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Global regularity for solutions of the minimal surface equation with continuous boundary values

by

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ABSTRACT. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$ having nonnegative mean curvature. We examine the regularity at the boundary of solutions u to the minimal surface equation having boundary values ϕ. If ϕ has modulus of continuity β we give a modulus of continuity for u which depends on β and the behaviour of the mean curvature of $\partial\Omega$. If ϕ is Lipschitz continuous then we show that u is Hölder continuous with some exponent α (explicitly obtained) that depends on the Lipschitz constant for ϕ. Finally we give examples showing the above results are best possible.

RÉSUMÉ. — Supposons que Ω soit un ouvert borné de $\mathbb{R}^n$ dont le bord $\partial\Omega$ est de classe C^2 et a une courbure moyenne positive au mille. Nous examinons la régularité sur le bord de toute solution u à l'équation des surfaces minimales avec ϕ donné au bord. Si ϕ a un module de continuité β, nous dérivons un module de continuité pour u qui dépend de β et du comportement de la courbure moyenne de $\partial\Omega$. Si ϕ est lipschitzienne, nous démontrons que u est höldérienne d'exposant α (obtenu explicitement), dépendant de la constante de Lipschitz pour ϕ. Finalement, nous donnons des exemples démontrant que les résultats obtenus sont rigoureux.

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0. INTRODUCTION

We consider the Dirichlet problem for the minimal surface equation. Thus given Ω, a bounded open subset of $\mathbb{R}^n$ and ϕ, a function defined on $\partial\Omega$, we seek a function $u \in C^2(\Omega)$ such that $u = \phi$ on $\partial\Omega$ and

$$\sum_{i=1}^{n} D_i((1 + |Du|^2)^{-\frac{1}{2}} D_i u) = 0 \quad \text{in } \Omega.$$

Jenkins and Serrin ([JS]) showed that if we wish to solve this problem for every continuous function ϕ then we must demand that $\partial\Omega$ has non-negative mean curvature everywhere. Furthermore with this condition on $\partial\Omega$ they showed that the problem is solvable for every continuous ϕ. Consequently throughout this paper we shall assume that $\partial\Omega$ has non-negative mean curvature. For existence considerations when $\partial\Omega$ has negative mean curvature the reader is referred to [JS] and [W2].

If we know that ϕ is more than just continuous then we should expect that the solution u will also have greater regularity. This is indeed true. Thus if $\partial\Omega$ and ϕ are $C^{k,\alpha}$ $k \geqq 2$, $0 < \alpha < 1$ then the solution u is in $C^{k,\alpha}(\Omega)$ (For example see [GT].) The case $k = 1$ has also recently been studied by Lieberman [L1] and Giaquinta and Giusti [GG] who have shown that the corresponding result is true. That is if $\partial\Omega$ is C^2 (and has nonnegative mean curvature) and ϕ is $C^{1,\alpha}$ then we have u in $C^{1,\alpha}(\overline{\Omega})$. In this paper we study the case $k = 0$. We shall assume that $\partial\Omega$ is C^2 and that $\phi \in C^{0,\alpha}(\partial\Omega)$ for some α, $0 < \alpha \leqq 1$. Results for this problem have been proved by Giusti [G2] who showed that if $\partial\Omega$ has strictly positive mean curvature then $u \in C^{0,\alpha/2}(\overline{\Omega})$. (Lieberman [L2] has also proved similar types of results for more general classes of equations.) Furthermore he gave an example due to Weinberger (see also the example in [G3]) in which $\phi \in C^{0,1}(\partial\Omega)$ and $u \notin C^{0,\alpha}(\Omega)$ for any $\alpha > \frac{1}{2}$. Thus the exponent $\alpha/2$ is, in general, best possible. In the second section of the paper we generalize these results and show that the same result holds in a local form. Thus Giusti demanded that $\partial\Omega$ have strictly positive mean curvature everywhere and $\phi \in C^{0,\alpha}(\partial\Omega)$ while we show that if these things hold in a neighbourhood of $x_0 \in \partial\Omega$ then the solution u satisfies a Hölder condition at x_0 with the required exponent. More generally we show that if ϕ has modulus of continuity $\beta(t)$ at x_0 and the mean curvature of $\partial\Omega$ grows like $|x - x_0|^\gamma$, $\gamma \geqq 0$, then, u has modulus of continuity $C\beta(Ct^{\frac{1}{2+\gamma}})$ at x_0 for some constant C. The reader should also see [S1] where modulus of continuity estimates are proved without any restrictions on the smoothness or the curvature of $\partial\Omega$. In section 5, given a set Ω, we show how to construct examples of boundary values ϕ so that the corresponding solution has exactly growth $C\beta(C\sqrt{t})$ at some point. This shows that the previous results are best possible.

It should be noted that, at this stage, a small change in the regularity of ϕ from $C^{1,\varepsilon}$ $(\varepsilon > 0)$ to $C^{0,1}$ produces a large change in the known regularity of u from $C^{1,\varepsilon}$ to at best $C^{0,\frac{1}{2}}$. In section 3 we show that this gap can be filled by taking into account the value of the Lipschitz constant of the boundary data. We give a function $K(\alpha)$ defined on $(0, 1)$ such that if the Lipschitz constant of ϕ is less than $K(\alpha)$ then $u \in C^{0,\alpha}(\overline{\Omega})$. Furthermore in section 4 we show that this is best possible in that, for any $K > K(\alpha)$ there is boundary data ϕ with Lipschitz constant K but such that $u \notin C^{0,\alpha}(\overline{\Omega})$. The function $K(\alpha)$ is obtained by looking at the zeros of certain ordinary differential equations and various properties are obtained. It is shown that as $\alpha \to 1$, $K(\alpha) \to 0$, as $\alpha \to 0$, $K(\alpha) \to \infty$ and $K\left(\frac{1}{2}\right) = \frac{1}{\sqrt{n-1}}$ (It is worth noting that the critical value $\frac{1}{\sqrt{n-1}}$ occurred in the existence work [W2] where $\partial\Omega$ has possibly negative mean curvature.) For the case $n = 2$ a particularly simple expression is obtained for $K(\alpha)$, namely, $K(\alpha) = \text{cotangent}\left(\frac{\pi\alpha}{2}\right)$.

All our results are local ones so that we only require the conditions to hold in a neighbourhood of a point x_0 and so they apply to generalized solutions (see [G1] or [G3]). Additionally they may also be applied to the equation of constant mean curvature

$$\sum_{i=1}^{n} D_i\left(\frac{D_i u}{\sqrt{1 + |Du|^2}}\right) = nH$$

except that conditions about the mean curvature of $\partial\Omega$ must be replaced by conditions about the mean curvature minus $\frac{n}{n-1}|H|$.

All of our proofs involve the construction of appropriate barriers and in some cases we make use of an idea of Simon [S2] which involves writing the barrier as a function over the tangent plane to the boundary cylinder $\partial\Omega \times \mathbb{R}$ instead of over Ω. This means that barriers, which become vertical over Ω at $\partial\Omega$, in the new setting have gradients tending to zero, greatly simplifying the calculations involved. This same idea was also used in [W1] and [W2].

