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A proof of Weinstein's conjecture in $\mathbb{R}^{2n}$

by

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ABSTRACT. — We prove that a hypersurface of contact type in $(\mathbb{R}^{2n}, \sum dx_i \wedge dy^i)$ has a closed characteristic. A geometric trick is used to reduce this problem to finding T-periodic solutions of a Hamiltonian system. This system is studied using the Clarke-Ekeland-Lasry dual action functional.

RÉSUMÉ. — On démontre que toute hypersurface de genre contact de $(\mathbb{R}^{2n}, \sum dx_i \wedge dy^i)$admet au moins une caractéristique fermée. Une astuce géométrique ramène notre problème à la recherche d'orbites T périodiques d'un système hamiltonien. Ce système est analysé en utilisant la fonctionnelle d'action duale de Clarke-Ekeland-Lasry.

INTRODUCTION

In his paper "On the hypothesis of Rabinowitz' periodic orbit theorem" (denoted by [W. 2] in the sequel) A. Weinstein made the following conjecture.

CONJECTURE. — If $\Sigma \subset (M, \omega)$ is a compact hypersurface of contact type in a symplectic manifold, satisfying $H^1(\Sigma; \mathbb{R})=0$, then Σ has a closed characteristic.

The definition of being of contact type is given in section one (Definition 1.1). We recall that a characteristic is a curve everywhere tangent to the line field $\ker \omega|_\Sigma$.

As the reader might have hinted from the title, our aim is to prove this conjecture for $(M, \omega)=(\mathbb{R}^{2n}, \omega_0)\left(\text{where } \omega_0=\sum_{i=1}^{n} dx_i \wedge dy^i\right)$. Thus, we state.

THEOREM. — *If $\Sigma \subset (\mathbb{R}^{2n}, \omega_0)$ is a compact hypersurface of contact type, then Σ has at least one closed characteristic.* □

Let us mention that we dropped the hypothesis $H^1(\Sigma; \mathbb{R})=0$; for our proof we only need that Σ has an interior in $\mathbb{R}^{2n}$, which is automatic.

Recall that if J is the standard symplectic matrix, and $N(x)$ denotes the outward normal to Σ at x, then the closed characteristics of Σ correspond to periodic solutions of

$$\left.\begin{aligned} \dot{x} &= JN(x) \\ x(t) &\in \Sigma. \end{aligned}\right\} \qquad (\mathcal{N})$$

The standard approach to $(\mathcal{N})$ is to transform it into a fixed period Hamiltonian system, that is, to find a function H on $\mathbb{R}^{2n}$ such that the non-trivial solutions of

$$\left.\begin{aligned} \dot{x} &= J\nabla H(x) \\ x(0) &= x(T) \end{aligned}\right\} \qquad (\mathcal{H})$$

correspond to periodic solutions of $(\mathcal{N})$ (here T is fixed).

Let us recall shortly the historical background. In 1948, Seifert proved existence of closed characteristics for some special class of convex hypersurfaces (*cf.* [S]). Thirty years later, Weinstein (*cf.* [W. 1]) extended this result to general C^2 convex hypersurfaces, and Rabinowitz (in [R]) to strictly starshaped hypersurfaces. Let us mention also the work of Bahri (*cf.* [B. 2] or a sketchy description in [B. 1]).

Rabinowitz' idea to construct H is, assuming Σ is starshaped with respect to the origin, to set

$$H=1 \quad \text{on } \Sigma$$

$$H(\lambda x)=\varphi(\lambda) \qquad \text{for all} \quad x\in\Sigma, \quad \lambda\geqq 0$$

where φ is some well chosen function.

In chapter one, by a modification of this idea we get a function H such that non-trivial solutions of $(\mathscr{H})$ yield periodic solutions of $(\mathscr{N})$.

In chapter two, we define the dual action functional, according to an idea of Clarke and Ekeland (*cf.* [C-E]) later modified by Berestycki, Lasry, Mancini and Ruf (*cf.* [B-L-M-R]).

The finite dimensional reduction seems to be needed in order to prove the Palais-Smale condition: all the known proofs of the (P.S.) condition use the fact that $\nabla H(z).z\geqq\alpha H(z)>0$ for some positive α, that is the level hypersurfaces of H are strictly starshaped. Let us remark that this condition is needed whether the direct action functional or the dual one are used.

