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Gradient theory of phase transitions with boundary contact energy

by

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ABSTRACT. — We study the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the variational problems for the Van der Waals-Cahn-Hilliard theory of phase transitions in a fluid. We assume that the internal free energy, per unit volume, is given by $\varepsilon^2 |\nabla \rho|^2 + W(\rho)$ and the contact energy with the container walls, per unit surface area, is given by $\varepsilon\sigma(\rho)$, where ρ is the density. The result is that such solutions approximate a two-phases configuration satisfying a variational principle related to the equilibrium configuration of liquid drops.

Key words : Phase transitions, variational thermodynamic principles, variational convergence.

RÉSUMÉ. — Nous étudions ici le comportement asymptotique pour $\varepsilon \to 0^+$ des solutions des problèmes variationnels qui viennent de la théorie de Van der Waals-Cahn-Hilliard sur les transitions de phase des fluides. Nous faisons l'hypothèse que l'énergie libre de Gibbs, pour unité de volume, est donnée par $\varepsilon^2 |\nabla \rho|^2 + W(\rho)$ et que l'énergie de contact avec la surface intérieure du containeur, pour unité de surface, est donnée par $\varepsilon\sigma(\rho)$, où ρ est la densité. Le résultat est que ces solutions approchent

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une configuration à deux phases qui satisfait un principe variationnel lié aux configurations à l'équilibre des gouttes.

INTRODUCTION

We continue in this paper the asymptotic analysis of the Van der Waals-Cahn-Hilliard theory of phase transitions in a fluid, by taking also into account, with respect to our earlier results [10], the contact energy between the fluid and the container walls. Our results give a positive answer to some conjectures by M. E. Gurtin [8].

Let us describe briefly the problem we are concerned with; we refer to [10] for further information and bibliography. Consider a fluid, under isothermal conditions and confined to a bounded container $\Omega \subset \mathbb{R}^n$, and assume that the Gibbs free energy, per unit volume, $W = W(u)$ and the contact energy, per unit surface area, $\sigma = \sigma(u)$ between the fluid and the container walls $\partial\Omega$ are prescribed functions of the density distribution (or composition) $u \geqq 0$ of the fluid. According to the Van der Waals-Cahn-Hilliard theory, and in particular to the Cahn's approach [2], the stable configurations of the fluid are determined by solving the variational problem

$$(*) \qquad \min\left\{\int_\Omega [\varepsilon^2 |Du|^2 + W(u)]\,dx + \int_{\partial\Omega} \varepsilon\theta(u)\,d\mathcal{H}_{n-1}\right\},$$

where $\varepsilon > 0$ is a small parameter, and the minimum is taken among all functions $u \geqq 0$ satisfying the constraint

$$\int_\Omega u\,dx = m,$$

m being the prescribed total mass of the fluid. The function $W(t)$ is supposed to vanish only at two points $t = \alpha$ and $t = \beta$ ($\alpha < \beta$), and to be strictly positive everywhere else. Of course, $\mathcal{H}_{n-1}$ denotes the Hausdorff $(n-1)$-dimensional measure.

Our goal is to study the asymptotic behavior as $\varepsilon \to 0^+$ of solutions u_ε of $(*)$ by looking for a variational problem solved by the limit point (or points) of u_ε in $L^1(\Omega)$. As conjectured by Gurtin [8], this limit problem does exist and agrees with the so-called liquid-drop problem.

Namely (*cf.* Theorem 2.1 for a precise statement), if the function u_0 is the limit of u_ε in $L^1(\Omega)$ as $\varepsilon \to 0^+$, then u_0 takes only the values α and β (i. e., u_0 corresponds to a two-phases configuration of the fluid), and the portion E_0 of the container occupied by the phase $u_0 = \alpha$ minimizes the geometric area-like quantity

$$\mathscr{H}_{n-1}(\partial E \cap \Omega) + \gamma \mathscr{H}_{n-1}(\partial E \cap \partial\Omega)$$

among all subsets E of Ω having the same volume as E_0. The number γ depends only on W and σ, and it can be explicitly calculated:

$$\gamma = \frac{\hat{\sigma}(\alpha) - \hat{\sigma}(\beta)}{2 c_0},$$

where

$$c_0 = \int_\alpha^\beta W^{1/2}(s)\, ds,$$

and $\hat{\sigma}$ represents a modified contact energy between the fluid and the container walls, whose definition involves the values of $\sigma(t)$ and $W(t)$ for every $t \geqq 0$. One has $|\gamma| \leqq 1$ in correspondence with the geometrical meaning of γ, which is the cosine of the contact angle between the fluid phase α and the walls of the container.

The presence of such $\hat{\sigma}$ instead of σ disproves a part of the Gurtin's conjecture but, what is more interesting, it is perfectly in accord with theory and experiments by J. W. Cahn and R. B. Heady ([2], [3]) about critical point wetting. They discovered that, in a range of temperatures below the critical one for a binary system, the phase α does not wet the container (i. e. $\gamma = 1$) and a layer of phase β, which is, on the contrary, perfectly wetting, appears between the phase α and the container walls. A theoretical explanation of such phenomenon was given by Cahn in the case $\varepsilon > 0$.

We confirm in this paper, under very general assumptions and by a fully mathematical proof, the existence of the critical point wetting phenomenon in the asymptotic case $\varepsilon \to 0$. Indeed, we show that $\gamma = 1$ and $\hat{\sigma}(\alpha) = \hat{\sigma}(\beta) + \sigma_{\alpha\beta}$ ($\sigma_{\alpha\beta}$ denotes the energy, per unit surface area, associated to the interface between the phases α and β), for σ and W having the same global behavior exhibited in the semi-empirical figures of [2]. It now suffices to remark that the balance of energy $\hat{\sigma}(\alpha) = \hat{\sigma}(\beta) + \sigma_{\alpha\beta}$ can be interpreted as the contact energy on $\partial E_0 \cap \partial\Omega$ coming from an infinitely thin layer of the phase β interposed between the phase α and the container walls (*cf.* Section 3 for details).

We think that other very interesting experimental evidences, discussed by Cahn in [2], would deserve a similar careful mathematical treatment.

Finally, we would like to thank Morton Gurtin for stimulating and friendly discussions.

1. SOME PRELIMINARY RESULTS

Throughout this paper Ω will be an open, bounded subset of $\mathbb{R}^n$ $(n \geqq 2)$ with smooth boundary $\partial\Omega$; W and σ will be two non-negative continuous functions defined on $[0, +\infty[$. The function $W(t)$ is supposed to have exactly two zeros at the points $t=\alpha$ and $t=\beta$, with $0<\alpha<\beta$.

For every $\varepsilon>0$ and for every non-negative function u in the Sobolev space $H^1(\Omega)$, we define

$$\mathscr{E}_\varepsilon(u)=\int_\Omega [\varepsilon^2 |Du(x)|^2+W(u(x))]\,dx+\varepsilon\int_{\partial\Omega}\sigma(\tilde{u}(x))\,d\mathscr{H}_{n-1}(x) \quad (1)$$

where Du denotes the gradient of u, $\tilde{u}$ denotes the trace of u on $\partial\Omega$, and $\mathscr{H}_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

1.1. PROPOSITION. — *For every* $\varepsilon>0$ *and for every* $m \geqq 0$ *the minimization problem*

$$(\mathrm{P}_\varepsilon) \qquad \min\left\{\mathscr{E}_\varepsilon(u): u\in H^1(\Omega), u\geqq 0, \int_\Omega u(x)\,dx=m\right\}$$

admits (at least) one solution.

Proof. — The proof is standard. Let

$$U=\left\{u\in H^1(\Omega): u\geqq 0, \mathscr{E}_\varepsilon(u)\leqq c, \int_\Omega u(x)\,dx=m\right\},$$

with $c\in\mathbb{R}$ large enough so that $U\neq\varnothing$. It suffices to prove that $\mathscr{E}_\varepsilon$ is lower semicontinuous on U and U is compact with respect to the topology of $L^2(\Omega)$.

