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The Wiener criterion and quasilinear uniformly elliptic equations

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ABSTRACT. — In this paper the Dirichlet problem is considered for a wide class of quasilinear elliptic equations with C^1 coefficients in divergence form. The main result is that if Ω is an arbitrary bounded open subset of R^n and ψ is a continuous function defined on the boundary of Ω, then there is a solution to the equation in Ω which assumes the boundary data continuously at all points at which a Wiener condition is satisfied. This condition is satisfied at all points of the boundary except for a set of capacity zero.

Key words : Dirichlet problem, Wiener condition, elliptic equations.

RÉSUMÉ. — Dans ce travail nous considérons le problème de Dirichlet pour une large classe d'équations elliptiques avec des coefficients de classe C^1. Le résultat principal est le suivant. Soit Ω un ensemble ouvert borné arbitraire dans R^n, et soit ψ une fonction continue sur la frontière de Ω. Alors il y a une solution de l'équation dans Ω qui est continue et égale à ψ dans tous les points où la condition de Wiener est satisfaite.

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Cette condition est satisfaite dans tous les points de la frontière, à l'exception d'un ensemble de capacité nulle.

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1. INTRODUCTION

Let Ω be a bounded non-empty open set in $\mathbb{R}^n$ (where $n \geqq 2$). The following equation, in divergence form, will be studied:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x} a_i(x, u(x), \mathrm{D}u(x)) + b(x, u(x), \mathrm{D}u(x)) = 0 \tag{1}$$

for $x \in \Omega$, where $a_1, \ldots, a_n$ and b are locally Lipschitz functions on $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

We denote by

$$\mathscr{K}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$$

the vector space consisting of all functions f on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ with the property:

for each compact subset E of $\Omega \times \mathbb{R} \times \mathbb{R}^n$,

there exists a $\mathrm{K} \geqq 0$ and a $\delta \in (0, 1)$, such that

$$|f(w) - f(w')| \leqq \mathrm{K} |w - w'|^{\delta}$$

for all $w, w' \in \mathrm{E}$.

Let E denote the set of all those points of $\partial\Omega$ at which a prescribed Wiener condition is satisfied. The main aim of the paper is to prove [under the assumptions (7) and (8) on (*a*) and (*b*)] the following two Dirichlet results:

(i) For every continuous function ψ on $\partial\Omega$, there exists a bounded continuous function u on $\Omega \cup \mathrm{E}$, such that u agrees with ψ on E, u is locally $\mathrm{C}^{1,\delta}$ on Ω and u satisfies (1) weakly on Ω.

(ii) Suppose that, in addition to the structure conditions (7) and (8), we assume that a is C^1 on Ω,

$$\frac{\partial a_i}{\partial p_j} \in \mathscr{K}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \tag{2}$$

for all i, j and

$$\sum_{i=1}^{n} \left[\frac{\partial a_i}{\partial x_i} + p_i \frac{\partial a_i}{\partial z} \right] \in \mathscr{K}(\Omega \times \mathbb{R} \times \mathbb{R}^n). \tag{3}$$

Then, for every continuous function ψ on $\partial\Omega$, there exists a bounded continuous function u on $\Omega \cup E$, such that u agrees with ψ on E, u is C^2 on Ω and u satisfies (1) for all $x \in \Omega$. By a result of Hedberg and Wolf [HW], it turns out that the Wiener condition is satisfied everywhere on $\partial\Omega$ except possibly for a small exceptional set (a set of capacity zero). Thus, whenever Ω is a bounded non-empty open set, the above set E necessarily occupies all of $\partial\Omega$ except for this exceptional set and therefore the function u in (i) and (ii) continuously assumes the values of ψ everywhere on $\partial\Omega$ except possibly for a set of capacity zero. When $\alpha > n$ [where α is determined in the structure (7) and (8) below] and Ω is an arbitrary bounded open set of R^n, it is shown (2.17) that the Wiener criterion is satisfied at every point of $\partial\Omega$; hence for every continuous function ψ on $\partial\Omega$ and in each of the two cases discussed above, there exists a solution u of (1) which agrees with ψ everywhere on $\partial\Omega$.

Although the Wiener criterion has been used by several authors, there have been very few general results on the existence of solutions. The original papers [W1 and W2] by N. Wiener, of course, dealt with the existence and boundary regularity of solutions to Laplace's equation and in [Z], existence of a solution is established when (1) is the Euler equation of a variational integral. In [MA] Maz'ja dealt with the existence and boundary regularity of weak solutions, but only for equations whose structure is similar to

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

In [GZ] it was assumed that a weak solution was given to an equation in divergence form with general structure and continuity was established at those boundary points where the Wiener criterion is satisfied.

In the present paper, the functions a_i and b are assumed to satisfy the following growth conditions. In these α, μ, ν are constants, such that

$$\alpha > 1 \quad \text{and} \quad \mu \geqq \nu > 0. \tag{6}$$

The conditions are required to hold for all $x \in \Omega$, $z \in \mathrm{R}^n$, $p \in \mathrm{R}^n$ and $\xi \in \mathrm{R}^n$.

$$\nu(1+|p|^2)^{(1/2)(\alpha-2)}|\xi|^2 \leqq \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial p_j}(x, z, p)\xi_i\xi_j \leqq \mu(1+|p|^2)^{(1/2)(\alpha-2)}|\xi|^2, \tag{7}$$

$$\sum_{i=1}^{n} |a_i(x, z, p)| + |b(x, z, p)| + |p| \sum_{i=1}^{n} \left|\frac{\partial a_i}{\partial z}(x, z, p)\right| + \sum_{i,j=1}^{n} \left|\frac{\partial a_i}{\partial x_j}(x, z, p)\right| \leqq \mu(1+|p|^2)^{(1/2)(\alpha-1)}. \tag{8}$$

Conditions (7) and (8) are similar to those used by other authors to investigate the Dirichlet problem for equations in divergence form, *cf.* [GT], [LU]. However, their primary concern is to find solutions smooth up to the boundary for the Dirichlet problem. Thus, if their methods and results are particularized to the problem of finding solutions which only need to be continuous at the boundary, their hypotheses regarding boundary regularity are unnecessarily strong. Typically, Ω is assumed to satisfy an exterior sphere condition at every point of the boundary [GT], Theorem 15.19.

We use the same α as in (7) and (8) in the definitions of capacity and Wiener integral. Suppose $\alpha < n$. When E is a bounded subset of R^n, we define the capacity $\Gamma(\mathrm{E})$ of E by

$$\Gamma(\mathrm{E}) = \inf\left\{\int_{\mathrm{R}^n} |\mathrm{D}\varphi(x)|^\alpha dx\right\} \tag{9}$$

where the infimum is taken over all non-negative C^∞ functions φ on R^n, with compact support and $\varphi(x) \geqq 1$ when $x \in \mathrm{E}$.

For $x_0 \in \partial\Omega$ and $\delta \in (0, 1)$, the Wiener integral is

$$\mathrm{W}_\delta(\Omega, x_0) = \int_\delta^1 [\Gamma(\mathrm{B}_\rho(x_0) \sim \Omega)\, \rho^{\alpha-n}]^{1/\alpha-1} \frac{d\rho}{\rho}, \tag{10}$$

where $\mathrm{B}_\rho(x_0)$ is the open ball of radius ρ and center x_0.

These definitions would be unsatisfactory when $\alpha \geqq n$, so for $\alpha \geqq n$, we define the Wiener integral by

$$W_\delta(\Omega, x_0) = \int_\delta^1 [\Gamma_{2\rho}(B_\rho(x_0) \sim \Omega)\, \rho^{\alpha-n}]^{1/\alpha-1} \frac{d\rho}{\rho}, \tag{11}$$

where $\Gamma_{2\rho}$ denotes the capacity relative to a ball of radius 2ρ and centre x_0; i.e.

$$\Gamma_{2\rho}(B_\rho(x_0) \sim \Omega) = \inf \left\{ \int_{\mathbf{R}^n} |D\varphi(x)|^\alpha \, dx \right\}, \tag{12}$$

where the infimum is taken over all C^∞ functions φ on $\mathbf{R}^n$ with compact support, $\operatorname{spt} \rho \subset B_{2\rho}(x_0)$ and $\varphi(x) \geqq 1$ when $x \in B_\rho(x_0) \sim \Omega$.

We say that the Wiener criterion is satisfied at x_0, when

$$W_\delta(\Omega, x_0) \to \infty \tag{13}$$

as $\delta \to 0^+$.

In the case where $\alpha = 2$, this criterion is precisely the one used by Wiener in [W1 and W2]. The criterion for general $\alpha > 1$, was introduced by Maz'ja in [MA] and was also used in [GZ].

2. SPECIAL SETS OF FUNCTIONS

We now develop the properties of certain special functions which occur when investigating continuity at the boundary of solutions to (1). The discussion given here is perhaps more general than is needed in the present paper, but the full generality will be needed for subsequent articles on obstacle problems.