1. NOTATION

In this section we introduce some notation to be used in later sections. Similar ideas and notation were used in [W1] and [W2] and the original idea of using a different coordinate system (the y-coordinate system below) to help in the construction of barriers was given by Simon in [S2].

We shall always suppose that Ω is a bounded open subset of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$ and ϕ is a given function in $L^1(\partial\Omega)$.

DEFINITION 1.1. — ([G1], [G3]); A function $u \in BV(\Omega)$ is said to be *a generalized solution of the Dirichlet problem* for the minimal surface equation in Ω with boundary data ϕ if

$$\int_\Omega \sqrt{1+|Du|^2} + \int_{\partial\Omega} |u-\phi|\, dH_{n-1} \leqq \int_\Omega \sqrt{1+|Dv|^2} + \int_{\partial\Omega} |v-\phi|\, dH_{n-1}$$

for every $v \in BV(\Omega)$.

We note that with the given conditions on Ω and ϕ a generalized solution will always exist.

In most of the theorems in this paper we will be given a point $x_0 \in \partial\Omega$ and a neighbourhood $\mathcal{N}$ of x_0 such that $\partial\Omega$ is C^2 in $\mathcal{N}$. It will then be convenient to introduce special coordinate systems to simplify calculations.

DEFINITION 1.2. — Suppose x_0, $\mathcal{N}$ and Ω are as above. An *x-coordinate system* for $\partial\Omega$ at x_0 is a Cartesian coordinate system having x_0 as origin and such that the positive x_n-axis has the same direction as the inner normal to $\partial\Omega$ at x_0. We denote $x' = (x_1, \ldots, x_{n-1})$ and $x = (x', x_n)$. Since $\partial\Omega$ is C^2 near x_0 there is $\delta_0 > 0$ and a function $w : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\partial\Omega \cap \{ x : |x'| < \delta_0, |x_n| < \delta_0 \} = \{ (x', w(x')) : |x'| < \delta_0 \}.$$

Furthermore w is C^2, $w(0) = 0$ and $Dw(0) = 0$.

DEFINITION 1.3. — Given the x-coordinate system of Definition 1.2 for $\mathbb{R}^n$ we define the *y-coordinate system* for $\mathbb{R}^{n+1}$ by setting $y_i = x_i$, $i = 1, \ldots, n-1$, $y_n = x_{n+1}$ and $y_{n+1} = x_n$. We denote $y' = (y_1, \ldots, y_{n-1})$ and $y = (y', y_n)$ so that $x' = y'$. In the same manner that we used the graph of w to describe $\partial\Omega$ near x_0 we can also describe $\partial\Omega \times \mathbb{R}$. Thus

$$(\partial\Omega \times \mathbb{R}) \cap \{ (x, x_{n+1}) : |x'| < \delta_0, |x_n| < \delta_0 \} = \{ (y, y_{n+1}) : y_{n+1} = w(y'), |y'| < \delta_0 \}$$

where w is the same function as in Definition 1.2.

Finally we note that the *minimal surface equation* can be written as $Mu = 0$ or equivalently as $M_0u = 0$ where

$$Mu = \sum_{i=1}^{n} D_i\left(\frac{D_iu}{\sqrt{1 + |Du|^2}}\right)$$

and

$$M_0u = \sqrt{1 + |Du|^2}Mu = \Delta u - \frac{1}{1 + |Du|^2} \sum_{i,j=1}^{n} D_iuD_juD_{ij}u\,.$$

2. CONTINUOUS BOUNDARY DATA

In this section we show how to find a modulus of continuity, $\eta(t)$, for the solution u on $\overline{\Omega}$, in terms of the modulus of continuity, $\beta(t)$, for the boundary data ϕ and the growth of the mean curvature of $\partial\Omega$.

THEOREM 1. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$. Suppose $\phi \in L^1(\partial\Omega)$ and u is a generalized solution of the Dirichlet problem. Suppose $x_0 \in \partial\Omega$ and there is a neighbourhood $\mathcal{N}$ of x_0, a function $\beta(t) : [0, \infty) \to [0, \infty)$ and constants $\gamma \geqq 0$, $a > 0$, and A such that

i) $\partial\Omega$ is C^2 in $\mathcal{N}$ and, if $H(x)$ is the mean curvature of $\partial\Omega$ at x, then $H(x) \geqq a\,|x - x_0|^\gamma$ for $x \in \partial\Omega \cap \mathcal{N}$,

ii) $\phi(x) \leqq A$, $x \in \partial\Omega$,

iii) β is an increasing subadditive function with $\lim_{t\to 0} \beta(t) = 0$,

iv) $\phi(x) \leqq \phi(x_0) + \beta(|x - x_0|)$, $x \in \mathcal{N} \cap \partial\Omega$.

Then there is a constant C depending on ϕ and Ω such that

$$u(x) - \phi(x_0) \leqq C\beta(C\,|x - x_0|^{\frac{1}{\gamma+2}}),\ x \in \Omega\,.$$

Proof. — Introduce x- and y-coordinate systems at x_0 and let w be the function describing $\partial\Omega$ as in section 1. We may suppose $\phi(x_0) = 0$. We may also assume that for $|x'| < \delta_0 \leqq 1$ we have

$$(2.1)\qquad |Dw(x')| \leqq 1 \quad \text{and} \quad |w(x')| \leqq |x'|\,.$$

For β as in *iii)* we can take the concave envelope and obtain an equivalent modulus of continuity which is concave (see [LO]). Then setting

$\tilde{\beta}(t) = 2\int_0^1 \beta(st)ds$ gives an equivalent C^1 concave modulus. Repeating this process we may assume $\beta \in C^2(0,\infty) \cap C^0[0,\infty)$ and β concave in addition to the properties of *iii*).

Suppose $K \geqq 1$, $\delta \leqq \delta_0$, $0 \leqq \alpha \leqq 1$ and that η is the inverse of the function β. Now define

$$f(y) = (\eta(K^{-1}y_n))^{\gamma+2} - (2\,|\,y'\,|)^{\gamma+2}$$
$$v(y) = w(y') + \alpha f(y)$$

on the set

$$D = \{\, y \in \mathbb{R}^n : 2\delta \geqq \eta(K^{-1}y_n) \geqq 2\,|\,y'\,|\,\}.$$

Note that $v(y) \geqq w(y')$ in D and furthermore $v(y) = w(y')$ if and only if $\eta(K^{-1}y_n) = 2\,|\,y'\,|$, that is, when $y_n = K\beta(2\,|\,y'\,|)$. Also note that $D_n v(y) = \alpha D_n f(y) = K^{-1}(\gamma+2)\eta^{\gamma+1}\eta' > 0$. These two facts imply that the graph of v can be written as the graph of a function $\tilde{v}$ defined in the x-coordinates. Moreover there will be a neighbourhood $\mathcal{M}$ of x_0 (depending on the choice of α, K and δ) such that $\tilde{v}$ is defined on $\mathcal{M} \cap \Omega$ and

$$\mathcal{M} \cap \partial\Omega = \{\,(x', w(x')) : |\,x'\,| < \delta\,\}, \tag{2.2}$$
$$\tilde{v}(x) = K\beta(2\,|\,x'\,|) \geqq \beta(|\,x\,|) \geqq \phi(x), \quad x \in \mathcal{M} \cap \partial\Omega, \tag{2.3}$$
$$\tilde{v}(x) = K\beta(2\delta), \quad x \in \mathcal{M} \cap \partial\Omega. \tag{2.4}$$