Let $u_\infty \in \mathrm{U}$ and (u_h) be a sequence in U converging to u_∞ in $\mathrm{L}^2(\Omega)$: we have to prove that

$$\mathscr{E}_\varepsilon(u_\infty) \leqq \liminf_{h \to +\infty} \mathscr{E}_\varepsilon(u_h). \tag{2}$$

Without loss of generality we can assume that there exists the limit of $\mathscr{E}_\varepsilon(u_h)$ as $h \to +\infty$ and it is finite. Since $\mathrm{W} \geqq 0$ and $\sigma \geqq 0$, we have that

$$\int_\Omega |\mathrm{D}u|^2 \, dx \leqq c/\varepsilon^2, \qquad \forall u \in \mathrm{U}; \tag{3}$$

hence, modulo replacing (u_h) by a subsequence, (u_h) and $(\tilde{u}_h)$ converge pointwise to u_∞ and $\tilde{u}_\infty$, respectively almost everywhere on Ω and $\mathscr{H}_{n-1}$-almost everywhere on $\partial\Omega$ [recall that the trace operator is compact between $\mathrm{H}^1(\Omega)$ and $\mathrm{L}^2(\partial\Omega, \mathscr{H}_{n-1})$]. Then (2) follows from lower semicontinuity of the Dirichlet integral and from continuity of W and σ, by applying Fatou's Lemma.

Lower semicontinuity of $\mathscr{E}_\varepsilon$ implies now that U is closed in $\mathrm{L}^2(\Omega)$; on the other hand, by (3) and by Poincaré Inequality, U is bounded in $\mathrm{H}^1(\Omega)$. Then Rellich's Theorem gives that U is compact in $\mathrm{L}^2(\Omega)$ and the proof is complete. ■

The aim of the present paper is to study the asymptotic behavior as $\varepsilon \to 0^+$ of (P_ε). We shall prove in Section 2 that such asymptotic behavior is related with the following geometric minimization problem:

$$(\mathrm{P}_0) \qquad \min\{\mathrm{P}_\Omega(\mathrm{E}) + \gamma \mathscr{H}_{n-1}(\partial^* \mathrm{E} \cap \partial\Omega) : \mathrm{E} \subseteq \Omega, |\mathrm{E}| = m_1\}.$$

Here $\gamma \in [-1, 1]$, $m_1 \in [0, |\Omega|]$ are fixed real constants; $|\mathrm{E}|$, $\mathrm{P}_\Omega(\mathrm{E})$, $\partial^* \mathrm{E}$ denote respectively the Lebesgue measure of E, the perimeter of E in Ω, and the reduced boundary of E. We refer to the book by E. Giusti [6] for these concepts, which go back to the De Giorgi's approach to the minimal surfaces theory. Anyhow, for reader's convenience, we recall that $\mathrm{P}_\Omega(\mathrm{E}) = \mathscr{H}_{n-1}(\partial \mathrm{E} \cap \Omega)$ and $\partial^* \mathrm{E} = \partial \mathrm{E}$, provided that the boundary of E is locally Lipschitz continuous; hence (P_0) consists in finding a subset E of Ω, with prescribed volume m_1, which minimizes a quantity related with the $(n-1)$-dimensional measure of its boundary.

The problem (P_0) is known as the liquid-drop problem (*cf.* E. Giusti [5]). Since Ω is bounded and $|\gamma| \leqq 1$, it always admits (at least) one solution. Such existence result could also be obtained by the following proposition, which we need later.

1.2. PROPOSITION. — *Let* $\tau : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ *be a Borel function and define, for* $u \in \mathrm{BV}(\Omega)$,

$$F(u) = \int_\Omega |Du| + \int_{\partial\Omega} \tau(x, \tilde{u}(x))\, d\mathscr{H}_{n-1}(x) \quad (^1),$$

where $\tilde{u}$ *denotes the trace of* u *on* $\partial\Omega$. *If*

$$\text{(i)} \qquad \begin{cases} |\tau(x, s_1) - \tau(x, s_2)| \leqq |s_1 - s_2|, \\ \forall x \in \partial\Omega, \qquad \forall s_1, s_2 \in \mathbb{R} \end{cases}$$

then the functional F *is lower semicontinuous on* $\mathrm{BV}(\Omega)$ *with respect to the topology of* $L^1(\Omega)$.

Proof. — Fix $u_\infty \in \mathrm{BV}(\Omega)$ and let (u_h) be a sequence in $\mathrm{BV}(\Omega)$ converging to u_∞ in $L^1(\Omega)$. We want to prove that

$$\limsup_{h \to +\infty} [F(u_\infty) - F(u_h)] \leqq 0. \tag{4}$$

By (i) we deduce that

$$F(u_\infty) - F(u_h) \leqq \int_\Omega |Du_\infty| - \int_\Omega |Du_h| + \int_{\partial\Omega} |\tilde{u}_\infty - \tilde{u}_h|\, d\mathscr{H}_{n-1}.$$

Let $\delta > 0$ and define $v_\delta = (1 - \chi_\delta)\,(u_\infty - u_h)$, where χ_δ is the usual cut-off function, i. e. $\chi_\delta \in C_0^1(\Omega)$, $0 \leqq \chi_\delta \leqq 1$, $\chi_\delta(x) = 1$ if $\mathrm{dist}(x, \partial\Omega) \geqq \delta$, $|D\chi_\delta| \leqq 2/\delta$. The trace inequality for BV functions (*cf.* G. Anzellotti and M. Giaquinta [1]), applied to v_δ, gives that

$$\int_{\partial\Omega} |\tilde{u}_\infty - \tilde{u}_h|\, d\mathscr{H}_{n-1}$$
$$\leqq c_1 \int_{\Omega'_\delta} |D(u_\infty - u_h)| + (2c_1/\delta) \int_{\Omega'_\delta} |u_\infty - u_h|\, dx + c_2 \int_{\Omega'_\delta} |u_\infty - u_h|\, dx,$$

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(1) For $u \in \mathrm{BV}(\Omega)$ and E measurable subset of Ω, we denote by $\int_E |Du|$ the value of the measure $|Du|$ at the set E. Of course, if Du is a Lebesgue integrable vector function, then $\int_E |Du|$ agrees with the ordinary integral $\int_E |Du(x)|\, dx$.

where $\Omega_\delta=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\delta\}$ and $\Omega'_\delta=\Omega\setminus\Omega_\delta$. Let us remark that $c_1=1$ because $\partial\Omega$ is smooth (see [1]), and that

$$\int_{\Omega'_\delta}|D(u_\infty-u_h)|\leqq\int_{\Omega'_\delta}|Du_\infty|+\int_{\Omega'_\delta}|Du_h|+\int_{\partial\Omega_\delta}|D(u_\infty-u_h)|.$$

Since $u_\infty-u_h\in BV(\Omega)$, we have that

$$\int_{\partial\Omega_\delta}|D(u_\infty-u_h)|=0,\qquad\forall h\in\mathbb{N}$$

for a set of $\delta>0$ of full measure; hence

$$F(u_\infty)-F(u_h)$$
$$\leqq\int_\Omega|Du_\infty|+\int_{\Omega'_\delta}|Du_\infty|-\int_{\Omega_\delta}|Du_h|+\left(\frac{2}{\delta}+c_2\right)\int_{\Omega'_\delta}|u_\infty-u_h|\,dx$$

and, by lower semicontinuity in $L^1(\Omega_\delta)$ of the functional

$$u\mapsto\int_{\Omega_\delta}|Du|,$$

we conclude that

$$\limsup_{h\to+\infty}[F(u_\infty)-F(u_h)]\leqq 2\int_{\Omega'_\delta}|Du_\infty|$$

for almost all $\delta>0$. By taking $\delta\to 0^+$, the inequality (4) is proved. ■

1.3. *Remark*. – The previous proposition fails to be true if $\partial\Omega$ is not smooth, or if the function τ has in (i) a Lipschitz constant $L>1$. For example, in the case $\Omega=]0,1[\times]0,1[$ and $\tau(x,s)=-\lambda s$ with $\lambda>\sqrt{2}/2$, the corresponding functional F is not lower semicontinuous at the point $u_\infty=0$; it is enough to check lower semicontinuity on the sequence (u_h) given by $u_h(x,y)=0$ for $x+y\geqq 1/h$, $u_h(x,y)=h$ for $x+y<1/h$. Analogously, in the case $\Omega=\{x\in\mathbb{R}^2:|x|<1\}$ and $\tau(x,s)=\lambda|s|$ with $\lambda>1$, the corresponding functional F is not lower semicontinuous at the point $u_\infty(x)=|x|$: one can choose $u_h(x)=\min\{|x|,(h-1)(1-|x|)\}$.

