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Critical points of convex perturbations of some indefinite quadratic forms and semi-linear boundary value problems at resonance

by

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ABSTRACT. — Critical points of convex perturbations of indefinite quadratic forms are obtained from the dual least action principle. The main result leads to necessary and sufficient conditions for the existence of a critical point when the corresponding Euler equation is scalar or when the perturbation is strictly convex. Applications are given to periodic solution of hamiltonian systems and to systems of semi-linear beam equations. A global version of the averaging method is given.

Mots-clés : Dual least action principle, periodic solutions, hamiltonian systems, semi-linear beam equations, averaging method.

RÉSUMÉ. — On prouve l'existence de points critiques de perturbations convexes de formes quadratiques indéfinies par le principe de moindre action duale. Si l'équation d'Euler est scalaire ou si la perturbation est strictement convexe, on obtient des conditions nécessaires et suffisantes pour l'existence d'un point critique. Des applications sont données aux solutions périodiques de systèmes hamiltoniens et à des systèmes d'équations de poutre semi-linéaires. On obtient une version globale de la méthode de la moyenne.

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1. INTRODUCTION

The dual least action principle of Clarke-Ekeland [*9*] (see also [*8*] for a general formulation and other applications) has been used by Mawhin, Willem and Ward [*13*] [*15*] [*16*] [*17*] to obtain necessary and sufficient conditions for the existence of solutions for the (scalar) Neumann problem

$$\Delta u + f(x, u) = 0 \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

and the (scalar) Dirichlet problem

$$u'' + u + f(x, u) = 0 \quad \text{in } [0, \pi]$$
$$u(0) = u(\pi) = 0\,,$$

when $f(x, .)$ is nondecreasing and $F(x, .) = \int_0^{\cdot} f(x, v)dv$ satisfies a suitable asymptotic quadratic growth condition.

The aim of this paper is to state and prove an existence result for semilinear equations, with *non-invertible* linear part and *convex* potential $F(x, .)$,

$$\text{(1)} \qquad Lu = \nabla F(x, u) \qquad (\text{with} \quad \nabla F = D_u F)$$

in a closed subspace V of $L^2(\Omega, \mathbb{R}^N)$, where Ω is a bounded domain of $\mathbb{R}^m$. The linear self-adjoint operator $L : D(L) \subset V \to V$ and the nonlinear potential $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ must satisfy some regularity assumptions. Moreover, F is subjected to some asymptotic conditions, namely $F(x, u)$ is strictly dominated at infinity by $\lambda_1 |u|^2/2$, with λ_1 the first positive eigenvalue of L, and the mapping

$$w \mapsto \int_\Omega F(x, w(x))dx$$

is coercive on ker L. This last condition is of the type introduced by Ahmad-Lazer-Paul [*2*] in variational elliptic problems with bounded nonlinearity and then used by Rabinowitz [*19*] and others. In contrast with those results, we allow unbounded nonlinearities, and can express our assumptions on the nonlinearity in terms of F and not of ∇F, which makes the verification of a Palais-Smale condition very unlikely. On the other hand, we must restrict ourself to the case of a convex F.

Section 2 is devoted to some preliminary results on the positive-definiteness of some quadratic forms and, in Section 3, we first study like in [*8*] the solvability of a *perturbed problem* and show how the existence of a solution of the original one is ensured by the obtention of *a posteriori* *estimates* on the solutions of the perturbed problem found in the first step. Explicit conditions for having those *a posteriori* estimates are then given in Section 4 and lead to our basic existence result (Theorem 1). Notice that ker L needs not to be finite-dimensional and L needs not to have a compact resolvant so that Theorem 1 covers not only systems of ordinary or elliptic partial differential equations, but also some hyperbolic problems. This is shown in Section 5 where we give applications to periodic solutions of Hamiltonian systems and to systems of semi-linear beam equations, generalizing earlier results of Bahri and Sanchez [*4*].

In the case of scalar equations, we show in Section 5 how to deduce from Theorem 1 a necessary and sufficient condition for the existence of a solution (Theorem 2) which contains as special cases the results on the Neumann and Dirichlet problems with resonance at the second eigenvalue given in [*13*] [*14*] [*15*] [*16*] [*17*]. Finally, when $F(x, .)$ is strictly convex and dim ker L is finite, we obtain in Section 6 necessary and sufficient conditions of existence in the line of those given by Berger and Schechter [*6*] in a narrower setting and with unnecessary regularity assumptions.

Notice that parts of the results of the present paper have been anounced in [*14*].

2. SOME LEMMAS FOR QUADRATIC FORMS

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, V a closed subspace of $L^2(\Omega, \mathbb{R}^N)$ with the usual inner product

$$(u, v) = \int_\Omega (u(x) \mid v(x))dx$$

($(.\,|\,.)$ denotes the inner product in $\mathbb{R}^N$) and the corresponding norm $\| u \| = (u, u)^{1/2}$. Let $L : D(L) \subset V \to V$ be a linear self-adjoint operator with closed range, so that $V = \ker L \oplus R(L)$ (orthogonal direct sum). We make the following assumptions upon the spectrum $\sigma(L)$ of L:

(S_1) $0 \in \sigma(L)$

(S_2) $\sigma(L) \cap \,]0, +\infty[\, \neq \phi$ and consists of isolated eigenvalues having finite multiplicity.

Let us denote by λ_1 the smallest positive eigenvalue of L and by $K : R(L) \to R(L)$ the right inverse of $-L$ defined by

$$K = (-L_{|D(L)\cap R(L)})^{-1}\,.$$

Thus by the closed graph theorem K is a bounded linear operator on R(L) and $\mu \in \sigma(K)\backslash\{0\}$ if and only if $-1/\mu \in \sigma(L)$. Let $\{P_\lambda : \lambda \in \mathbb{R}\}$ be the spectral resolution of

$$-L,\ P^- = \int_{-\infty}^{-\lambda_1/2} dP_\lambda,\ P^+ = \int_{-\lambda_1/2}^{+\infty} dP_\lambda,\ H^- = P^-(R(L)),\ H^+ = P^+(R(L)).$$

Then, P^- and P^+ are orthogonal projectors, $R(L) = H^- \oplus H^+$ (orthogonal direct sum), $KH^{\mp} \subset H^{\mp}$, KP^+ is semi-positive definite on R(L) and, by assumption (S_2), KP^- is compact on R(L). K being self-adjoint, it follows also from (S_2) that

$$(2) \qquad (Kv, v) \geqslant -(1/\lambda_1)\|v\|^2$$

for all $v \in R(L)$ and hence the quadratic form defined by

$$\gamma : v \mapsto \frac{1}{2}\left[(Kv, v) + \int_\Omega [|v(x)|^2/\alpha]dx\right]$$

will be strongly positive definite whenever

$$0 < \alpha < \lambda_1 .$$

This simple result is generalized in the following

LEMMA 1. — *Let $\alpha \in L^\infty(\Omega)$ be such that $\inf\operatorname{ess}_\Omega \alpha > 0$ and the quadratic form γ defined by*

$$(3) \qquad \gamma(v) = \frac{1}{2}\left[(Kv, v) + \int_\Omega (|v(x)|^2/\alpha(x))dx\right]$$

is positive definite on R(L). *Then there exists $\delta > 0$ such that*

$$\gamma(v) \geqslant \delta \|v\|^2$$

whenever $v \in R(L)$.