We shall choose α, K and δ so that in addition to (2.1)-(2.4) we have

$$K\beta(2\delta) \geqq A \geqq \sup_{\partial\Omega} \phi, \tag{2.5}$$
$$M\tilde{v} \leqq 0, \quad x \in \mathcal{M} \cap \Omega. \tag{2.6}$$

(Note that (2.6) holds if $M_0 v \geqq 0$ in D.) The comparison principle will then imply that

$$u(x) \leqq \tilde{v}(x) \leqq K\beta(C\,|\,x\,|^{\frac{1}{\gamma+2}}) \quad \text{in } \mathcal{M} \cap \Omega$$

for some constant C and the result is proved. To check (2.6) we use Lemma 1 of [S2] and (2.1) to obtain that, provided

$$\alpha\,|\,Df\,| \leqq 1 \quad \text{on D} \tag{2.7}$$

we have

$$M_0 v = M_0 w + \alpha\Delta f + \alpha E$$

where

$$\begin{aligned}|\,E\,| &\leqq C(|\,Df\,|\,|\,D^2 w\,| + |\,D^2 f\,|)(\alpha\,|\,Df\,| + |\,Dw\,|)\\ &\leqq C(|\,y'\,|^{\gamma} + K^{-2}\eta^{\gamma}((\eta')^2(\gamma+1) + \eta\eta'') + K^{-1}\eta^{\gamma+1}\eta')(|\,y'\,| + \alpha K^{-1}\eta^{\gamma+1}\eta').\end{aligned}$$

Also

$$\Delta f = (\gamma+2)[K^{-2}\eta^{\gamma}(\eta')^2(\gamma+1) + K^{-2}\eta^{\gamma+1}\eta'' - |\,y'\,|^{\gamma}(n+\gamma-1)]$$

and so

$$\begin{aligned} M_0 v \geqq |y'|^\gamma\left[\frac{a}{2} - \alpha(\gamma+2)(n+\gamma-1) - \alpha C(|y'| + \alpha K^{-1}\eta^{\gamma+1}\eta')\right] \\ + \alpha K^{-2}\eta^\gamma((\gamma+1)(\eta')^2 + \eta\eta'')((\gamma+2) - C(|y'| + \alpha K^{-1}\eta^{\gamma+1}\eta')) \\ - \alpha C\eta^{\gamma+1}\eta' K^{-1}(|y'| + \alpha K^{-1}\eta^{\gamma+1}\eta') . \end{aligned}$$

Now since $\eta(0) = 0$ and $\eta'' > 0$ we have $\eta(t) \leqq t\eta'(t)$ and so

$$\eta(K^{-1}y_n) \leqq K^{-1}y_n\eta'(K^{-1}y_n) \leqq K\beta(2\delta)K^{-1}\eta'(K^{-1}y_n) .$$

Hence provided

$$K\beta(2\delta) \leqq C \tag{2.8}$$

we may absorb the last term into the previous one. Now choosing δ sufficiently small, then K so that (2.5) and (2.8) hold and finally α sufficiently small we may conclude that (2.7), (2.5) and (2.6) all hold. □

An important choice for the function β is $\beta(t) = Kt^\alpha$ where $K \geqq 0$ and $0 < \alpha \leqq 1$. The condition on ϕ is then Hölder continuity. The case $\gamma = 0$, that is strictly positive mean curvature, was treated by Giusti [G2].

COROLLARY 1. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$ and let $H(x)$ be the mean curvature of $\partial\Omega$ at x. Suppose that $\phi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leqq 1$, and that there exist constants $\gamma \geqq 0$ and $a > 0$ such that for each $x_0 \in \partial\Omega$ we have $H(x) \geqq a|x - x_0|^\gamma$ in a neighbourhood of x_0. Then there is a function $u \in C^2(\overline{\Omega}) \cap C^0(\Omega)$ such that $Mu = 0$ in Ω and $u = \phi$ on $\partial\Omega$. Furthermore u is Hölder continuous with exponent $\dfrac{\alpha}{\gamma+2}$ on $\overline{\Omega}$.

There are of course many other possibilities for β. One example would be $\beta = -\dfrac{K}{\log at}$ in which case u has modulus of continuity $-\dfrac{K'}{\log a't}$ for some constants a' and K'.

Remarks. — *i*) It should be noted that the subadditivity condition on β could be relaxed. However if β is the modulus of continuity for a continuous function on an open set it is necessarily subadditive and so there is little to be gained by relaxing this assumption.

ii) The condition $H(x) \geqq a|x - x_0|^\gamma$ is a fairly strong one, in particular in the case $\gamma = 1$ it would imply that H is not differentiable at x_0. This condition has been relaxed even to the extent of allowing H to be negative at some places near x_0 (See [S2] and [W1].) The Hölder exponent for the solution u depends in the same manner on the growth of H.

iii) The reader should note that Simon [S1] has shown that if the boundary values of u have a modulus of continuity β then the solution u has some modulus of continuity ψ on $\overline{\Omega}$. The modulus ψ is in general much worse than the ones we have obtained above, however Simon's results hold without any restrictions on the smoothness of $\partial\Omega$ or the behaviour of its mean curvature.

3. LIPSCHITZ CONTINUOUS BOUNDARY

If we consider the particular case of Corollary 1 when we have Lipschitz continuous data and strictly positive mean curvature we see that the solution $u \in C^{0,\frac{1}{2}}(\overline{\Omega})$. Furthermore examples in [G2] and [G3] show that, in general, this is best possible (The results of sections 4 and 5 show how to construct numerous examples where the solutions grow like $|x|^{\frac{1}{2}}$.) If the assumption that $\partial\Omega$ has strictly positive mean curvature is relaxed then the regularity for u given by Theorem 1 is correspondingly less. Again examples (see section 4) show this is best possible. If we only assume nonnegative mean curvature then Giusti [G1] has shown that the solution is Hölder continuous with some exponent which will in general be much less than $\frac{1}{2}$. Techniques like those used in Theorem 1 can also be used to prove a local version of this result. In this section we show that it is possible to improve all these regularity results by taking into account bounds for the Lipschitz constant of the boundary data. More particularly we show that given α, $0 < \alpha < 1$, then there is a number $K(\alpha)$ such that if the Lipschitz constant of the boundary data is less than $K(\alpha)$ then the solution is in $C^{0,\alpha}(\overline{\Omega})$. The constant $K(\alpha)$ is obtained explicitly (at least in terms of the zeros of certain solution of well known ordinary differential equations) and it should be noted that it depends only on n and α and not on $\partial\Omega$ at all (although we must assume $\partial\Omega$ has nonnegative mean curvature). In the next section we show that, in general, the number $K(\alpha)$ is best possible (See, however, the more general result in Theorem 3.)