Throughout the paper, we will employ the familiar notation $C^{k,\alpha}(\Omega)$ to denote the space of functions whose kth order derivatives are Holder continuous with exponent α on Ω. In particular, $C^{0,1}(\Omega)$ is the space of Lipschitz funtions on Ω. Finally, $C_0^{k,\alpha}(\Omega) = C^{k,\alpha}(\Omega) \cap \{u : \operatorname{spt} u \subset \Omega\}$.

We shall recall the following definition from [MZ2].

2.1. Definition. — Let Γ be a non-empty open set of $\mathbf{R}^n$, $1 < \alpha < \infty$, $C > 0$ and $\lambda \geqq 0$. Let ε denote the symbol $+$ or the symbol $-$. We denote by

$$S^{\varepsilon}(\Gamma, \alpha, C, \lambda)$$

the set of all non-negative functions v of $W^{1,\alpha}(\Gamma)$ such that

$$\int_{\Gamma} |D(v(x)-k)^{\varepsilon}|^{\alpha} \eta(x)^{\alpha} dx \leqq C \int_{\Gamma} \{(v(x)-k)^{\varepsilon}\}^{\alpha} \{\eta(x)^{\alpha} + |D\eta(x)|^{\alpha}\} dx + C(k^{\alpha}+\lambda) \int_{\{(v(x)-k)^{\varepsilon} > 0\}} \eta(x)^{\alpha} dx \quad (14)$$

for all $k \geqq 0$ and all non-negative Lipschitz functions η on R^n with compact support and $\eta(x)=0$ when $x \notin \Gamma$.

The sets S^+ and S^- are appropriate for developing Harnack inequalities and establishing Hölder continuity in the interior of the domain for a solution to a partial differential equation or an obstacle problem. However, they appear to be inadequate for showing continuity at the boundary. Therefore we introduce special subsets T^+ and T^- of S^+ and S^-.

2.2. Definition. – Let Γ be a non-empty open set of R^n, $\alpha \in (1, \infty)$, $C > 0$ and $\lambda \geqq 0$. Let ε denote the symbol $+$ or the symbol $-$. We denote by

$$T^{\varepsilon}(\Gamma, \alpha, C, \lambda)$$

the set of all non-negative functions v of $W^{1,\alpha}(\Gamma)$ such that

$$\int_{\Gamma \cap \{(v-k)^{\varepsilon} > 0\}} |Dv(x)|^{\alpha} \eta(x)^{\alpha} dx \leqq C \int_{\Gamma \cap \{(v-k)^{\varepsilon} > 0\}} (v(x)-k)^{\varepsilon} |Dv(x)|^{\alpha-1} \eta(x)^{\alpha-1} |D\eta(x)| dx + C \int_{\Gamma \cap \{(v-k)^{\varepsilon} > 0\}} [v(x)^{\alpha}+\lambda] \eta(x)^{\alpha} dx + C \int_{\Gamma \cap \{(v-k)^{\varepsilon} > 0\}} [v(x)^{\alpha-1}+\lambda] (v(x)-k)^{\varepsilon} \eta(x)^{\alpha-1} |D\eta(x)| dx \quad (15)$$

for all $k \geqq 0$ and all non-negative Lipschitz functions η on R^n with compact support and $\eta(x)=0$ when $x \notin \Gamma$.

Some simple applications of Young's inequality to (15) yield the following:

2.3. THEOREM. — *Let* $\alpha \in (1, \infty)$, *and* $C > 0$. *There exists a constant* C', *depending only on* α *and* C *and such that*

$$T^\varepsilon(\Gamma, \alpha, C, \lambda) \subset S^\varepsilon(\Gamma, \alpha, C', \sup\{\lambda, \lambda^{\alpha/\alpha-1}\}),$$

whenever Γ *is a non-empty open subset of* R^n, $\varepsilon = +$ *or* $-$ *and* $\lambda \geqq 0$.

2.4 DEFINITION. — Let Γ be a non-empty bounded open set of R^n, $\alpha \in (1, \infty)$, $\mu > 0$ and $\nu \geqq 0$. Let ε denote the symbol $+$ or the symbol $-$. We denote by

$$U^\varepsilon(\Gamma, \alpha, \mu, \nu)$$

the set of all functions of $W^{1,\alpha}(\Gamma)$, for which there exist Borel measurable functions $A: \Gamma \times R \times R^n \to R^n$ and $B: \Gamma \times R \times R^n \to R^1$ with the properties:

(i)

$$|A(x, z, p)| \leqq \mu |p|^{\alpha-1} + \mu |z|^{\alpha-1} + \nu \tag{16}$$

$$p.A(x, z, p) \geqq |p|^\alpha - \mu |z|^\alpha - \nu \tag{17}$$

and

$$|B(x, z, p)| \leqq \mu |p|^{\alpha-1} + \mu |z|^{\alpha-1} + \nu \tag{18}$$

for $x \in \Gamma$, $z \in R$ and $p \in R^n$;

(ii)

$$\sum_{i=1}^{n} \int_\Gamma A(x, u(x), Du(x)) \cdot \nabla \varphi(x) + \int B(x, u(x), Du(x)) \varphi(x)\, dx \leqq 0 \tag{19}$$

for all $\varphi \in W_0^{1,\alpha}(\Gamma)$ with $\varepsilon\varphi \geqq 0$.

The functions of U^+ are simply weak subsolutions of certain equations and the functions of U^- are supersolutions.

2.5. THEOREM. — *Let* $\alpha \in (1, \infty)$, $\mu > 0$, $\nu \geqq 0$ *and* $M_0 \geqq 0$. *Then there exist constants* $C > 0$, $\lambda \geqq 0$, *depending only on* n, α, μ, ν *and* M_0 *such that*

$$(u - M)^+ \in T^+(\Gamma, \alpha, C, \lambda),$$

whenever Γ *is an open subset of* R^n, M *is a real number with* $|\mathrm{M}| \leqq |\mathrm{M}_0|$ *and*

$$u \in \mathrm{U}^+(\Gamma, \alpha, \mu, \nu).$$

Proof. — Let Γ, M and u be as described. Put

$$v = (u - \mathrm{M})^+.$$

Let $k \geqq 0$ and put

$$w = (v - k)^+.$$

Then

$$w = (u - \mathrm{M} - k)^+.$$

Let $\eta \in \mathrm{C}_0^{0,1}(\mathrm{R}^n)$, be a non-negative function such that $\eta(x) = 0$ when $x \notin \Gamma$. Define

$$\varphi(x) = w(x)\,\eta(x)^\alpha.$$

By employing φ in (19) and using the structural conditions (16), (17) and (18), it follows that

$$\int_{\{u > \mathrm{M}+k\}} (|\mathrm{D}u|^\alpha - \mu |u|^\alpha - \nu)\,\eta^\alpha\,dx$$

$$\leqq \alpha \int_{\{u > \mu+k\}} (\mu |\mathrm{D}u|^{\alpha-1} + \mu |u|^{\alpha-1} + \nu)\,w\,\eta^{\alpha-1} |\mathrm{D}\eta|\,dx$$

$$+ \int_{\{u > \mathrm{M}+k\}} (\mu |\mathrm{D}u|^{\alpha-1} + \mu |u|^{\alpha-1} + v)\,w\,\eta^{\alpha-1}\,dx. \quad (20)$$

By applying Young's inequality to the integral

$$\mu \int |\mathrm{D}u|^{\alpha-1}\,w\,\eta^\alpha\,dx$$

and rearranging (20), we obtain the desired result.

2.6. THEOREM. — *If* $v \in \mathrm{T}^+(\Gamma, \alpha, \mathrm{C}, \lambda)$ *and* $v(x) \leqq \mathrm{M}$ *for almost all* $x \in \Gamma$, *then*

$$\mathrm{M} - v \in \mathrm{T}^-(\Gamma, \alpha, \mathrm{C}, \lambda + \sup\{\mathrm{M}^{\alpha-1}, \mathrm{M}^\alpha\}).$$

Proof. – Put

$$w = M - v.$$

Let $k \geqq 0$. Then

$$(w-k)^- = (M-v-k)^- = (v-(M-k))^+.$$

If $k \leqq M$, then by

$$\int_{\{v > M-k\}} |Dv|^\alpha \eta^\alpha \, dx$$

$$\leqq C \int_{\{v > M-k\}} (v-(M-k))^+ |Dv|^{\alpha-1} \eta^{\alpha-1} |D\eta| \, dx$$

$$+ C \int_{\{v > M-k\}} (v^\alpha + \lambda) \eta^\alpha \, dx$$

$$+ C \int_{\{v > M-k\}} (v^{\alpha-1} + \lambda)(v-(M-k))^+ \eta^{\alpha-1} |D\eta| \, dx, \quad (21)$$

where η is a non-negative Lipschitz function with compact support and $\eta(x) = 0$ when $x \notin \Gamma$. When $k > M$, (21) follows from the case $k = M$ since v is non-negative. The desired result now follows easily from (21).

