}t}W_{\mu_{N+1}}+v_{N+1}$, one has (recall that $\Lambda_{k_N}\leq\mu_N<\mu_{N+1}<\Lambda_{k_N+1}$) :

$$(R_1+\dots+R_{k_N})v_{N+1}(t)=(R_1+\dots+R_{k_N})v_N(t)$$
$$-\frac{1}{v}e^{-v\mu_{N+1}t}\sum_{j=1}^{k_N}\frac{1}{\mu_{N+1}-\Lambda_j}\sum_{\mu_l+\mu_m=\mu_{N+1}}R_jB(W_{\mu_l},W_{\mu_m})$$

and by (2.50) we obtain

$$(2.58)\qquad |(R_1+\dots+R_{k_N})v_{N+1}(t)|=O(e^{-v(\mu_{N+1}+\varepsilon_{N+1})t}).$$

Let now $Q=I-(R_1++++R_{k_N})$. Obviously, $Qv_{N+1}=Qv_N=e^{-v\mu_{N+1}t}S_{\mu_{N+1}}$ and we get with (2.51)

$$(2.59)\quad Qv_{N+1}(t)=\sum_{k\geq k_N+1}\Bigg[e^{-v\Lambda_k t}R_kv_N(0)+\int_0^t e^{-v\Lambda_k(t-\tau)}R_kh_N(\tau)d\tau$$
$$+\frac{1}{v(\Lambda_k-\mu_{N+1})}e^{-v\Lambda_k t}R_k\Big\{\sum_{\mu_i+\mu_j=\mu_{N+1}}B(W_{\mu_i},W_{\mu_j})\Big\}\Bigg].$$

Since, in particular, $|h_N(t)|=O(e^{-v(\mu_{N+1}+\varepsilon_{N+1})t})$, (2.59) yields

$$(2.60)\qquad |Qv_{N+1}(t)|=O(e^{-v(\mu_{N+1}+\varepsilon_{N+1})t}).$$

Let us now write the equation for v_{N+1}:

$$(2.61)\quad \frac{dv_{N+1}}{dt} + \nu A v_{N+1} + B(v_{N+1}, v_{N+1}) + \sum_{j=1}^{N+1} e^{-\nu\mu_j t}[B(W_{\mu_j}, v_{N+1}) + B(v_{N+1}, W_{\mu_j})] + \sum_{\substack{\mu_i+\mu_j>\mu_{N+1}\\ \mu_i,\mu_j\leqq\mu_{N+1}}} e^{-\nu(\mu_i+\mu_j)t} B(W_{\mu_i}, W_{\mu_j}) = 0\,.$$

We introduce the time derivatives $v_{N+1}^{(j)} = \frac{d^j v_{N+1}}{dt^j}$, $j = 1, 2, 3, \ldots$ which satisfy the equation (similar to (2.42))

$$(2.62)\quad \frac{dv_{N+1}^{(j)}}{dt} + \nu A v_{N+1}^{(j)} + \sum_{k=0}^{j} \binom{j}{k} B(v_{N+1}^{(k)}, v_{N+1}^{(j-k)}) + \sum_{\substack{\mu_k+\mu_l>\mu_{N+1}\\ \mu_k,\mu_l\leqq\mu_{N+1}}} [-\nu(\mu_k+\mu_l)]^j e^{-\nu(\mu_k+\mu_l)t} B(W_{\mu_k}, W_{\mu_l}) + \sum_{i=1}^{N+1}\sum_{k=0}^{j} \binom{j}{k} (-\nu\mu_i)^k e^{-\nu\mu_i t}\left[B(W_{\mu_i}, v_{N+1}^{(j-k)}) + B(v_{N+1}^{(j-k)}, W_{\mu_i})\right] = 0\,.$$

From this point on we proceed as in the proof of Lemma 1 and Lemma 2. Thus by induction on j, one obtains (2.61), (2.62) and the decay estimates (consequence of the induction hypothesis) that

$$(2.63)\qquad \| v_{N+1}^{(j)}(t) \|_m = O(e^{-\nu(\mu_N+\varepsilon_N)t})$$

first for $m = 0$, then by a repetitive use of the Cattabriga-Yudovitch-Solonnikov theorem and (1.8) for $m = 1, 2, \ldots$ We get after taking the scalar product of (2.62) with $Qv_{N+1}^{(j)}$,

$$(2.64)\qquad | Qv_{N+1}^{(j)}(t) | = O(e^{-\nu(\mu_{N+1}+\varepsilon_{N+1})t}), \qquad j = 1, 2, \ldots$$

Using (2.61), (2.62), (2.64), the estimates (1.3), (1.8) and again the regularity results for the Stokes operator, one deduces

$$(2.65)\qquad \| Qv_{N+1}(t) \|_m = O(e^{-\nu(\mu_{N+1}+\varepsilon_{N+1})t}) \qquad \forall m \geqq 0\,,$$

and Lemma 3 is proved. □

3. LINEARIZATION OF THE NAVIER-STOKES EQUATIONS (NONRESONANT CASE)

In this paragraph we construct a nonlinear functional mapping which linearizes the Navier-Stokes equation, and we give some of its properties.

We recall that $\mathscr{S}_\Lambda$ denotes the Fréchet space $R_1H \oplus R_2H \oplus \dots$ equipped with the topology of convergence of components.

THEOREM 3. — The mapping $U : \mathscr{R} \to \mathscr{S}_\Lambda$ defined by

$$U(u_0) = W_{\Lambda_1}(u_0) \oplus W_{\Lambda_2}(u_0) \oplus \dots$$

is analytic and one to one.

Proof. — Let us prove first the second part of the Theorem.

Let $u_0, v_0 \in \mathscr{R}$ such that $U(u_0) = U(v_0)$. Then, Theorem 2 shows that $u(t) = S(t)u_0$ and $v(t) = S(t)v_0$ have the same asymptotic expansion (2.2).

Let $w(t) = u(t) - v(t)$. It follows at once that:

$$\frac{dw}{dt} + \nu A w + B(u, w) + B(w, v) = 0. \tag{3.1}$$

We notice that, for some constant C, u and v satisfy the estimates

$$\| u(t) \|, \quad \| v(t) \| \leq C e^{-\nu\Lambda_1 t} \qquad \forall t \geq 0$$
$$| Au(t) |, \quad | Av(t) | \leq C e^{-\nu\Lambda_1 t}, \qquad \forall t \geq t_0 > 0.$$

Thus, equation (3.1) is exactly of the type discussed in [*8*] § 2, and from [*8*] we deduce that, unless $w_0 = u_0 - v_0 = 0$, $w(t)$ decays *exactly* as $e^{-\nu\Lambda(w_0)t}$, where $\Lambda(w_0)$ is some element in the spectrum of A. But, since u and v have the same expansions, $w(t)$ must decay faster than e^{-kt}, $\forall k > 0$, so $w_0 = 0$. This proves that U is one to one.

To prove the analyticity of the mapping U, we shall use the following smoothness property due to Foias [*6*].

LEMMA 4. — The mapping:

$$(t, u_0) \to u(t)$$

is analytic from $(0, \infty) \times \mathscr{R}$ *into* $D(A)$. ■

It suffices to prove that each component W_{Λ_k} of U is analytic in u_0. We proceed by induction on k.

By (2.5), one has

$$W_{\Lambda_1} = R_1 u_0 - \int_0^\infty e^{\nu\Lambda_1 t} R_1 B(u(t), u(t)) dt,$$

and Lemma 4 implies that $u_0 \to W_{\Lambda_1}(u_0)$ is analytic.

Assume now that W_{Λ_j} is analytic for $j \leq k_N$.

We recall (cf. (2.37)) that $W_{\Lambda_{k_N+1}}$ has the form:

$$(3.2)\quad W_{\Lambda_{k_N+1}} = R_{k_N+1}u_0 - \sum_{i=1}^{N} R_{k_N+1}W_{\mu_i} - \int_0^\infty \Big\{ \sum_{j=1}^{N} e^{\nu\,\Lambda_{k_N+1}-\mu_j)t} \times R_{k_N+1}[B(W_{\mu_j}, v_N) + B(v_N, W_{\mu_j})] + \sum_{\substack{\mu_1 \leqslant \mu_i, \mu_j \leqslant \mu_N \\ \mu_i+\mu_j > \Lambda_{k_N}+1}} e^{-\nu(\mu_i+\mu_j-\Lambda_{k_N+1})t} R_{k_N+1} B(W_{\mu_i}, W_{\mu_j}) \Big\}\, dt - \int_0^\infty e^{\nu\Lambda_{k_N+1}t} R_{k_N+1} B(v_N, v_N)\, dt .$$

On the other hand, $\Lambda_{k_N} \leq \mu_N$ and

$$(3.3)\qquad v_N(t) = u(t) - \sum_{j=1}^{N} W_{\mu_j} e^{-\nu\mu_j t} .$$

By (2.3) and the induction hypothesis, W_{μ_j} is analytic in u_0 for $1 \leq j \leq N$. Therefore by (3.3) and Lemma 4, the mapping $(t, u_0) \to v_N(t)$ is analytic from $(0, \infty) \times \mathscr{R}$ to D(A), and the analyticity of $W_{\Lambda_{k_N+1}}$ results from (3.2). ■

COROLLARY 1. — The mapping U linearizes the Navier-Stokes equations in the following sense:

$$(3.4)\qquad U(u(t)) = e^{-\nu A t} U(u_0) \qquad \forall u_0 \in \mathscr{R}, \qquad \forall t \geq 0 .$$

Proof. — Taking $u(t)$ as initial value, one has by Theorem 2

$$(3.5)\qquad u(t+s) = e^{-\nu\mu_1 s} W_{\mu_1}^t + e^{-\nu\mu_2 s} W_{\mu_2}^t + \ldots + e^{-\nu\mu_N s} W_{\mu_N}^t + v_N^t(s)$$

where

$$\| v_N^t(s) \|_m = O(e^{-\nu(\mu_N+\varepsilon_N)s})$$

for all $N, m \geqq 1$. Of course,

$$(3.6)\qquad U(u(t)) = W_{\Lambda_1}^t \oplus W_{\Lambda_2}^t \oplus \ldots .$$

On the other hand,

$$(3.7)\qquad u(t+s) = e^{-\nu\mu_1(t+s)} W_{\mu_1} + e^{-\nu\mu_2(t+s)} W_{\mu_2} + \ldots + e^{-\nu\mu_N(t+s)} W_{\mu_N} + v_N(t+s)$$

where

$$\| v_N(t+s) \|_m = O(e^{-\nu(\mu_N+\varepsilon'_N)(t+s)})$$

for all $N, m \geqq 1$. Also,

$$(3.8)\qquad U(u_0) = W_{\Lambda_1}(u_0) \oplus W_{\Lambda_2}(u_0) \oplus \ldots .$$

But the obvious uniqueness of the asymptotic expansion (2.2) implies

$$W^t_{\Lambda_j} = e^{-\nu\Lambda_j t} W_{\Lambda_j},$$

which proves Corollary 1. ■

The mapping U linearizes the nonlinear spectral manifolds M_k in the following sense:

COROLLARY 2. — $u \in M_k$ if and only if the k first components of $U(u_0)$ vanish.