Chapter three is concerned with the proof of the (P.S.) condition for the finite dimensional reduction.

Finally by a cohomological argument, we prove that our functional has non-trivial critical points, that is done in chapter four.

I am glad to thank Leila Lassoued for interesting discussions during a stay at the University of Tunis. François Laudenbach for introducing me to symplectic geometry, and for attempting (sometimes unsuccessfully as it is the case in this paper) to get from me "geometric proofs". Abbas Bahri for useful comments, Helmut Hofer for reading the manuscript, finding mistakes, and simplifying the proof (*cf.* [H-Z]).

And of course special thanks to Ivar Ekeland. He introduced me to Hamiltonian systems. The reader will easily trace his influence in this work. Let me mention how enjoyable it is to work under his direction.

NOTATIONS AND STANDARD DEFINITIONS

$(.,.)$; $|\ .\ |$ scalar product and norm in euclidean space;
$\langle .,. \rangle$; $\|\ \ \|$ scalar product and norm in L^2 space;

"strictly convex function": f such that

$$\forall x, y \in \mathbb{R}^q (f'(x) - f'(y), x - y) \geqq \varepsilon |x - y|^2$$

for some positive ε;

"conformal diffeomorphism": φ such that $\varphi^* \omega = c\omega$ (c a non-zero constant) $H_*(A)$, $H^*(A)$: homology group, cohomology ring of A (rational coefficients) $H_{S^1,*}(A)$, $H^*_{S^1}(A)$: equivariant homology group and cohomology ring of A in the sense of Borel (*cf.* [Bo]) (with rational coefficients);

[x]: integer part of x;

$\square$: end of a statement;

$\odot$: end of a proof.

1. REDUCTION TO A HAMILTONIAN SYSTEM IN $\mathbb{R}^{2n}$

Let (M^{2n}, ω) be a symplectic manifold, Σ a hypersurface of contact type of (M^{2n}, ω), that is:

DEFINITION 1.1. — Σ is said to be of *contact type* if and only if there is a 1-form θ on Σ such that

(i) $d\theta = j^* \omega$ (where $j: \Sigma \to M$ is the inclusion map);

(ii) $\theta \wedge (d\theta)^{n-1}$ is a volume form on Σ (i. e. does not vanish on Σ). $\square$

We now have (*cf.* [W.2], p. 354, Lemma 2).

LEMMA 1.2. — *Σ is of contact type if and only if there is a vector field η, defined in a neighborhood of Σ, which is:*

$$\textit{transverse to } \Sigma \tag{1.3}$$

$$\textit{satisfies } L_\eta \omega = \omega. \ \square \tag{1.4}$$

Proof. — Consider the form θ of Definition 1. By Poincaré's lemma, we can extend θ to $\tilde{\theta}$, defined in a neighborhood of Σ, such that $d\tilde{\theta} = \omega$.

Set $\tilde{\theta} = i_\eta \omega$ (η exists since ω is non-degenerate), then $\eta(x)$ is not in $T_x\Sigma$ otherwise we would have $\eta(x) = dj(x)\xi$, with $\xi \in T_x\Sigma$, and

$$j^*(\tilde{\theta} \wedge (d\tilde{\theta})^{n-1}(x) = j^*(i_\eta \omega^n)(x) = i_\xi j^*(\omega^n)(x)$$

which is zero because $j^*(\omega^n)$ is zero. So η is transverse to Σ.

Checking (1.4) is straightforward:

$$L_\eta \omega = di_\eta \omega + i_\eta d\omega = d\tilde{\theta} = \omega. \odot$$

From the lemma, we infer:

PROPOSITION 1.5. — *For some positive* ε, *there is a symplectic diffeomorphism*

$$\varphi: \quad (\Sigma \times]1-\varepsilon, 1+\varepsilon[, d(\operatorname{tr}^*\theta)) \to (U, \omega)$$

where r (resp. t) is the projection of $\Sigma \times]1-\varepsilon, 1+\varepsilon[$ *on the first (resp. second) factor, and* U *is a tubular neighborhood of* Σ *in* (M, ω).