2.7. THEOREM. – *Let v be a bounded function of $T^+(\Gamma, \alpha, C, \lambda)$, let G be a non-empty bounded open set of R^n and suppose that for every $\theta \in C^{0,1}(R^n)$ with $\theta(x) = 0$ when $x \notin G$, the function*

$$(v \,.\, \theta) \mid G \cap \Gamma$$

belongs to $W_0^{1,\alpha}(G \cap \Gamma)$.

Define

$$w(x) = \begin{cases} v(x) & \text{when } x \in G \cap \Gamma \\ 0 & \text{when } x \in G - \Gamma \end{cases}$$

Then

$$w \in T^+(G, \alpha, C, \lambda).$$

Proof. — Define

$$w_i(x)=\frac{\partial v}{\partial x_i}(x) \quad \text{when } x\in G\cap\Gamma,$$

$$=0 \quad \text{when } x\in G\sim\Gamma.$$

Then w and each w_i belong to $L^\alpha(\Omega)$. It is easily seen that the w_i's form generalized derivatives for w and that $w\in W^{1,\alpha}(\Omega)$.

Let $\eta\in C^{0,1}(G)$ be a non-negative function which vanishes outside G. For $t>0$ and $x\in G$, define

$$w_t(x)=\frac{1}{t}\inf\{t,\ w(x)\}$$

and

$$\xi_t(x)=w_t(x)\cdot\eta(x).$$

Since

$$\xi_t(x)=\frac{1}{t}\inf\{t\,\eta(x),\ w(x)\cdot\eta(x)\},$$

it follows that

$$\xi_t \mid G\cap\Gamma\in W_0^{1,\alpha}(G\cap\Gamma). \tag{22}$$

Since v and ξ_t are bounded, it follows from (14) and Fatou's lemma that, when $k\geqq 0$,

$$\begin{aligned}\int_{G\cap\Gamma\cap\{v>k\}} &|Dv|^\alpha\xi_t^\alpha\,dx\\ &\leqq C\int_{G\cap\Gamma\cap\{v>k\}}(v(x)-k)^+\,|Dv|^{\alpha-1}\,\xi_t^{\alpha-1}\,|D\xi_t|\,dx\\ &\quad+C\int_{G\cap\Gamma\cap\{v>k\}}[v^\alpha+\lambda]\,\xi_t^\alpha\,dx\\ &\quad+C\int_{G\cap\Gamma\cap\{v>k\}}[v^\alpha+\lambda]\,(v-k)^+\,\xi_t^{\alpha-1}\,|D\xi_t|\,dx.\end{aligned} \tag{23}$$

When $0<t<k$ and $v(x)>k$, we have

$$\xi_t(x)=\eta(x)$$

so that by (23)

$$\int_{G \cap \{w > k\}} |Dw|^\alpha \eta^\alpha \, dx$$

$$\leqq C \int_{G \cap \{w > k\}} (w-k)^+ |Dw|^{\alpha-1} \eta^{\alpha-1} |D\eta| \, dx$$

$$+ C \int_{G \cap \{w > k\}} (w^\alpha + \lambda) \eta^\alpha \, dx$$

$$+ C \int_{G \cap \{w > k\}} (w^{\alpha-1} + \lambda)(w-k)^+ \eta^{\alpha-1} |D\eta| \, dx. \quad (24)$$

By letting $k \to 0^+$, we see that (24) still holds when $k=0$. Thus

$$w \in T^+(G, \alpha, C, \lambda).$$

2.8. THEOREM. — *Let* $\alpha \in (1, \infty)$ *and* $C > 0$. *There exists a constant* C', *depending only on* n, α *and* C *and such that*

$$\int_{\Gamma \cap \{v > k\}} v(x)^{-\beta-1} |Dv(x)|^\alpha \eta(x)^\alpha \, dx$$

$$\leqq C'(1+\beta^{-\alpha}) \int_{\Gamma \cap \{v > k\}} v(x)^{\alpha-\beta-1} \{\eta(x)^\alpha + |D\eta(x)|^\alpha\} \, dx \quad (25)$$

whenever Γ *is a non-empty open subset of* R^n, $v \in T^-(\Gamma, \alpha, C, 0)$, $\beta > 0$ *and* $\eta \in C_0^{0,1}(R^n)$ *is such that* $\eta(x) = 0$ *when* $x \notin \Gamma$.

Proof. — Let Γ, v, β and μ be as described. Multiply both sides of (15) by $k^{-\beta-2}$ and integrate with respect to k over $(0, \infty)$. Using Youngs's inequality and the fact that $(v(x)-k)^- \leqq k$, yields the desired result.

2.9. LEMMA. — *Let* $\rho \in (0, 1)$, $x_0 \in R^n$ *and* $v \in T^\varepsilon(B_\rho(x_0), \alpha, C, \lambda)$. *Define*

$$w(\xi) = \rho^{-1} v(\rho\xi + x_0)$$

for $\xi \in B_1(0)$. *Then*

$$w \in T^\varepsilon(B_1(0), \alpha, C, \lambda).$$

This is proved by a straightforward change of variables in the integrals involved.

2.10. LEMMA. — *Let* $v \in \mathrm{T}^{\varepsilon}(\Gamma, \alpha, \mathrm{C}, \lambda)$ *and set* $\mathrm{E} = \sup\{\lambda^{1/\alpha - 1}, \lambda^{1/\alpha}\}$. *Then* $v + \mathrm{E} \in \mathrm{T}^{\varepsilon}(\Gamma, \alpha, 2\mathrm{C}, 0)$.

Proof. — Set

$$w = v + \mathrm{E}.$$

Suppose $k \geqq 0$. Then

$$\begin{aligned}\int_{\{(w-k)^{\varepsilon} > 0\}} |\mathrm{D}w|^{\alpha} \eta^{\alpha}\, dx &= \int_{\{(v-(k-\mathrm{E})^{+})^{\varepsilon} > 0\}} |\mathrm{D}v|^{\alpha} \eta^{\alpha}\, dx \\ &\leqq \mathrm{C} \int_{\{(v-(k-\mathrm{E})^{+})^{\varepsilon} > 0\}} (v-(k-\mathrm{E})^{+})^{\varepsilon} |\mathrm{D}v|^{\alpha-1} \eta^{\alpha-1} |\mathrm{D}\eta|\, dx \\ &\quad + \mathrm{C} \int_{\{(v-(k-\mathrm{E})^{+})^{\varepsilon} > 0\}} (v^{\alpha} + \lambda) \eta^{\alpha}\, dx \\ &\quad + \mathrm{C} \int_{\{(v-(k-\mathrm{E})^{+})^{\varepsilon} > 0\}} (v^{\alpha-1} + \lambda)(v-(k-\mathrm{E})^{+})^{\varepsilon} \eta^{\alpha-1} |\mathrm{D}\eta|\, dx.\end{aligned}$$

The result easily follows since $(v-(k-\mathrm{E})^{+})^{\varepsilon} \leqq (w-k)^{\varepsilon}$.

In the following let

$$\begin{aligned}\chi &= \frac{n}{n-\alpha} \quad \text{when } \alpha < n, \\ &= 2 \quad \text{when } \alpha \geqq n.\end{aligned}$$

2.11. LEMMA. — *Let* $\alpha \in (1, \infty)$, $\mathrm{C} > 0$ *and* $\theta \in (0, \chi(\alpha - 1))$. *Then there exists a positive constant* E, *depending only on* n, α, C *and* θ *and such that for* $x_0 \in \mathbf{R}^n$, $\rho \in (0, 1]$, $\lambda \geqq 0$ *and*

$$v \in \mathrm{T}^{-}(\mathrm{B}_{\rho}(x_0), \alpha, \mathrm{C}, \lambda),$$

we have the inequality

$$\operatorname*{ess\,inf}_{\mathrm{B}_{\rho/2}(x_0)} v(x) \geqq \mathrm{E} \left[\fint_{\mathrm{B}_{\rho/2}(x_0)} v(x)^{\theta}\, dx \right]^{1/\theta} - \mathrm{E}\rho \sup\{\lambda^{1/\alpha}, \lambda^{1/a-1}\} \tag{26}$$

Proof. — It follows from (2.9) that the theorem need only be proved for the case where $x_0 = 0$ and $\rho = 1$.