Proof. — This is obvious from (2.2) and the following characterization of $\Lambda(u_0)$ (see [7])

$$\lim_{t\to\infty} e^{\nu\Lambda_j t} \,|\, u(t) \,|\, = 0 \qquad \text{for} \qquad \Lambda_j < \Lambda(u_0)$$

and

$$\lim_{t\to\infty} e^{\nu\Lambda(u_0)t} \,|\, u(t) \,| \quad \text{exists and is not zero.}$$

Let us point out again that the first non zero term in (2.2) is $W_{\Lambda(u_0)}$.

4. ASYMPTOTIC EXPANSION: THE GENERAL CASE

Throughout this chapter we shall assume that resonances can occur in the spectrum of the Stokes operator A. In particular this will cover the space periodic boundary condition (0.3) or the case of the flow on a sphere of $\mathbb{R}^3$ (cf. [*12*]). Let us just recall that an eigenvalue Λ_k of A is said to be resonant if

$$\Lambda_k = \alpha_1\Lambda_1 + \ldots + \alpha_{k-1}\Lambda_{k-1}, \quad \text{for some} \quad \alpha_1, \ldots, \alpha_{k-1} \in \mathbb{N}\,.$$

The following elementary lemma will be useful in the sequel. Let us also recall that $\mathcal{S}$ denotes the additive semi-group generated by $\Lambda_1, \Lambda_2, \Lambda_3, \ldots$

LEMMA 5. — For $\mu_i \in \mathcal{S}$, we define inductively $d_i = d_i(\mu_i) \in \mathbb{N}$ by $d_i = 0$ if μ_i is a nonresonant eigenvalue of A $d_i = 1 + \underset{\mu_j+\mu_k=\mu_i}{\text{Max}} (d_j + d_k)$ if μ_i is a resonant eigenvalue and $d_i = \underset{\mu_j+\mu_k=\mu_i}{\text{Max}} (d_j + d_k)$ if $\mu_i \in \mathcal{S}\backslash\sigma(A)$. Then d_i satisfies:

$$(4.0) \qquad 0 \leq d_i \leq i - 1, \qquad i = 1, 2, 3, \ldots.$$

Proof. — Let us first notice that for every μ_i, μ_j in $\mathcal{S}$ we have $\mu_i + \mu_j \geq \mu_{i+j}$ (observe that at least $i+j$ elements in $\mathcal{S}$ are $\leq \mu_i + \mu_j$). We prove (4.0) by induction on i, the inequality being trivial for $i=1$. If μ_i is a nonresonant eigenvalue, there is nothing to prove. If $\mu_i = \mu_j + \mu_k j$, $k \geq 1$, is such that $d_i = d_j + d_k$ or $d_i = 1 + d_j + d_k$, one has by the preceding observation $j + k \leq i$, and on the other hand, $d_j + d_k \leq j - 1 + k - 1 = j+k-2$; this proves that, in each case, $d_i \leq j + k - 2 + 1 \leq i - 1$. The equality is achieved in the « purely resonant case » where $\Lambda_k = k\Lambda_1$, $k=1, 2, 3, \ldots$. ■

The main result of this paragraph is the following result which is a counterpart of Theorem 2 in § 2.

THEOREM 4. — For each $N \in \mathbb{N}$, the solution u of (1.8) admits the expansion in H:

$$u(t) = W_{\mu_1} e^{-\nu\mu_1 t} + W_{\mu_2}(t) e^{-\nu\mu_2 t} + \ldots + W_{\mu_N}(t) e^{-\nu\mu_N t} + v_N(t) \qquad \forall t > 0, \tag{4.1}$$

where $W_{\mu_j}(t)$ $(1 \leq j \leq N)$ is a $E^\infty(\Omega) \cap V$-valued polynomial in t and $v_N \in C([0, \infty); V) \cap C^\infty([t_0, \infty); E^\infty(\Omega) \cap V)$, $t_0 > 0$.

This expansion satisfies the following properties:

i) $\| v_N(t) \|_m = O(e^{-\nu(\mu_N + \varepsilon_N)t})$, for some $\varepsilon_N > 0$, and all $m = 0, 1, 2, \ldots$

ii) $d_j^0 \underset{\text{def}}{=} \deg W_{\mu_j} \leq j - 1$, $j = 1, \ldots, N$.

iii) If $\Lambda_j \leq \mu_N$ is a nonresonant eigenvalue, W_{Λ_j} is a constant in t and $R_j W_{\Lambda_j} = W_{\Lambda_j}$.

iv) If $\mu_j \leq \mu_N$ is not a nonresonant eigenvalue, $W_{\mu_j}(t)$ satisfies the equation

$$\frac{dW_{\mu_j}}{dt} + \nu(A - \mu_j)W_{\mu_j} + \sum_{\mu_l + \mu_k = \mu_j} B(W_{\mu_l}(t), W_{\mu_k}(t)) = 0. \tag{4.2}$$

If Λ_j is a resonant eigenvalue, one has

$$\deg W_{\Lambda_j} \leqq \operatorname*{Max}_{\mu_l + \mu_k = \Lambda_j} (d_l^0 + d_k^0) + 1. \tag{4.3}$$

Moreover, $R_k W_{\Lambda_j}(t)$, for $k \neq j$ and the coefficients of order $\geqq 1$ in $R_j W_{\Lambda_j}(t)$ are obtained from $R_1 W_{\Lambda_1}(0), \ldots, R_{j-1} W_{\Lambda_{j-1}}(0)$, after the successive integrations of some explicit elementary functions ([2]).

If $\mu_j \notin \sigma(A)$, $W_{\mu_j}(t)$ can be obtained from $R_1 W_{\Lambda_1}(0), R_2 W_{\Lambda_2}(0), \ldots, R_{k_j} W_{\Lambda_{k_j}}(0)$ after the successive integrations of some explicit elementary functions ([2]), where

$$\Lambda_{k_j} \underset{\text{def}}{=} \operatorname{Max} \{ \Lambda \in \sigma(A) ; \Lambda < \mu_j \}.$$

One has also $\deg W_{\mu_j} = d_j^0 \leqq \sup_{\mu_l + \mu_k = \mu_j} (d_l^0 + d_k^0)$.

REMARK 2. — The first coefficient $W_{\mu_j}(t)$ which is not identically zero in (4.1) corresponds to $\mu_j = \Lambda(u_0)$. In this case, it is constant in t and belongs to $R_{\Lambda(u_0)}H$.

([2]) See the proof below as well as Lemma 7 in § 6.

Proof of Theorem 4. — We shall proceed by induction on N. The construction of the first coefficient in (4.1) was given in the proof of Theorem 2 in § 2.

Before going to the general induction argument, we shall construct the second coefficient in the case where, for instance, $\mu_2 = 2\Lambda_1 = \Lambda_2$.

We start with

$$u(t) = e^{-\nu\Lambda_1 t}W_{\Lambda_1} + v_1(t), \tag{4.3}$$

where

$$\| v_1(t) \|_m = O(e^{-\nu(\Lambda_1+\varepsilon_1)t}), \qquad \forall m \geq 0.$$

In fact, cf. (2.19), v_1 satisfies the equation

$$\frac{dv_1}{dt} + \nu A v_1 + e^{-2\nu\Lambda_1 t}B(W_{\Lambda_1}, W_{\Lambda_1}) = h(t) \tag{4.4}$$

where $\| h(t) \|_m = O(e^{-\nu(2\Lambda_1+\varepsilon_2)t})\ \forall m \geq 0$.

From (4.4) we proceed as in the nonresonant case (cf. derivation of (2.20)), to get

$$R_1 v_1(t) = \frac{1}{\nu\Lambda_1} e^{-2\nu\Lambda_1 t}R_1 B(W_{\Lambda_1}, W_{\Lambda_1}) + O(e^{-\nu(2\Lambda_1+\varepsilon_2)t}) \tag{4.5}$$

We multiply the equation (4.4) by $e^{\nu\Lambda_2 t}$ and apply the projection R_2 to obtain

$$\frac{d}{dt}(e^{\nu\Lambda_2 t}R_2 v_1(t) + tR_2 B(W_{\Lambda_1}, W_{\Lambda_1})) = O(e^{-\nu\varepsilon_2 t}) \tag{4.6}$$

which proves that

$$e^{\nu\Lambda_2 t}R_2 v_1(t) + tR_2 B(W_{\Lambda_1}, W_{\Lambda_1}) \to W_{2,0} \quad \text{as} \quad t \to \infty;$$

$$e^{\nu\Lambda_2 t}R_2 v_1(t) + tR_2 B(W_{\Lambda_1}, W_{\Lambda_1}) - W_{2,0} = \int_t^\infty O(e^{-\nu\varepsilon_2 \tau})d\tau = O(e^{-\nu\varepsilon_2 t}).$$

Finally,

$$R_2 v_1(t) = e^{-\nu\Lambda_2 t}(W_{2,0} - tR_2 B(W_{\Lambda_1}, W_{\Lambda_1})) + O(e^{-\nu(\Lambda_2+\varepsilon_2)t}). \tag{4.7}$$

Continuing, we apply the projection R_k (for $k \geqq 3$) to (4.4) obtaining

$$\frac{d}{dt}R_k v_1(t) + \nu\Lambda_k R_k v_1(t) = - e^{-2\nu\Lambda_1 t}R_k B(W_{\Lambda_1}, W_{\Lambda_1}) + R_k h(t),$$

and

$$\begin{aligned}\frac{d}{dt} e^{2\nu\Lambda_1 t}R_k v_1(t) &+ \nu(\Lambda_k - 2\Lambda_1)e^{2\nu\Lambda_1 t}R_k v_1(t) \\ &= - R_k B(W_{\Lambda_1}, W_{\Lambda_1}) + e^{2\nu\Lambda_1 t}R_k h(t).\end{aligned}$$

At this point the proof is similar to the derivation of (2.23) in § 2. We find

$$R_k v_1(t) = - \frac{R_k B(W_{\Lambda_1}, W_{\Lambda_1})}{2(\Lambda_k - 2\Lambda_1)} e^{-2\nu\Lambda_1 t} + O(e^{-\nu(2\Lambda_1+\varepsilon_2)t}), \tag{4.8}$$

the O term in (4.8) being similar to that in the formulas (2.23), (2.52). We then define $W_{\mu_2}(t) \in H$ by

$$(4.9) \qquad W_{\mu_2}(t) = \frac{1}{\nu\Lambda_1} R_1 B(W_{\Lambda_1}, W_{\Lambda_1}) + W_{2,0} - t R_2 B(W_{\Lambda_1}, W_{\Lambda_1}) - \sum_{k \geq 3} \frac{R_k B(W_{\Lambda_1}, W_{\Lambda_1})}{\nu(\Lambda_k - 2\Lambda_1)}.$$

Let us notice that the « new part » in $W_{\mu_2}(t)$ is given by $R_2 W_{\mu_2}(0) = W_{2,0}$.