In order to define the numbers $K(\alpha)$ and construct the appropriate barriers it is necessary to consider solutions for Laplace's equation on conical domains in $\mathbb{R}^n$. In particular we shall look for positive solutions of the form $u = r^\lambda f(\theta)$, $|\theta| < \theta_0$ and $r > 0$ where r is the distance from the origin and θ is the angle with the x_n-axis. Since

$$\Delta(r^\lambda f(\theta)) = r^{\lambda-2}[f''(\theta) + (n-2)\operatorname{Cot}\theta f'(\theta) + \lambda(\lambda+n-2)f]$$

we are led to looking at the following problem:

$$(3.1)\quad f'' + (n-2)\operatorname{Cot}\theta f' + \lambda(n+\lambda-2)f = 0, \quad f'(0) = 0, \quad f(0) = 1.$$

We shall be concerned only with the cases $n \geqq 2$, $\lambda \geqq 1$ and $0 \leqq \theta \leqq \frac{\pi}{2}$.

For the most part the results are classical and so proofs are omitted. Similar considerations but for more general operators than the Laplacian are made by Miller in [M].

DEFINITION 3.1. — Suppose $\lambda \geqq 1$, $n \geqq 2$, $0 \leqq \theta \leqq \frac{\pi}{2}$. Then

i) $f_{n,\lambda}(\theta)$ is the solution of (3.1),

ii) $\psi_n(\lambda)$ is the first value of θ in $\left[0, \frac{\pi}{2}\right]$ such that $f_{n,\lambda}(\theta) = 0$.

Note that $f_{n,\lambda}(\theta)$ and $\psi_n(\lambda)$ exist. For example, using the method of Frobenius we may easily find a representation for $f_{n,\lambda}$ as a power series in $S = 1 - \cos\theta$ or see the more general results in [M].

Properties.

i) if $\lambda = 1$ and $n \geqq 2$, $f_{n,1}(\theta) = \operatorname{Cos}\theta$, $\psi_n(1) = \frac{\pi}{2}$,

ii) if $\lambda = 2$ and $n \geqq 2$, $f_{n,2}(\theta) = \operatorname{Cos}^2\theta - \frac{1}{n-1}\operatorname{Sin}^2\theta$, $\psi_n(2) = \arctan(\sqrt{n-1})$,

iii) if $\lambda \geqq 1$ and $n = 2$, $f_{2,\lambda}(\theta) = \operatorname{Cos}\lambda\theta$, $\psi_2(\lambda) = \frac{\pi}{2\lambda}$,

iv) if $0 < \theta < \psi_n(\lambda)$, $f'_{n,\lambda}(\theta) < 0$,

v) $\psi_n(\lambda)$ is strictly decreasing in λ.

DEFINITION 3.2. — For $n \geqq 2$ and $\alpha \in (0, 1)$ we define

$$K_n(\alpha) = \text{cotangent}\left(\psi_n\left(\frac{1}{\alpha}\right)\right).$$

Properties.

i) K_n is strictly decreasing in α,

ii) $$K_n\left(\frac{1}{2}\right) = \frac{1}{\sqrt{n-1}},$$

iii) $$\lim_{\alpha \to 0} K_n(\alpha) = \infty, \quad \lim_{\alpha \to 1} K_n(\alpha) = 0,$$

iv) $$K_2(\alpha) = \text{cotangent}\left(\frac{\pi\alpha}{2}\right).$$

With the aid of these definitions we can now state the main result of this section.

THEOREM 2. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$. Suppose $\phi \in L^1(\partial\Omega)$ and u is the generalized solution of the Dirichlet problem. Suppose there is a neighbourhood $\mathcal{N}$ of $x_0 \in \partial\Omega$ and numbers A, K and α, $0 < \alpha < 1$, such that

i) $\partial\Omega$ is C^2 and has nonnegative mean curvature in $\mathcal{N}$, *ii*) $$K < K_n(\alpha),$$

iii) $$\phi(x) - \phi(x_0) \leqq K\,|\,x - x_0\,|, \quad x \in \mathcal{N} \cap \partial\Omega,$$

iv) $$\phi(x) \leqq A, \quad x \in \partial\Omega,$$

then there is a constant C such that

$$u(x) - \phi(x_0) \leqq C\,|\,x - x_0\,|^\alpha.$$

Proof. — Since increasing ϕ will increase the corresponding solution u it is sufficient to prove the result in the case where $\phi(x) = K\,|\,x - x_0\,|$, $x \in \mathcal{N} \cap \partial\Omega$. Introduce x- and y-coordinate systems at x_0 and let w be a function whose graph gives $\partial\Omega$ near x_0. We may assume that the neighbourhood $\mathcal{N}$ has the form, for some $\delta_0 > 0$, $\mathcal{N} = \{ x : |\,x'\,| < \delta_0, |\,x_n\,| < \delta_0 \}$, that $\partial\Omega \cap \mathcal{N} \{ (x', w(x')) : |\,x'\,| < \delta_0 \}$ and that $K\,|\,x\,| \leqq \frac{1}{4}(3K + K_n(\alpha))\,|\,x'\,|$ for $x \in \mathcal{N} \cap \partial\Omega$.

Now set $J = \frac{1}{2}(K + K_n(\alpha))$, $\lambda = \frac{1}{\alpha}$ and $\theta_1 = \text{arcotangent}\,(J) > \psi_n(\lambda)$. We consider

$$f(y) = r^\lambda(f_{n,\lambda}(\theta) + b),$$
$$v(y) = w(y') + \beta f(y),$$

where $r = |\,y\,|$, $\theta = \arccos(r^{-1}y_n)$, $b = -\ f_{n,\lambda}(\theta_1) > 0$ and $1 > \beta > 0$ is to be chosen. Suppose $0 < \delta < \delta_0$ and let

$$D = \{ y \in \mathbb{R}^n : J\delta \geqq y_n \geqq J\,|\,y'\,| \}.$$

Then on D, $|\,Df\,| \leqq Cr^{\lambda-1}$, $|\,D^2 f\,| \leqq Cr^{\lambda-2}$, $\Delta f = \lambda(\lambda + n - 2)r^{\lambda-2}b$, $f(y) \geqq 0$, $f(y) = 0$ if and only if $y_n = J|\,y'\,|$ and $D_n f = r^{\lambda-1}\ [\lambda \operatorname{Cos}\theta f_{n,\lambda}(\theta) - \operatorname{Sin}\theta f'_{n,\lambda}(\theta)] > 0$. Consequently as in the proof of Theorem 1 there is a function $\tilde{v}(x)$ and a neighbourhood $\mathcal{G}$ of x_0 (depending on the choice of δ and β) such that graph $\tilde{v}$ = graph v, $\partial\Omega \cap \mathcal{G} = \{ (x', w(x')) : |\,x'\,| < \delta \}$, $\tilde{v}(x) = J\,|\,x'\,| \geqq K\,|\,x\,|$ if $x \in \mathcal{G} \cap \partial\Omega$ and $\tilde{v}(x) = J\delta$ if $x \in \partial\mathcal{G} \cap \Omega$. We need to show that for some choice of β and δ we have $Mv \geqq 0$ in D (and hence $M\tilde{v} \leqq 0$ in $\mathcal{G} \cap \Omega$) and $u \leqq J\delta$ on $\partial\mathcal{G}$. In this case we will have $u \leqq \tilde{v}$ on $\partial(\mathcal{G} \cap \Omega)$ and the comparison principle will then give that $u \leqq \tilde{v}$ in $\mathcal{G} \cap \Omega$.