Proof. — If it is not the case, it follows from assumptions on K that we can find a sequence (v_k) in R(L), with $\|v_k\| = 1$ $(k \in \mathbb{N}^*)$, and some $v_0 \in R(L)$ such that

$$(4) \qquad \begin{aligned} 0 \leqslant \gamma(v_k) \leqslant \frac{1}{k} \qquad (k \in \mathbb{N}^*) \\ v_k \rightharpoonup v_0 \\ KP^-v_k \to KP^-v \end{aligned}$$

as $k \to \infty$. We can write

$$\begin{aligned} \gamma(v) &= \frac{1}{2}\left[(KP^-v, v) + (KP^+v, v) + \int_\Omega [|v(x)|^2/\alpha(x)]dx\right] = \\ &= \gamma_1(v) + \gamma_2(v) + \gamma_3(v), \qquad v \in R(L) \end{aligned}$$

and, by (4),

$$(5) \qquad \gamma_2(v_k) + \gamma_3(v_k) \to -\gamma_1(v_0) = -\frac{1}{2}((KP^-v_0, v_0)),$$

as $k \to \infty$. Now, γ_2 and γ_3, convex and continuous, are weakly lower semi-continuous and hence the same is true for $\gamma_2 + \gamma_3$. Consequently, by (5),

$$\gamma_2(v_0) + \gamma_3(v_0) \leqslant -\gamma_1(v_0),$$

i. e.

$$\gamma(v_0) \leqslant 0$$

which, by the positive-definiteness of γ, implies that

$$v_0 = 0$$

and hence, by (5) and the non-negativeness of γ_2 and γ_3,

$$\gamma_j(v_k) \to 0 \qquad (j = 2, 3).$$

As, for all $k \in \mathbb{N}^*$, we have

$$\gamma_3(v_k) = \int_\Omega [\,|v_k(x)|^2/\alpha(x)\,]dx \geqslant \|v_k\|^2/\sup_\Omega \operatorname{ess} \alpha = (1/\sup_\Omega \operatorname{ess} \alpha) > 0$$

we obtain a contradiction.

We now prove a sufficient condition for γ to be positive definite on R(L).

LEMMA 2. — *Let* $\alpha \in L^\infty(\Omega)$ *with* $\operatorname{ess\,inf}_\Omega \alpha > 0$ *be such that*

(6) $$\alpha(x) \leqslant \lambda_1$$

for a. e. $x \in \Omega$ *and*

(7) $$\int_\Omega (\lambda_1 - \alpha(x))\,|v^1(x)|^2 dx > 0$$

for all $0 \neq v^1 \in \ker(K + \lambda_1^{-1} I) = \ker(L - \lambda_1 I)$. *Then the quadratic form* γ *defined by (3) is positive definite on* R(L).

Proof. — It follows from (2) and (6) that

$$\gamma(v) \geqslant \frac{1}{2}\left[(Kv, v) + \lambda_1^{-1}\|v\|^2\right] \geqslant 0$$

for all $v \in R(L)$. Therefore, if $\gamma(v) = 0$, then

(8) $$(Kv + \lambda_1^{-1}v, v) = 0.$$

Let $v = v^1 + v^2$, with $v^1 \in \ker(K + \lambda_1^{-1}I)$ and $v^2 \perp \ker(K + \lambda_1^{-1}I)$. Then (8) implies

(9) $$(Kv^2 + \lambda_1^{-1}v^2, v^2) = 0.$$

Let $\lambda_2 > \lambda_1$ be the next element of $\sigma(L) \cap]0, \infty[$ (with $\lambda_2 = +\infty$ if $\sigma(L) \cap]0, \infty[= \{\lambda_1\}$). Then,

$$(Kw, w) \geqslant -(1/\lambda_2)\|w\|^2$$

for all $w \in [\ker(K + \lambda_1^{-1})]^\perp$ and hence, by (9),

$$0 \geqslant \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \| v^2 \|^2$$

so that $v^2 = 0$. Thus $v = v^1 \in \ker(K + \lambda_1^{-1} I)$ and

$$0 = \gamma(v_1) = \int_\Omega \left(\frac{1}{\alpha(x)} - \frac{1}{\lambda_1}\right) | v^1(x) |^2 dx = \int_\Omega \frac{[\lambda_1 - \alpha(x)]}{\alpha(x)\lambda_1} | v^1(x) |^2 dx .$$

Thus, if $c_1 = \lambda_1 \operatorname{ess\,sup}_\Omega \alpha$, $c_2 = \lambda_1 \operatorname{ess\,inf}_\Omega \alpha$, we get

$$\frac{1}{c_1} \int_\Omega (\lambda_1 - \alpha(x)) | v^1(x) |^2 dx \leqslant 0 \leqslant \frac{1}{c_2} \int_\Omega (\lambda_1 - \alpha(x)) | v^1(x) |^2 dx$$

which by (7) implies that $v^1 = 0$ and completes the proof.

REMARK 1. — If nontrivial elements of $\ker(L - \lambda_1 I)$ have the unique continuation property (i. e. if they vanish at most a subset of measure zero of Ω) then, when (6) holds, (7) is obviously equivalent to $\alpha(x) < \lambda_1$ *on a subset of* Ω *with positive measure.*

3. *A POSTERIORI* ESTIMATES ON SOLUTIONS OF A PERTURBED PROBLEM AND THEIR LINK WITH THE SOLVABILITY OF (1)

Let now $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $(x, u) \mapsto F(x, u)$ be such that $F(x, .)$ is continuous and convex for a. e. $x \in \Omega$ and $F(., u)$ is measurable for each $u \in \mathbb{R}^N$. Assume moreover that there exist $\beta \in L^2(\Omega ; \mathbb{R}_+)$ and $l \in L^2(\Omega, \mathbb{R}^N)$ such that

(10) $$F(x, u) \geqslant (l(x) \mid u) - \beta(x)$$

for a. e. $x \in \Omega$ and all $u \in \mathbb{R}^N$. Finally, let us assume that $\nabla F = (D_{u_1}F, \ldots, D_{u_N}F)$ exists for a. e. $x \in \Omega$ and all $u \in \mathbb{R}^N$, and is such that $\nabla F(., u(.)) \in V$ whenever $u \in D(L)$.

For $\varepsilon > 0$, we define F_ε by

$$F_\varepsilon(x, u) = (\varepsilon/2) | u |^2 + F(x, u)$$

and the Legendre (or Fenchel) transform $F_\varepsilon^*(x, .)$ of $F_\varepsilon(x, .)$ by

$$F_\varepsilon^*(x, v) = \sup_{u \in \mathbb{R}^N} [(u \mid v) - F_\varepsilon(x, u)] .$$

The condition (10) easily implies that

(11) $$F_\varepsilon^*(x, v) \leqslant \beta(x) + (2\varepsilon)^{-1} | v - l(x) |^2 ,$$

so that F_ε^* takes values in $\mathbb{R}$ and classical results on the Legendre transform imply that $\nabla F_\varepsilon^*(x, .)$ exists and is continuous for a. e. $x \in \Omega$. We define on R(L) the functional φ_ε by

$$\varphi_\varepsilon(v) = \int_\Omega [(1/2)(Kv(x) \mid v(x)) + F_\varepsilon^*(x, v(x))]dx .$$

We can now state and prove the following important lemma.