It is simple to check (4.2) in Theorem 4. In fact,

$$\frac{d}{dt} W_{\mu_2} + \nu(A - 2\Lambda_1) W_{\mu_2} + B(W_{\Lambda_1}, W_{\Lambda_1}) = - R_2 B(W_{\Lambda_1}, W_{\Lambda_1}) - R_1 B(W_{\Lambda_1}, W_{\Lambda_1}) - \sum_{k \geq 3} R_k B(W_{\Lambda_1}, W_{\Lambda_1}) + B(W_{\Lambda_1}, W_{\Lambda_1}) = 0.$$

This shows in particular that $W_{\mu_2}(t)$ is smooth for $t \geq 0$. We have finally shown that

$$u(t) = W_{\mu_1} e^{-\nu\mu_1 t} + W_{\mu_2}(t) e^{-\nu\mu_2 t} + v_2(t),$$

where $| v_2(t) | = O(e^{-\nu(\mu_2 + \varepsilon_2)t})$ for some suitable $\varepsilon_2 > 0$. It can be shown (cf. Lemma 6 below) that $\| v_2(t) \|_m = O(e^{-\nu(\mu_2 + \varepsilon_2)t})$, $\forall m \geqq 0$.

The general induction argument

Let us suppose that we have obtained the expansion

$$(4.10) \qquad u(t) = e^{-\nu\mu_1 t} W_{\mu_1} + \ldots + e^{-\nu\mu_N t} W_{\mu_N}(t) + v_N(t),$$

where the coefficient W_{μ_j} and the remainder v_N satisfy the properties of Theorem 4.

Let us set $\Lambda_{k_N+1} = \operatorname{Min} \{ \Lambda \in \sigma(A); \Lambda > \mu_N \}$.

We shall distinguish 2 cases

CASE I. — For $j, k = 1, \ldots, N$, one has either

$$\mu_j + \mu_k \leqq \mu_N \qquad \text{or} \qquad \mu_j + \mu_k \geqq \Lambda_{k_N+1}.$$

i. e. $\mu_{N+1} = \Lambda_{k_N+1}$. This eigenvalue may or may not be resonant.

From (4.10) and the induction hypothesis on the coefficients $W_{\mu_j}(t)$ we deduce the equation satisfied by v_N:

$$(4.11) \qquad \frac{dv_N}{dt} + \nu A v_N + B(v_N, v_N) + \sum_{i=1}^{N} e^{-\nu\mu_i t} [B(v_N, W_{\mu_i}) + B(W_{\mu_i}, v_N)] + \sum_{\mu_i + \mu_j \geq \Lambda_{k_N+1}} e^{-\nu(\mu_i + \mu_j)t} B(W_{\mu_i}, W_{\mu_j}) = 0.$$

Let us assume first of all that Λ_{k_N+1} is nonresonant. Then (4.11) reduces to

$$\frac{dv_N}{dt} + \nu A v_N = h_N(t), \tag{4.12}$$

where

$$\| h_N(t) \|_m = O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}) \qquad \forall m \geq 0. \tag{4.13}$$

At this stage the proof is similar to the proof of the corresponding step in the purely nonresonant case (cf. § 2). In particular, we define

$$W_{\mu_{N+1}} = W_{\Lambda_{k_N+1}} \underset{\text{def}}{=} \lim_{t\to\infty} e^{\nu\Lambda_{k_N+1}t} R_{k_N+1} v_N(t). \tag{4.14}$$

This coefficient belongs obviously to $E^\infty(\Omega)$ and satisfies *iii*) in Theorem 4. We set

$$u(t) = \sum_{i=1}^{N+1} W_{\mu_i}(t) e^{-\nu\mu_i t} + v_{N+1}(t) \tag{4.15}$$

and postpone until Lemma 6 below the proof that $v_{N+1}(t)$ satisfies the property *i*) in Theorem 4.

Let us now assume that Λ_{k_N+1} is resonant. In this case, (4.11) reduces to

$$\frac{dv_N}{dt} + \nu A v_N + \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} e^{-\nu\Lambda_{k_N+1}t} B(W_{\mu_i}(t), W_{\mu_j}(t)) = h_N(t), \tag{4.16}$$

where h_N satisfies (4.13). We obtain from (4.16):

$$\frac{d}{dt} e^{\nu\Lambda_k t} R_k v_N = - \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} e^{-\nu(\Lambda_{k_N+1}-\Lambda_k)t} R_k B(W_{\mu_i}, W_{\mu_j}) + O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1}-\Lambda_k)t}),$$

which yields, for $\Lambda_k < \Lambda_{k_N+1}$,

(4.17)

$$R_k v_N(t) = e^{-\nu\Lambda_{k_N+1}t} \int_t^\infty \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} e^{-\nu(\Lambda_{k_N+1}-\Lambda_k)(\tau-t)} R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau)) d\tau + O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}).$$

By the induction hypothesis, $W_{\mu_j}(t)$ is a polynomial in t, of degree $d_j^0 \leqq j-1$. Therefore, $R_k B(W_{\mu_i}(t), W_{\mu_j}(t))$ is a polynomial in t of degree $\leqq d_i^0 + d_j^0$. By integration, we obtain from (4.17):

$$R_k v_N(t) = e^{-\nu\Lambda_{k_N+1}t} R_k W^k_{k_N+1,0}(t) + O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}) \quad (k \leqq k_N) \tag{4.18}$$

where $W^k_{k_N+1,0}(t)$ is a polynomial in t of degree $\leqq d_i^0 + d_j^0$.

By the induction hypothesis, the coefficients of $W^k_{k_N+1,0}(t)$ can be explicitely expressed from $R_j W_{\Lambda_j}(0)$, $1 \leqq j \leqq k_N$.

On the other hand, one gets from (4.16)

$$\frac{d}{dt}(e^{\nu\Lambda_{k_N+1}t}R_{k_N+1}v_N) + \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_{k_N+1}B(W_{\mu_i}(t), W_{\mu_j}(t)) = O(e^{-\nu\varepsilon_{N+1}t}). \tag{4.19}$$

But $W_{\mu_i}(t)$, $W_{\mu_j}(t)$ are polynomials in t of the form

$$\begin{cases} W_{\mu_i}(t) = \sum_{l=0}^{d_i^0} a_{i,l}t^l \\ W_{\mu_j}(t) = \sum_{m=0}^{d_j^0} a_{j,m}t^m . \end{cases} \tag{4.20}$$

where $a_{i,l}$, $a_{j,m}$ are elements in $E^\infty(\Omega) \cap V$, which by the induction hypothesis are determined by $R_jW_{\Lambda_j}(0)$, $j \leq k_N$. Consequently,

$$\sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_{k_N+1}B(W_{\mu_i}(t), W_{\mu_j}(t)) = \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_{k_N+1} \sum_{l=0}^{d_i^0}\sum_{m=0}^{d_j^0} t^{l+m}B(a_{i,l}, a_{j,m}),$$

and (4.19) can be rewritten as follows

$$\frac{d}{dt}\left[e^{\nu\Lambda_{k_N+1}t}R_{k_N+1}v_N + \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_{k_N+1}\sum_{l=0}^{d_i^0}\sum_{m=0}^{d_j^0}\frac{t^{l+m+1}}{l+m+1} \times B(a_{i,l}, a_{j,m})\right] = O(e^{-\nu\varepsilon_{N+1}t}). \tag{4.21}$$

This equation shows that

$$W_{k_N+1,1} \underset{\text{def}}{=} \lim_{t\to\infty} \left[e^{\nu\Lambda_{k_N+1}t}R_{k_N+1}v_N(t) + \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_{k_N+1}\sum_{l=0}^{d_i^0}\sum_{m=0}^{d_j^0}\frac{t^{l+m+1}}{l+m+1}B(a_{i,l}, a_{j,m})\right] \tag{4.22}$$

exists.

Moreover,

$$e^{\nu\Lambda_{k_N+1}t}R_{k_N+1}v_N + \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_{k_N+1}\sum_{l=0}^{d_i^0}\sum_{m=0}^{d_j^0}\frac{t^{l+m+1}}{l+m+1}B(a_{i,l}, a_{j,m}) - W_{k_N+1,1} = \int_t^\infty O(e^{-\nu\varepsilon_{N+1}\tau})d\tau = O(e^{-\nu\varepsilon_{N+1}t}). \tag{4.23}$$

Thus, we have shown that

(4.24) $R_{k_N+1}v_N(t) = [W_{k_N+1,1} + W_{k_N+1,2}(t)]e^{-\nu\Lambda_{k_N+1}t} + O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t})$

where

$$(4.25)\quad W_{k_N+1,2}(t) \underset{\text{def}}{=} \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_{k_N+1} \sum_{l=0}^{d_i^0}\sum_{m=0}^{d_j^0} \frac{t^{l+m+1}}{l+m+1} B(a_{i,l}, a_{j,m})$$

is a polynomial of degree $1 + \underset{\mu_i+\mu_j=\Lambda_{k_N+1}}{\text{Max}} (d_i^0 + d_j^0)$ and such that $W_{k_N+1,2}(0) = 0$.

In order to achieve the construction of the $(N + 1)^{\text{th}}$ coefficient in our first case, it remains to deal with the behavior of the projection of v_N on $[I - (R_1 + \ldots + R_{k_N+1})]H$.