(i) Suppose first of all that $\lambda=0$. Let

$$v \in T^{-}(B_1(0), \alpha, C, 0).$$

By (2.3) of this paper and (2.14) of [MZ2], there exists a positive constant E and a constant $\gamma \in (0, 1)$, depending only on n, α and C and such that

$$\operatorname*{ess\,inf}_{B_{1/2}(0)} v(x) \geqq E' \left[\int_{B_{3/4}(0)} v(x)^{\gamma} \, dx \right]^{1/\gamma}. \tag{27}$$

Let $0 < t < u \leqq \frac{3}{4}$ and let η be a Lipschitz function on R^n, such that $0 \leqq \eta \leqq 1$, $\eta(x)=1$ when $|x| \leqq t$, $\eta(x)=0$ when $|x| \geqq u$ and

$$|D\psi(x)| = \frac{1}{u-t}$$

when $t < |x| < u$. By 2.8,

$$\int_{\{v > 0\} \cap B_1(0)} v^{-\beta-1} |Dv|^{\alpha} \eta^{\alpha} \, dx$$

$$\leqq C_1 (1+\beta^{-\alpha}) \int_{B_1(0)} v^{\alpha-\beta-1} (\eta^{\alpha} + |D\eta|^{\alpha}) \, dx, \tag{28}$$

for $0 < \beta < \alpha - 1$. Let

$$w(x) = v(x)^{(\alpha-1-\beta)/\alpha} \eta(x).$$

Then by (28) and Sobolev's inequality,

$$\left[\int_{B_1(0)} v^{\chi(\alpha-\beta-1)} \eta^{\chi\alpha} \, dx \right]^{1/\chi\alpha}$$

$$\leqq C_3 (1+\beta^{-\alpha})^{1/\alpha} \left[\int_{B_1(0)} v^{\alpha-\beta-1} (\eta^{\alpha} + |D\eta|^{\alpha}) \, dx \right]^{1/\alpha}$$

Now by setting $\theta = \alpha - \beta - 1$ and performing a standard iteration one can show that

$$\left[\int_{B_{1/2}(0)} v^{\theta} \, dx \right]^{1/\theta} \leqq C_5 \left[\int_{B_{3/4}(0)} v^{\delta} \, dx \right]^{1/\delta},$$

where $\sigma < \delta \leqq \gamma$. A simple application of Hölder's inequality and (27) now yields

$$\operatorname*{ess\,inf}_{B_{1/2}(0)} v(x) \geqq E\left[\fint_{B_{1/2}(0)} v(x)^\theta \, dx \right]^{1/\theta}.$$

(ii) Suppose $\lambda > 0$. Put

$$\Lambda = \{\sup \lambda^{1/\alpha}, \lambda^{1/(\alpha-1)}\}.$$

By (i) and Lemma 2.10

$$\operatorname*{ess\,inf}_{B_{1/2}(0)} \{v(x) + \Lambda\} \geqq E' \left[\fint_{B_{1/2}(0)} (v + \Lambda)^\theta \, dx \right]^{1/\theta}. \tag{30}$$

Since $\theta > 0$, (26) follows from (30).

2.12. LEMMA. – *Let $\alpha \in (1, \infty)$ and $C > 0$. Then there exists a constant C', depending only on n, α and C such that for $x_0 \in R^n$, $\rho \in (0, 1]$, $\lambda \geqq 0$ and*

$$v \in T^- (B_\rho(x_0), \alpha, C, \lambda),$$

we have

$$\rho^{\alpha - n - 1} \int_{B_{\rho/4}(x_0)} |Dv(x)|^{\alpha - 1} \, dx \leqq C' [\operatorname*{ess\,inf}_{B_{\rho/2}(x_0)} v(x)]^{\alpha - 1} + C' \rho^{\alpha - 1} \sup \{\lambda^{(\alpha - 1)/\alpha}, \lambda\} \tag{31}$$

Proof. – It follows from (2.9) that we can assume $x_0 = 0$ and $\rho = 1$.
(i) Suppose first of all that $\lambda = 0$. Let

$$v \in T^- (B_1(0), \alpha, C, 0).$$

Let θ be a positive number, such that

$$1 < (1 - \theta) \alpha < \chi. \tag{32}$$

Put

$$m = \operatorname*{ess\,inf}_{|x| < 1/2} v(x).$$

Now

$$\int_{B_{1/4}(0)} |Dv|^{\alpha-1}\,dx$$
$$\leqq \int_{\{v>0\}\cap B_{1/4}(0)} (v^{-(1-\theta)}|Dv|)^{\alpha-1}(v^{(1-\theta)(\alpha-1)})\,dx$$
$$\leqq \left[\int_{\{v>0\}\cap B_{1/4}(0)} v^{-(1-\theta)\alpha}|Dv|^{\alpha}\,dx\right]^{(\alpha-1)/\alpha}$$
$$\times\left[\int_{B_{1/4}(0)} v^{(1-\theta)(\alpha-1)\alpha}\,dx\right]^{1/\alpha} \qquad (33)$$

It follows from (25) that the first factor is bounded by

$$C_0\left[\int_{B_{1/2}(0)} v^{\alpha\theta}\right]^{(\alpha-1)/\alpha}.$$

Then, by (2.11) we have

$$\int_{B_{1/4}(0)} |Dv|^{\alpha-1}\,dx \leqq C' m^{(\alpha-1)/\alpha\,(\alpha-(1-\theta)\alpha)}\, m^{(1/\alpha)\,(1-\theta)\,(\alpha-1)\,\alpha} = C' m^{\alpha-1}.$$

(ii) Suppose $\lambda > 0$. Set

with $v(x) \leqq M$ *for almost all* $x \in B_\rho(x_0)$, *the following inequalities hold:*

$$\rho^{\alpha-n}\int_{B_{1/8}(x_0)} |Dv(x)|^\alpha dx \leqq C' M\left(\frac{1}{4}\rho\right)\left[m\left(\frac{1}{2}\rho\right)+\rho\right]^{\alpha-1}+C'\rho^\alpha \quad (34)$$

$$\rho^{-n}\int_{B_{1/2}(x_0)} v(x)^\alpha dx \leqq C' M\left(\frac{1}{2}\rho\right)\left[m\left(\frac{1}{2}\rho\right)+\rho\right]^{\alpha-1} \quad (35)$$

where

$$m(\delta)=\text{ess inf}\{v(x): |x-x_0|<\delta\}$$

and

$$M(\delta)=\text{ess sup}\{v(x): |x-x_0|<\delta\}.$$

Proof: Let x_0, v and ρ be as described. Let $\eta \in C^{0,1}(R^n)$, $\eta \geqq 0$, be such that $\eta(x)=0$ when $|x-x_0| \geqq \frac{1}{4}\rho$, $\eta(x)=1$ when $|x-x_0| \leqq \frac{1}{8}\rho$ and $|D\eta(x)|=\frac{8}{\rho}$ when $\frac{1}{8}\rho<|x-x_0|<\frac{1}{4}\rho$. Using this η in (15) with $k=M\left(\frac{1}{4}\rho\right)$, we obtain

$$\int_{B_{1/8}(x_0)} |Dv|^\alpha dx$$
$$\leqq C_1 M\left(\frac{1}{4}\rho\right)\rho^{-1}\int_{B_{\rho/4}(x_0)} |Dv|^{\alpha-1} dx + C_1\rho^n + C_1 M\left(\frac{1}{4}\rho\right)\rho^{n-1},$$

so that by (2.12),

$$\rho^{\alpha-n}\int_{B_{1/8}(x_0)} |Dv|^\alpha dx \leqq C' M\left(\frac{1}{4}\rho\right)\left[m\left(\frac{1}{2}\rho\right)+\rho\right]^{\alpha-1}+C'\rho^\alpha.$$

Now

$$\rho^{-n}\int_{B_{1/2}(x_0)} v^\alpha dx \leqq M\left(\frac{1}{2}\rho\right)\rho^{-n}\int_{B_{1/2}(x_0)} v^\alpha dx$$

and by (2.11),

$$\leqq C'' M\left(\frac{1}{2}\rho\right)\left[m\left(\frac{1}{2}\rho\right)+\rho\right]^{\alpha-1}$$

2.14. THEOREM. — *Let* $\alpha\in(1, \infty)$, $C>0$, $\lambda\geqq 0$ *and* $M\geqq 0$. *Then there exists a constant* C', *depending only on* n, α, C, λ *and* M *such that for* $x_0\in R^n$, $\rho\in(0, 1]$ *and*

$$v\in T^+(B_\rho(x_0), \alpha, C, \lambda)$$

with $v(x)\leqq M$ *for almost all* $x\in B_\rho(x_0)$, *the inequalities*

$$\rho^{\alpha-n}\int_{B_{\rho/8}(x_0)}|Dv(x)|^\alpha dx\leqq C'[M(\rho)+\rho][M(\rho)-M\left(\frac{1}{2}\rho\right)+\rho]^{\alpha-1} \quad (36)$$

$$\rho^{-n}\int_{B_{\rho/2}(x_0)}[M(\rho)+\rho-v(x)]^\alpha dx$$

$$\leqq C'[M(\rho)+\rho]\left[M(\rho)-M\left(\frac{1}{2}\rho\right)+\rho\right]^{\alpha-1}, \quad (37)$$

hold, where $m(\delta)$ *and* $M(\delta)$ *are as defined in* (2.13).