LEMMA 3. — *Assume that there exists $\alpha \in L^\infty(\Omega)$ as in Lemma 1 such that, for each $\eta > 0$, one can find $\beta_\eta \in L^2(\Omega ; \mathbb{R}_+)$ such that*

$$F(x, u) \leqslant (\alpha(x) + \eta)(|u|^2/2) + \beta_\eta(x) \tag{12}$$

for a. e. $x \in \Omega$ and all $u \in \mathbb{R}^N$. Then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in]0, \varepsilon_0]$, the equation

$$Lu = \nabla F_\varepsilon(x, u) \tag{13}$$

has a solution u_ε such that $v_\varepsilon = Lu_\varepsilon$ minimizes φ_ε on R(L).

Proof. — Let us choose $\varepsilon_0 > 0$ such that

$$(\alpha(x) + 2\varepsilon_0)^{-1} \geqslant [\alpha(x)]^{-1} - \delta$$

where $\delta > 0$ is given by Lemma 1. For each $\varepsilon \in]0, \varepsilon_0]$, a. e. $x \in \Omega$ and each $u \in \mathbb{R}^N$ we have

$$F_\varepsilon(x, u) \leqslant (\alpha(x) + 2\varepsilon_0)(|u|^2/2) + \beta_{\varepsilon_0}(x)$$

and hence

(14) $F_\varepsilon^*(x, v) \geqslant |v|^2/2\alpha(x) - \delta |v|^2/2 - \beta_{\varepsilon_0}(x)$. Notice that (12) and (14) imply also that

$$|\nabla F_\varepsilon^*(x, v)| \leqslant 2\varepsilon^{-1}[|v| + |l(x)| + \beta_{\varepsilon_0}(x) + \beta(x)] .$$

Thus $\varphi_\varepsilon : R(L) \to \mathbb{R}$ is well defined and of class $\mathscr{C}^1$ for each $\varepsilon \in]0, \varepsilon_0]$ and, by Lemma 1 and (14),

$$\varphi_\varepsilon(v) \geqslant (\delta/2) \| v \|^2 - \int_\Omega \beta_{\varepsilon_0}(x)dx \tag{15}$$

for all $v \in R(L)$. On the other hand,

$$\begin{aligned}\varphi_\varepsilon(v) &= (1/2)(KP^- v, v) + (1/2)(KP^+ v, v) + \int_\Omega F_\varepsilon^*(x, v(x))dx = \\ &= \varphi^1(v) + \varphi^2(v) + \varphi_\varepsilon^3(v)\end{aligned}$$

with φ^1 sequentially weakly continuous (as KP^- is compact), φ^2 and φ^3 w. l. s. c. (as continuous and convex). Thus φ_ε is w. l. s. c. and coercive and hence has a minimum v_ε for each $\varepsilon \in]0, \varepsilon_0]$. Consequently,

$$(Kv_\varepsilon + \nabla F_\varepsilon^*(., v_\varepsilon(.)), h) = 0$$

for all $h \in R(L)$, i. e.

$$Kv_\varepsilon + \nabla F_\varepsilon^*(., v_\varepsilon(.)) = \bar{u}_\varepsilon \in \ker L.$$

Letting $u_\varepsilon = \bar{u}_\varepsilon - Kv_\varepsilon \in D(L)$, we deduce from the above equality, by duality,

$$Lu_\varepsilon = v_\varepsilon = \nabla F_\varepsilon(., u_\varepsilon(.))$$

and the proof is complete.

REMARK 2. — Condition (12) will be abreviated by saying that

$$\limsup_{|u| \to \infty} |u|^{-2} F(x, u) \leqslant \alpha(x)/2$$

uniformly a. e. in Ω.

The following result shows that we can get a solution to the problem (1) from the found solutions u_ε of the modified problem (13) if we have *a posteriori* estimates on u_ε independent of $\varepsilon \in]0, \varepsilon_0]$.

LEMMA 4. — *Under the assumptions of Lemma 3, if there exists constants* C_1, C_2 *such that*

$$(16) \qquad \| Lu_\varepsilon \| \leqslant C_1, \quad \| u_\varepsilon \| \leqslant C_2, \quad \varepsilon \in]0, \varepsilon_0],$$

then problem (1) has at least one solution.

Proof. — By (16) there exists $u \in V$, $v \in V$ and a sequence (ε_k) in $]0, \varepsilon_0]$ converging to 0 such that

$$(17) \qquad u_k = u_{\varepsilon_k} \rightharpoonup u, \quad Lu_k = Lu_{\varepsilon_k} \rightharpoonup v$$

as $k \to \infty$. By the weak closedness of the graph of L, it follows that $u \in D(L)$ and

$$v = Lu.$$

$F(x, .)$ being convex, $\nabla F(x, .)$ is monotone and hence, for all $w \in D(L)$ we shall have

$$(\nabla F(., u_k(.)) - \nabla F(., w(.)), u_k - w) \geqslant 0$$

and hence, by (13),

$$(Lu_k - \varepsilon_k u_k - \nabla F(., w(.)), u_k - w) \geqslant 0$$

or

$$(18) \quad (LP^- u_k, P^- u_k) + (LP^+ u_k, P^+ u_k) - (Lu_k - \varepsilon_k u_k, w) - \\ - \varepsilon_k \| u_k \|^2 - (\nabla F(., w(.)), u_k - w) \geqslant 0, \qquad k \in \mathbb{N}^*, \quad w \in D(L).$$

Now, $P^- u_k = P^- K(-L) u_k = -KP^- Lu_k$ and, as KP^- is compact,

$$P^- u_k = -KP^-(Lu_k) \to -KP^- Lu = P^- u$$

and hence

$$(19) \qquad (LP^- u_k, P^- u_k) \to (LP^- u, P^- u)$$

if $k \to \infty$. Consequently, as (18) implies

$$(LP^{-}u_k, P^{-}u_k)) - (Lu_k - \varepsilon_k u_k, w) - (\nabla F(.,w(.)), u_k - w) \geqslant \\ \geqslant (-LP^{+}u_k, P^{+}u_k), \qquad k \in \mathbb{N}^*, \quad w \in D(L),$$

we obtain, by (17) and (19)

$$(20) \quad (LP^{-}u, P^{-}u) - (Lu, w) - (\nabla F(.,w(.)), u - w) \geqslant \\ \geqslant \liminf_{k\to\infty} (-LP^{+}u_k, P^{+}u_k), \qquad w \in D(L).$$

But, from the obvious relation

$$0 \leqslant (-LP^{+}(u - u_k), u - u_k) = \\ = -(LP^{+}u, u) + (LP^{+}u, u_k) + (LP^{+}u_k, u) - (LP^{+}u_k, u_k) = \\ = -(LP^{+}u, u) + 2(LP^{+}u, u_k) - (LP^{+}u_k, u_k), \qquad k \in \mathbb{N}^*$$

we obtain

$$0 \leqslant (LP^{+}u, u) + \liminf_{k\to\infty} (-LP^{+}u_k, u_k),$$

and hence

$$\liminf_{k\to\infty} (-LP^{+}u_k, P^{+}u_k) = \liminf_{k\to\infty} (-LP^{+}u_k, u_k) \geqslant -(LP^{+}u, u) = \\ = -(LP^{+}u, P^{+}u).$$

Thus, by (20),

$$(Lu - \nabla F(.,w(.)), u - w) \geqslant 0, \qquad w \in D(L).$$

We use now the Minty's trick by taking

$$w = u - tz, \qquad (z \in D(L), \quad t > 0),$$

which gives

$$(Lu - \nabla F(.,u(.) - tz(.)), z) \geqslant 0, \qquad t > 0, \quad z \in D(L),$$

hence if $t \to 0_+$,

$$(Lu - \nabla F(.,u(.)), z) \geqslant 0, \qquad z \in D(L),$$

so that, as D(L) is dense in V,

$$Lu = \nabla F(.,u(.)),$$

and the proof is complete.