Let $\Lambda_k \geq \Lambda_{k_N+2}$. We obtain from (4.16):

$$\frac{d}{dt} e^{\nu\Lambda_{k_N+1}t}R_k v_N(t) + \nu(\Lambda_k - \Lambda_{k_N+1})e^{\nu\Lambda_{k_N+1}t}R_k v_N(t) = - \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_k B(W_{\mu_i}, W_{\mu_j}) + e^{\nu\Lambda_{k_N+1}t}R_k h_N(t);$$

integrating this differential system yields

$$(4.26)\quad e^{\nu\Lambda_{k_N+1}t}R_k v_N(t) = e^{-\nu(\Lambda_k-\Lambda_{k_N+1})t}R_k v_N(0) + \int_0^t e^{-\nu(\Lambda_k-\Lambda_{k_N+1})(t-\tau)}e^{\nu_{k_N+1}\tau}R_k h_N(\tau)d\tau - \int_0^t e^{-\nu(\Lambda_k-\Lambda_{k_N+1})(t-\tau)} \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau))d\tau\,.$$

Let us now express more precisely the product $P_k(t)$ of $e^{-\nu\Lambda_{k_N+1}t}$ by the last integral in (4.26), i. e. the expression:

$$- \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} e^{-\nu\Lambda_k t} \int_0^t e^{\nu\tau(\Lambda_k-\Lambda_{k_N+1})}R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau))d\tau\,.$$

One has (cf. (4.20)):

$$\int_0^t e^{\nu\tau(\Lambda_k-\Lambda_{k_N+1})}R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau))d\tau = \sum_{l=0}^{d_i^0}\sum_{m=0}^{d_j^0} R_k B(a_{i,l}, a_{j,m}) \int_0^t e^{\nu(\Lambda_k-\Lambda_{k_N+1})\tau}\tau^{l+m}d\tau\,.$$

Hence, by elementary computations, we get

$$(4.27)\quad P_k(t) = - \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} \sum_{l=0}^{d_i^0} \sum_{m=0}^{d_j^0} \left\{ R_k B(a_{i,l}, a_{j,m}) \left\{ e^{-\nu\Lambda_{k_N+1}t} \left[\frac{1}{\nu(\Lambda_k - \Lambda_{k_N+1})} t^{l+m} - \frac{(l+m)}{\nu^2(\Lambda_k - \Lambda_{k_N+1})^2} t^{l+m-1} \right.\right.\right.$$
$$\left. + \ldots + (-1)^{l+m-1} \frac{(l+m)!}{\nu^{l+m}(\Lambda_k - \Lambda_{k_N+1})^{l+m}} t + (-1)^{l+m} \frac{(l+m)!}{\nu^{l+m+1}(\Lambda_k - \Lambda_{k_N+1})^{l+m+1}} \right]$$
$$\left.\left. + e^{-\nu\Lambda_k t}(-1)^{l+m+1} \frac{(l+m)!}{\nu^{l+m+1}(\Lambda_k - \Lambda_{k_N+1})^{l+m+1}} \right\} \right\}.$$

This expression shows that $P_k(t)$ can be written as

$$e^{-\nu\Lambda_{k_N+1}t} R_k W^k_{k_N+1,3}(t) - e^{-\nu\Lambda_k t} R_k W^k_{k_N+1,3}(0),$$

where $W^k_{k_N+1,3}(t)$ is an $E^\infty(\Omega) \cap V$-valued polynomial in t, of degree $\leq \underset{\mu_i+\mu_j=\Lambda_{k_N+1}}{\mathrm{Max}} (d_i^0 + d_j^0)$, whose coefficients are uniquely determined by $R_j W_{\Lambda_j}(0)$, $j = 1, \ldots, k_N$.

It follows that

$$(4.28)\quad R_k v_N(t) = e^{-\nu\Lambda_{k_N+1}t} R_k W^k_{k_N+1,3}(t) + O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t})$$

where the O term behaves as in (4.8), i. e. it is similar to that in (2.23), (2.52). Now we define $W_{\mu_{N+1}}(t) \in H$ by

$$(4.29)\quad W_{\mu_{N+1}}(t) = \sum_{k=1}^{k_N} R_k W^k_{k_N+1,0}(t) + W_{k_N+1,1} + W_{k_N+1,2}(t) + \sum_{k \geq k_N+2} R_k W^k_{k_N+1,3}(t).$$

Let us notice that, due to the expression of $P_k(t)$, the series in (4.29) do converge in H and D(A). We insist on the fact that in (4.29), only the term $W_{k_N+1,1}$ ($\in R_{k_N+1}H$) is « new », since the coefficients of the other polynomials involved are determined from $R_1 W_{\Lambda_1}(0), \ldots, R_{k_N} W_{\Lambda_{k_N}}(0)$.

It remains to check the properties *i)-iv)* in Theorem 4. Property *ii)* follows immediately from Lemma 5; property *iii)* follows from the fact that if Λ_{k_N+1} is nonresonant, then $W_{\mu_{N+1}} \equiv W_{k_N+1,1}$. Let us verify *iv)*:

first it is easily seen from (4.29), (4.17), (4.38), (4.23), (4.24), (4.25), (4.28), (4.19), (4.26) that

$$(4.30)\quad \frac{d}{dt}W_{\mu_{N+1}}(t) = -\sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} B(W_{\mu_i}(t), W_{\mu_j}(t))$$
$$+ \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} \Bigg\{ \sum_{k=1}^{k_N} \nu(\Lambda_{k_N+1} - \Lambda_k)e^{\nu(\Lambda_{k_N+1}-\Lambda_k)t}$$
$$\times \int_t^{\infty} e^{-\nu(\Lambda_{k_N+1}-\Lambda_k)\tau} R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau))d\tau$$
$$+ \sum_{k\geq k_N+2} \nu(\Lambda_k - \Lambda_{k_N+1})e^{\nu(\Lambda_{k_N+1}-\Lambda_k)t}$$
$$\times \int_0^t e^{-\nu(\Lambda_{k_N+1}-\Lambda_k)\tau} R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau))d\tau \Bigg\}.$$

On the other hand,

$$(4.30')\quad \nu(A - \Lambda_{k_N+1})W_{\mu_N+1}$$
$$= \sum_{\mu_i+\mu_j=\Lambda_{k_N+1}} \Bigg\{ \sum_{k=1}^{k_N} \nu(\Lambda_k - \Lambda_{k_N+1}) \int_t^{\infty} e^{-\nu(\Lambda_{k_N+1}-\Lambda_k)(\tau-t)} R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau))d\tau$$
$$- \sum_{k\geqq k_N+2} \nu(\Lambda_k - \Lambda_{k_N+1}) \int_0^t e^{-\nu(\Lambda_k-\Lambda_{k_N+1})(t-\tau)} R_k B(W_{\mu_i}(\tau), W_{\mu_j}(\tau))d\tau \Bigg\}.$$

Thus equation (4.2) follows immediately from (4.30), (4.30'). It proves in particular that $W_{\Lambda_{k_N+1}}(t)$ has smooth coefficients. The other assertions in *iv*) have already been proved. We postpone until Lemma 6 the fact that $v_{N+1}(t)$ satisfies also property *i*).

CASE II. — There exist $j, k \leq N$ such that

$$(4.31)\quad \mu_N < \mu_j + \mu_k < \Lambda_{k_N+1}.$$

In this case, we set

$$\mu_{N+1} = \text{Min}\,\{ \mu_j + \mu_k \ (j, k \leq N) \text{ such that (4.31) holds} \}.$$

The construction of $W_{\mu_{N+1}}(t)$ is the same as that of $W_{\Lambda_{k_N+1}}(t)$ in the resonant case. The remainder $v_N(t)$ satisfies (4.16) with Λ_{k_N+1} replaced by μ_{N+1}, and $h_N(t)$ satisfies (4.13) with Λ_{k_N+1} replaced by μ_{N+1}. For $k \leq k_N$ where $\Lambda_{k_N} = \text{Max}\,\{ \Lambda \in \sigma(A);\ \Lambda < \mu_{N+1} \}$, we obtain as in Case I, formulas for $R_k v_N(t)$ which are exactly (4.17), (4.18), with Λ_{k_N+1} replaced by μ_{N+1}. The coefficients of the polynomial $W_{N+1,0}(t)$ are again determined by $R_j W_{\Lambda_j}(0)$, $j \leqq k_N$.

For $k \geq k_N + 1$, we obtain formulas similar to (4.26), (4.27), always with Λ_{k_N+1} replaced by μ_{N+1}. Finally one set

$$(4.32) \qquad W_{\mu_{N+1}}(t) = \sum_{k=1}^{k_N} R_k W^k_{N+1,0}(t) + \sum_{k \geq k_N+1} R_k W^k_{N+1,3}(t) .$$

The coefficients of $W^k_{N+1,0}(t)$, $W^k_{N+1,3}(t)$ are determined from $R_j W_{\Lambda_j}(0)$, $1 \leq j \leq k_N$. By Lemma 5, one has d^0_{N+1} = degree of

$$W_{\mu_{N+1}}(t) = \operatorname*{Max}_{\mu_i + \mu_j = \mu_{N+1}} (d^0_i + d^0_j) \leq N$$

(there is no shift of the degree by one since μ_{N+1} is not a resonant eigenvalue). The verification of (4.2) is as in Case I.

It remains in both cases to check property *i*) in Theorem 4. This will be the aim of the following lemma.

LEMMA 6. — One has

$$(4.33) \qquad \| v_{N+1}(t) \|_m = O(e^{-\nu(\mu_{N+1}+\varepsilon_{N+1})t}), \qquad m \geq 0 .$$

Proof. — We shall give the proof in the case where $\mu_{N+1} = \Lambda_{k_N+1}$, since the case $\mu_{N+1} \notin \sigma(A)$ is similar to the resonant eigenvalue case.

Let us first assume that Λ_{k_N+1} is *nonresonant*. As in the purely nonresonant case, one proves (cf. the derivation of (2.40) in § 2 that

$$(4.34) \qquad | (R_1 + \ldots + R_{k_N+1}) v_{N+1}(t) | = O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{n+1})t}) .$$

In a similar manner one deduces from (4.11) (see the derivation of (2.41) in § 2 that

$$(4.35) \qquad | Q v_{N+1}(t) | = O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}),$$

where $Q = I - (R_1 + \ldots + R_{k_N+1})$. On the other hand, setting $v^{(j)}_N = \dfrac{d^j v_N}{dt^j}$, one easily checks that this function satisfies an equation similar to (2.42). The only difference lies in the appearance of polynomials in t which do not affect the exponential rates of decay. The analysis is now identical to that in the purely nonresonant case; one obtains the estimates:

$$\| v^{(j)}_N \|_m = O(e^{-\nu(\mu_N+\varepsilon_N)t}), \qquad j = 1, 2, \ldots, \qquad \forall m \geqq 0 ,$$

and then

$$(4.36) \qquad | Q v^{(j)}_N(t) | = O(^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t}), \qquad j = 1, 2, \ldots ,$$

and (4.33) follows from these estimates.