Proof. — Let x_0, ρ and v be as described. Put

$$w=M(\rho)+\rho v$$

By (2.6)

$$w\in T^-(B_\rho(x_0), \alpha, C, \lambda+(M+1)^\alpha)$$

and the inequalities now follow from (2.13).

We conclude this section with a discussion of the relation between the Wiener integral and the functions of T^+.

We observe that in (9) and (12), the infimum can be taken over Lipschitz functions φ instead of C^∞ functions.

2.15 THEOREM. — *Let* $C\geqq 0$, $\lambda\geqq 0$, $M\geqq 0$, $\sigma\in(0, 1)$ *and* $1<\alpha<\infty$. *Then there exist positive constants* C_1 *and* C_2, *such that*

$$\operatorname*{ess\,sup}_{B_\delta(x_0)} v(x)\leqq C_1\exp(-C_2 W_\delta(\Omega, x_0)),$$

whenever Ω *is a non empty open subset of* R^n, $\delta \in \left(0, \frac{1}{16}\sigma\right)$, $x_0 \in \partial\Omega$ *and*

$$v \in \mathrm{T}^+(\mathrm{B}_\sigma(x_0) \cap \Omega, \alpha, \mathrm{C}, \lambda)$$

for which

$$v(x) \leqq \mathrm{M}$$

when $x \in \mathrm{B}_\sigma(x_0) \cap \Omega$ *and for which there exists a sequence* $\{\theta_r\}$ *of non-negative Lipschitz functions on* R^n, *such that, whenever* $\eta \in \mathrm{C}_0^{0,1}(\mathrm{B}_\sigma(x_0))$, $\theta_r \cdot \eta$ *vanishes outside* $\mathrm{B}_\sigma(x_0) \cap \Omega$ *for all* r *and* $\{\theta_r \cdot \eta\}$ *converges to* $v \cdot \eta$ *in* $\mathrm{W}^{1,\alpha}(\mathrm{B}_\sigma(x_0) \cap \Omega)$.

Proof. — Let Ω, x_0, δ and v be as described. Define

$$w(x) = \begin{cases} v(x) & \text{when } x \in \mathrm{B}_\sigma(x_0) \cap \Omega \\ 0 & \text{when } x \in \mathrm{B}_\sigma(x_0) - \Omega \end{cases}$$

By 2.7,

$$w \in \mathrm{T}^+(\mathrm{B}_\sigma(x_0), \alpha, \mathrm{C}, \lambda).$$

Let $0 < t \leqq \frac{1}{16}\sigma$ and let $\eta \in \mathrm{C}^{0,1}(\mathrm{R}^n)$, $\eta \geqq 0$, be such that $\eta(x) = 1$ when $|x - x_0| \leqq t$, $\eta(x) = 0$ when $|x - x_0| \geqq \frac{3}{2}t$ and $|\mathrm{D}\eta(x)| \leqq 2/t$ when $t < |x - x_0| < 3t/2$.

Let

$$\mathrm{M}(\rho) = \operatorname{ess\,sup}\{w(x) : |x - x_0| < \rho\}$$

for $0 < \rho \leqq \sigma$ and put

$$\varphi = (\mathrm{M}(16t) + 16t + 16t - w) \cdot \eta$$

By 2.14,

$$\int |\mathrm{D}\varphi(x)|^\alpha \, dx$$

$$\leqq \mathrm{C}_3 t^{n-\alpha} [\mathrm{M}(16t) + 16t] [\mathrm{M}(16t) - \mathrm{M}(8t) + 16t]^{\alpha - 1} \quad (38)$$

Put

$$\varphi_r = (M(16t) + 16t - \theta_r) \cdot \eta.$$

Then φ_r converges to φ in $W^{1,\alpha}(\Omega)$ and

$$\varphi_r(x) = M(16t) + 16t$$

when $x \in B_t(x_0) \sim \Omega$. Then the function

$$[M(16t) + 16t]^{-1} \varphi_r$$

is Lipschitz on R^n, has the value 1 on $B_t(x_0) \sim \Omega$, has compact support and has its support contained in $B_{2t}(x_0)$. Hence, for all r,

$$\gamma(B_t(x_0) \sim \Omega \leqq [M(16t) + 16t]^{-\alpha} \int |D\varphi_r|^\alpha dx,$$

where $\gamma = \Gamma$ when $\alpha < n$ and $\gamma = \Gamma_{2t}$, when $\alpha \geqq n$. By letting $r \to \infty$, applying (38) and putting $C_4 = 2 C_3^{1/(\alpha-1)}$, we see that

$$[\gamma(B_t(x_0) \sim \Omega) t^{\alpha - n}]^{1/\alpha - 1}$$
$$\leqq C_4 [M(16t) + 16t]^{-1} [M(16t) - M(8t) + 8t] \quad (39)$$

and this holds for all $t \in \left(0, \frac{1}{16}\sigma\right]$.

For $t \in \left(0, \frac{1}{16}\sigma\right]$, put

$$f(t) = M(16t) + 16t \quad (40)$$

and

$$A(t) = C_4^{-1} [\gamma(B_t(x_0) \sim \Omega) t^{\alpha - n}]^{1/(\alpha - 1)}. \quad (41)$$

By (39)

$$f\left(\frac{1}{2}t\right) \leqq (1 - A(t)) f(t) \quad (42)$$

for all $t \in \left(0, \frac{1}{16}\sigma\right]$. It now follows from (42) that

$$f(2^{-m}\sigma') \leqq \prod_{i=0}^{m-1} [1 - A(2^{-i}\sigma')] f(\sigma'),$$

where $\sigma' = \frac{1}{16}\sigma$. Hence

$$\ln f(2^{-n}\sigma') \leqq \ln f(\sigma') + \sum_{i=0}^{m-1} \ln[1 - A(2^{-i}\sigma')].$$

But $\ln(1-\tau) \leqq -\tau$, when $\tau \in [0, 1)$, so that

$$\ln f(2^{-m}\sigma') \leqq \ln f(\sigma') - \sum_{i=0}^{m-1} A(2^{-i}\sigma')$$

and therefore

$$f(2^{-m}\sigma') \leqq f(\sigma') \exp\left[-\sum_{i=0}^{m-1} A(2^{-i}\sigma')\right], \tag{43}$$

for $m = 1, 2, \ldots$ But when $\xi \leqq t \leqq 2\xi$, it follows from (41), that

$$A(t) \leqq 2^{(|n-\alpha|)/(\alpha-1)} C_4^{-1} [\gamma(B_{2\xi}(x_0) \sim \Omega)(2\xi)^{\alpha-n}]^{1/(\alpha-1)}$$
$$\leqq 2^{(|n-\alpha|)/(\alpha-1)} A(2\xi),$$

so that

$$\int_{\xi}^{2\xi} A(t)\frac{dt}{t} \leqq 2^{(|n-\alpha|)/(\alpha-1)} A(2\xi).$$

Hence

$$\int_{2^{-m}\sigma'}^{\sigma'} A(t)\frac{dt}{t} \leqq 2^{(|n-\alpha|)/(\alpha-1)} \sum_{i=0}^{m-1} A(2^{-i}\sigma').$$

and therefore, by (43)

$$f(2^{-m}\sigma') \leqq f(\sigma') \exp\left[2^{(|n-\alpha|)/(\alpha-1)} \int_{2^{-m}\sigma'}^{\sigma'} A(t)\frac{dt}{t}\right] \tag{44}$$

for $m=0, 1, 2, \ldots$ Since f is increasing the required result now follows from (44).

We will be interested in those points x_0 where

$$(W_\delta(\Omega, x_0)) \to \infty$$

as $\delta \to 0^+$. The following theorem gives information of this nature.

2.16. THEOREM. — *Let $\alpha > n$ and Ω be an arbitrary open set of $\mathbb{R}^n$. Then, for all $x_0 \in \partial\Omega$,*

$$W_\delta(\Omega, x_0) \to \infty$$

as $\delta \to 0^+$.

Proof. — Let $x_0 \in \partial\Omega$. It will be sufficient to show that,

$$\Gamma_\sigma(\{x_0\})\,\sigma^{\alpha-n} \geqq m \tag{45}$$

for all $\sigma > 0$, where m is a positive constant.

Let $\sigma > 0$ and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be non-negative with $\varphi(x_0) = 1$, and with spt $\varphi \subset B_\sigma(x_0)$. By letting $\delta \to 0^+$ in 3.1 we see that

$$\varphi(x_0) \leqq ⨍_{|y-x_0|<\sigma} \varphi(y)\,dy + \frac{1}{n} ⨍_{|y-x_0|<\sigma} |D\varphi(y)|.|y-x_0|\,dy + E_0 ⨍_{|y-x_0|<\sigma} |D\varphi(y)|.|y-x_0|^{1-n}\,dy \tag{46}$$

where E_0 depends only on n. Let I_1, I_2 and I_3 denote the three terms on the right-hand side of (46).