However, it is worth noticing that, in the particular case $\tau(x,s)=|s-\psi(x)|$ with $\psi\in L^1(\partial\Omega,\mathscr{H}_{n-1})$, the functional F defined in Proposition 1.2 is lower semicontinuous on $L^1(\Omega)$ even for Lipschitz continuous $\partial\Omega$. Indeed, by choosing an open, bounded set $\Omega' \supseteqq \bar{\Omega}$ and a function $\hat{\psi} \in \mathrm{BV}(\Omega')$ whose trace on $\partial\Omega$ is ψ, we have that

$$\mathrm{F}(u) = \int_\Omega |\mathrm{D}u| + \int_{\partial\Omega} |\tilde{u}(x) - \psi(x)| \, d\mathscr{H}_{n-1} = \int_{\Omega'} |\mathrm{D}v_u| - \int_{\Omega' \setminus \bar{\Omega}} |\mathrm{D}\hat{\psi}|,$$

where the function v_u is defined by $v_u(x) = u(x)$ for $x \in \Omega$, $v_u(x) = \hat{\psi}(x)$, for $x \in \Omega' \setminus \Omega$. Since the first addendum of the right-hand side is lower semicontinuous with respect to u in $\mathrm{L}^1(\Omega)$, F also is lower semicontinuous in $\mathrm{L}^1(\Omega)$.

From now on, we let, for $t \geqq 0$,

$$\varphi(t) = \int_0^t \mathrm{W}^{1/2}(s) \, ds, \tag{5}$$

$$\hat{\sigma}(t) = \inf\{\sigma(s) + 2|\varphi(s) - \varphi(t)| : s \geqq 0\}, \tag{6}$$

and, for $u \in \mathrm{BV}(\Omega)$,

$$\mathscr{E}_0(u) = 2\int_\Omega |\mathrm{D}(\varphi \circ u)| + \int_{\partial\Omega} \hat{\sigma}(\tilde{u}(x)) \, d\mathscr{H}_{n-1}, \tag{7}$$

where, as above, $\tilde{u}$ denotes the trace of u on $\partial\Omega$.

1.4. Proposition. — *Let (u_h) be a sequence of functions of class C^1 on Ω. If (u_h) converges in $\mathrm{L}^1(\Omega)$ to a function u_∞ and there exists a real constant c such that*

$$\int_\Omega |\mathrm{D}(\varphi \circ u_h)| \, dx \leqq c$$

for every $h \in \mathbb{N}$, then $\varphi \circ u_\infty \in \mathrm{BV}(\Omega)$ and

$$\mathscr{E}_0(u_\infty) \leqq \liminf_{h \to +\infty} \mathscr{E}_0(u_h).$$

Proof. — Let us denote $v_h(x) = \varphi(u_h(x))$ and fix an open subset Ω' of Ω such that $\overline{\Omega'} \subset \Omega$. If we consider the smooth function $\bar{v}_h(x) = v_h(x) - \vartheta_h$, where

$$\vartheta_h = \int_{\Omega'} v_h \, dx,$$

Poincaré Inequality gives

$$\int_{\Omega'} |\bar{v}_h| \, dx \leqq c_1(\Omega) \int_{\Omega'} |D\bar{v}_h| \, dx \leqq c_1(\Omega)\, c$$

for every $h \in \mathbb{N}$ and for a real constant $c_1(\Omega)$ depending on Ω but independent of $\Omega' \subseteq \Omega$. It follows that the sequence $(\bar{v}_h)$ is bounded in $BV(\Omega)$; hence, by Rellich's Theorem, there exists a subsequence $(\bar{v}_{\sigma(h)})$ which converges in $L^1(\Omega)$ to a function $\bar{v}_\infty$.

Since it is not restrictive to assume that $(\bar{v}_{\sigma(h)})$ and $(v_{\sigma(h)})$ both converge almost everywhere in Ω, we infer that $(\vartheta_{\sigma(h)})$ converges in $\mathbb{R}$ to ϑ_∞, and finally that $(v_{\sigma(h)})$ converges in $L^1(\Omega)$ to $\bar{v}_\infty + \vartheta_\infty$. We have of course $\bar{v}_\infty + \vartheta_\infty = \varphi \circ u_\infty$, so we conclude that the whole (v_h) converges in $L^1(\Omega)$ to $v_\infty = \varphi \circ u_\infty$ and, by semicontinuity, that

$$\int_\Omega |D v_\infty| \leqq \liminf_{h \to +\infty} \int_\Omega |D v_h| \leqq c < +\infty.$$

We now consider the inverse function φ^{-1} of φ; note that φ^{-1} exists because $\varphi'(t) = W(t) > 0$ except for $t = \alpha, \beta$. Denoting $\tau(s) = \hat{\sigma}(\varphi^{-1}(s))$, we have that

$$|\tau(s_1) - \tau(s_2)| \leqq 2 |s_1 - s_2|$$

for every s_1, s_2 in the domain of φ^{-1}; then Proposition 1.2 yields that

$$\begin{aligned} \mathscr{E}_0(u_\infty) &= 2 \int_\Omega |D v_\infty| + \int_{\partial\Omega} \tau(\tilde{v}_\infty)\, d\mathscr{H}_{n-1} \\ &\leqq \liminf_{h \to +\infty} \left[2 \int_\Omega |D v_h| \, dx + \int_{\partial\Omega} \tau(\tilde{v}_h)\, d\mathscr{H}_{n-1} \right] = \liminf_{h \to +\infty} \mathscr{E}_0(u_h) \end{aligned}$$

and Proposition 1.4 is proved. ■

We now turn to the liquid-drop problem (P_0) by proving that the class of competing sets can be restricted to smooth sets.

1.5. Proposition. — *Suppose* $0 < m_1 < |\Omega|$ *and* $|\gamma| \leqq 1$. *If* λ *is a fixed real number such that*

$$\lambda \leqq P_\Omega(A) + \gamma \mathscr{H}_{n-1}(\partial(A \cap \Omega) \cap \partial\Omega)$$

for every open, bounded subset A *of* $\mathbb{R}^n$ *which has smooth boundary and satisfies* $\mathcal{H}_{n-1}(\partial A \cap \partial\Omega)=0$, $|A \cap \Omega|=m_1$, *then*

$$\lambda \leqq \min\{P_\Omega(E)+\gamma\mathcal{H}_{n-1}(\partial^* E \cap \partial\Omega) : E \subseteq \Omega, |E|=m_1\}.$$

Proof. — We omit the details because we closely follow the proof of the analogous result proved for the case $\gamma=0$ in Lemmas 1 and 2 of [10].

Let E_0 be the set which realizes the minimum of (P_0). By a theorem of E. Gonzalez, U. Massari and I. Tamanini ([7], Th. 1), which was stated for $\gamma=0$ but holds also in our situation because of its local character, we have that both E_0 and $\Omega\setminus E_0$ contain a non-empty open ball. Then, arguing as in Lemma 1 of [10], one can construct a sequence (E_h) of open, bounded, smooth subsets of $\mathbb{R}^n$ such that $|E_h \cap \Omega|=m_1$, $\mathcal{H}_{n-1}(\partial E_h \cap \partial\Omega)=0$ for every $h\in\mathbb{N}$, and

$$\lim_{h\to+\infty} |(E_h \cap \Omega) \triangle E_0|=0, \tag{8}$$

$$\lim_{h\to+\infty} P_\Omega(E_h)=P_\Omega(E_0), \tag{9}$$

$$\lim_{h\to+\infty} \mathcal{H}_{n-1}(\partial(E_h\cap\Omega)\cap\partial\Omega)=\mathcal{H}_{n-1}(\partial^* E_0\cap\partial\Omega). \tag{10}$$

The last assertion is not actually contained in Lemma 1 of [10] but it easily follows from (8) and from

$$\mathcal{H}_{n-1}(\partial(E_h\cap\Omega)\cap\partial\Omega)=\int_{\partial\Omega}\tilde{\chi}_{E_h\cap\Omega}\,d\mathcal{H}_{n-1},$$

$$\mathcal{H}_{n-1}(\partial^* E_0\cap\partial\Omega)=\int_{\partial\Omega}\tilde{\chi}_{E_0}\,d\mathcal{H}_{n-1},$$

where $\tilde{\chi}_T$ denotes the trace on $\partial\Omega$ of the characteristic function of T for $T=E_h\cap\Omega$ and $T=E_0$.

The proof of the proposition is now a straightforward consequence of (9) and (10). ■

The next result, stated here without proof, was proved in [10] (Lemma 4).