Let us now assume that Λ_{k_N+1} is resonant. The proof is modelled on the proof of Lemma 3 in § 2. From (4.17) and the definition of $W_{\Lambda_{k_{N+1}}}$, it comes immediately:

(4.37) $$|(R_1 + \ldots + R_{kN})v_{N+1}(t)|\, O(e^{-\nu(\Lambda_{k_{N+1}}+\varepsilon_{N+1})t}).$$

Similarly, from (4.24), one obtains

(4.38) $$|R_{k_N+1}v_{N+1}(t)| = O\, e^{-\nu(\Lambda_{k_{N+1}}+\varepsilon_{N+1})t}).$$

Let us now consider the « infinite dimensional » part of v_{N+1}. We set

$$Q' = R_{k_N+2} + R_{k_N+3} + \ldots .$$

Obviously,

$$Q'v_{N+1} = Q'v_N - \sum_{k \geq k_N+2} R_k W^k_{k_N+1,3}(t) e^{-\nu\Lambda_{k_N+1}t}$$

and we get from (4.26), (4.28):

$$Q'v_{N+1}(t) = \sum_{k \geq k_N+2} \left\{ e^{-\nu\Lambda_k t} R_k v_N(0) + \int_0^t e^{-\nu\Lambda_k(t-\tau)} R_k h_N(\tau) d\tau \right\},$$

which proves that $|Q'v_{N+1}(t)| = O(e^{-\nu(\Lambda_{k_N+1}+\varepsilon_{N+1})t})$.

Now the differential equation for v_{N+1} is:

(4.39) $$\frac{dv_{N+1}}{dt} + \nu A v_{N+1} + B(v_{N+1}, v_{N+1}) + \sum_{j=1}^{N+1} e^{-\nu\mu_j t}[B(W_{\mu_j}, v_{N+1}) + B(v_{N+1}, W_{\mu_j})] + \sum_{\mu_i+\mu_j > \Lambda_{k+1}} e^{-\nu(\mu_i+\mu_j)t} B(W_{\mu_i}, W_{\mu_j}) = 0 .$$

Denoting as in the proof of Lemma 3, $v^{(j)}_{N+1} = \dfrac{d^j v_{N+1}}{dt^j}$, $j = 1, 2, 3, \ldots$ and similarly $W^{(j)}_{\mu_i} = \dfrac{d^j}{dt^j} W_{\mu_i}$, one obtains the following equation for $v^{(j)}_{N+1}$:

(4.40) $$\frac{dv^{(j)}_{N+1}}{dt} + \nu A v^{(j)}_{N+1} + \sum_{i=1}^{N+1} \sum_{k=0}^{j} \binom{j}{k} (-\nu\mu_i)^k e^{-\nu\mu_i t} \left\{ \sum_{l=0}^{j-k} \binom{j-k}{l} B(W^{(l)}_{\mu_i}, v^{(j-k-l)}_{N+1}) + B(v^{(l)}_{N+1}, W^{(j-k-l)}_{\mu_i}) \right\} + \sum_{k=0}^{j} \binom{j}{k} B(v^{(k)}_{N+1}, v^{(j-k)}_{N+1}) + \sum_{\mu_k+\mu_l > \Lambda_{k_N+1}} \sum_{m=0}^{j} \binom{j}{m} [-\nu(\mu_k+\mu_l)]^m \times e^{-\nu(\mu_k+\mu_l)t} \left\{ \sum_{s=0}^{j-m} \binom{j-m}{s} B(W^{(s)}_{\mu_k}, W^{(j-m-s)}_{\mu_l}) \right\} = 0.$$

At this level, one can proceed as in Lemmas 2 and 3 above, since the appearance of polynomials in t in (4.40) does not affect the exponential rates of decay. One obtains:

$$(4.41) \quad \| v_{N+1}^{(j)}(t) \|_m = O(e^{-\nu(\mu_N + \varepsilon_N)t}), \qquad \forall m \geqq 0, \qquad j = 1, 2, 3 \ldots$$

Then, taking the scalar product of (4.40) with $Q_{k_N+2} v_{N+1}^{(j)}$, one gets

$$(4.42) \quad Q_{k_N+2} v_{N+1}^{(j)}(t) = O(e^{-\nu(\Lambda_{k_N+1} + \varepsilon_{N+1})t}), \qquad j = 1, 2, 3, \ldots$$

(cf. the proof of Lemmas 2 and 3 above). The proof is completed along the same lines of those of Lemmas 2 and 3. □

5. THE EXTENSION OF THE LINEARIZATION MAP TO THE GENERAL CASE

In this paragraph, we construct, in the general case, a nonlinear map which is the counterpart of the linearizing map U.

To begin with, it will be convenient to denote by $W_{\mu_i}(t; v)$ the i^{th} coefficient of the asymptotic expansion at time t of the solution with initial data v. The following theorem defines the nonlinear transformation and states some of its basic properties.

THEOREM 5. — The mapping $W: \mathscr{R} \to \mathscr{S}_A$ given by

$$(5.1) \quad W(u_0) = R_1 W_{\Lambda_1}(0; u_0) \oplus R_2 W_{\Lambda_2}(0; u_0) \oplus R_3 W_{\Lambda_3}(0; u_0) \oplus \cdots$$

is analytic and one to one.

Proof. — The proof is analogous to the one of Theorem 3. Let us prove the second part of theorem. Let $u_0, v_0 \in \mathscr{R}$ be such that $W(u_0) = W(v_0)$. Then the construction of the coefficients $W_{\mu_i}(t; u_0)$, $W_{\mu_i}(t; v_0)$ (see property *iv*) in Theorem 4) shows that $u(t) = S(t)u_0$ and $v(t) = S(t)v_0$ have the same asymptotic expansion (4.1). The same argument used in Theorem 3 proves that $u(t) = v(t)$, $\forall t \geqq 0$.

Let us now show the analyticity of W.

It suffices to prove that every component $R_k W_{\Lambda_k}(0; u_0)$ of W is analytic in u_0. We proceed by induction on k. The fact that W_{Λ_1} is analytic in u_0 was already proven in [7]. Assume now that $R_j W_{\Lambda_j}(0; u_0)$ is analytic for $j \leqq k_N$. We have $R_{k_N+1} W_{\Lambda_{k_N+1}}(0; u_0) = W_{k_N+1,1}$, where $W_{k_N+1,1}$ is given by the limit (4.22). It is easily seen that $W_{k_N+1,1}$ can be expressed in a somewhat complicated but explicit expression as:

$$(5.2)\quad W_{k_N+1,1} = R_{k_N+1}\left[u_0 - \sum_{i=1}^{N} W_{\mu_i}(0)\right] - \int_0^\infty \left[\sum_{j=1}^{N} e^{\nu(\Lambda_{k_N+1}-\mu_j)t}\right.$$

$$\times [R_{k_N+1}[B(W_{\mu_j}(t), v_N(t)) + B(v_N(t), W_{\mu_j}(t))] + \sum_{\substack{\mu_i,\mu_j \leqq \mu_N \\ \mu_i+\mu_j > \Lambda_{k_N+1}}} e^{-\nu(\mu_i+\mu_j-\Lambda_{k_N+1})t}$$

$$\times R_{k_N+1}B(W_{\mu_i}(t), W_{\mu_j}(t))]dt - \int_0^\infty e^{\nu\Lambda_{k_N+1}t}R_{k_N+1}B(v_N(t), v_N(t))dt.$$

We recall also that

$$(5.3)\qquad v_N(t) = u(t) - \sum_{j=1}^{N} W_{\mu_j}(t)e^{-\nu\mu_j t}.$$

By Theorem 4, the coefficients $W_{\mu_j}(t)$ ($\mu_j \leqq \mu_N$) are polynomials in t, the coefficients of which are continuous multilinear functions of $R_jW_{\Lambda_j}(0, u_0)$, $1 \leqq j \leqq N$. This observation, used together with (5.2), (5.3), the induction assumption of Lemma 5 in § 3, show that $W_{k_N+1,1}$ is analytic in u_0 and the proof is complete.

REMARK 3. — Of course, W reduces to U in the nonresonant case. Because of the dependence of $W_{\mu_j}(t)$ in t, the nonlinear transform W does not (as it was the case in the nonresonant situation) linearize equation (1.9) into the linear Navier-Stokes equation $\frac{du}{dt} + \nu Au = 0$.

In order to give a weaker but similar property we start with an analog of Corollary 1 in § 3.

PROPOSITION 1. — One has for $s \geqq 0$, $t \geqq 0$, $u_0 \in \mathscr{R}$

$$(5.4)\qquad e^{-\nu s\mu_i}W_{\mu_i}(t+s; u_0) = W_{\mu_i}(t; u(s)), \qquad i = 1, 2, \ldots$$

Proof. — By Theorem 4 we have for all $N \geq 0$

$$(5.5)\quad u(t+s) = W_{\mu_1}(t+s; u_0)e^{-\nu(t+s)\mu_1}$$
$$+ W_{\mu_2}(t+s; u_0)e^{-\nu(t+s)\mu_2} + \ldots + W_{\mu_N}(t+s; u_0)e^{-\nu(t+s)\mu_N}$$
$$+ v_N(t+s) = W_{\mu_1}(t; u(s))e^{-\nu t\mu_1} + W_{\mu_2}(t; u(s))e^{-\nu t\mu_2} + \ldots$$
$$+ W_{\mu_N}(t; u(s))e^{-\nu t\mu_N} + v'_N(t),$$

where $v'_N(t)$ has the « right » exponential decay. The formula (5.4) is now a consequence of the uniqueness of the asymptotic expansion (4.2).

Let us now define $\mathscr{W}: [0, \infty) \times \mathscr{R} \to \mathscr{S}_A$ by

$$\mathscr{W}(s; v) = R_1 W_{A_1}(s; v) \oplus R_2 W_{A_2}(s; v) \oplus \dots$$

in such a way that

$$\mathscr{W}(0; v) = W(v).$$

Proposition 1 leads immediately to the « linearization » process

$$e^{-\nu s A}\mathscr{W}(s; u_0) = W(u(s)), \qquad \forall s \geqq 0, \qquad u_0 \in \mathscr{R} \tag{5.6}$$

to be compared with formula (3.4).

We recall once again that the polynomials $W_{A_j}(s; u_0)$ ($j = 1, 2, \dots$) are determined by elementary integration from $W(u_0)$.