By the Sobolev inequality

$$I_1 \leqq E_1\,\sigma^{1\,(n/\alpha)} \left[\int_{\mathbb{R}^n} |D\varphi(y)|^\alpha\,dy\right]^{1/\alpha} \tag{47}$$

where E_1 dependes only on n and α. By Hölder's inequality

$$I_2 \leqq E_2 \sigma^{-n} \left[\int_{|y-x_0|<\sigma} |y-x_0|^{\alpha/(\alpha-1)} dy \right]^{(\alpha-1)/\alpha} \left[\int_{R^n} |D\varphi(y)|^\alpha dy \right]^{1/\alpha} \quad (48)$$

where E_2 depends only on n, so that

$$I_2 \leqq E_2' \sigma^{1-(n/\alpha)} \left[\int_{R^n} |D\varphi(y)|^\alpha dy \right]^{1/\alpha}, \quad (49)$$

where E_2' depends only on n and α. It again follows Hölder's inequality that

$$I_3 \leqq E_3 \left[\int_{|y-x_0|<\sigma} |y-x_0|^{(1-n)\alpha/(\alpha-1)} dy \right]^{(\alpha-1)/\alpha} \left[\int_{R^n} |D\varphi(y)|^\alpha dy \right]^{1/\alpha} dy,$$

where E_3 depends only on n, so that

$$I_3 \leqq E_3' \sigma^{1-(n/\alpha)} \left[\int_{R^n} |D\varphi|^\alpha \right]^{1/\alpha} \quad (50)$$

where E_3' depends only on n and α. It now follows from (46), (47), (49), and (50) that

$$1 \leqq E^1 \sigma^{1-(n/\alpha)} \left[\int_{R^n} |D\varphi(y)|^\alpha dy \right]^{1/\alpha}, \quad (51)$$

where E' depends only on α and n. The inequality (45) now follows from (51).

3. THE MAIN RESULTS

In this section we discuss the existence theorem for the Dirichlet problem. We begin by introducing the type of operator to be considered.

3.1. DEFINITION. — Let Ω be a non-empty bounded open set of R^n, let $1<\alpha<\infty$ and let $\mu \geqq v 0$. We denote by

$$\mathscr{M}(\Omega, \alpha, \mu' v)$$

the set of all $(n+1)$-triples

$$(a_1, \ldots, a_n, b)$$

of locally Lipschitz functions on $\Omega \times \mathrm{R} \times \mathrm{R}^n$, which satisfy (7) for almost all $(x, z, p) \in \Omega \times \mathrm{R} \times \mathrm{R}^n$ and all $\xi \in \mathrm{R}^n$ and (8) for almost all $(x, y, z) \in \Omega \times \mathrm{R} \times \mathrm{R}^n$.

When $(a, b) \in \mathcal{M}(\Omega, \alpha, \mu' \nu)$, $a \in \mathrm{C}^1(\Omega)$, and $u \in \mathrm{C}^2(\Omega)$, we denote by

$$\mathrm{L}_{(a,\, b)} u$$

the function defined on Ω by

$$\mathrm{L}_{(a,\, b)} u(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u(x), \mathrm{D}u(x)) + b(x, u(x), \mathrm{D}u(x)). \quad (52)$$

The following is a simple consequence of 3.1.

3.2. Lemma. – *If* $(a, b) \in \mathcal{M}(\Omega, \alpha, \mu, \nu)$ *and* $u \in \mathrm{C}^2(\Omega)$ *with*

$$\mathrm{L}_{(a,\, b)} u(x) = 0$$

for $x \in \Omega$, *then there exists*

$$(a^*, b^*) \in \mathcal{M}(\Omega, \alpha, \mu, \nu)$$

such that

$$\mathrm{L}_{(a^*,\, b^*)}(-u)(x) = 0$$

for $x \in \Omega$.

The following theorem can easily be established by using mollifiers on a and b.

3.3. Theorem. – Let $(a, b) \in \mathcal{M}(\Omega, \alpha, \mu, \nu)$, $0 < \varepsilon < \nu$, $\mathrm{M} > 0$ *and* Ω^* *be a non-empty open set with* $\overline{\Omega^*} \subset \Omega$. *Then there exists*

$$(a^*, b^*) \in \mathcal{M}(\Omega^*, \alpha, \mu + \varepsilon, \nu - \varepsilon)$$

such that $a^*, b^* \in \mathrm{C}^\infty(\Omega^*)$ *and*

$$|a^*(x, z, p) - a(x, z, p)| + |b^*(x, z, p) - b(x, z, p)| < \varepsilon$$

when $x \in \Omega^*$, $z \in [-\mathrm{M}, \mathrm{M}]$, $p \in \mathrm{R}^n$ and $|p| \leqq \mathrm{M}$.

3.4. DEFINITION. — Let $(a, b) \in \mathcal{M}(\Omega, \alpha, \mu, \nu)$ and f be a function on Ω. In the usual way, we say $u \in C^1(\Omega)$ weakly satisfies the inequalities

$$L u \geqq f. \quad L u \leqq f$$

or the equation

$$L u = 0$$

on Ω, according as

$$-\sum_{i=1}^{n} \int_{\Omega} a_i(x, u(x), Du(x)) \frac{\partial \varphi}{\partial x_i}(x)\, dx + \int_{\Omega} b(x, u(x), Du(x))\, \varphi(x)\, dx \geqq \int_{\Omega} f(x)\, \varphi(x)\, dx, \quad \leqq \int_{\Omega} f(x)\, \varphi(x)\, dx \quad \text{or} \quad = 0,$$

for every C^1 function on R^n with compact support, with spt $\varphi \subset \Omega$ and with $\varphi \geqq 0$, in the case of the two inequalities.

We need a relation between weak solutions of the inequalities

$$L u \geqq 0, \qquad L u \leqq 0$$

and the functions of $U^{\pm}(\Omega, \alpha, \mu, \nu)$ that were introduced in 2.4. The following results gives this.

3.5. THEOREM. — *Let* $\alpha > 1$ *and* $\mu \geqq \nu > 0$, *then there exist constants* $\mu' > 0$ *and* $\nu' \geqq 0$, *such that*

$$p \,.\, a(x, z, p) \geqq \mu' |p|^{\alpha} - \nu' \tag{53}$$

where Ω *is a bounded, non-empty open subset of* R^n.

$$(a, b) \in \mathcal{M}(\Omega, \alpha, \mu', \nu'),$$

and $(x, z, p) \in \Omega \times R \times R^n$.

Proof. — Let Ω and (a, b) be as described. Let $x \in \Omega$, $z \in R$ and $p \in R^n$. Because of 3.3, we can assume $a \in C^{\infty}$.

For $t \in [0, 1]$, put

$$\varphi(t) = \sum_{i=1}^{n} [a_i(x, z, tp) - a_i(x, z, 0)]\, p_i.$$

Then

$$\varphi(1)=\int_0^1 \varphi'(t)\,dt=\int_0^1\left[\sum_{i,j=1}^{n}\frac{\partial a_i}{\partial p_j}(x, z, tp)\,p_i p_j\right]$$

$$\geqq \nu |p|^2\int_0^1 [1+t^2|p|^2]^{(1/2)(\alpha-2)}\,dt$$

When $\alpha \geqq 2$, we have

$$\varphi(1)\geqq \nu |p|^2\int_0^1 [t^2|p|^2]^{(1/2)(\alpha-2)}\,dt=\frac{\nu}{\alpha-1}|p|^{\alpha}$$

and when $\alpha<2$, we have

$$\varphi(1)\geqq \nu |p|^2\int_0^1 [1+|p|^2]^{(1/2)(\alpha-2)}\,dt=\nu|p|^2[1+|p|^2]^{(1/2)(\alpha-2)},$$

so that, if $|p|\geqq 1$, we have

$$\varphi(1)\geqq 2^{(1/2)(\alpha-2)}|p|^{\alpha}$$

and, if $|p|<1$, then

$$\varphi(1)\geqq |p|^{\alpha}-1.$$

Thus

$$a(x, z, p)\cdot p-a(x, z, 0)\cdot p\geqq \nu'(|p|^{\alpha}-1). \tag{54}$$

But

$$|a(x, z,)\cdot p|\leqq \mu|p|$$

so that by Young's inequality

$$|a(x, z, 0)\cdot p|\leqq \frac{1}{2}\nu'|p|^{\alpha}+C \tag{55}$$

The inequality (53), now follows from (54) and (55).

The following comparison theorem will be needed.