1.6. PROPOSITION. — *Let* A *be an open subset of* $\mathbb{R}^n$ *with smooth, non-empty, compact boundary* ∂A *such that* $\mathcal{H}_{n-1}(\partial A\cap\partial\Omega)=0$. *Define the function* $h:\mathbb{R}^n\to\mathbb{R}$ *by* $h(x)=\operatorname{dist}(x,\partial A)$ *for* $x\in A$, $h(x)=-\operatorname{dist}(x,\partial A)$ *for* $x\notin A$. *Then* h *is Lipschitz continuous,* $|Dh(x)|=1$ *for almost all* $x\in\mathbb{R}^n$, *and*

$$\lim_{t \to 0} \mathcal{H}_{n-1}(S_t \cap \Omega) = \mathcal{H}_{n-1}(\partial A \cap \Omega)$$

where $S_t = \{x \in \mathbb{R}^n : h(x) = t\}$.

2. THE MAIN RESULT

We recall that Ω denotes an open, bounded subset of $\mathbb{R}^n$ ($n \geqq 2$) with smooth boundary, and W, σ: $[0, +\infty[\to \mathbb{R}$ denote two non-negative continuous functions. We assume also that $W(t) = 0$ only for $t = \alpha$ or $t = \beta$ with $0 < \alpha < \beta$.

2.1. THEOREM. — *Fix* $m \in [\alpha|\Omega|, \beta|\Omega|]$ *and, for every* $\varepsilon > 0$, *let* u_ε *be a solution of the minimization problem* (P_ε). *If each* u_ε *is of class* C^1 *and there exists a sequence* (ε_h) *of positive numbers, converging to zero, such that* (u_{ε_h}) *converges in* $L^1(\Omega)$ *to a function* u_0, *then*

(i) $W(u_0(x)) = 0$ [*i. e.* $u_0(x) = \alpha$ *or* $u_0(x) = \beta$] *for almost all* $x \in \Omega$;

(ii) *the set* $E_0 = \{x \in \Omega : u_0(x) = \alpha\}$ *is a solution of the minimization problem* (P_0) *with*

$$\gamma = \frac{\hat{\sigma}(\alpha) - \hat{\sigma}(\beta)}{2 c_0}, \qquad m_1 = \frac{\beta|\Omega| - m}{\beta - \alpha},$$

where [see (5) *and* (6)]

$$\hat{\sigma}(t) = \inf\left\{\sigma(s) + 2\left|\int_t^s W^{1/2}(y)\, dy\right| : s \geqq 0\right\}$$

for $t = \alpha$, β, *and*

$$c_0 = \int_\alpha^\beta W^{1/2}(y)\, dy;$$

(iii) $$\lim_{h \to +\infty} \varepsilon_h^{-1} \mathcal{E}_{\varepsilon_h}(u_{\varepsilon_h})$$

$$= 2 c_0 P_\Omega(E_0) + \hat{\sigma}(\alpha) \mathcal{H}_{n-1}(\partial^* E_0 \cap \partial\Omega) + \hat{\sigma}(\beta) \mathcal{H}_{n-1}(\partial\Omega \setminus \partial^* E_0).$$

For some comments about this statement we refer to Remarks 2.5. The proof of Theorem 2.1 is similar to that one of the result with $\sigma=0$ given in [10]. Neverthless the extension is not trivial, because in the asymptotic ($\varepsilon=0$) boundary behavior, given by $\hat{\sigma}$, both the boundary and the interior behavior for $\varepsilon > 0$, given by W and σ, are involved.

In the language of Γ-convergence theory, the proof of Theorem 2.1 consists in verifying that $(\varepsilon^{-1}\mathscr{E}_\varepsilon + I_m)$ converges as $\varepsilon \to 0^+$, in the sense of $\Gamma(L^1(\Omega))$-convergence, to the functional $\mathscr{E}_0 + I_m$, at the points $u \in L^1(\Omega)$ such that $W(u(x))=0$ for almost all $x \in \Omega$ (*cf.* Section 3 in [10]). The functional $\mathscr{E}_0$ was defined in (7); I_m denotes here the $0/+\infty$ characteristic function of the constraint $\int_\Omega u(x)\,dx = m$.

The main steps in the proof of Theorem 2.1 are the following propositions.

2.2. Proposition. — *Suppose that $(v_\varepsilon)_{\varepsilon>0}$ is a family in $\{u \in C^1(\Omega) : u \geqq 0\}$ which converges in $L^1(\Omega)$ as $\varepsilon \to 0^+$ to a function v_0. If*

$$\liminf_{\varepsilon \to 0^+} \varepsilon^{-1}\mathscr{E}_\varepsilon(v_\varepsilon) < +\infty,$$

then $v_0 \in BV(\Omega)$, $W(v_0(x))=0$ for almost all $x \in \Omega$, and

$$\mathscr{E}_0(v_0) \leqq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1}\mathscr{E}_\varepsilon(v_\varepsilon). \tag{11}$$

2.3. Proposition. — *Let A be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary such that $\mathscr{H}_{n-1}(\partial A \cap \partial\Omega)=0$. Define the function $v_0 : \Omega \to \mathbb{R}$ by $v_0(x)=\alpha$ for $x \in A \cap \Omega$, $v_0(x)=\beta$ for $x \in \Omega \setminus A$. For every $r > 0$ denote*

$$U_r = \left\{ v \in H^1(\Omega) : v \geqq 0,\ \|v - v_0\|_{L^2(\Omega)} < r,\ \int_\Omega v\,dx = \int_\Omega v_0\,dx \right\}.$$

Then, for every $r > 0$, we have that

$$\limsup_{\varepsilon \to 0^+} \inf_{v \in U_r} \varepsilon^{-1}\mathscr{E}_\varepsilon(v) \leqq \mathscr{E}_0(v_0). \tag{12}$$

2.4. *Remark.* — For the connection between (12) and the corresponding inequality in the usual definition of Γ-convergence, see Proposition 1.14 of [4].

Proof of Proposition 2.2. — By the continuity of W and by Fatou's Lemma we have that

$$\int_\Omega W(v_0)\,dx \leqq \liminf_{\varepsilon\to 0^+} \int_\Omega W(v_\varepsilon)\,dx \leqq \liminf_{\varepsilon\to 0^+} \mathscr{E}_\varepsilon(v_\varepsilon)=0;$$

since $W \geqq 0$, we have at once proved that $W(v_0(x))=0$ for almost all $x\in\Omega$.

Now

$$\begin{aligned}\int_\Omega |D(\varphi\circ v_\varepsilon)| &= \int_\Omega |\varphi'(v_\varepsilon(x))|.|Dv_\varepsilon(x)|\,dx \\ &= \int_\Omega W(v_\varepsilon(x))\,|Dv_\varepsilon(x)|\,dx \\ &\leqq \int_\Omega [\varepsilon|Dv_\varepsilon|^2+\varepsilon^{-1}W(v_\varepsilon)]\,dx \leqq \varepsilon^{-1}\mathscr{E}_\varepsilon(v_\varepsilon),\end{aligned}$$

so Proposition 1.4 and $\hat{\sigma} \leqq \sigma$ apply for obtaining

$$\begin{aligned}\mathscr{E}_0(v_0) &\leqq \liminf_{\varepsilon\to 0^+} \mathscr{E}_0(v_\varepsilon) \\ &\leqq \liminf_{\varepsilon\to 0^+}\left\{\int_\Omega [\varepsilon|Dv_\varepsilon|^2+\varepsilon^{-1}W(v_\varepsilon)]\,dx \right.\\ &\qquad \left. +\int_{\delta\Omega} \hat{\sigma}(v_\varepsilon)\,d\mathscr{H}_{n-1}\right\} \leqq \liminf_{\varepsilon\to 0^+} \varepsilon^{-1}\mathscr{E}_\varepsilon(v_\varepsilon).\end{aligned}$$

It remains to prove that $v_0\in BV(\Omega)$. This is obvious because v_0 takes only the values α and β, and $\varphi\circ v_0\in BV(\Omega)$; hence the proof of Proposition 2.2 is complete. ■

Proof of Proposition 2.3. — Let us fix $r>0$ and also, for further convenience, $L \geqq 0$, $M \geqq 0$ and $\delta > 0$. We shall not often indicate in the following the dependence on r, L, M, δ as well as on the other data n, Ω, W, α, β, σ, A; in particular we shall denote by c_1, c_2, . . . real positive constants depending on all such data.

The following lemma contains a purely technical part of the proof.