In fact the nonlinear transform $\mathscr{W}$ reduces the nonlinear Navier-Stokes equation to a linear Navier-Stokes equation with an extra-time parameter, as shows the

COROLLARY 3. — For every $u_0 \in \mathscr{R}$, $s \geqq 0$, $t \geqq 0$, the function $\mathscr{W}(s; u(t))$ satisfies

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}\right)\mathscr{W}(s; u(t)) + \nu A \mathscr{W}(s; u(t)) = 0. \tag{5.7}$$

Proof. — From (5.4) one gets

$$\mathscr{W}(s; u(t)) = e^{-\nu t A}\mathscr{W}(t + s; u_0), \tag{5.8}$$

and by derivation:

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathscr{W}(s; u(t)) &= \frac{\partial}{\partial t} e^{-\nu t A}\mathscr{W}(t + s; u_0) = -\nu A e^{-\nu t A}\mathscr{W}(t + s; u_0) \\
&\quad + e^{-\nu t A}\frac{\partial}{\partial t}\mathscr{W}(t + s; u_0) \\
&= -\nu A \mathscr{W}(s; u(t)) + e^{-\nu t A}\frac{\partial}{\partial s}\mathscr{W}(t + s; u_0) \\
&= -\nu A \mathscr{W}(s; u(t)) + \frac{\partial}{\partial s}(e^{-\nu t A}\mathscr{W}(t + s; u_0)) \\
&= -\nu A \mathscr{W}(s; u(t)) + \frac{\partial}{\partial s}\mathscr{W}(s; u(t)),
\end{aligned}$$

which is exactly (5.7). ■

Concerning the linearization of the nonlinear spectral manifolds M_k, Corollary 2 extends to the general case:

COROLLARY 4. — $u_0 \in M_k$ if and only if the k first components of $W(u_0)$ vanish.

Proof. — Is if similar to the corresponding proof of Corollary 2 and lies on the same characterization of M_k. It suffices to observe that if $R_1 W_{\Lambda_1}(0; u_0), \ldots, R_k W_{\Lambda_k}(0; u_0)$ vanish, then, by the properties of the coefficients of our expansion, one has

$$W_{\Lambda_1}(t; u_0) \equiv W_{\mu_2}(t; u_0) \equiv \ldots \equiv W_{\Lambda_k}(t; u_0) \equiv 0, \qquad \forall t \geq 0$$

and therefore $\Lambda(u_0) \geq \Lambda_{k+1}$.

REMARK 4. — In the case of the spatial periodic boundary condition (0.3), it has been noticed in [7], Remark 7, that each spectral manifold M_k contains an unbounded, infinite dimensional linear submanifold L_k. It is easily seen from the definition of L_k that W reduces to the identity operator on L_k.

6. THE NORMALIZATION MAP AND THE NORMAL FORM

In this chapter we shall prove that the mapping W reduces the Navier-Stokes equation to an autonomous normal form and can therefore be called the *normalization map.* This can be thought as a global, non formal (3), extension of the Poincaré-Dulac theory to the Navier-Stokes equations with potential forces.

To begin with, we state a proposition which makes more precise the behavior of W at the origin.

PROPOSITION 2. — The derivative of W at 0 is I, where I denotes the canonical embedding $V \to \mathscr{S}_A$.

Proof. — It suffices to prove that, for every N, the derivative of the map (cf. (5.2)):

$$u_0 \to W_{k_N+1,1}$$

at $u_0 = 0$ is the projection $v \to R_{k_N+1} v$. First, we notice that the derivative at 0 of the integral term in (5.2) vanishes. On the other hand, one has, for $\mu_i \leqq \Lambda_{k_N}$,

$$R_{k_N+1} W_{\mu_i}(0) = \begin{cases} 0 & \text{if } \mu_i \text{ is a nonresonant eigenvalue} \\ R_{k_N+1} W^{k_N+1}_{k_i+1,3}(0) & \text{otherwise}. \end{cases}$$

In this last case, if $\mu_i \in \sigma(A)$, then by using Theorem 4 and (4.26) (4.27) (4.28), we obtain easily that the derivative of $R_{k_N+1} W^{k_N+1}_{k_i+1,3}(0)$ at $u_0 = 0$

(3) By this we mean that our transformation is well defined on $\mathscr{R}$. In the classical Poincare-Dulac theory, the normalization is defined by a formal series.

is zero. The same argument holds also if $\mu_i \notin \sigma(A)$. Now the conclusion follows readily from (5.2). □

REMARK 5. — One infers from Proposition 2 that for $N = 1, 2, \ldots$, the range of $\pi_N W$ contains a ball centered at 0, where π_N denotes the canonical projection $\pi_N = \mathscr{S}_A \to R_1 H \oplus \ldots \oplus R_N H$.

Our next goal is to look for the equation satisfied by $W(u(t))$. This will lead to a normal form of the Navier-Stokes equation, which of course is the linear Navier-Stokes equation in the nonresonant case.

First we will clarify the dependence of $W_{\mu_j}(0; u_0)$ in $W_{\Lambda_1}(u_0), \ldots, W_{\Lambda_{k_j}}(u_0)$.

LEMMA 7. — For every $j = 1, 2, 3, \ldots$, there exists a $E^\infty(\Omega) \cap V$-valued multilinear function P_j, defined on $R_1 H \oplus \ldots \oplus R_{k_j} H$, depending on $\sigma(A)$, B, ν, such that

$$W_{\mu_j}(0; u_0) = P_j(W_1(u_0), W_2(u_0), \ldots, W_{k_j}(u_0))$$

where $W_l(u_0) \underset{\text{def}}{=} R_l W_{\Lambda_l}(0; u_0)$ (l^{th} component of the mapping W evaluated at u_0). If $\mathscr{M}(X_1, \ldots, X_{k_j})$ is a monomial in $P_j(X_1, \ldots, X_{k_j})$ of degree $m_1, \ldots, m_{k_j}$ in $X_1, \ldots, X_{k_j}$ respectively, then $m_1 \Lambda_1 + \ldots + m_{k_j} \Lambda_{k_j} = \mu_j$. If moreover $\mu_j = \Lambda_k$, then $P_j(X_1, \ldots, X_k) = X_k +$ higher order terms.

Proof. — *i*) $\mu_j = \Lambda_{k_j}$ is a nonresonant eigenvalue. Then

$$R_{k_j} W_{\Lambda_{k_j}}(0, u_0) = W_{k_j}(u_0) \quad \text{and} \quad P_j(X_1, \ldots, X_{k_j}) = X_{k_j}.$$

ii) $\mu_j = \Lambda_{k_j}$ is a resonant eigenvalue. Then, by (4.29), one has

$$W_{\Lambda_{k_j}}(0) = \sum_{k=1}^{k_j - 1} R_k W^k_{k_j,0} + W_{k_j,1} + \sum_{k \geq k_j + 1} R_k W^k_{k_j,3}(0). \tag{6.1}$$

Here we used the fact that $W_{k_j,2}(0) = 0$. But

$$W_{k_j,1} = R_{k_j} W_{k_j,1} = R_{k_j} W_{\Lambda_{k_j}}(0; u_0) \underset{\text{def}}{=} W_{k_j}(u_0).$$

Moreover the other terms in (6.1) are multilinear functions of

$$W_1(u_0), \ldots, W_{k_j - 1}(u_0)$$

(use induction and formulas (4.18), (4.27), (4.28), (4.29)). An obvious analog of this argument proves also the remaining case:

iii) $\mu_j \notin \sigma(A)$.

Let us prove the last assertion in Lemma 7. By (5.4) and the beginning of the present proof, one has for $\mu_j \in \mathscr{S}$

$$\begin{aligned} e^{-\nu \mu_j t} W_{\mu_j}(t; u_0) &= W_{\mu_j}(0; S(t, u_0)) \\ &= P_j(W_1(S(t, u_0)), \ldots, W_{k_j}(S(t, u_0))), \end{aligned}$$

where $W_l(S(t, u_0)) = R_l W_{\Lambda_l}(0; S(t, u_0))$. Again by (5.4) we have

$$W_{\Lambda_l}(0, S(t, u_0)) = e^{-\nu t \Lambda_l} W_{\Lambda_l}(t, u_0)$$

and finally

$$e^{-\nu \mu_j t} W_{\mu_j} = P_j(e^{-\nu t \Lambda_1} R_1 W_{\Lambda_1}(t, u_0), \ldots, e^{-\nu t \Lambda_{k_j}} R_{k_j} W_{\Lambda_{k_j}}(t, u_0)).$$

So if $\mathcal{M}$ is any monomial of P_j of degree m_l in X_l, $1 \leqq l \leqq k_j$, one must have $\mu_j = m_1 \Lambda_1 + \ldots + m_{k_j} \Lambda_{k_j}$. This proves moreover that

$$W_{\mu_j}(t, u_0) = P_j(R_1 W_{\Lambda_1}(t, u_0), \ldots, R_{k_j} W_{\Lambda_{k_j}}(t, u_0)). \tag{6.1$'$}$$

If $\mu_j = \Lambda_{k_j}$ is a resonant eigenvalue, formulas (6.1) and (4.29), together with a simple induction argument, show that $P_j(X_1, \ldots, X_{k_j}) = X_{k_j} +$ higher order terms depending only on $X_1, \ldots, X_{k_j - 1}$. □

The following theorem gives the normal form of the Navier-Stokes equations in the resonant case. Let $v(t) = S(t)u_0$, $u_0 \in \mathcal{R}$:

THEOREM 6. — The ($\mathcal{S}_A$-valued) function $v(t) = W(u(t))$ satisfies the equation

$$\frac{dv(t)}{dt} + \nu A v(t) + \mathcal{B}(v(t)) = 0 \tag{6.2}$$

where

$$\mathcal{B}_k(v) \underset{\text{def}}{=} R_k \mathcal{B}(v) = \sum_{\mu_l + \mu_j = \Lambda_k} R_k B(P_l(v_1, \ldots, v_{k_l}), P_j(v_1, \ldots, v_{k_j}), \tag{6.3}$$

and $v = v_1 \oplus v_2 \oplus \ldots \in \mathcal{S}_A$.

REMARK 6. — It must be noticed that equation (6.2), although nonlinear in $\mathcal{S}_A$, can be solved by integrating successively an infinite sequence of *linear* nonhomogeneous differential equations in $R_k H$, $k = 1, 2, \ldots$, each of which having already known nonhomogeneous part. □

Proof. — It is based on the following trivial observation: if one has a dynamical system $\begin{cases} \dfrac{dv(t)}{dt} = \mathcal{N}(v(t)) \\ v(0) = v \end{cases}$, then $\mathcal{N}(v) = \dfrac{d}{dt} S(t; v)|_{t=0}$, where $S(t; v) = v(t)$.