As in Theorem 1 we use Lemma 1 of [S2] to obtain that, provided

$$\beta\,|\,Df\,| \leqq 1, \quad \text{in } D, \tag{3.2}$$

we have

$$M_0 v = M_0 w + \beta\Delta f + \beta E$$

where

$$\begin{aligned} |\,E\,| &\leqq C(|\,Df\,|\,|\,D^2 w\,| + |\,D^2 f\,|)(\beta\,|\,Df\,| + |\,Dw\,|) \\ &\leqq C(r^{\lambda-1} + r^{\lambda-2})(r^{\lambda-1} + r) \\ &\leqq Cr^{\lambda-2}(y_n^{\lambda-1} + y_n). \end{aligned}$$

Now since $\partial\Omega$ has nonnegative mean curvature in $\mathcal{N}$ we have $M_0 w \geqq 0$ and so $M_0 v \geqq \beta r^{\lambda-2}[\lambda(\lambda + n - 2)b - C(y_n^{\lambda-1} + y_n)]$. Since $\lambda > 1$ we can choose δ, $0 < \delta \leqq \delta_0$, such that $M_0 v \geqq 0$ on D. Furthermore with this choice of δ, $\beta \,|\, Df \,| \leqq \beta C \delta^{\lambda-1}$ and so by a suitable choice of β we can ensure that (3.2) holds.

To check that $u \leqq J\delta$ on $\partial\mathcal{G} \cap \Omega$ we first note that, provided $\beta f \leqq \delta_0$ on D,

$$\partial\mathcal{G} \cap \Omega = \{ x : x_n = w(x') + \beta f(x', J\delta), |x'| < \delta \}$$

so that if $x \in \partial\mathcal{G} \cap \Omega$, $x_n - w(x') \leqq C\beta\delta^\lambda$. Now since $\phi(x) = K\,|\,x - x_0\,|$ for $x \in \partial\Omega \cap \mathcal{N}$ we may apply the result of Giusti [G2] mentioned at the start of this section to conclude that u is Hölder continuous with some exponent, η, up to $\partial\Omega \cap \mathcal{G}$. Thus there is a constant a such that in $\mathcal{G} \cap \Omega$

$$u(x) \leqq \frac{1}{2}(J + K)\,|\,x'\,| + a\,|\,x_n - w(x')\,|^\eta .$$

Hence on $\partial\mathcal{G} \cap \Omega$ we have

$$u(x) \leqq \frac{1}{2}(J + K)\delta + a(C\beta\delta^\lambda)^\eta$$

and so, by possibly decreasing β, we can ensure that $u(x) \leqq J\delta = \tilde{v}(x)$ on $\partial\mathcal{G} \cap \Omega$.

We now have that $u \leqq \tilde{v}$ in $\mathcal{G}$ and so if we can show that $\tilde{v}(x) \leqq C\,|\,x\,|^\alpha$ the proof is complete. In the construction of $\tilde{v}$ we have $\tilde{v}(x) = y_n$, $x_n = y_{n+1}$, $x_i = y_i$, $i = 1, \ldots, n-1$ and $y_{n+1} = w(y') + \beta(|\,y'\,|^2 + y_n^2)^{\lambda/2}(f_{n,\lambda}(\theta) + b)$. If $y_n > 2J\,|\,y'\,|$ then $f_{n,\lambda}(\theta) + b \geqq C > 0$ and so $y_{n+1} - w(y') \geqq C\beta y_n^\lambda$, or, recalling that $\lambda = \dfrac{1}{\alpha}$, $\tilde{v}(x) \leqq C(x_n - w(x'))^\alpha$. On the other hand, if $y_n \leqq 2J\,|\,y'\,|$ then we have $\tilde{v}(x) \leqq 2J\,|\,x'\,|$. Consequently

$$\tilde{v}(x) \leqq 2J\,|\,x'\,| + C\,|\,x_n - w(x')\,|^\alpha \qquad \text{in } \mathcal{G} \cap \Omega . \qquad \square$$

COROLLARY 2. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$ having nonnegative mean curvature everywhere. Suppose $\phi \in C^{0,1}(\partial\Omega)$ and $|\,\phi(x) - \phi(y)\,| \leqq K\,|\,x - y\,|$ for all $x, y \in \partial\Omega$. Then there is a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $Mu = 0$ in Ω, $u = \phi$ on $\partial\Omega$ and $u \in C^{0,\alpha}(\overline{\Omega})$ for every α such that $K_n(\alpha) > K$.

It should be noted that in some cases a better exponent than that given in Theorem 2 may be obtained from Theorem 1. For example in the case of strictly positive mean curvature we obtain

COROLLARY 3. — Suppose Ω, ϕ and u are as in Corollary 2 and that $\partial\Omega$ has strictly positive mean curvature. If $K \geqq \dfrac{1}{\sqrt{n-1}}$ then $u \in C^{0,\frac{1}{2}}(\overline{\Omega})$, otherwise $u \in C^{0,\alpha}(\overline{\Omega})$ for every α such that $K_n(\alpha) > K$.

We now show that the Lipschitz constant of the boundary data is not really the crucial quantity determining the regularity of the solution u. For example if the boundary data is differentiable at each point then $u \in C^{0,\alpha}(\bar{\Omega})$ for every α, $0 < \alpha < 1$, even though the derivatives of ϕ and hence its Lipschitz constant may be arbitrarily large. Rather it is the size of the angle of any corners that there may be in the graph of the boundary data that determine the Hölder exponent for the solution u. Thus we replace the condition $\phi(x) \leqq \phi(x_0) + K|x - x_0|$ by a condition that the graph of ϕ near the point $(x_0, \phi(x_0))$ lies below a cone having slope K and vertex at $(x_0, \phi(x_0))$. This cone need no longer have vertical axis which would be equivalent to the case already considered). However since ϕ is assumed Lipschitz continuous at x_0 we may assume that the cone contains the positive x_{n+1}-axis.