2.5. LEMMA. — *Consider, for every* $\varepsilon > 0$, *the first-order ordinary differential equation*

$$|y'|=\varepsilon^{-1}(\delta+W(y))^{1/2}. \tag{13}$$

Then there exist three constants c_1, c_2, c_3, independent of ε, and a Lipschitz continuous function $\chi_\varepsilon(s, t)$, defined on the upper half-plane $\mathbb{R} \times [0, +\infty[$, satisfying the following properties:

$$\begin{aligned} \chi_\varepsilon(s, t) = \alpha \quad & \textit{for} \quad s \geqq c_1 \varepsilon, \quad t \geqq c_1 \varepsilon, \\ \chi_\varepsilon(s, t) = \beta \quad & \textit{for} \quad s \leqq 0, \quad t \geqq c_1 \varepsilon, \\ \chi_\varepsilon(s, t) = \mathrm{L} \quad & \text{for} \quad s \leqq 0, \\ \chi_\varepsilon(s, t) = \mathrm{M} \quad & \text{for} \quad s \geqq c_1 \varepsilon; \end{aligned} \tag{14}$$

$$0 \leqq \chi_\varepsilon \leqq c_2, \qquad |\mathrm{D}\chi_\varepsilon| \leqq c_3/\varepsilon; \tag{15}$$

(16) *on the strip $\{s \leqq 0,\ t \leqq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on t and fulfils the equation* (13) *in the set $\{\chi_\varepsilon(t) \neq \beta\}$; on the strip $\{s \geqq c_1 \varepsilon,\ t \leqq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on t and fulfils* (13) *in the set $\{\chi_\varepsilon(t) \neq \alpha\}$; on the strip $\{0 \leqq s \leqq c_1 \varepsilon,\ t \geqq c_1 \varepsilon\}$ the function $\chi_\varepsilon(s, t)$ depends only on s and fulfils* (13) *in the set $\{\chi_\varepsilon(s) \neq \alpha\}$.*

Proof. – We have to determine c_1, c_2, c_3 and to complete the definition of χ_ε on the strips

$$\mathrm{S}_1 = \{s \leqq 0,\ t \leqq c_1 \varepsilon\}, \qquad \mathrm{S}_2 = \{s \geqq c_1 \varepsilon,\ t \leqq c_1 \varepsilon\},$$
$$\mathrm{S}_3 = \{0 \leqq s \leqq c_1 \varepsilon,\ t \geqq c_1 \varepsilon\},$$

and on the square $\mathrm{Q} = [0, c_1 \varepsilon[\times [0, c_1 \varepsilon[$.

Let us begin by S_1, where we have the prescribed boundary values $\chi_\varepsilon(s, c_1 \varepsilon) = \beta$, $\chi_\varepsilon(s, 0) = \mathrm{L}$. If $\beta = \mathrm{L}$, we define $\chi_\varepsilon(t) = \beta$; if $\beta > \mathrm{L}$, we solve the Cauchy problem

$$y'(t) = \varepsilon^{-1} (\delta + \mathrm{W}(y(t)))^{1/2}, \qquad y(0) = \mathrm{L},$$

and we define $\chi_\varepsilon(t) = \min\{\beta, y(t)\}$; if $\beta < \mathrm{L}$, we solve the same Cauchy problem with $-y'$ instead of y' and we define $\chi_\varepsilon(t) = \max\{\beta, y(t)\}$. Since

$$|\chi'_\varepsilon(t)| = \varepsilon^{-1} (\delta + \mathrm{W}(\chi_\varepsilon(t)))^{1/2} \geqq \varepsilon^{-1} \delta^{1/2}$$

provided that $\chi_\varepsilon(t) \neq \beta$, we have $\chi_\varepsilon(t) = \beta$ for $t \geqq \varepsilon |\beta - \mathrm{L}|/\delta$; then, in order that χ_ε takes the prescribed boundary values $\chi_\varepsilon(s, c_1 \varepsilon) = \beta$, we need $c_1 \geqq |\beta - \mathrm{L}|/\delta$. The same holds on S_2 and S_3, so we are led to define

$$c_1 = \max\{|\beta - \mathrm{L}|/\delta,\ |\alpha - \beta|/\delta,\ |\alpha - \mathrm{M}|/\delta\}.$$

Define also $c_2 = \max\{\alpha, \beta, L, M\}$, so that

$$0 \leqq \chi_\varepsilon \leqq c_2$$

and

$$|D\chi_\varepsilon| \leqq \varepsilon^{-1}(\delta + \max\{W(s): 0 \leqq s \leqq c_2\})^{1/2}$$

on $(\mathbb{R} \times [0, +\infty[) \setminus Q$. Finally, as we know χ_ε on three sides of the square Q, we can extend χ_ε on Q in such a way that χ_ε becomes Lipschitz continuous on the whole upper half-plane and (15) is satisfied with

$$c_3 = 3c_1(\delta + \max\{W(s): 0 \leqq s \leqq c_2\})^{1/2}.$$

The proof of Lemma 2.5 is now complete. ■

Let us return to the proof of Proposition 2.3. The first part of the proof consists in constructing a family (v_ε) in U_r such that v_ε converges to v_0 as $\varepsilon \to 0^+$, and

$$\inf_{v \in U_r} \mathscr{E}_\varepsilon(v)$$

is approximatively equal to $\mathscr{E}_\varepsilon(v_\varepsilon)$.

Define

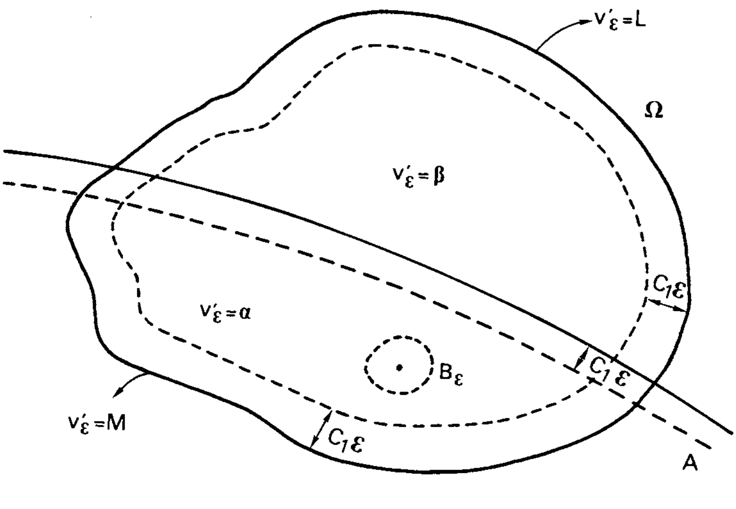

FIG. 1.

$$d_\Omega(x)=\operatorname{dist}(x, \partial\Omega), \qquad d_A(x)=\operatorname{dist}(x, \partial A) \quad \text{for } x\in A,$$
$$d_A(x)=-\operatorname{dist}(x, \partial A) \qquad \text{for} \quad x\notin A,$$

and let χ_ε be the function constructed in Lemma 2.5. Let, for $x\in\Omega$,

$$v'_\varepsilon(x)=\chi_\varepsilon(d_A(x), d_\Omega(x)).$$

Look at Figure 1 for understanding the meaning of our construction.

Denoting

$$S_s=\{x\in A\cap\Omega: d_A(x)=s\},$$
$$\Sigma_t^\alpha=\{x\in\Omega\cap A: d_\Omega(x)=t\},$$
$$\Sigma_t^\beta=\{x\in\Omega\setminus A: d_\Omega(x)=t\},$$

Federer's coarea formula and $|Dd_\Omega|=|Dd_A|=1$ (see Proposition 1.6) yield

$$\int_\Omega |v'_\varepsilon-v_0|\,dx$$
$$\leqq c_4[|\{x\in\Omega: d_\Omega(x)\leqq c_1\varepsilon\}|+|\{x\in A\cap\Omega: d_A(x)\leqq c_1\varepsilon\}|]$$
$$=c_4\int_0^{c_1\varepsilon}[\mathcal{H}_{n-1}(\Sigma_t^\alpha\cup\Sigma_t^\beta)+\mathcal{H}_{n-1}(S_t)]\,dt;$$

hence, as ∂A and $\partial\Omega$ are smooth, Proposition 1.6 implies

$$\int_\Omega |v'_\varepsilon-v_0|\,dx\leqq c_5\,\varepsilon$$

for ε small enough. It follows that v'_ε converges to v_0 in $L^1(\Omega)$ as $\varepsilon\to 0^+$ and, defining

$$\eta_\varepsilon=\int_\Omega v'_\varepsilon\,dx-\int_\Omega v_0\,dx,$$

we have that

$$|\eta_\varepsilon|\leqq c_5\,\varepsilon \tag{17}$$

for ε small enough.