Thus if $v(t) = W(u(t)) = W(S(t)u_0)$ ($\forall u_0 \in \mathcal{R}$) is a solution of such an autonomous differential equation in $\mathcal{S}_A$, then $\mathcal{N}$ must be defined by

$$\mathcal{N}(W(u_0)) \underset{\text{def}}{=} \frac{d}{dt} W(S(t)u_0)|_{t=0}.$$

For the k-coordinate $\mathcal{N}_k$ of $\mathcal{N}$, we have

$$\mathcal{N}_k(W(u_0)) = \frac{d}{dt} R_k W_{\Lambda_k}(0; S(t)u_0)|_{t=0},$$

hence by (5.4) and (4.2)

$$(6.4)\quad \mathcal{N}_k(\mathrm{W}(u_0)) = \frac{d}{dt} e^{-\nu\Lambda_k t}\mathrm{R}_k\mathrm{W}_{\Lambda_k}(t, u_0)\big|_{t=0}$$

$$= -\nu\Lambda_k \mathrm{W}_k(u_0) - \sum_{\mu_l+\mu_j=\Lambda_k} \mathrm{R}_k\mathrm{B}(\mathrm{W}_{\mu_j}(0; u_0), \mathrm{W}_{\mu_l}(0; u_0)).$$

Obviously, if Λ_k is not a resonant eigenvalue, the nonlinear term in (6.4) is zero. The theorem follows now from Lemma 7. □

COROLLARY 5. — A necessary and sufficient condition that our normal form (6.2) be linear, i. e. $\mathcal{B} \equiv 0$ is that

$$(6.5)\quad \nu(\mathrm{A} - \mu_j)\mathrm{P}_j + \sum_{\mu_l+\mu_k=\mu_j} \mathrm{B}(\mathrm{P}_l, \mathrm{P}_k) = 0, \qquad \text{for} \qquad j = 1, 2, \ldots$$

Proof. — If (6.5) holds, for any k, choose j such that $k_j = k$. Then, applying R_k to (6.5) for this j, we obtain $\mathcal{B}_k \equiv 0$.

On the other hand, from (4.2) (with an obvious interpretation if μ_j is a nonresonant eigenvalue) and from (6.1′) we obtain:

$$(6.6)\quad \sum_{i=2}^{k_j} (\mathrm{D}_i\mathrm{P}_j(\mathrm{R}_1\mathrm{W}_{\Lambda_1}, \ldots, \mathrm{R}_{k_j}\mathrm{W}_{\Lambda_{k_j}}))(\mathrm{R}_i\dot{\mathrm{W}}_{\Lambda_i})$$

$$+ \nu(\mathrm{A} - \mu_j)\mathrm{P}_j(\mathrm{R}_1\mathrm{W}_{\Lambda_1}, \ldots, \mathrm{R}_{k_j}\mathrm{W}_{\Lambda_{k_j}})$$

$$+ \sum_{\mu_l+\mu_k=\mu_j} \mathrm{B}(\mathrm{P}_l(\mathrm{R}_1\mathrm{W}_{\Lambda_1}, \ldots, \mathrm{R}_{k_j}\mathrm{W}_{\Lambda_{k_j}}), \mathrm{P}_k(\mathrm{R}_1\mathrm{W}_{\Lambda_1}, \ldots, \mathrm{R}_{k_j}\mathrm{W}_{\Lambda_{k_j}}))$$

where D_i denotes the i^{th} partial derivative and the dot the time derivative. Taking in (4.2) $\mu_j = \Lambda_i$, and applying the projection R_i we deduce

$$\mathrm{R}_i\dot{\mathrm{W}}_{\Lambda_i} + \mathcal{B}_i = 0.$$

So if $\mathcal{B} \equiv 0$, the first sum in (6.6) vanishes. (6.5) follows now from (6.6) and Remark 5 with $\mathrm{N} = k_j$. □

We now turn to the problem of the uniqueness of our normalizing mapping W.

We shall start with a very simple observation.

REMARK 7. — Let $\Theta = (\Theta_k)_{k=1}^{\infty}$, $\Theta : \mathcal{R} \to \mathcal{S}_\mathrm{A}$ be a C^1 mapping such that $\Theta'(0) = \mathrm{I}d$ and which linearizes the Navier-Stokes equation, i. e. such that $\Theta(u(t)) = e^{-\nu t\mathrm{A}}\Theta(u_0)$ for every solution u of (1.9). Then, one has $\Theta_1(v) = \mathrm{W}_1(v)$, $\forall v \in \mathcal{R}$.

Proof. — In fact, one has in particular for every regular solution u of (1.9)

$$\Theta_1(u(t)) = R_1 u(t) + T_1(u(t))$$

where $T_1(u(t)) \in R_1 H$ and satisfies $| T_1(u(t)) | = o(| u(t) |)$. Thus,

$$\lim_{t \to \infty} e^{\nu \Lambda_1 t} T_1(u(t)) = 0 .$$

But

$$e^{-\nu \Lambda_1 t} \Theta_1(u_0) = \Theta_1(u(t)) = R_1 u(t)) + T_1(u(t)) .$$

Finally, $\Theta_1(u_0) = \lim_{t \to \infty} e^{\nu \Lambda_1 t} R_1 u(t) = W_{\Lambda_1}(u_0) = W_1(u_0)$. □

A more complete picture is available when Θ is analytic. This will be the aim of the next theorem.

THEOREM 7. — Let $\Theta : \mathscr{R} \to \mathscr{S}_A$ be an analytic mapping satisfying $\Theta'(0) = Id$ and linearizing the Navier-Stokes equation. Then

i) If Λ_k is a nonresonant eigenvalue, one has $\Theta_k = W_k$. In particular, if A is nonresonant, $\Theta = W$.

ii) If Λ_k is a resonant eigenvalue, one has for all $u_0 \in \mathscr{R}$

$$(6.5) \quad \Theta_k(u_0) = W_k(u_0) + \sum_{\substack{m \geq 2 \\ \mu_{\sigma_1}, \ldots, \mu_{\sigma_m} \in \mathscr{S} \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} = \Lambda_k}} \left(\frac{1}{m!} D^m \Theta_k(0) \right) (P_{\sigma_1}, \ldots, P_{\sigma_m}) \text{ (4)},$$

where the $P_{\sigma_i} = P_{\sigma_i}(W_1(u_0), \ldots, W_{k\sigma_i}(u_0))$'s were defined in Lemma 7.

REMARK 8. — 1) If Θ is known explicitely, Theorem 7 provides effective formulas to compute W. This fact will be used in § 7 below in connection with the Cole-Hopf transform for the Burgers equation and the Minea system.

2) Taking $\Theta = W$, formulas (6.5) yield the relation (with a slight abuse of notation):

$$\sum_{\substack{m \geq 2 \\ \mu_{\sigma_1}, \ldots, \mu_{\sigma_m} \in \mathscr{S} \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} = \Lambda_k}} \frac{1}{m!} (D^m W_k(0))(P_{\sigma_1}(u_0), \ldots, P_{\sigma_m}(u_0)) = 0, \qquad \forall u_0 \in \mathscr{R} .$$

when Λ_k is a resonant eigenvalue (see also the proof of Theorem 7 for other related properties of W in the resonant case).

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(4) $D^m \Theta_k(0)$ denotes the m^{th} derivative of Θ_k at 0. We refer to [3] for the elementary theory of analytic mapping between functional spaces.

3) As a by product of the proof of point i) we shall obtain more informations on W; in particular it will be proven that, in the purely nonresonant case:

$$\sum_{\substack{\mu_2 \leqq \mu_j < \Lambda_k, \mu_j \notin \sigma(\Lambda) \\ \mu_{\sigma 1} + \dots + \mu_{\sigma m} = \mu_j}} R_k W_{\mu_j}(u_0) + \frac{1}{m!}(D^m W_k(0))(W_{\mu_{\sigma_1}}(u_0), \dots, W_{\mu_{\sigma m}}(u_0)) = 0, \qquad \forall u_0 \in \mathscr{R}.$$

Proof of Theorem 7. — For every $N \geqq 1$, we have for every solution u of (1.9)

$$(6.6) \quad u(t) = e^{-\nu\Lambda_1 t} W_{\Lambda_1} + e^{-\nu\mu_2 t} W_{\mu_2}(t) + \dots + e^{-\nu\mu_N t}(t) W_{\mu_N}(t) + v_N(t))$$

where v_N satisfies the right exponential decay (see Theorem 4 i)).

On the other hand, the assumptions made on Θ yield for $k \geqq 2$

$$(6.7) \quad e^{-\nu\Lambda_k t}\Theta_k(u_0) = \Theta_k(u(t)) = R_k u(t) + \sum_{m \geqq 2} \left(\frac{1}{m!} D^m \Theta_k(0)\right) \underbrace{(u(t), \dots, u(t))}_{m \text{ times}}$$

(this holds provided that t is large enough to make $u(t)$ so small that the series converges).

Applying the projection R_k to (6.6) we get:

$$(6.8) \qquad R_k u(t) = \sum_{\mu_i < \Lambda_k} \{ e^{-\nu\mu_i t} R_k W_{\mu_i}(t) + e^{-\nu\Lambda_k t} R_k W_{\Lambda_k}(t) + O(e^{-\nu(\Lambda_k + \varepsilon_k)t}) \}.$$

Introducing (6.6) and (6.8) into (6.7), we obtain:

$$\begin{aligned}(6.9) \quad e^{-\nu\Lambda_k t}\Theta_k(u_0) = & \sum_{2 \leqq j \leqq i(k)-1} \{ R_k W_{\mu_j}(t) e^{-\nu\mu_j t} \} + R_k W_{\Lambda_k}(t) e^{-\nu\Lambda_k t} \\ & + \sum_{\substack{m \geqq 2 \\ \mu_{\sigma 1}, \dots, \mu_{\sigma m} \in \mathscr{S} \\ \mu_{\sigma 1} + \dots + \mu_{\sigma m} < \Lambda_k}} e^{-\nu(\mu_{\sigma 1} + \dots + \mu_{\sigma m})t} P_{k,m}(W_{\mu_{\sigma_1}}(t), \dots, W_{\mu_{\sigma m}}(t)) \\ & + \sum_{\substack{m \geqq 2 \\ \mu_{\sigma 1} + \dots + \mu_{\sigma m} = \Lambda_k}} e^{-\nu\Lambda_k t} P_{k,m}(W_{\mu_{\sigma_1}}(t), \dots, W_{\mu_{\sigma m}}(t)) + O(e^{-\nu(\Lambda_k + \varepsilon_k)t}).\end{aligned}$$

where we have set $P_{k,m} = \dfrac{1}{m!}(D^m \Theta_k)(0)$ and we have define the integer $i(k)$ by $\mu_{i(k)} = \Lambda_k$. Note that in the last two sums, the elements μ_{σ_i} may not be distinct; also the corresponding m's are such that $m \leqq \dfrac{\Lambda_k}{\Lambda_1}$.