4. THE BASIC EXISTENCE THEOREM

Condition (12) limits the asymptotic interaction between $2F(x,u)/|u|^2$ and the first positive eigenvalue λ_1 of L. The following lemma show that some *a posteriori* estimates on u_ε can be obtained if we add a condition on the interaction of F with the kernel of L, i. e. with the eigenspace of the preceding eigenvalue 0 of L.

LEMMA 5. — *Under the assumptions of Lemma 3, if the functional*

$$\overline{\mathrm{F}} : \ker \mathrm{L} \to \mathbb{R},\ \overline{u} \mapsto \int_\Omega \mathrm{F}(x, \overline{u}(x))dx$$

has a critical point, then there exist positive numbers C_i $(i = 1, 2, 3)$ *such that*

$$\| v_\varepsilon \| = \| \mathrm{L}u_\varepsilon \| \leqslant \mathrm{C}_1,\ \| \tilde{u}_\varepsilon \| \leqslant \mathrm{C}_2,\ \int_\Omega \mathrm{F}(x, \overline{u}_\varepsilon(x)/2)dx \leqslant \mathrm{C}_3 .$$

Proof. — By assumption, the function $\tilde{v}$ defined by

$$\tilde{v}(x) = \nabla \mathrm{F}(x, w(x))$$

belongs to $(\ker \mathrm{L})^{\perp} = \mathrm{R}(\mathrm{L})$. Therefore, by duality

$$\mathrm{F}^*(x, \tilde{v}(x)) = (\tilde{v}(x) \mid w(x)) - \mathrm{F}(x, w(x)) \tag{21}$$

for a. e. $x \in \Omega$. Notice that from the obvious inequality

$$\mathrm{F}(x, u) \leqslant \mathrm{F}_\varepsilon(x, u)$$

we deduce the inequality

$$\mathrm{F}^*(x, v) \geqslant \mathrm{F}_\varepsilon^*(x, v) \tag{22}$$

where the left hand member can be $+\infty$.

As v_ε minimizes φ_ε on R(L) and $\tilde{v} \in \mathrm{R(L)}$, we have, using (15), (21) and (22),

$$\begin{aligned}(\delta/2) \| v_\varepsilon \|^2 - \int_\Omega \beta_{\varepsilon_0}(x)dx &\leqslant \varphi_\varepsilon(v_\varepsilon) \leqslant \varphi_\varepsilon(\tilde{v}) \leqslant \\ &\leqslant \int_\Omega [(1/2)(\mathrm{K}\tilde{v}(x) \mid \tilde{v}(x)) + \mathrm{F}^*(x, \tilde{v}(x))]dx = \\ &= \int_\Omega [(1/2)(\mathrm{K}\tilde{v}(x) \mid \tilde{v}(x)) + (\tilde{v}(x) \mid \overline{w}(x)) - \mathrm{F}(x, \overline{w}(x))]dx = \mathrm{C}_0,\end{aligned}$$

and hence, for all $0 < \varepsilon \leqslant \varepsilon_0$.

$$\| \mathrm{L}u_\varepsilon \|^2 = \| v_\varepsilon \|^2 \leqslant (2/\delta)\left[\int_\Omega \beta_{\varepsilon_0}(x)dx + \mathrm{C}_0\right] = \mathrm{C}_1^2 .$$

Consequently,

$$\| \tilde{u}_\varepsilon \| = \| \mathrm{KL}u_\varepsilon \| \leqslant \| \mathrm{K} \| \| \mathrm{L}u_\varepsilon \| \leqslant \| \mathrm{K} \| \mathrm{C}_1 = \mathrm{C}_2 .$$

Now, by the convexity of $F(x, .)$, we obtain

$$\begin{aligned}F(x, \bar{u}_\varepsilon(x)/2) &= F(x, u_\varepsilon(x)/2 - \tilde{u}_\varepsilon(x)/2) \leqslant \\ &\leqslant \frac{1}{2} F(x, u_\varepsilon(x)) + \frac{1}{2} F(x, -\tilde{u}_\varepsilon(x)) \leqslant \\ &\leqslant \frac{1}{2} [F(x, 0) + (\nabla F(x, u_\varepsilon(x)) | u_\varepsilon(x)) + (\alpha(x)+1) | \tilde{u}_\varepsilon(x)|^2 + \beta_1(x)] = \\ &= \frac{1}{2} [F(x, 0) + (Lu_\varepsilon(x) | u_\varepsilon(x)) - \varepsilon | u_\varepsilon(x)|^2 + (\alpha(x)+1) | \tilde{u}_\varepsilon(x)|^2 + \beta_1(x)] \leqslant \\ &\leqslant \frac{1}{2} [F(x, 0) + (Lu_\varepsilon(x) | u_\varepsilon(x)) + (\alpha(x)+1) | \tilde{u}_\varepsilon(x)|^2 + \beta_1(x)].\end{aligned}$$

Hence, as $(Lu_\varepsilon, u_\varepsilon) = (Lu_\varepsilon, \tilde{u}_\varepsilon) \leqslant \| Lu_\varepsilon \| \| \tilde{u}_\varepsilon \|$, we get

$$\int_\Omega F(x, \bar{u}_\varepsilon(x)/2) dx \leqslant \frac{1}{2}\left[\int_\Omega F(x, 0) dx + C_1 C_2 + \operatorname{ess}\sup_\Omega (\alpha + 1) C_2^2 + \right. \\ \left. + \int_\Omega \beta_1(x) dx \right] = C_3 .$$

We can now state and prove a rather general existence theorem for (1).

THEOREM 1. — *Let* $\Omega \subset \mathbb{R}^m$ *be a bounded domain,* $V \subset L^2(\Omega, \mathbb{R}^N)$ *a closed subspace with the induced inner product* $(.,.)$, $L : D(L) \subset V \to V$ *a linear self-adjoint operator with closed range and* $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $(x, u) \mapsto F(x, u)$ *a function such that* $F(x, .)$ *is convex and differentiable for a. e.* $x \in \Omega$, *satisfies the regularity assumptions listed at the beginning of the section and is such that* $\nabla F(., u(.)) \in V$ *whenever* $u \in D(L)$. *Assume moreover that the following conditions are satisfied:*

(S_1) $0 \in \sigma(L)$

(S_2) $\sigma(L) \cap]0, +\infty[\neq \phi$ *and consists in isolated eigenvalues with finite multiplicity.*

(S_3) *If* $\lambda_1 > 0$ *is the smallest positive eigenvalue of* L *and* $K : R(L) \to D(L) \cap R(L)$ *is the right-inverse* $(-L|_{D(L) \cap R(L)})^{-1}$ *of* $-L$ *there exists* $\alpha \in L^\infty(\Omega)$ *with* $\inf\operatorname{ess}_\Omega \alpha > 0$ *such that*

a) *the quadratic form on* $R(L)$

$$v \mapsto \frac{1}{2}\left[(Kv, v) + \int_\Omega [| v(x)|^2/\alpha(x)] dx \right]$$

is positive definite

b) *for each* $\eta > 0$ *there is a* $\beta_\eta \in L^2(\Omega, \mathbb{R}_+)$ *such that*

$$F(x, u) \leqslant (\alpha(x) + \eta)(| u |^2/2) + \beta_\eta(x)$$

for a. e. $x \in \Omega$ and all $u \in \mathbb{R}^N$.