Let us choose a point $x_0\in\Omega\setminus\partial A$ and, for fixing the ideas, assume that $x_0\in\Omega\cap A$. In the case $\Omega\cap A=\varnothing$ or $x_0\in\Omega\setminus A$ the changes in the proof are trivial. Note that the closed ball $B_\varepsilon = B(x_0, \varepsilon^{1/n})$ is contained, for ε small enough, in the set $\{v'_\varepsilon = \alpha\}$; then the function v_ε, defined on Ω by $v_\varepsilon = v'_\varepsilon$ for $x \notin B_\varepsilon$, and by

$$v_\varepsilon(x) = \alpha + h_\varepsilon (1 - \varepsilon^{-1/n} |x - x_0|),$$

for $x \in B_\varepsilon$, is Lipschitz continuous whenever $h_\varepsilon \in \mathbb{R}$.

We now choose

$$h_\varepsilon = -n\,\omega_{n-1}^{-1}\,\eta_\varepsilon\,\varepsilon^{(1-n)/n},$$

with ω_{n-1} equal to the volume of the unit ball in $\mathbb{R}^{n-1}$, so that

$$\int_{B_\varepsilon} (v_\varepsilon - v'_\varepsilon)\,dx = \int_{B_\varepsilon} h_\varepsilon (1 - \varepsilon^{-1/n} |x - x_0|)\,dx = -\eta_\varepsilon,$$

and, by the definition of η_ε and v_ε,

$$\int_{B_\varepsilon} v_\varepsilon\,dx = \int_{B_\varepsilon} v_0\,dx \tag{18}$$

for ε small enough. Since, by (17),

$$|h_\varepsilon| \leqq c_6\,\varepsilon^{1/n}, \tag{19}$$

we have, for ε small enough,

$$0 \leqq v_\varepsilon \leqq c_7, \tag{20}$$

and

$$\lim_{\varepsilon \to 0^+} \int_\Omega |v_\varepsilon - v_0|^2\,dx = 0; \tag{21}$$

hence

$$\lim_{\varepsilon \to 0^+} \inf_{v \in U_r} \varepsilon^{-1} \mathscr{E}_\varepsilon(v) \leqq \limsup_{\varepsilon \to 0^+} \varepsilon^{-1} \mathscr{E}_\varepsilon(v_\varepsilon). \tag{22}$$

The second part of the proof consists in a sharp estimate of the right-hand side of such inequality. For the sake of simplicity, let

$$\varepsilon^{-1} \mathscr{E}_\varepsilon(v_\varepsilon) = \mathscr{E}'_\varepsilon(v_\varepsilon; \Omega) + \mathscr{E}''_\varepsilon(v_\varepsilon)$$

with

$$\mathscr{E}'_\varepsilon(v_\varepsilon;\ C)=\int_C [\varepsilon\,|\,Dv_\varepsilon|^2+\varepsilon^{-1}\,W(v_\varepsilon)]\,dx \qquad (C\subseteqq\Omega),$$

and

$$\mathscr{E}''_\varepsilon(v_\varepsilon)=\int_{\delta\Omega}\sigma(\tilde{v}_\varepsilon)\,d\mathscr{H}_{n-1}.$$

By (20) and (21), and by the continuity of σ and of the trace operator, we at once obtain

$$\limsup_{\varepsilon\to 0^+}\mathscr{E}''_\varepsilon(V_\varepsilon)\leqq\int_{\delta\Omega}\sigma(\tilde{v}_0)\,d\mathscr{H}_{n-1}$$
$$=\sigma(L)\,\mathscr{H}_{n-1}(\partial\Omega\setminus A)+\sigma(M)\,\mathscr{H}_{n-1}(\partial\Omega\cap A). \quad (23)$$

The evaluation of $\mathscr{E}'_\varepsilon(v_\varepsilon;\ \Omega)$ is more complicated. Let us divide Ω in seven parts, corresponding to the construction of χ_ε in Lemma 2.5 and of v_ε (see *Fig.* 1) :

$$B_\varepsilon=B(x_0,\ \varepsilon^{1/n}),$$
$$\Omega^\varepsilon_\alpha=\{\,x\in\Omega: d_A(x)>c_1\,\varepsilon,\ d_\Omega(x)>c_1\,\varepsilon,\ x\notin B_\varepsilon\,\},$$
$$\Omega^\varepsilon_\beta=\{\,x\in\Omega: d_A(x)\leqq 0;\ d_\Omega(x)>c_1\,\varepsilon\,\},$$
$$\Omega^\varepsilon_{\alpha\beta}=\{\,x\in\Omega: 0<d_A(x)\leqq c_1\,\varepsilon,\ d_\Omega(x)>c_1\,\varepsilon\,\},$$
$$\Omega^\varepsilon_{\beta\,L}=\{\,x\in\Omega: d_A(x)\leqq 0,\ d_\Omega(x)\leqq c_1\,\varepsilon\,\},$$
$$\Omega^\varepsilon_{\alpha\,M}=\{\,x\in\Omega: d_A(x)>c_1\,\varepsilon,\ d_\Omega(x)\leqq c_1\,\varepsilon\,\},$$
$$\Omega^\varepsilon_0=\{\,x\in\Omega: 0<d_A(x)\leqq c_1\,\varepsilon,\ d_\Omega(x)\leqq c_1\,\varepsilon\,\}.$$

On B_ε we have, by (19),

$$\mathscr{E}'_\varepsilon(v_\varepsilon;\ B_\varepsilon)$$
$$=\varepsilon\,|\,h_\varepsilon|^2\,\varepsilon^{-2/n}\,|\,B_\varepsilon|+\varepsilon^{-1}\int_{B_\varepsilon}W(\alpha+h_\varepsilon(1-\varepsilon^{-1/n}\,|\,x-x_0\,|))\,dx$$
$$\leqq c_7\left[\varepsilon^2+\int_0^1 W(\alpha+h_\varepsilon(1-r))\,r^{n-1}\,dr\right];$$

hence

$$\limsup_{\varepsilon \to 0^+} \mathscr{E}'_\varepsilon(v_\varepsilon;\, B_\varepsilon) = 0. \tag{24}$$

On $\Omega^\varepsilon_\alpha$ and Ω^ε_β the function v_ε equals respectively α and β, so that

$$\mathscr{E}'_\varepsilon(v_\varepsilon;\, \Omega^\varepsilon_\alpha) + \mathscr{E}'_\varepsilon(v_\varepsilon;\, \Omega^\varepsilon_\beta) = 0. \tag{25}$$

On $\Omega^\varepsilon_{\alpha\beta}$ we have $v_\varepsilon(x) = \chi_\varepsilon(d_A(x),\, d_\Omega(x))$; moreover, by (16), $\chi_\varepsilon(s, t) = \chi_\varepsilon(s)$ depends only on the first variable and satisfies the equation

$$-\chi'_\varepsilon(s) = \varepsilon^{-1}(\delta + W(\chi_\varepsilon(s)))^{1/2}$$

on an interval $]0, \tau_\varepsilon[$, with $0 < \tau_\varepsilon < c_1 \varepsilon$, while $\chi_\varepsilon(s) = \alpha$ for $s \geqq \tau_\varepsilon$. Then, applying Federer's coarea formula and $\chi_\varepsilon(0) = \beta$, we obtain that

$$\begin{aligned}
\mathscr{E}'_\varepsilon(v_\varepsilon;\, \Omega^\varepsilon_{\alpha\beta}) &= \int_0^{\tau_\varepsilon} [\varepsilon \chi'^2_\varepsilon(s) + \varepsilon^{-1} W(\chi_\varepsilon(s))]\, \mathscr{H}_{n-1}(S_s)\, ds \\
&\leqq (\sup_{0 \leqq s \leqq \tau_\varepsilon} \mathscr{H}_{n-1}(S_s)] \int_0^{\tau_\varepsilon} 2(-\chi'_\varepsilon)(\delta + W(\chi_\varepsilon))^{1/2}\, ds \\
&= (\sup_{0 \leqq s \leqq \tau_\varepsilon} \mathscr{H}_{n-1}(S_s)) \left(2 \int_\alpha^\beta (\delta + W(t))^{1/2}\, dt \right),
\end{aligned}$$

and therefore, by Proposition 1.6,

$$\limsup_{\varepsilon \to 0^+} \mathscr{E}'_\varepsilon(v_\varepsilon;\, \Omega^\varepsilon_{\alpha\beta}) \leqq 2\, \mathscr{H}_{n-1}(\partial A \cap \Omega) \int_\alpha^\beta (\delta + W(t))^{1/2}\, dt. \tag{26}$$