3.6. THEOREM. — *Let* $\mu \geqq \nu > 0$, $\alpha > 1$ *and* Ω *be a bounded non-empty open set of* R^n. *Let* $\varphi \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ *be such that*

$$\sum_{i,j=1}^{n} \gamma_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) < -\mu [1 + |D\varphi(x)|^2]^{1/2}, \tag{56}$$

Whenever $x \in \Omega$ *and* (γ_{ij}) *is a real symmetric matrix with*

$$\nu |\xi|^2 \leqq \sum_{i,j=1}^{n} \gamma_{ij} \xi_i \xi_j \leqq \mu |\xi|^2 \tag{57}$$

for all $\xi \in R^n$. *Let* $(a, b) \in \mathcal{M}(\Omega, \alpha, \mu' \nu)$, $a \in C^1(\Omega)$ *and let* $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ *be such that*

$$L_{(a,b)} u(x) \geqq 0 \tag{58}$$

for $x \in \Omega$. *Then*

$$u(x) - \varphi(x) < \sup_{\xi \in \partial\Omega} [u(\xi) - \varphi(\xi)] \tag{59}$$

for all $x \in \Omega$.

Proof. — Put

$$w(x) = u(x) - \varphi(x) \tag{60}$$

for $x \in \bar{\Omega}$ and

$$\Lambda = \sup_{x \in \bar{\Omega}} w(x). \tag{61}$$

It will be sufficient to show that

$$w(x) < \Lambda \tag{62}$$

for $x \in \Omega$. Suppose there is an $x' \in \Omega$ with

$$w(x') = \Lambda. \tag{63}$$

To simplify the notation, we put

$$A_{ij}(x, z, p) = \frac{\partial a_i}{\partial p_j}(x, z, p) \tag{64}$$

and

$$B(x, z, p)=\sum_{i=1}^{n}\left[\frac{\partial a_i}{\partial x_i}(x, z, p)+\frac{\partial a_i}{\partial z}(x, z, p)p_i\right]+b(x, z, p). \quad (65)$$

Then

$$\sum_{i,j=1}^{n} A_{ij}(x, u(x), Du(x))\frac{\partial^2 u}{\partial x_i \partial x_j}(x)+B(x, u(x), Du(x))\geqq 0 \quad (66)$$

for $x\in\Omega$. Since $Du(x')=D\varphi(x')$, it follows from (66), that

$$\sum_{i,j=1}^{n} A_{ij}(x', u(x'), Du(x'))\frac{\partial^2 w}{\partial x_i \partial x_j}(x')$$
$$\geqq -B(x', u(x'), D\varphi(x'))-\sum_{i,j=1}^{n} A_{ij}(x', u(x'), D\varphi(x'))\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x')$$
$$> -\mu[1+|D\varphi(x')|^2]^{(1/2)(\alpha-1)}+\mu[1+|D\varphi(x')|^2]^{(1/2)(\alpha-1)}.$$

This contradicts (63).

This result will now be used to obtain uniform bounds to solutions of equations of the form

$$L_{(a,\, b)}u(x)=0.$$

3.7. THEOREM. — *Let* $\mu\geqq\nu>0$ *and* $\alpha>1$. *Let* $E>0$. *Then, there exists a positive constant* M, *such that*

$$\sup_{x\in\Omega}|u(x)|\leqq M+\sup_{x\in\partial\Omega}|u(x)| \quad (67)$$

whenever Ω *is a bounded open set with diameter* $\leqq E$,

$$(a, b)\in\mathcal{M}(\Omega, \alpha, \mu, \nu).$$

$a\in C^1(\Omega)$, *and* $u\in C^0(\bar\Omega)\cap C^2(\Omega)$ *with*

$$L_{(a,\, b)}u(x)=0$$

for $x\in\Omega$.

Proof. — Let Ω, (a, b) and u be as described. Put

$$\Lambda=\sup_{x\in\partial\Omega}u(x)$$

and let $x' \in \mathrm{R}^n$ be such that $d(x', \Omega) = 1$. Define

$$\varphi(x) = \Lambda + \mathrm{M} - \mathrm{M} \exp(-\mathrm{K} |x - x'|^2)$$

for $x \in \mathrm{R}^n$, where K and M are positive constants to be determined. Then

$$\frac{\partial \varphi}{\partial x_i} = 2\,\mathrm{KM} \exp(-\mathrm{K} |x - x'|^2)(x_i - x'_i)$$

and

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \{-4\,\mathrm{K}^2 \mathrm{M}(x_i - x'_i)(x_j - x'_j) + 2\,\mathrm{KM}\,\delta_{ij}\} \exp(-\mathrm{K} |x - x'|^2.$$

Suppose (γ_{ij}) is a real symmetric matrix with

$$\nu(|\xi|^2| \leqq \sum_{i,j=1}^{n} \gamma_{ij} \xi_i \xi_j \leqq \mu |\xi|^2$$

for all $\xi \in \mathrm{R}^n$. Then

$$\sum_{i,j=1}^{n} \gamma_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) + \mu(1 + |\mathrm{D}\varphi(x)|)$$
$$\leqq \{-4\nu \mathrm{K}^2 \mathrm{M} + 2n\mu \mathrm{KM} + 2\mu \mathrm{KM}(1 + \mathrm{E})\} \exp(-\mathrm{K} |x - x'|^2)$$

for $x \in \Omega$. Choose K so that

$$-4\nu \mathrm{K}^2 + 2n\mu \mathrm{K} + 2\mu \mathrm{K}(1 + \mathrm{E}) = -1$$

and then choose M so that

$$-\mathrm{M} \exp(-\mathrm{K}(1 + \mathrm{E})) + \mu < 0.$$

Then

$$\sum_{i,j=1}^{n} \gamma_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) + \mu(1 + |\mathrm{D}\varphi(x)|^2)^{1/2} < 0$$

for all $x \in \Omega$. Hence by 3.6,

$$u(x) \leqq \varphi(x) \leqq \Lambda + \mathrm{M} \tag{68}$$

for all $x \in \Omega$.

It follows from 3.2 that the above arguments also apply to $-u$, so that

$$-u(x) \leqq M + \sup_{x \in \partial\Omega} \{-u(x)\} \tag{69}$$

for all $x \in \Omega$. The required result now follows from (68) and (69).

The following theorem can be proved by using standard estimates from quasi-linear and linear theory.

3.8. THEOREM. — *Let Ω be a non-empty bounded open set of R^n and $\{\Omega_r\}$ an increasing sequence of non-empty open sets with limit Ω. Let $\mu \geqq \nu > 0$, let $\alpha > 1$ and $\{(a^{(r)}, b^{(r)})\}$ be a sequence with*

$$(a^{(r)}, b^{(r)}) \in \mathcal{M}(\Omega_r, \alpha, \mu, \nu)$$

for all r and with the properties:

(i) *for each r, $a^{(r)}$ and $b^{(r)}$ are C^1 on Ω_r;*

(ii) *for each i, $\{a_i^{(r)}\}$ converges C^1 on the interior of compact subsets of $\Omega \times R \times R^n$ to a C^1 function a_i and $\{b^{(r)}\}$ converges uniformly on compact subsets of $\Omega \times R \times R^n$ to a locally Lipschitz function b;*

(iii) *for each compact subset E of $\Omega \times R \times R^n$, there exists a $\delta \in (0.1)$ and a $K \geqq 0$, such that*

$$\left| \frac{\partial a_i^{(r)}}{\partial p_j}(w) - \frac{\partial a_i^{(r)}}{\partial p_j}(w') \right| \leqq K \,|\, w - w' \,|^{\delta}$$

for all i, j, r and all $w, w' \in E$,

$$\left| \sum_{i=1}^{n} \left\{ \frac{\partial a_i}{\partial x_i}(x, z, p) + p_i \frac{\partial a_i}{\partial z}(x, z, p) - \frac{\partial a}{\partial x_i}(x', z', p') - p'_i \frac{\partial a_i}{\partial z}(x', z', p') \right\} \right| \leqq K \,|\,(x, z, p) - (x', z', p')\,|^{\delta}$$

for all r and all $(x, z, p), (x', z', p') \in E$ and

$$|\, b^{(r)}(w) - b^{(r)}(w') \,| \leqq K \,|\, w - w' \,|$$

for all r and all $w, w' \in E$.

Suppose $\{u_r\}$ is a sequence of C^2 functions with domains Ω_r, which is uniformly bounded and has

$$L_{(a^{(r)}, b^{(r)})} u_r(x) = 0$$

for all $x \in \Omega_r$ *and all r. Then*

$$(a, b) \in \mathcal{M}(\Omega, \alpha, \mu, \nu)$$

and there is a subsequence $\{u_{r_s}\}$ *which converges* C^2 *on the interior of compact subsets of* Ω *to a* C^2 *function u, with*

$$L_{(a, b)} u(x) = 0$$

for $x \in \Omega$.