(S_4) $\displaystyle\int_\Omega F(x, \bar{u}(x))dx \to +\infty$ *if* $\|\bar{u}\| \to \infty, \bar{u} \in \ker L$.

Then the problem

$$\text{(23)} \qquad Lu = \nabla F(x, u)$$

has at least one solution u such that

$$\text{(24)} \qquad v = Lu$$

minimizes the dual action $\varphi : R(L) \to \mathbb{R} \cup \{+\infty\}$ *defined by*

$$\varphi(w) = \frac{1}{2}(Kw, w) + \int_\Omega F^*(x, w(x))dx .$$

Proof. — $G, \ker L \to \mathbb{R}$, $\bar{v} \mapsto \displaystyle\int_\Omega F(x, \bar{v}(x))dx$ is convex, continuous and coercive, and hence it has a minimum, say at $\bar{w}$. Then, by Lemmas 5 with $w = \bar{w}$ and condition (S_4), all conditions of Lemma 4 are satisfied and (23) has a solution $u = \lim_{k\to\infty} u_k$ where $v_k = Lu_k$ minimizes on R(L) the functional

$$\varphi_{\varepsilon_k} : v \mapsto \frac{1}{2}(Kv, v) + \int_\Omega F^*_{\varepsilon_k}(x, v(x))dx$$

for some sequence (ε_k) of positive numbers tending to zero. Therefore, if $h \in R(L)$, we have

$$\varphi_{\varepsilon_k}(v_k) \leqslant \varphi_{\varepsilon_k}(h) \leqslant \varphi(h) .$$

From the properties of (v_k) we can assume, without loss of generality, that

$$\text{(25)} \qquad \begin{aligned} v_k &\rightharpoonup v = Lu , \\ KP^- v_k &\to KP^- v \end{aligned}$$

as $k \to \infty$. Therefore,

$$(KP^- v_k, v_k) \to (KP^- v, v)$$

and

$$\liminf_{k\to\infty} (KP^+ v_k, v_k) \geqslant (KP^+ v, v) .$$

Thus,

$$\text{(26)} \qquad \begin{aligned} \varphi(h) \geqslant \liminf_{k\to\infty} \varphi_{\varepsilon_k}(v_k) &= \liminf_{k\to\infty} \Big[\frac{1}{2}(KP^- v_k, v_k) + \\ &+ \frac{1}{2}(KP^+ v_k, v_k) + \int_\Omega F^*_{\varepsilon_k}(x, v_k(x))dx\Big] \geqslant \\ &\geqslant \frac{1}{2}(KP^- v, v) + \frac{1}{2}(KP^+ v, v) + \liminf_{k\to\infty} \int_\Omega F^{*\tau}_{\varepsilon_k}(x, v_k(x))dx. \end{aligned}$$

Now, for a. e. $x \in \Omega$,

$$F^*_{\varepsilon_k}(x, v_k(x) \geqslant (v_k(x) \mid u(x)) - F_{\varepsilon_k}(x, u(x)) = \\ = (v_k(x) \mid u(x)) - \varepsilon_k \mid u(x) \mid^2/2 - F(x, u(x)),$$

so that

$$\int_\Omega F^*_{\varepsilon_k}(x, v_k(x))dx \geqslant \int_\Omega [(v_k(x) \mid u(x)) - \varepsilon_k \mid u(x) \mid^2 - F(x, u(x))]dx$$

and hence, as $v_k \rightharpoonup v$,

$$\liminf_{k \to \infty} \int_\Omega F^*_{\varepsilon_k}(x, v_k(x))dx \geqslant \int_\Omega [(v(x) \mid u(x)) - F(x, u(x))]dx .$$

Now, by (23), (24) and duality, we have

$$v(x) = \nabla F(x, u(x))$$

and hence

$$F^*(x, v(x)) = (v(x) \mid u(x)) - F(x, u(x))$$

a. e. on Ω, so that

$$\liminf_{k \to \infty} \int_\Omega F^*_{\varepsilon_k}(x, v_k(x))dx \geqslant \int_\Omega F^*(x, v(x))dx . \tag{27}$$

Introduced in (26), (27) implies that

$$\varphi(h) \geqslant \frac{1}{2}(Kv, v) + \int_\Omega F^*(x, v(x))dx = \varphi(v)$$

for all $h \in R(L)$, and the proof is complete.

5. APPLICATION TO THE PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS AND HYPERBOLIC SEMILINEAR EQUATIONS

Let us first consider the periodic problem

$$\begin{aligned} J\dot{u} - A(t)u &= \nabla H(t, u) \\ u(0) &= u(2\pi) \end{aligned} \tag{28}$$

for Hamiltonian systems, where $H : [0, \pi] \times \mathbb{R}^{2M} \to \mathbb{R}$ satisfies the regularity and convexity conditions listed for F at the beginning of section 3, A is a measurable mapping from $[0, 2\pi]$ into the space of real $2M \times 2M$ symmetric matrices which is dominated a. e. on $[0, 2\pi]$ by a L^1-function and $J = \begin{pmatrix} O_M & I_M \\ -I_M & O_M \end{pmatrix}$ is symplectic matrix. Taking $V = L^2(0, 2\pi; \mathbb{R}^{2M})$, $D(L) = \{ u : [0, 2\pi] \to \mathbb{R}^{2M} \mid u$ is absolutely continuous, $u(0) = u(2\pi)$,

$u' \in L^2(0, 2\pi ; \mathbb{R}^{2M}) \}$, $Lu = J\dot{u} - A(.)u(.)$, it is well known that L is self-adjoint, has closed range and a discrete spectrum $\{ \ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots \}$ unbounded from below and from above and made of eigenvalues having finite multiplicity. Let us assume that $\lambda_0 = 0 \in \sigma(L)$, so that λ_1 is the smallest positive eigenvalue of L. We deduce immediately from Theorem 1 the following existence result.

COROLLARY 1. — *Assume that there exists* $\alpha \in L^{\infty}(0, 2\pi)$ *with* $\inf\operatorname{ess}_{[0,2\pi]} \alpha > 0$ *such that*

$$\alpha(t) \leqslant \lambda_1 \text{ a. e. on } [0, 2\pi], \tag{29}$$

$$\alpha(t) < \lambda_1 \tag{30}$$

on a subset of $[0, 2\pi]$ *with positive measure and such that*

$$\limsup_{|u| \to \infty} |u|^{-2} H(t, u) \leqslant \alpha(t)/2 \tag{31}$$

uniformly a. e. in $[0, 2\pi]$. *Then, if*

$$\int_0^{2\pi} H(t, \bar{u}(t))dt \to \infty \tag{32}$$

when $\|\bar{u}\| \to \infty$ *and* $\bar{u}$ *is solution of*

$$J\dot{\bar{u}} + A(t)\bar{u} = 0$$
$$\bar{u}(0) = \bar{u}(2\pi),$$

problem (28) has at least one solution u such that $v = J\dot{u} - A(.)u$ *minimizes on* R(L) *the corresponding dual action.*

An interesting special case is the one where

$$A(t) = kI_{2M}$$

for some integer $k \in \mathbb{Z}$. Then, it is easy to check that

$$\lambda_j = j, \quad j \in \mathbb{Z}$$
$$\ker L = \{ (\cos kt)c - (\sin kt)Jc : c \in \mathbb{R}^{2M} \}$$

so that (32) becomes

$$\int_0^{2\pi} H(t, (\cos kt)c - (\sin kt)Jc)dt \to +\infty \tag{33}$$

if $|c| \to \infty$.