The same argument leads to

$$\limsup_{\varepsilon \to 0^+} \mathscr{E}'_\varepsilon(v_\varepsilon;\, \Omega^\varepsilon_{\beta\, L}) \leqq 2\, \mathscr{H}_{n-1}(\partial\Omega \cap A) \left| \int_\beta^L (\delta + W(t))^{1/2}\, dt \right|, \tag{27}$$

and to

$$\limsup_{\varepsilon \to 0^+} \mathscr{E}'_\varepsilon(v_\varepsilon;\, \Omega^\varepsilon_{\alpha\, M}) \leqq 2\, \mathscr{H}_{n-1}(\partial\Omega \cap A) \left| \int_\alpha^M (\delta + W(t))^{1/2}\, dt \right|. \tag{28}$$

Finally, on Ω^ε_0 we have, by (15),

$$\mathscr{E}'_\varepsilon(v_\varepsilon;\, \Omega^\varepsilon_0) \leqq c_8\, \varepsilon^{-1} |\Omega^\varepsilon_0|.$$

Note that, again by coarea formula,

$$|\Omega_0^\varepsilon| = \int_0^{c_1\varepsilon} \mathcal{H}_{n-1}\Big(\{x\in\Omega : d_A(x)=s,\ d_\Omega(x)\leqq c_1\varepsilon\}\Big)\,ds$$
$$\leqq c_1\big(\sup_{0\leqq s\leqq c_1\varepsilon} \mathcal{H}_{n-1}(S_s\setminus\Omega_{c_1\varepsilon})\big),$$

where Ω_ρ denotes here the set $\{x\in\Omega : d_\Omega(x)>\rho\}$. Since we have $\mathcal{H}_{n-1}(\partial A\cap\partial\Omega_\rho)=0$ for almost all $\rho>0$, Proposition 1.6 gives

$$\limsup_{\varepsilon\to 0^+}\big(\sup_{0\leqq s\leqq c_1\varepsilon}\mathcal{H}_{n-1}(S_s\setminus\Omega_{c_1\varepsilon})\big)$$
$$\leqq \limsup_{\varepsilon\to 0^+}\big(\sup_{0\leqq s\leqq c_1\varepsilon}\mathcal{H}_{n-1}(S_s\setminus\Omega_\rho)$$
$$=\mathcal{H}_{n-1}(\partial A\cap\partial(\Omega\setminus\Omega p))$$

for almost all $\rho>0$; by taking the infimum for $\rho>0$, we conclude that

$$\limsup_{\varepsilon\to 0^+}\mathcal{E}'_\varepsilon(v_\varepsilon;\ \Omega_0^\varepsilon)=0. \tag{29}$$

Now, by collecting (22) to (29), we have that

$$\limsup_{\varepsilon\to 0^+}\ \inf_{v\in U_r}\ \varepsilon^{-1}\mathcal{E}_\varepsilon(v)\leqq 2\mathcal{H}_{n-1}(\partial A\cap\Omega)\int_\alpha^\beta(\delta+W(t))^{1/2}\,dt$$
$$+\mathcal{H}_{n-1}(\partial\Omega\cap A)\left(2\left|\int_\alpha^M(\delta+W(t))^{1/2}\,dt\right|+\sigma(M)\right)$$
$$+\mathcal{H}_{n-1}(\partial\Omega\cap A)\left(2\left|\int_\beta^L(\delta+W(t))^{1/2}\,dt\right|+\sigma(L)\right).$$

The left-hand side does not depend on δ, L, and M, so, by taking first the infimum for $\delta>0$, and then the infima for $M\geqq 0$ and for $L\geqq 0$ of the right-hand side, we obtain, by the definition of $\hat\sigma$ and c_0, that

$$\limsup_{\varepsilon\to 0^+}\ \inf_{v\in U_r}\ \varepsilon^{-1}\mathcal{E}_\varepsilon(v)$$
$$\leqq 2c_0\mathcal{H}_{n-1}(\partial A\cap\Omega)+\hat\sigma(\alpha)\mathcal{H}_{n-1}(\partial\Omega\cap A)$$
$$+\hat\sigma(\beta)\mathcal{H}_{n-1}(\partial\Omega\setminus A)$$
$$=2c_0\mathcal{H}_{n-1}(\partial A\cap\Omega)+\int_{\delta\Omega}\hat\sigma(\tilde v_0)\,d\mathcal{H}_{n-1}. \tag{30}$$

Remarking that the Fleming-Rishel formula yields

$$2\int_{\Omega}|D(\varphi\circ v_0)|=2\int_{\mathbb{R}}P_{\Omega}(\{x\in\Omega:\varphi(v_0(x))>t\})\,dt$$

$$=2\int_{\varphi(\alpha)}^{\varphi(\beta)}P_{\Omega}(A\cap\Omega)\,dt=2c_0\mathscr{H}_{n-1}(\partial A\cap\Omega),\quad(31)$$

the right-hand side of (30) agrees with $\mathscr{E}_0(v_0)$ and the proof of Proposition 2.3 is complete. ■

Now, we can prove Theorem 2.1.

Proof of Theorem 2.1. — Assume for simplicity that all (u_ε) converges, as $\varepsilon\to 0^+$, to u_0. By constructing, as in the proof of Theorem I of [10], a suitable family of comparison piecewise affine functions, we first obtain that

$$\liminf_{\varepsilon\to 0^+}\varepsilon^{-1}\mathscr{E}_\varepsilon(u_\varepsilon)<+\infty;\quad(32)$$

hence Proposition 2.2 gives $W(u_0(x))=0$ and

$$\mathscr{E}_0(u_0)\leqq\liminf_{\varepsilon\to 0^+}\varepsilon^{-1}\mathscr{E}_\varepsilon(u_\varepsilon).$$

Now, let $\mathscr{A}$ be the class of all open, bounded subsets A of $\mathbb{R}^n$, with smooth boundary, such that $\mathscr{H}_{n-1}(\partial A\cap\partial\Omega)=0$ and $|A\cap\Omega|=|E_0|=m_1$. For every $A\in\mathscr{A}$, we define $v_0^A(x)=\alpha$ for $x\in A\cap\Omega$, $v_0^A(x)=\beta$ for $x\in\Omega\setminus A$; applying Proposition 2.3 with $r=1$, we infer that

$$\limsup_{\varepsilon\to 0^+}\inf_{v\in U}\varepsilon^{-1}\mathscr{E}_\varepsilon(v)\leqq\mathscr{E}_0(v_0^A),$$

where

$$U=\left\{v\in H^1(\Omega):v\geqq 0,\int_{\Omega}|v-v_0^A|^2\,dx<1,\int_{\Omega}v\,dx=\int_{\Omega}v_0^A\,dx\right\}$$

Since

$$\int_{\Omega}v_0^A\,dx=m,$$

we have, by the minimality of u_ε, that

$$\mathscr{E}_\varepsilon(u_\varepsilon) \leqq \mathscr{E}_\varepsilon(v), \qquad \forall v \in \mathrm{U},$$

and we conclude that

$$\mathscr{E}_0(u_0) \leqq \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \mathscr{E}_\varepsilon(u_\varepsilon) \leqq \limsup_{\varepsilon \to 0^+} \varepsilon^{-1} \mathscr{E}_\varepsilon(u_\varepsilon) \leqq \mathscr{E}_0(v_0^{\mathrm{A}}) \qquad (33).$$

for every $\mathrm{A} \in \mathscr{A}$. Arguing as for (30) and (31), we obtain

$$\mathscr{E}_0(u_0) = 2 c_0 \mathrm{P}_\Omega(\mathrm{E}_0) + \hat{\sigma}(\alpha) \mathscr{H}_{n-1}(\partial^* \mathrm{E}_0 \cap \partial\Omega) + \hat{\sigma}(\beta) \mathscr{H}_{n-1}(\partial\Omega \setminus \partial^* \mathrm{E}_0) \quad (34)$$

and

$$\mathscr{E}_0(v_0^{\mathrm{A}}) = 2 c_0 \mathrm{P}_\Omega(\mathrm{A}) + \hat{\sigma}(\alpha) \mathscr{H}_{n-1}(\partial\Omega \cap \mathrm{A}) + \hat{\sigma}(\beta) \mathscr{H}_{n-1}(\partial\Omega \setminus \mathrm{A}),$$

so that

$$\mathrm{P}_\Omega(\mathrm{E}_0) + \gamma \mathscr{H}_{n-1}(\partial^* \mathrm{E}_0 \cap \partial\Omega) \leqq \mathrm{P}_\Omega(\mathrm{A}) + \gamma \mathscr{H}_{n-1}(\partial(\mathrm{A} \cap \Omega) \cap \partial\Omega)$$

for every $\mathrm{A} \in \mathscr{A}$. Then the required minimality property (ii) of E_0 follows from Proposition 1.5. Finally, by employing again (33) and Proposition 1.5, with

$$\lambda = \limsup_{\varepsilon \to 0^+} \varepsilon^{-1} \mathscr{E}_\varepsilon(u_\varepsilon),$$

we have that

$$\mathscr{E}_0(u_0) = \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathscr{E}_\varepsilon(u_\varepsilon);$$

hence the result (iii) follows from (34) and this concludes the proof of Theorem 2.1. ■

2.5. *Remarks.* — (*a*) The assumption that $\partial\Omega$ is smooth in Theorem 2.1 cannot be easily replaced by $\partial\Omega$ Lipschitz continuous, except for $\sigma = 0$ (*cf.* [10]). In fact, as we already observed in Remark 1.3, the liquid-drop problem (P_0) in bounded domains with angles requires a particular treatment.