3.9 THEOREM. – *Let* Ω *be a non-empty bounded open set in* $\mathbf{R}^n$ *and let* $\Omega' \subset\subset \Omega$ *be an open set such that* $\partial\Omega' \in C^{2, \gamma}$, *where* $0 < \gamma < 1$. *Let* $\alpha > 1$, $\mu \geqq \nu > 0$ *and* $\psi \in C^0(\partial\Omega)$. *Finally, let* $(a, b) \in \mathcal{M}(\Omega, \alpha, \mu, \nu)$. *Then, there exists a function* $u \in C^0(\overline{\Omega'}) \cap C^2(\Omega')$ *such that* $u = \psi$ *on* $\partial\Omega$ *and*

$$L_{(a, b)} u(x) = 0$$

for all $x \in \Omega$.

Proof. – The proof is based on an approximation procedure. First, by using mollifers one can construct a sequence

$$\{(a^{(r)}, b^{(r)})\}$$

such that each of

$$a_1^{(r)}, a_2^{(r)}, \ldots, a_n^{(r)}, b^{(r)}$$

is C^∞ on $\Omega \times \mathbf{R} \times \mathbf{R}^n$,

$$(a^{(r)}, b^{(r)}) \in \mathcal{M}\left(\Omega', \alpha, 2\mu \frac{1}{2}\nu\right)$$

for all r, $a^{(r)}$ converges C^1 to a on the interior of compact subsets of Ω and $b^{(r)}$ converges to b uniformly on compact subsets of Ω.

Next, approximate ψ uniformly on $\partial\Omega'$ by functions $\psi_r \in C^{2, \gamma}(\Omega')$. (Of course, ψ_r can be assumed to be C^∞, but this is not required.) Now apply [GT], Theorem 15.11, with [GT], Theorem 10.9, replaced by Theorem 3.7, to find $u_r \in C^{2, \gamma}(\bar{\Omega}')$ such that $u_r = \psi_r$ on $\partial\Omega'$ and $L_{(a^r, b^r)} u_r(x) = 0$ for all $x \in \Omega'$. Now apply Theorem 3.8 to conclude that there is a subsequence $\{u_{r_s}\}$ which converges C^2 on the interior of compact subsets of Ω' to a function $u \in C^2(\Omega')$ with $L_{(a, b)} u(x) = 0$. Moreover, $u = \psi$

on $\partial\Omega'$. The solution u actually is an ellement of $C^{2,\gamma}(\Omega')$, but this will not be needed and cannot be applied in the sequel.

3.10. THEOREM. — *Let Ω be a non-empty bounded open set of R^n and ψ a continuous function on $\partial\Omega$. Let $\alpha > 1$, $\mu \geqq \nu > 0$ and*

$$(a, b) \in \mathcal{M}(\Omega, \alpha, \mu, \nu).$$

Let E be the subset of $\partial\Omega$ consisting of all points x_0, for which

$$W_\delta(\Omega, x_0) \to \infty$$

as $\delta \to 0^+$.

(i) *Then there exists a bounded continuous function u on $\Omega \cup E$, such that $u \in C^1(\Omega)$, $u(x) = \psi(x)$ when $x \in E$.*

$$L_{(a,\,b)}\, u = 0$$

weakly on Ω and for each compact subset Q of Ω, there is an $\eta > 0$ and such that each $\dfrac{\partial u}{\partial x_i}$ is Hölder continuous with exponent η on Q.

(ii) *If $a \in C^1(\Omega)$, $\dfrac{\partial a_i}{\partial p_j} \in \mathcal{H}(\Omega \times R \times R^n)$ for all i, j and*

$$\sum_{i=1}^{n} \left[\frac{\partial a_i}{\partial x_i} + p_i \frac{\partial a_i}{\partial z} \right] \in \mathcal{H}(\Omega \times R \times R^n),$$

then there exists $u \in C^0(\Omega \cup E) \cap C^2(\Omega)$ such that $u(x) = \psi(x)$ when $x \in E$ and

$$L_{(a,\,b)}\, u(x) = 0$$

for all $x \in \Omega$.

Proof. — Extend ψ to a continuous function on $\bar{\Omega}$. By standard methods, it is possible to construct an increasing sequance $\{\Omega_r\}$ of C^∞ smooth domains such that $\Omega_r \subset\subset \Omega$ with limit Ω.

Suppose that the conditions of (ii) are satisfied. It follows from Theorem 3.9 that for each r, there exists $u_r \in C^0(\bar{\Omega}_r) \cap C^2(\Omega_r)$ such that $u_r = \psi$ on $\partial\Omega_r$ and

$$L_{(a,\,b)}\, u_r(x) = 0$$

for $x \in \Omega_r$. By Theorem 3.7, there exists a constant M, such that

$$|u_r(x)| \leqq M$$

for all $x \in \bar{\Omega}_r$ and all r. By Theorem 3.8, there exists a subsequence $\{u_{r_s}\}$ which converges C^2 on the interior of compact subsets of Ω to a C^2 function u with

$$L_{(a,\ b)} u(x) = 0$$

for $x \in \Omega$. Then

$$|u(x)| \leqq M$$

for $x \in \Omega$.

We now discuss the case where the conditions of (ii) are not assumed to hold. By using 3.3, we can obtain a C^∞ sequence $\{(a^{(r)}, b^{(r)}\}$ such that

$$(a^{(r)}, b^{(r)}) \in \mathcal{M}\left(\Omega_r, \alpha, \left(1+\frac{1}{2r}\right)\mu\left(1-\frac{1}{2r}\right)\nu\right)$$

for all r and each $a_i^{(r)}$ and $b^{(r)}$ converge uniformly to a_i and b on compact subsets of $\Omega \times R \times R^n$. Again by Theorem 3.9, there exists $u_r \in C^0(\bar{\Omega}_r) \cap C^2(\Omega_r)$ which agress with ψ on $\partial\Omega_r$, and

$$L_{(a^{(r)},\ b^{(r)})} u_r(x) = 0$$

for all $x \in \Omega_r$. Referring to Theorem 3.7 again, there exists a constant M such that

$$|u_r(x)| \leqq M$$

for all $x \in \Omega_r$ and all r. By using standard estimates, one can now obtain a subsequence $\{u_{r_s}\}$ which converges C^1 on the interior of compact subsets of Ω to a C^1 function u such that

$$L_{(a,\ b)} u = 0$$

weakly on Ω and for each compact subset Q of Ω, there is an $\eta > 0$ and such that each $\frac{\partial u}{\partial x_i}$ is Hölder continuous with exponent η on Q. Then

$$|u(x)| \leqq M$$

for $x \in \Omega$. It remains to prove that in each case u can be extended to a continuous function on $\Omega \cup E$.

Let $x_0 \in E$ and let $\Lambda > \psi(x_0)$. Choose $\rho > 0$ and such that

$$\psi(x) < \Lambda$$

when $|x - x_0| \leqq r$. Let r_1 be such that

$$\Omega_r \cap B_r(x_0) \neq 0$$

when $r \geqq r_1$. Suppose $r \geqq r_1$. By Theorems 3.5 and 2.5,

$$(u_r - \Lambda)^+ \in T^+(B_r(x_0) \cap \Omega_r, \alpha, C, \lambda) \tag{70}$$

where C, λ are independent of r. Whenever $\eta \in C^{0,1}(R^n)$ vanishes outside $B_r(x_0)$, it is clear that

$$(u_r - \Lambda)^+ \cdot \eta$$

can be extended to a Lipschitz function on R^n, with compact support and

$$\operatorname{spt}\{(u_r - \Lambda)^+ \cdot \eta\} \subset B_r(x_0) \cap \Omega_r.$$

Define

$$w_r(x) = (u_r(x) - \Lambda)^+ \quad \text{when } x \in \Omega_r,$$
$$= 0 \quad \text{when } x \in \{B_r(x_0) \cap \Omega\} \sim \Omega_r.$$

By Theorem 2.7

$$w_r \in T^+(B_r(x_0) \cap \Omega, \alpha, C, \lambda)$$

and by Theorem 2.16

$$\sup_{|x - x_0| \leqq \delta} (u_r(x) - \Lambda)^+ \leqq C_1 \exp(-C_2 W_\delta(\Omega, x_0))$$

for $\delta \in (0, \rho/16)$, where C_1, C_2 are independent of r. Thus

$$\sup_{|x - x_0| \leqq \delta} (u(x) - \Lambda)^+ \leqq C_1 \exp(-C_2 W_\delta(\Omega, x_0))$$

for $\delta \in (0, \rho/16)$. Hence

$$\lim_{x \to x_0} \sup\{-u(x)\} \leqq -\psi(x_0),$$

where the lim sup is taken relative to Ω.

The same arguments apply to $-u$, so that

$$\lim_{x \to x_0} \sup u(x) \leqq \psi(x_0),$$

Since x_0 was arbitrary, u can be extended to a continuous function on $\Omega \cup E$.

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