In particular, when $k = 0$, (33) reduces to

$$\int_0^T H(t, c)dt \to +\infty$$

if $|c| \to \infty$. This special case can be taken as a starting point to prove a number of results on the existence of subharmonics for non-autonomous Hamiltonian systems and on the existence of periodic solutions with fixed period of fixed energy in the autonomous case (see e. g. [5] [16]).

Now let us consider the existence of weak solutions $\Omega =]0,\pi[\times]0,2\pi[$ for the problems

$$(34_\pm)\quad \begin{aligned} \pm (u_{tt} + u_{xxxx}) &= \nabla H(x,t,u) \\ u(0,t) &= u(\pi,t) = 0 \\ \text{i. e.}\quad u_{xx}(0,t) &= u_{xx}(\pi,t) = 0 \\ u(0,x) - u(2\pi,x) &= u_t(0,x) - u_t(2\pi,x) = 0 \end{aligned}$$

where $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ measurable in (x,t) and continuous in u is such that for each $R > 0$, $|H(x,t,u)| \leqslant \gamma_R(x,t)$ for some $\gamma_R \in L^\infty(\Omega)$ and a. e. $(x,t) \in \Omega$ (L^∞-Caratheodory condition). If $L_\pm$ denotes the abstract realization in $L^2(\Omega;\mathbb{R}^N)$ of $\pm(D_t^2 + D_x^4)$ with the above boundary conditions, it is well known (see e. g. [4]) that $L_\pm$ is self-adjoint, has closed range and a discrete spectrum

$$\pm\{j^4 - k^2 : j \in \mathbb{N}^*, k \in \mathbb{Z}\}.$$

Thus, 0 is the only eigenvalue with infinite multiplicity. On the other hand

$$\ker L = \left\{ \sum_{\substack{j\in\mathbb{N}^* \\ \varepsilon=\pm 1}} v_{j\varepsilon} \sin jx \exp(\varepsilon i j^2 t) \;\middle|\; \sum_{\substack{j=1 \\ \varepsilon=\pm 1}}^{\infty} |v_{j\varepsilon}|^2 < \infty \right\}$$

and the following result is proved in [4]:

LEMMA 6. — *The* L^1 *and* L^4*-norms are equivalent on* $\ker L$.

Now, if $\sigma(L_\pm) = \{\lambda_k^\pm : k \in \mathbb{Z}\}$, we see that $\lambda_1^+ = 1$ and $\lambda_1^- = 3$. Thus, we shall assume that

$$\limsup_{|u|\to\infty} |u|^{-2} H(x,t,u) \leqslant \alpha(x,t)/2$$

uniformly a. e. in Ω, with, according to the considered case,

$$\alpha(x,t) \leqslant 1 \text{ or } 3$$

a. e. on Ω with strict inequality on a subset of positive measure of Ω. Let us finally assume that

$$H(x,t,u) \to +\infty$$

as $|u| \to \infty$, uniformly a. e. in Ω. Then, by convexity and the L^∞-Caratheodory conditions mentioned above, there will exist $\delta > 0$ and $\gamma \in L^\infty(\Omega)$ such that

$$H(x,t,u) \geqslant \delta |u| - \gamma(x,t)$$

for a. e. $(x, t) \in \Omega$ and all $u \in \mathbb{R}^N$. Consequently, for $\bar{u} \in \ker L$,

$$\int_\Omega H(x, t, \bar{u}(x, t))dxdt \geqslant \delta \int_\Omega |\bar{u}(x, t)| dxdt - \int_\Omega \gamma(x, t)dxdt$$

and, as the L^1 and L^2-norms are equivalent on ker L by Lemma 6, we see that condition (S_4) of Theorem 1 is satisfied. We have therefore proved the following

COROLLARY 2. — *Assume that* $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *satisfies the* L^∞*-Caratheodory condition, that*

$$H(x, t, u) \to +\infty$$

as $|u| \to \infty$ *uniformly a. e. in* Ω *and that*

$$\limsup_{|u| \to \infty} |u|^{-2} H(x, t, u) \leqslant \alpha(x, t)/2 \tag{35}$$

uniformly a. e. in Ω *for some* $\alpha \in L^\infty(\Omega)$ *such that*

$$\alpha(x, t) \leqslant 1 \text{ (resp. 3)} \tag{36}$$

a. e. on Ω, *with strict inequality on a subset of* Ω *with positive measure. Then problem* (34_+) *(resp.* (34_-)*) has at least one weak solution.*

In the special case where $N = 1$,

$$\pm (u_{tt} + u_{xxxx}) = f(x, t, u) \tag{$37_\pm$}$$

the conditions in [*4*] are that $f(x, t, .)$ is nondecreasing,

$$|f(x, t, u)| \leqslant \gamma |u| + C \tag{38}$$

for a. e. $(x, t) \in \Omega$ and all $u \in \mathbb{R}$, with $\gamma < 1$ for (37_+) and $\gamma < 3$ for (37_-), and that there exist $\varphi \in R(L) \cap L^\infty(\Omega)$, $M > 0$ and $\delta > 0$ such that

$$\begin{aligned} \varphi(x, t) + f(x, t, M) &\geqslant \delta \\ \varphi(x, t) + f(x, t, -M) &\leqslant -\delta \end{aligned} \tag{39}$$

a. e. on Ω. Denoting by $\tilde{u}$ the unique solution in $R(L) \cap D(L)$ of

$$Lu = -\varphi,$$

so that $\tilde{u} \in L^\infty(\Omega)$ (see e. g. [*4*]) and letting

$$u = \pm \tilde{u} + w,$$

we obtain the equivalent problem

$$\pm Lw = f(x, t, \tilde{u} + w) + \varphi(x, t) = D_w H(x, t, w)$$

where

$$H(x, t, w) = \int_0^w [f(x, t, \tilde{u}_\pm(x, t) + s) + \varphi(x, t)]ds.$$

It follows easily from (38) that (35) and (36) are satisfied with $\alpha(x, t) = \gamma$ and (39) together with the monotonicity of $f(x^1, t, .)$ imply that

$$H(x, t, w) \geqslant \int_0^M [f(x, t, \tilde{u}(x, t) + s) + \varphi(x, t)]ds + \delta(w - M)$$

when $w \geqslant M$ and

$$- H(x, t, w) \leqslant \int_{-M}^0 [f(t, x, \tilde{u}(x, t) + s) + \varphi(x, t)]ds + \delta(M + w)$$

when $w \leqslant - M$, so that

$$H(x, t, w) \geqslant \delta | w | - \eta(x, t)$$

with $\eta \in L^\infty(\Omega)$ for a. e. $(x, t) \in \Omega$ and all $u \in \mathbb{R}$. This shows that the Bahri-Sanchez theorem is a special case of Corollary 2.