(*b*) Well-known growth conditions at infinity on W guarantee that the minimizers u_ε are of class C^1. Of course, if $u_\varepsilon \in \mathrm{L}^\infty(\Omega)$, then u_ε is smooth.

(c) The (relative) compactness of (u_ε) in $L^1(\Omega)$ may be studied as in Proposition 4 of [10]. It is ensured either by equiboundedness of (u_ε) (cf. [9]), or again by a growth condition at infinity on W.

3. A DISCUSSION ABOUT CRITICAL POINT WETTING

We make here more precise some statements of Introduction, about the connection between Theorem 2.1 and the critical point wetting theory by J. W. Cahn [2].

According to this author, and looking in particular at page 3668 and Figure 4 of [2], we assume that the contact energy σ is a non-negative, convex, decreasing function of class C^1. Moreover we denote by W_T the Gibbs free energy at the temperature T (recall that we are concerned with isothermal phenomena), by α_T and β_T the corresponding zeros, by M_T the maximum height of the hump between α_T and β_T. We assume that $W_T(t)$ increases for $t \geqq \beta_T$. By thermodynamic and experimental reasons (cf. [2], page 3669), we assume also that β_T and M_T are decreasing in T, α_T is increasing in T and $(\beta_T - \alpha_T) \to 0$, $M_T \to 0$ when T increases towards a critical temperature T_0 (critical point of a binary system). The φ and $\hat{\sigma}$ corresponding to σ and W_T will be denoted by φ_T and $\hat{\sigma}_T$.

Let us compute now $\hat{\sigma}_T(t)$ for $t \geqq \alpha_T$. Since σ is decreasing and

$$\lim_{t \to +\infty} \varphi_T(t) = +\infty,$$

we obtain that the minimum of $s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|$ is attained at a point $s = \lambda_{t,T} \geqq t$. Moreover, either $\lambda_{t,T} = t$, or

$$-\sigma'(\lambda_{t,T}) = 2\varphi'(\lambda_{t,T}) = 2W^{1/2}(\lambda_{t,T}).$$

For T_0-T small enough, that is for a temperature T below and close to the critical one, the hump in the graph of $2W_T^{1/2}$ between α_T and β_T does not intersect the graph of $-\sigma'$ in the same interval; on the other hand, since σ is convex, the decreasing function $-\sigma'$ does intersect the increasing function $2W_T^{1/2}$ at a single point $\lambda_T \geqq \beta_T$ (see Fig. 2).

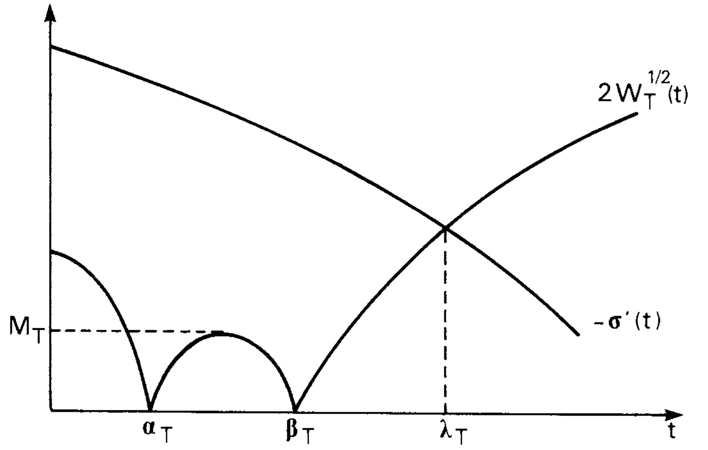

FIG. 2

It is easy to check that λ_T (independent of t) is actually the minimum point of $s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|$; hence we conclude that

$$\hat{\sigma}_T(t) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(t)), \quad \forall t \geqq \alpha_T;$$

hence

$$\gamma_T = \frac{\hat{\sigma}_T(\alpha_T) - \hat{\sigma}_T(\beta_T)}{2(\varphi_T(\beta_T) - \varphi_T(\alpha_T))} = 1$$

in correspondence with the phenomenon of the perfectly wetting phase β quoted in Introduction. If one prefers not to consider the modified energy $\hat{\sigma}_T$, it could be alternatively thought that a very thin layer of a third phase of the fluid, with density $\lambda_T > \beta_T$, appears on the whole boundary of the container.

When the temperature T is much more below T_0, a possible relative behavior of $-\sigma'$ and $2W^{1/2}$ is shown in Figure 3, with both μ_T and λ_T relative minima of

$$s \mapsto \sigma(s) + 2|\varphi_T(t) - \varphi_T(s)|$$

for every $t \geqq \alpha_T$.

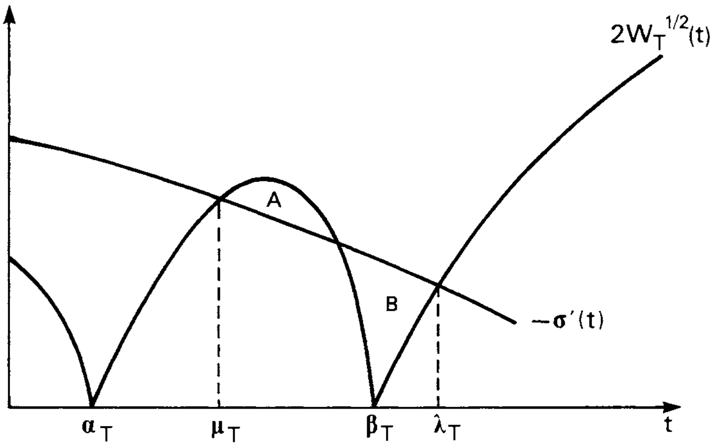

FIG. 3

Note that

$$\hat{\sigma}_T(\beta_T) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\beta_T)),$$

while the value of $\sigma_T(\alpha_T)$ depends on the areas A and B. Indeed, if $A \leqq B$, then

$$\hat{\sigma}_T(\alpha_T) = \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\alpha_T))$$

and $\gamma_T = 1$ as above. On the contrary, if $A > B$, then

$$\hat{\sigma}_T(\alpha_T) = \sigma(\mu_T) + 2(\varphi_T(\mu_T) - \varphi_T(\alpha_T)) < \sigma(\lambda_T) + 2(\varphi_T(\lambda_T) - \varphi_T(\alpha_T))$$

and $\gamma_T < 1$; since we have analogously $\gamma_T > -1$, this means that both the fluid phases wet the container walls. Or, alternatively, two thin layers of fluid, with densities μ_T and λ_T, are interposed between the phases α_T and β_T and the container.

Finally, we want to remark that the equation $\hat{\sigma} = \sigma$ is equivalent to the inequality

$$|\sigma(s_1) - \sigma(s_2)| \leqq 2 |\varphi(s_1) - \varphi(s_2)|, \quad \forall\, 0 \leqq s_1 \leqq s_2, \tag{35}$$

which gives in particular

$$\sigma'(\alpha) \geqq \varphi'(\alpha) = W^{1/2}(\alpha) = 0$$

and analogously $\sigma'(\beta) \geqq 0$; hence (35) cannot be satisfied in the case $\sigma' < 0$. It would be interesting to know whether the inequality (35), and then the equality $\sigma = \hat{\sigma}$, are verified in some other thermodynamic situation, different from the phenomenon studied in [2] by Cahn.

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