DEFINITION 3.1. — Suppose $K > 0$ and $v \in \mathbb{R}^{n+1}$ satisfy $|v| = 1$ $v_{n+1} > \sqrt{\dfrac{K^2}{K^2+1}}$. Then we define

$$\mathscr{C}_{K,v} = \left\{ z \in \mathbb{R}^{n+1} : z \cdot v \geqq \sqrt{\frac{K^2}{K^2+1}}\,|z| \right\}.$$

Remark. — $\mathscr{C}_{K,v}$ is a cone of slope K over the hyperplane through the origin which has normal v. The condition $v_{n+1} > \sqrt{\dfrac{K^2}{K^2+1}}$ ensures that $\mathscr{C}_{K,v}$ contains $\{ z \in \mathbb{R}^{n+1} : z_{n+1} \geqq 0 \}$. It also ensures that the boundary of $\mathscr{C}_{K,v}$ can be written as the graph of some function defined over $\mathbb{R}^n$.

DEFINITION 3.2. — Suppose $\mathscr{C}_{K,v}$ is as in Definition 3.1. Then $F_{K,v}$ is the unique function such that

$$\mathscr{C}_{K,v} = \{ (x, t) : x \in \mathbb{R}^n,\ t \geqq F_{K,v}(x) \}.$$

THEOREM 3. — The results of Theorem 2 and Corollaries 2 and 3 remain true if condition *iii*) is replaced by

iii) there exists $v \in \mathbb{R}^{n+1}$ with $|v| = 1$ and $v_{n+1} > \sqrt{\dfrac{K^2}{K^2+1}}$ such that

$$\phi(x) \leqq \phi(x_0) + F_{K,v}(x - x_0), \quad x \in \mathscr{N} \cap \partial\Omega.$$

Proof. — Since, after introducing x-coordinates, the tangent plane to $\partial\Omega$ at x_0 is $\{ x : x_n = 0 \}$ we may as well assume that $v_n = 0$. If this were not the case we could tilt $\mathscr{C}_{K,v}$ until $v_n = 0$ and so obtain a better value for K. Thus by disregarding the nth coordinate of v we obtain a vector (still called v) in the y-coordinate system. Now construct f as in Theorem 2 but then rotate so that the y_n-axis coincides with v and so obtain a function f_ν. Calculations can be made similarly to those before (see [W2] for the modifications) to show that there is a function $\tilde{v}_\nu$ and a neighbourhood $\mathcal{G}$ of x_0 such that $M\tilde{v}_\nu \leqq 0$ in $\mathcal{G} \cap \Omega$, $\tilde{v}_\nu \geqq u$ on $\partial(\mathcal{G} \cap \Omega)$ and $\tilde{v}_\nu$ has the right growth. The only difficulty is to check that v_ν really has graph which can be written as the graph of $\tilde{v}_\nu$. That is we must check that $D_n f_\nu > 0$, or since f_ν is obtained by rotation of f, that $D_\nu f > 0$ in D. Using the condition $\nu_{n+1} > \sqrt{\dfrac{K^2}{K^2+1}}$ and $|\nu| < 1$ it is seen to be sufficient to show that $\left(\sum_{i=1}^{n-1} (D_i f)^2\right)^{\frac{1}{2}} \Big/ D_n f \leqq K$. Or rather since we have a strict inequality in the condition on ν it will be sufficient to prove this inequality under the assumption that $J = K = K_n(\alpha)$ and $b = 0$. Hence we may assume that $f(y) = r^\lambda f_{n,\lambda}(\theta)$ and so $\Delta f = 0$. We set $v(y) = \left(\sum_{i=1}^{n-1} (D_i f)^2\right)^{\frac{1}{2}} \Big/ D_n f$ and an easy calculation shows that

$$v(y) = \frac{|\lambda \tan\theta f_{n,\lambda} + f'_{n,\lambda}|}{\lambda f_{n,\lambda} - \tan\theta f'_{n,\lambda}}$$

and so is independent of r. Furthermore $v = 0$ if $\theta = 0$ and $v = K$ if $\theta = \theta_1$. On the other hand it is not hard to show that $\Delta v + \sum_{i=1}^{n} b_i D_i v \geqq 0$ for some functions b_i so that v cannot have an interior maximum point. Hence $0 \leqq v \leqq K$ in D and so $D_n f_\nu > 0$ in D. □

COROLLARY 4. — The results of Theorem 2 and Corollaries 2 and 3 remain true if condition *iii)* is replaced by

iii)″ $\phi(x) = \phi_1(x) + \phi_2(x)$ for $x \in \partial\Omega \cap \mathcal{N}$ where ϕ_1 is differentiable at x_0 and $\phi_2(x) \leqq \phi_2(x_0) + K|x - x_0|$ for $x \in \partial\Omega \cap \mathcal{N}$.

COROLLARY 5. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$ which is C^2 near $x_0 \in \partial\Omega$. Suppose that $\phi \in L^\infty(\partial\Omega)$ is differentiable at x_0 and u is the generalized solution of the Dirichlet problem. Then for every $\alpha \in (0, 1)$ there is a constant C such that

$$|u(x) - \phi(x_0)| \leqslant C|x - x_0|^\alpha, \quad x \in \Omega.$$

Remark. — This result appears in [L3] where it is shown to be best possible. That is, in general the Corollary is not true with $\alpha = 1$.

4. GROWTH WITH LIPSCHITZ DATA

In this section we show that the bound $K_n(\alpha)$ on the Lipschitz constant of the boundary data (given in the last section) cannot be increased if we wish to ensure Hölder continuity of the solution u with exponent α. We also show that without restricting the Lipschitz contant in any way then the best Hölder exponent is that given by Theorem 1 $\left(\text{namely } \dfrac{1}{\gamma+2}\right)$. The method is to take boundary data which has a particular form near a fixed point $x_0 \in \partial\Omega$ and then introduce the y-coordinate system of section 1. In this situation a result of Simon [S3] allows us to write the graph of the solution u as a C^1 graph in the y-coordinates, at least near x_0. We then use techniques like those in the last section to construct a lower barrier having the required growth at x_0.

THEOREM 4. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega$. Suppose $\phi \in L^1(\partial\Omega)$ and u is the generalized solution of the Dirichlet problem. Suppose $x_0 \in \partial\Omega$ and there is a neighbourhood $\mathcal{N}$ of x_0 and numbers $\gamma \geqq 0$, $a \geqq 0$, $\alpha \in \left[\dfrac{1}{\gamma+2}, 1\right)$ and $K > K_n(\alpha)$ such that

i) $\partial\Omega$ is C^2 in $\mathcal{N}$ and, if $H(x)$ is the mean curvature of $\partial\Omega$ at x, then $H(x) \leqq a\,|x - x_0|^\gamma$ for $x \in \partial\Omega \cap \mathcal{N}$,

ii) $\phi(x) - \phi(x_0) \geqq K\,|x - x_0|$, $x \in \mathcal{N} \cap \partial\Omega$,

iii) $\phi(x) \geqq \phi(x_0)$, $x \in \partial\Omega$.