REMARK 3. — Conditions (38) and (39) were motivated by a result of Bahri and Brézis [*3*] relative to the semi-linear wave equation

$$u_{tt} - u_{xx} = f(x, t, u)$$

with the Dirichlet-periodic boundary conditions on Ω. In the absence of a result like Lemma 6 for the wave operator, it is not clear that the Bahri-Brézis result is a special case of Theorem 1, although it is a consequence of Lemma 4 as shown by the proof given in [*8*].

6. THE CASE OF A SCALAR EQUATION

When $N = 1$, the results of the preceding sections can be sharpened and Theorem 1 leads to a necessary and sufficient condition already considered in particular situations in [*13*] [*14*] [*15*] [*16*] [*17*].

We first have the following generalizations of Lemmas 1 and 2 whose proof, very similar to the ones of Section 2, are left to the reader. We assume that L satisfies the conditions (S_1) and (S_2) and use the notations of Section 2. For a real function u on Ω, we write $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$, so that $u = u^+ - u^-$.

LEMMA 1'. — *Let* α_+, α_- *in* $L^\infty(\Omega)$ *be such that* $\inf\operatorname{ess}_\Omega \alpha_+ > 0$, $\inf\operatorname{ess}_\Omega \alpha_- > 0$ *and such that the quadratic form* γ *defined by*

$$\gamma(v) = (1/2)\left[(Kv, v) + \int_\Omega \left(\frac{(v^+(x))^2}{\alpha_+(x)} + \frac{(v^-(x))^2}{\alpha_-(x)}\right)dx\right] \tag{40}$$

is positive definite on R(L). *Then there exists* $\delta > 0$ *such that*

$$\gamma(v) \geqslant \delta \| v \|^2$$

whenever $v \in R(L)$.

LEMMA 2'. — *Let* α_+, α_- *in* $L^\infty(\Omega)$ *be like in Lemma 1 and such that*

$$\alpha_+(x) \leqslant \lambda_1, \alpha_-(x) \leqslant \lambda_1 \tag{41}$$

for a. e. $x \in \Omega$ *and*

$$\int_\Omega [(\lambda_1 - \alpha_+(x))(v_1^+(x))^2 + (\lambda_1 - \alpha_-(x))(v_1^-(x))^2]dx > 0 \tag{42}$$

for all $v^1 \in \ker(L - \lambda_1 I) \setminus \{0\}$. *Then the quadratic form* γ *defined by (40) is positive definite on* R(L).

An immediate consequence of those Lemmas is that *when* N = 1, *Theorem 1 holds with assumptions* (a) *and* (b) *replaced by the following one:*

There exist $\alpha_+, \alpha_- \in L^\infty(\Omega)$ *with* $\inf_\Omega \operatorname{ess} \alpha_+ > 0$, $\inf_\Omega \operatorname{ess} \alpha_- > 0$ *such that*

a') *the quadratic form (40) is positive definite on* R(L)

b') *for each* $\eta > 0$ *there is a* $\beta_- \in L^2(\Omega, \mathbb{R}_+)$ (resp. $\beta_+ \in L^2(\Omega, \mathbb{R}_+)$) *such that*

$$F(x, u) \leqslant (\alpha_-(x) + \eta)(u^2/2) + \beta_-(x)$$
$$(\text{resp. } F(x, u) \leqslant (\alpha_+(x) + \eta)(u^2/2) + \beta_+(x))$$

for a.e. $x \in \Omega$ *and* $u < 0$ (resp. $u > 0$).

We can therefore prove the following necessary and sufficient condition for the solvability of the equation

$$Lu = D_u F(x, u) \tag{43}$$

in $D(L) \subset V \subset L^2(\Omega, \mathbb{R})$, by combining Theorem 1 to some ideas of the proof of Theorem 2 of [*17*].

THEOREM 2. — *Assume that* N = 1 *and that the regularity conditions as well as assumptions* (S_1)-(S_2)-(S_3) *of Theorem 1 and the conditions* (*a'*)-(*b'*) *above hold. Assume moreover that*

$$\ker L = \operatorname{span} \{ \varphi \} \tag{44}$$

where $\varphi > 0$ *a. e. on* Ω *and that for each possible solution u of the problem* $Lu = \nabla_u F(x, v(x))$ *with* $v \in D(L)$, *there exist constants a, b such that*

$$a\varphi(x) \leqslant u(x) \leqslant b\varphi(x) \tag{45}$$

for a. e. $x \in \Omega$. *Then (43) has at least one solution if and only if the real function* $\overline{F} : c \mapsto \int_\Omega F(x, c\varphi(x))dx$ *has a critical point.*

Proof. — *Necessity.* — If (43) has a solution u, then $f(.,u(.))\in R(L)=(\ker L)^{\perp}$ and hence

$$\int_{\Omega} D_u F(x, u(x))\varphi(x)dx = 0$$

so that, using (45), the monotonicity of $D_u F(x, .)$ and the positivity of φ, we obtain

$$\int_{\Omega} D_u F(x, a\varphi(x))\varphi(x)dx \leqslant 0 \leqslant \int_{\Omega} D_u F(x, b\varphi(x))\varphi(x)dx$$

i. e.

$$F'(a) \leqslant 0 \leqslant F'(b).$$

The conclusion then follows from the intermediate value theorem.

Sufficiency. — Let $\bar{c} \in \mathbb{R}$ be a critical point of $\overline{F}$ and let us assume first that

$$\int_{\Omega} D_u F(x, c\varphi(x))\varphi(x)dx = 0 \tag{46}$$

for all $c \geqslant c$. Then, by the monotonicity of $D_u F(x, .)$, this implies that

$$D_u F(x, c\varphi(x)) = D_u F(x, \bar{c}\varphi(x))$$

for a. e. $x \in \Omega$ and all $c \geqslant \bar{c}$. Let now $\tilde{u}$ be a solution of the linear problem

$$Lu = D_u F(x, \bar{c}\varphi(x))$$

and $\tilde{a}$, $\tilde{b}$ be such that

$$\tilde{a}\varphi(x) \leqslant \tilde{u}(x) \leqslant \tilde{b}\varphi(x)$$

a. e. on Ω. Then, if we take $\tilde{c} \geqslant \bar{c} - \tilde{a}$ and

$$u(x) = \tilde{c}\varphi(x) + \tilde{u}(x)$$

we have

$$u(x) \geqslant (\tilde{c} + \tilde{a})\varphi(x) \geqslant \bar{c}\varphi(x)$$

a. e. on Ω and hence

$$Lu = L\tilde{u} = D_u F(x, \bar{c}\varphi(x)) = D_u F(x, u(x)),$$

i. e. u is a solution of (43). Similarly if (46) holds for all $c \leqslant \bar{c}$. It remains therefore to consider the case where there exist $c_1 < \bar{c} < c_2$ such that

$$\delta_1 = \int_{\Omega} D_u F(x, c_1\varphi(x))\varphi(x)dx < 0 < \int_{\Omega} D_u F(x, c_2\varphi(x))\varphi(x)dx = \delta_2 .$$

But then, for $c \geqslant c_2$, by convexity,

$$\overline{F}(c) = \int_{\Omega} F(x, c\varphi(x))dx \geqslant \int_{\Omega} F(x, c_2\varphi(x))dx + (c - c_2)\delta_2$$

so that $\overline{F}(c) \to +\infty$ if $c \to +\infty$. Similarly, $\overline{F}(c) \to +\infty$ if $c \to -\infty$ and condition (S_4) of Theorem 1 holds, which completes the proof.