Let us suppose first that Λ_k is not resonant. Then (6.9) implies

$$(6.10)\quad \Theta_k(u_0) = W_{\Lambda_k} + \sum_{2 \leq j \leq i(k)-1} R_k W_{\mu_j}(t) e^{\nu(\Lambda_k - \mu_j)t}$$

$$+ \sum_{\substack{m \geq 2 \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} < \Lambda_k}} e^{\nu(\Lambda_k - \mu_{\sigma_1} - \ldots - \mu_{\sigma_m})t} P_{k,m}(W_{\mu_{\sigma_1}}(t), \ldots, W_{\mu_{\sigma_m}}(t)) + O(e^{-\nu\varepsilon_k t}).$$

This equality can be written as

$$(6.11)\quad \Theta_k(u_0) = W_k(u_0)$$

$$+ \sum_{\substack{2 \leq j \leq i(k)-1 \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} = \mu_j}} e^{\nu(\Lambda_k - \mu_j)t} \{ R_k W_{\mu_j}(t) + P_{k,m}(W_{\mu_{\sigma_1}}(t), \ldots, W_{\mu_{\sigma_m}}(t) \} + O(e^{-\nu\varepsilon_k t}).$$

Letting $t \to +\infty$ one sees that the polynomials in t, $\{ \ldots \}$ must vanish identically (this gives in particular the point 3 of Remarks 8), and then

$$(6.12)\qquad \Theta_k(u_0) = W_k(u_0).$$

If Λ_k is a resonant eigenvalue we obtain

$$(6.13)\quad \Theta_k(u_0) = R_k W_{\Lambda_k}(t) + \sum_{\substack{m \geq 2 \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} = \Lambda_k}} P_{k,m}(W_{\mu_{\sigma_1}}(t), \ldots, W_{\mu_{\sigma_m}}(t))$$

$$+ \sum_{\substack{2 \leq j \leq i(k)-1 \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} = \mu_j}} e^{\nu(\Lambda_k - \mu_j)t} \{ R_k W_{\mu_j}(t) + P_{k,m}(W_{\mu_{\sigma_1}}(t), \ldots, W_{\mu_{\sigma_m}}(t)) \} + O(e^{-\nu\varepsilon_k t}).$$

Letting $t \to \infty$, one sees again that the polynomials $\{ \ldots \}$ must vanish identically, and that the polynomial

$$R_k W_{\Lambda_k}(t) + \sum_{\substack{m \geq 2 \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} = \Lambda_k}} P_{k,m}(W_{\mu_{\sigma_1}}(t), \ldots, W_{\mu_{\sigma_m}}(t))$$

must be constant. One obtains finally (recalling that $W_k(u_0) = R_k W_{\Lambda_k}(0; u_0)$):

$$(6.14)\quad \Theta_k(u_0) = W_k(u_0)$$

$$+ \sum_{\substack{m \geq 2 \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_m} = \Lambda_k}} \frac{1}{m!} (D^m \Theta_k(0))(W_{\mu_{\sigma_1}}(0, u_0), \ldots, W_{\mu_{\sigma_m}}(0, u_0)).$$

Using Lemma 7 and its notation we infer from (6.14) that

$$(6.15)\quad \Theta_k(u_0) = W_k(u_0) + \sum_{\substack{m \geqq 2 \\ \mu_{\sigma_1} + \ldots + \mu_{\sigma_2} = \Lambda_k}} \frac{1}{m!}(D^m\Theta_k(0)) \\ .(P_{\sigma_1}(W_1(u_0), \ldots, W_{k_{\sigma_1}}(u_0)), \ldots, (P_{\sigma_m}(W_1(u_0), \ldots, W_{k_{\sigma_m}}(u_0)))$$

and the proof of Theorem 7 is complete. □

7. SOME EXAMPLES: THE BURGERS EQUATION AND THE MINEA SYSTEM

As we mentionned before, our methods apply to a large class of equations, including the Navier-Stokes equations. To illustrate Theorem 7 we shall emphasize first the case of the Burgers equation.

$$(7.1)\quad \begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \qquad x \in \mathbb{R}, \qquad t \geqq 0, \qquad \nu > 0 \\ &u(x, 0) = u_0. \end{aligned}$$

complemented with the periodic boundary conditions (for $a > 0$ fixed)

$$(7.2)\quad u(a, t) = u(-a, t), \qquad \forall t \geqq 0.$$

The previous analysis is still valid for equations (7.1) (7.2) in the following functional setting.

We define $\mathscr{V} = \left\{ u = \text{trigonometric polynomials}, \int_{-a}^{a} u dx = 0, u \text{ satisfies (7.2)} \right\}$; $\mathscr{R} = V =$ closure of $\mathscr{V}$ in $H^1(-a, a)$; $H =$ closure of $\mathscr{V}$ in $L^2(-a, a)$. The operator $A \underset{\text{def}}{=} -\frac{d^2}{dx^2}$ is self-adjoint unbounded in H and has the pure point spectrum $\left\{ \Lambda_1 = \frac{\Pi^2}{a^2}, \Lambda_2 = \frac{4\Pi^2}{a^2}, \ldots, \Lambda_k = \frac{k^2\Pi^2}{a^2}, \ldots \right\}$ (all the eigenvalues are resonant: $\Lambda_k = k^2\Lambda_1$).

Finally, by setting $B(u, u) = u\frac{\partial u}{\partial x}$, the space periodic Burgers equation is reduced to our abstract form (see (1.9))

$$(7.3)\quad \frac{du}{dt} + \nu A u + B(u, u) = 0, \qquad u(0) = u_0.$$

The classical Cole-Hopf transform (cf. [*24*][*14*]) can be obviously extended to the space periodic case in the following way: for $u \in V$, let $\Theta(u) = v$ be defined by

$$v(x) = -2\nu \frac{d}{dx}\left(e^{-\frac{1}{2\nu}\int_{-a}^{x} u(\zeta)d\zeta}\right). \tag{7.4}$$

It is easily seen that v is still periodic and that $\int_{-a}^{a} v dx = 0$. Moreover the mapping $\Theta = V \to (V \subset) \mathscr{S}_A = R_1H \oplus R_2H \oplus \ldots$ (5) is clearly analytic. Also

$$\Theta'(0).v = -\frac{d}{d\varepsilon}\left[2\nu \frac{d}{dx} e^{-\frac{\varepsilon}{2\nu}\int_{-a}^{x} v(\zeta)d\zeta}\right]_{|\varepsilon=0} = v,$$

i. e. $\Theta'(0) = Id$.

Also it is straightforward that this Θ linearizes the Burgers equation: for every regular solution $u(t)$ of (7.3), one has

$$\Theta(u(t)) = e^{-\nu t A}\Theta(u_0).$$

One can therefore apply Theorem 7, which shows that in this case, our mapping W can be explicitely computed in terms of the Cole-Hopf transform.

Another instructive example is the following Minea system [5] [15].

$$\begin{aligned} &\frac{d\alpha_1}{dt} + \lambda_1\alpha_1 + \alpha_2^2 + \alpha_3^2 + \ldots = 0 \\ &\frac{d\alpha_j}{dt} + \lambda_j\alpha_j - \alpha_1\alpha_j = 0, \qquad j = 2, 3, \ldots \end{aligned} \tag{7.5}$$

where $u = (\alpha_j)_{j=1}^{\infty} \in H = l^2(\mathbb{N})$, $0 < \lambda_1 < \lambda_2 < \ldots \to \infty$.

This system enters in the class of equations of type (1.9) to which our previous considerations can be applied. In this case A is the diagonal operator with diagonal entries $(\lambda_1, \lambda_2, \ldots)$ and $\mathscr{S}_A$ is the space $\mathscr{S}(\mathbb{N})$ of all numerical sequences. In order to determine the normalization mapping and the normal form of (7.5), we will study the map from $H = l^2(\mathbb{N})$ to $\mathscr{S}_A = \mathscr{S}(\mathbb{N})$ defined by:

$$u \to \Theta(u) = \left(\alpha_1 - \int_0^{\infty} e^{\lambda_1\tau}(|S(t,u)|^2 - |S(t,u)_1|^2)d\tau\right), \quad \alpha_2 e^{\int_0^{\infty} S(\tau,u)_1 d\tau}, \ldots, \alpha_j e^{\int_0^{\infty} S(\tau,u)_1 d\tau}, \ldots) \tag{7.6}$$

where $S(t,.)$ as usual denotes the nonlinear semigroup associated to (7.5).

(5) Here we still denote R_i the projection on the eigenspace of Λ_i.

Then Θ is analytic from $l^2(\mathbb{N})$ to $\mathcal{S}(\mathbb{N})$, its derivative at $u=0$ being easily seen to be the canonical imbedding from $l^2(\mathbb{N})$ into $\mathcal{S}(\mathbb{N})$. Moreover $v(t)=\Theta(S(t;u))$ satisfies in $\mathcal{S}(\mathbb{N})$ the differential equation

$$\frac{d\beta_j}{dt}+\lambda_j\beta_j=0 \qquad (j=1,2,3,\ldots) \text{ where } v=(\beta_j)_{j=1}^{\infty}. \tag{7.7}$$

Let us check that Θ is one-to-one. For this aim let $u=(\alpha_j)_{j=1}^{\infty}$, $v=(\beta_j)_{j=1}^{\infty}$ be such that $\Theta(u)=\Theta(v)$. Then using (7.7)

$$\Theta(u(t))=e^{-\mathrm{A}t}\Theta(u)=e^{-\mathrm{A}t}\Theta(v)=\Theta(v(t))$$

from where we infer that

$$S(t,u)_1\equiv S(t,v)_1, \qquad \sum_{j\geqq 2} S(t,u)_j^2\equiv\sum_{j\geqq 2} S(t,v)_j^2. \tag{7.8}$$

From (7.8) and (7.5) we obtain

$$\sum_{j\geqq 2}\alpha_j^2e^{-2\lambda_j t}=\sum_{j\geqq 2}\beta_j^2e^{-2\lambda_j t} \qquad (\text{for all } t\geqq 0). \tag{7.9}$$

If there exists a $j\geqq 2$ such that $\alpha_j^2\neq\beta_j^2$, let j_0 be the first one with this property. Multiplying (7.9) with $e^{2\lambda_{j_0}t}$ and letting t go to $+\infty$ we obtain the contradiction $\alpha_{j_0}^2=\beta_{j_0}^2$. Hence $\alpha_j^2=\beta_j^2$, for all $j\geqq 2$, and thus by (7.6), $\alpha_j=\beta_j$ for all $j\geqq 2$. One can now apply Theorem 7 concluding that in the case of the Minea system our normalization mapping W can be explicitely computed from the Θ defined by (7.6). Moreover we can also deduce from Theorem 7 that if the spectrum $(\lambda_1<\lambda_2<\ldots)$ of A is nonresonant, then $W=\Theta$ and our normal from coincides with the linear system (7.7).

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