Then there are constants $C > 0$ and $t_0 > 0$ such that, if ν is the inward unit normal to $\partial\Omega$ at x_0, then

$$u(t\nu + x_0) \geqq \phi(x_0) + t^\alpha, \quad 0 \leqq t \leqq t_0\,.$$

Proof. — We may assume $\phi(x_0) = 0$ and introduce x- and y-coordinate systems at x_0. Furthermore we may assume, perhaps after decreasing $\mathcal{N}$ and K slightly (but so that we still have $K > K_n(\alpha)$) that $\phi(x) = K\,|x'|$ for $x \in \mathcal{N} \cap \partial\Omega$. Now by a result of Simon [S3], close to $(x_0, \phi(x_0))$ the graph of u can be represented as the graph of a function $v(y)$ in the y-coordinates. Thus for some δ, $0 < \delta < 1$, we have $Mv = 0$ in D and $v(y) = w(y')$ if $y_n = K\,|y'|$, where $D = \{\, y \in \mathbb{R}^n : K\delta > y_n > K\,|y'|\,\}$. The function w is the one which describes $\partial\Omega$ near x_0 as in section 1, and so $w(0) = 0$, $|w(y')| \leqq L\,|y'|^2$, $|Dw(y')| \leqq L\,|y'|$ and $|D^2w(y')| \leqq L$ for some constant L. Furthermore the results of [S3] give that v is C^1 in $\overline{D}$ and so

$$|v(y)| \leqq |y|\,\varepsilon(|y|) \tag{4.1}$$

where $\varepsilon(t) \to 0$ as $t \to 0$.

Now let $\lambda = \frac{1}{\alpha}$ (so that $1 < \lambda \leqq \gamma + 2$), $J = \frac{1}{2}(K_n(\alpha) + K)$ and $\theta_1 = \text{arcotangent}\,(J)$ (so that $\theta_1 < \Psi_n(\lambda)$). We consider

$$f(y) = r^\lambda (f_{n,\lambda}(\theta) - b)$$

where $r = |y|$, $\theta = \arccos(y_n r^{-1})$, $f_{n,\lambda}$ is defined in Definition 3.1 and $b = f_{n,\lambda}(\theta_1) > 0$. Then there are constants $\beta_0 > 0$ and C such that on D,

$$\begin{aligned} & f(y) \geqq \beta_0 r^\lambda \geqq \beta_0 r^{\gamma+2}, \\ (4.2) \qquad & \Delta f(y) = -b\lambda(\lambda + n - 2) r^{\lambda-2}, \\ & |Df(y)| \leqq Cr^{\lambda-1} \quad \text{and} \quad |D^2 f(y)| \leqq Cr^{\lambda-2}. \end{aligned}$$

Now we set

$$h(y) = w(y') + \beta f(y)$$

where β is yet to be chosen. Provided $\beta |Df| \leqq 1$ or in other words, provided

$$(4.3) \qquad \beta C \delta^{\lambda-1} \leqq 1$$

we have by Lemma 1 of [S2],

$$\begin{aligned} M_0 h &= M_0 w + \beta \Delta f + \beta E \\ &\leqq a r^\gamma - \beta b \lambda(\lambda + n - 2) r^{\lambda-2} + \beta |E| \end{aligned}$$

where

$$\begin{aligned} |E| &\leqq C(|D^2 f| + |Df|\,|D^2 w|)(\beta |Df| + |Dw|) \\ &\leqq C r^{\lambda-2}(\beta r^{\lambda-1} + r). \end{aligned}$$

Thus on D,

$$M_0 h \leqq a r^{\lambda-2} - \beta r^{\lambda-2}(\lambda b(\lambda + n - 2) - C(\beta \delta^{\lambda-1} + \delta)).$$

Consequently if

$$(4.4) \qquad C(\beta \delta^{\lambda-1} + \delta) \leqq \frac{1}{2} \lambda b(\lambda + n - 2),$$

$$(4.5) \qquad \beta \lambda b(\lambda + n - 2) \geqq 2a,$$

we will have that $M_0 h \leqq 0$ on D.

Now by (4.1)

$$|v(y) - w(y')| \leqq |y| \eta(|y|)$$

where $\eta(t) = \varepsilon(t) + Lt \to 0$ as $t \to 0$, and so if $\beta f(y) \geqq |y| \eta(|y|)$ when $y_n = K\delta$ and $y \in D$ we will have that $h \geqq v$ on ∂D. Hence we require, (using (4.2) and the definition of D), that

$$(4.6) \qquad \beta \beta_0 \delta^\lambda \geqq C \delta \eta(C\delta)$$

where C depends on K.

It is easily seen that by the correct choice of β and δ, (4.3)-(4.6) may be satisfied. For example assume $\beta = B \max\{1, \delta^{1-\lambda}\eta(C\delta)\}$ for some $B > 0$, Then (4.5) and (4.6) are satisfied for any choice of δ by taking B large enough. Fixing this B (4.3) and (4.4) can be satisfied if δ is sufficiently small. Thus we have that $M_0 h \leqq 0$ on D, and $h \geqq v$ on ∂D and so by the comparison principle.

$$(4.7) \qquad v(y) - w(y') \leqq h(y) - w(y') = \beta f(y) \leqq Cr^{\lambda} \leqq C y_n^{\lambda}.$$

Now recall that the graph of v over D is the same as the graph of u over $\mathcal{G} \cap \Omega$ for some neighbourhood $\mathcal{G}$ of x_0. Consequently if $x \in \mathcal{G} \cap \Omega$,

$$(x', x_n, u(x)) = (x', v(x', u(x)), u(x)).$$

Then with the special choice $x' = 0$ and so $x = x_n e_n$ and $w(x') = 0$ we have by (4.7)

$$x_n \leqq C u^{\lambda}(x_n e_n)$$

and the result is proved. □

COROLLARY 6. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ having zero mean curvature in a neighbourhood of $x_0 \in \partial\Omega$. Then for any $\alpha \in (0, 1)$ there are functions in $C^{0,\alpha}(\partial\Omega)$, arbitrarily small in norm, such that the corresponding solutions of the Dirichlet problem are not Hölder continuous at x_0 for any exponent.

5. GROWTH WITH CONTINUOUS DATA

In this final section we aim to show that the modulus of continuity in Theorem 1 cannot, in general, be improved. We note, however, that for the case of Lipschitz data the results of section 3 showed that it could be improved. Thus we cannot expect to construct examples showing that we have obtained the best modulus of continuity for all functions β satisfying the hypotheses of Theorem 1. The extra conditions we impose will still allow most examples of interest such as Hölder and logarithmic growth.

THEOREM 5. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ and $u \in C^2(\bar{\Omega}) \cap C^0(\Omega)$ satisfies $Mu = 0$ in Ω. Suppose there is a neighbourhood $\mathcal{N}$ of $x_0 \in \partial\Omega$, a function $\beta : [0, \infty) \to [0, \infty)$ and a number $\varepsilon > 0$ such that

i) $\partial\Omega$ is C^2 in $\mathcal{N}$,

ii) $u(x) \geqq u(x_0) + \beta(|x - x_0|), \quad x \in \partial\Omega \cap \mathcal{N}$

$u(x) \geqq u(x_0) + \varepsilon, \quad x \in \partial\Omega \sim \mathcal{N}$

iii) β is an increasing subadditive function with $\lim_{t\to 0} \beta(t) = 0$ and $\lim_{t\to 0} \beta(t)/t = \infty$.