REMARK 4. — Condition (45) is obviously a regularity assumption concerning $D_u F$ and the solutions of the linear problem $Lu = h$.

REMARK 5. — The condition for $\overline{F}$ to have a critical point is obviously equivalent to the existence of $\overline{c} \in \mathbb{R}$ such that

$$\int_\Omega D_u F(x, \overline{c}\varphi(x))\varphi(x)dx = 0 .$$

The assumptions of Theorem 2 suggest applications to the problems

$$(47) \qquad \begin{aligned} \Delta u + f(x, u) &= 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \text{ in } \partial\Omega \end{aligned}$$

and

$$(48) \qquad \begin{aligned} \Delta u + \lambda_1 u + f(x, u) &= 0 \text{ in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega \end{aligned}$$

where $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots$ denote the eigenvalues of $-\Delta$ with the Neumann or Dirichlet condition according to the considered problem ($\lambda_1 = 0$ and $\varphi \equiv 1$ in the Neumann case). Assuming f a Caratheodory function on $\Omega \times \mathbb{R}$ such that

$$| f(x, u) | \leqslant a | u | + b(x)$$

with $b \in L^q(\Omega)$, Ω smooth enough, $q > m/2$ if $m \geqslant 4$ and $q = 2$ for $m = 1, 2, 3$ in the Neumann case and $q > m$ in the Dirichlet case, one can check that the regularity conditions on L, the abstract realization of $-\Delta - \lambda_1 I$ in $L^2(\Omega)$ (with the boundary conditions included in the sense of traces in D(L)) and on $F(x, .) = \int_0^{.} f(x, v)dv$ as well as conditions (S_1)-(S_2)-(S_3), (44) and (45) are satisfied. Consequently, if we assume that F satisfies (b') with α_- and α_+ verifying (41) and (42), we obtain the necessary and sufficient conditions for the solvability of (47) or (48) given in [*13*] [*14*] and [*17*] and which give a sharp answer to a question raised in [*10*] (p. 574) about some pioneering results of Klingelhöfer [*11*].

In the case of the periodic problem for a second order scalar equation

$$\begin{gathered} - u''(t) = g(t, u(t)) \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \end{gathered}$$

with $g(t, .)$ nondecreasing, Theorem 2 answers positively (and even under weaker conditions) an open question stated in [*12*] about a possible improvement of a result of Ahmad-Lazer [*1*]. The case of a non-monotone g is still open.

7. THE CASE OF A STRICTLY CONVEX POTENTIAL

In the case of a strictly convex $F(x, .)$, one can deduce from Theorem 1 a necessary and sufficient condition for the solvability of (26).

THEOREM 3. — *Assume that the conditions of Theorem 1 are satisfied with* (S_4) *replaced by* (S_4') $\dim \ker L < \infty$ *and* $F(x, .)$ *is strictly convex for a. e.* $x \in \Omega$.

Then the following statements are equivalent:

a) *problem (23) has a solution.*

b) *there exists* $\overline{w} \in \ker L$ *such that*

$$\int_\Omega (\nabla F(x, \overline{w}(x)) \mid \overline{v}(x)) dx = 0$$

for every $\overline{v} \in \ker L$.

c) $\int_\Omega F(x, \overline{v}(x)) dx \to +\infty$ *as* $\|\overline{v}\| \to \infty$ *in* $\ker L$.

Proof. — $a \Rightarrow b$. If (23) has a solution u, then

$$\int_\Omega (\nabla F(x, u(x)) \mid \overline{v}(x)) dx = 0 \tag{49}$$

for all $\overline{v} \in \ker L$ as $V = \ker L \oplus R(L)$. Let us write $u = \overline{u} + \tilde{u}$ with $\overline{u} \in \ker L$ and $\tilde{u} \in R(L)$ and let us define on $\ker L$ the strictly convex continuous functions G and $\tilde{G}$ respectively by

$$G(\overline{v}) = \int_\Omega F(x, \overline{v}(x)) dx$$

$$\tilde{G}(\overline{v}) = \int_\Omega F(x, \overline{v}(x) + \tilde{u}(x)) dx .$$

Since, by (49), $\nabla\tilde{G}(\overline{u}) = 0$, the strict convexity of $\tilde{G}$ and the fact that $\dim \ker L < \infty$ imply that

$$\tilde{G}(\overline{v}) \to +\infty \quad \text{as} \quad \|\overline{v}\| \to \infty \text{ in } \ker L . \tag{50}$$

By the convexity of $F(x, .)$, we have

$$\tilde{G}(\overline{v}) \leqslant (1/2) \int_\Omega F(x, 2\overline{v}(x)) dx + (1/2) \int_\Omega F(x, 2\tilde{u}(x)) dx = (1/2) G(2\overline{v}) + C$$

and hence, by (50)

$$G(\overline{v}) \to +\infty \quad \text{as} \quad \|\overline{v}\| \to \infty \text{ in } \ker L .$$

Consequently, there will exist $\overline{w} \in \ker L$ such that $\nabla G(\overline{w}) = 0$, i. e. such that

$$\int_\Omega (\nabla F(x, \overline{w}(x)), \overline{v}(x))dx = 0$$

for all $\overline{v} \in \ker L$.

$b \Rightarrow c$. By assumption, G has a critical point and then in the same way as for $\tilde{G}$ above $G(\overline{v}) \to +\infty$ as $\|\overline{v}\| \to \infty$ in ker L.

$c \Rightarrow a$. This follows immediately from Theorem 1.

This result generalizes in several ways earlier theorems of Berger and Schechter [*6*] who, under more restrictive conditions upon L and F, have shown the equivalence between condition (*a*) of Theorem 3 and the following condition.

b′) there exists $u \in D(L)$ *such that*

$$\int_\Omega (\nabla F(x, u(x)) \mid \overline{v}(x))dx = 0$$

for every $\overline{v} \in \ker L$.

The interest of the equivalent condition (*b*) is to make clear its relation with the coercivity on ker L of the averaged potential $\int_\Omega F(x, .)dx$. It is well known that for periodic problems for systems of differential equations with a small parameter

$$\begin{aligned} \dot{u}(t) &= \varepsilon f(t, u(t)) \\ u(0) &= u(2\pi) \end{aligned} \tag{50}$$

(and in particular for the corresponding Hamiltonian systems

$$\begin{aligned} J\dot{u}(t) &= \varepsilon \nabla H(t, u(t)) \\ u(0) &= u(2\pi)) \end{aligned}$$

with sufficiently smooth right-hand members, the averaging method (see e. g. [*7*]) relates the existence of solutions of (50) to that of zeros of the averaged equation

$$\int_0^{2\pi} f(t, \overline{u})dt = 0 .$$

Thus, Theorem 3 can be considered, for abstract variational semi-linear equations, with a strictly convex nonlinear potential and a finite dimensional kernel, as a *global* version of this averaging method. In another direction, it exactly extends to nonlinear perturbations deriving from a strictly convex potential the usual Fredholm alternative for non-homogeneous linear equations.

REMARK 6. — One can notice that under (S_4') only, the condition *a*) implies *b*) and *c*).

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