Then there is a constant $C > 0$ such that if ν is the normal to $\partial\Omega$ at x_0,

$$u(x) \geqq u(x_0) + \beta(C\,|(x - x_0)\cdot\nu\,|^{\frac{1}{2}}), \quad x \in \Omega\,.$$

Proof. — Since $\partial\Omega$ is C^2 near x_0, there exists $R_0 > 0$ such that if $R \leqq R_0$ there is a ball B_R of radius R such that $\overline{B}_R \cap \overline{\Omega} = \{ x_0 \}$. We may assume $u(x_0) = 0$, the centre of B_R is 0 and $x_0 = Re_n$. Now consider the function

$$\delta(x) = |x| - R, \quad |x| \geqq R$$

which gives the distance from x to ∂B_R if x is outside B_R. As in Theorem 1 we can assume $\beta \in C^2(0, \infty) \cap C^0[0, \infty)$ and is concave. The additional assumption, $\dfrac{\beta(t)}{t} \to \infty$, means that $\beta'(t) \to \infty$ as $t \to 0$. We set $\eta(t) = \beta(\sqrt{t})$ and

$$v(x) = \eta(\alpha R\delta(x))$$

where α is a constant chosen so that

$$(5.1) \qquad v(x) \leqq \eta(\alpha R_0\delta(x)) \leqq \eta(\alpha R_0 d) \leqq \varepsilon$$

where d is the diameter of Ω. If $x_n = R$ we have

$$(5.2) \qquad \delta(x) = \sqrt{|x'|^2 + R^2} - R \leqq \frac{|x'|^2}{2R} \leqq \frac{|x|^2}{2R}$$

and so $v(x) \leqq \eta\left(\dfrac{\alpha}{2}|x'|^2\right)$ when $x_n = R$. But the plane $\{ x_n = R \}$ is the tangent plane to $\partial\Omega$ at x_0 and $\partial\Omega$ is C^2 near x_0 and so by possibly decreasing α and the neighbourhood $\mathcal{N}$ we have

$$(5.3) \qquad v(x) \leqq \eta(|x|^2) = \beta(|x|) \leqq u(x), \quad x \in \partial\Omega \cap \mathcal{N}\,.$$

Then by (5.1) and (5.3) we have that $v(x) \leqq u(x)$ if $x \in \partial\Omega$ and so, if we can show that $Mv \geqq 0$ in Ω we will have that $u \geqq v$ in Ω. But

$$\delta(x) = \frac{(|x'|^2 + x_n{}^2 - R^2)}{\delta(x) + 2R} \geqq C\,|x_n - R|, \quad \text{in } \Omega\,,$$

for some constant C depending on R, R_0 and d. Consequently we will have $v(x) \geqq \eta(C\,|x_n - R|)$ in Ω and the theorem is proved.

It only remains to check that given α we can choose $R < R_0$ such that $Mv \leqq 0$ in Ω Now $D_i v = \alpha R|x|^{-1}x_i$, $D_{ij}v = \alpha^2R^2|x|^{-2}x_ix_j\eta'' + (\delta_{ij} + |x|^{-2}x_ix_j)\alpha R|x|^{-1}\eta'$, $\Delta v = \alpha^2R^2\eta'' + \alpha R|x|^{-1}(n-1)\eta'$, $1 + |Dv|^2 = 1 + \alpha^2R^2(\eta')^2$ and (using the summation convention) $D_iv D_jv D_{ij}v = \alpha^4R^4(\eta')^2\eta''$. Thus

$$\begin{aligned}(1 + |Dv|^2)M_0v &= \alpha R|x|^{-1}\{\alpha R|x|\eta'' + (n-1)\eta'(1 + \alpha^2R^2(\eta')^2)\} \\ &= \alpha R|x|^{-1}\{tn''(t) + (n-1)\eta'(t)\} + \alpha^2R^3|x|^{-1}\{\eta''(t) + \alpha(n-1)(\eta'(t)^3\}.\end{aligned}$$

where we have put $t = \alpha R\delta(x) \leqq \alpha Rd$. Now by hypothesis *iii)* of the theorem both these terms are positive for t sufficiently small and so by taking R sufficiently small we can ensure $Mv \geqq 0$ in Ω. □

Examples. — *i)* $\beta(t) = \mathrm{K}t^\alpha$, $0 < \alpha < 1$.

ii) $\beta(t) = -\dfrac{\mathrm{K}}{\log at}$, $k > 0$, $a > 0$.

Remark. — The theorem shows that, in the mentioned examples, the strongest regularity given by Theorem 1 (that is when $\gamma = 0$ or we have strictly positive mean curvature) cannot be improved. In particular we have:

COROLLARY 5. — Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$. Then for any α, $0 < \alpha < 1$, there exist functions ϕ such that $\phi \in C^{0,\alpha}(\partial\Omega)$ but if u satisfies $\mathrm{M}u = 0$ in Ω and $u = \phi$ on $\partial\Omega$ then $u \notin C^{0,\gamma}(\overline{\Omega})$ for any $\gamma > \frac{\alpha}{2}$. If $\partial\Omega$ has strictly positive mean curvature then $u \in C^{0,\frac{\alpha}{2}}(\overline{\Omega})$.

Remark. — We note that by taking $\gamma = 0$ and $\mathrm{K} > \mathrm{K}_n\left(\frac{1}{2}\right) = \dfrac{1}{\sqrt{n-1}}$ in Theorem 4 this Corollary also holds for the case $\alpha = 1$. However the examples in Corollary 4 can be arbitrarily small in $C^{0,\alpha}(\partial\Omega)$ norm whereas in the Lipschitz case we need the Lipschitz constant at least $\dfrac{1}{\sqrt{n-1}}$. In fact this last result about Lipschitz data can also be proved using the same technique as in Theorem 5. Indeed if we take $v(x) = \sqrt{\alpha\mathrm{R}\delta(x)}$ where α is to be chosen (that is we take $\eta(t) = \sqrt{t}$) then the same calculation shows that

$$(1 + |\mathrm{D}v|^2)\mathrm{M}_0 v = \frac{1}{2}\alpha\mathrm{R}|x|^{-1}t^{-\frac{1}{2}}\left(n - 1\frac{1}{2}\right) + \frac{\alpha^2\mathrm{R}^3|x|^{-1}t^{-3/2}}{8}(-2 + \alpha(n-1)).$$

Hence if $\alpha \geqq \dfrac{2}{n-1}$ we have $\mathrm{M}v \geqq 0$. Also as in (5.3) we have that when $x_n = \mathrm{R}$

$$v(x) \leqq \eta\left(\frac{\alpha}{2}|x'|^2\right) = \frac{1}{\sqrt{n-1}}|x'|$$

so that if $\partial\Omega$ is C^2 near x_0 we may ensure $v(x) \leqq u(x)$ for $x \in \partial\Omega \cap \mathcal{N}$ provided $\mathrm{K} > \dfrac{1}{\sqrt{n-1}}$.

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