Moreover $\varphi|_{\Sigma\times\{1\}} = \mathrm{id}_\Sigma$. □

Proof. — Write ψ_s for the flow of η defined in Lemma 1.2, and set

$$\varphi(x, e^s) = \psi_s(x) \qquad \text{for} \qquad (x, s) \in \Sigma \times]-\varepsilon, \varepsilon[. \tag{1.6}$$

By (1.3), φ is a diffeomorphism provided ε is small. To show that it is symplectic, we shall compare $\varphi^*(\tilde{\theta})$ and $\operatorname{tr}^*(\theta)$ (the reader is invited to check that $d(\operatorname{tr}^*(\theta))$ is a symplectic form. Now,

$$\frac{\partial}{\partial s}\psi_s^*(\tilde{\theta}) = \psi_s^*(L_\eta \tilde{\theta}) = \psi_s^*(di_\eta \tilde{\theta} + i_\eta d\tilde{\theta}).$$

From $\tilde{\theta} = i_\eta \omega$ and $d\tilde{\theta} = \omega$ we infer $i_\eta \tilde{\theta} = 0$ and $i_\eta d\tilde{\theta} = \tilde{\theta}$, so $\frac{\partial}{\partial s}\psi_s^*(\tilde{\theta}) = \psi_s^*\tilde{\theta}$ which implies

$$\psi_s^*\tilde{\theta} = e^s\tilde{\theta}. \tag{1.7}$$

Let us write $\psi(x, s) = \psi_s(x)$, then

$$\begin{aligned}
\psi^*(\tilde{\theta}) &= \tilde{\theta}(\psi(x, s))\, d\psi(x, s) \\
&= \tilde{\theta}(\psi_s(x))\left(d\psi_s(x) + \frac{\partial}{\partial s}\psi_s(x)\, ds\right) \\
&= \psi_s^*(\tilde{\theta}) + \psi_s^*(i_\eta \tilde{\theta})\, ds \\
&= \psi_s^*(\tilde{\theta}) \quad \text{as } i_\eta \tilde{\theta} = 0 \\
&= e^s \tilde{\theta} \quad \text{by (1.5)}
\end{aligned}$$

The change of variable $t = e^s$ yields, using (1.6),

$$\varphi^*(\tilde{\theta}) = \operatorname{tr}^*(\theta).$$

Finally, the last assertion of (1.5) is obvious. ⊙

We now assume the origin to be an interior point of Σ. Our aim is to prove.

PROPOSITION 1.8. – *For any a, positive constant, there is a* C^∞ *function* H *on* $\mathbb{R}^{2n}$ *satisfying:*

(i) $H(0)>0$ *is the absolute minimum of* H, *and* H *is constant in a neighborhood of the origin.*

(ii) H'' *is bounded.*

(iii) $H(z) \geqq \frac{a}{2}|z|^2$ *for* $|z|$ *large enough.*

(iv) *If* $(\mathscr{H})$ *has a nonconstant solution, then* $(\mathscr{N})$ *has a periodic orbit.* □

Proof. – We construct H explicitly.

Let $k>1$ be some sufficiently large number so that U and $k.U$ are disjoint [U is as in (1.5), $k.U$ is the image of U by a dilation of ratio k].

This implies that the $k^p.U$, for p positive integers, will be pairwise disjoint.

We first define H on $\bigcup_{p \geqq 1} k^p.U$

by

$$H(k^p.\varphi(x, t)) = k^{2p}\lambda(t) \tag{1.9}$$

where λ is some increasing function on $[1-\varepsilon, 1+\varepsilon]$ that shall be defined more precisely later on.

From (1.9) we can check that in $k^p.U$ the level hypersurfaces of H are the $k^p\psi_s(\Sigma)$. Since by (1.7) ψ_s is a conformal map, and the dilations are also conformal, the $k^p\psi_s(\Sigma)$ are conformally diffeomorphic to Σ. Hence on $\bigcup_{p \geqq 1} k^p.U$, (iv) holds.

Also on $\bigcup_{p \geqq 1} k^p.U$ (i) to (iii) hold provided

$$aR^2 \geqq \inf_t \lambda(t) = \lambda(1-\varepsilon) \tag{1.10}$$

where R is a real number such that U is contained in the ball of radius R, centered at the origin.

Now let us extend H to $kD-\Delta$ where D is the union of U and the interior of Σ, and $\Delta = D - U$.

We can assume that H is constant on $k\Delta - D$ provided $k^2 \lambda(1-\varepsilon) = \lambda(1+\varepsilon)$ which we will assume henceforth.

We then extend H to $\mathbb{R}^{2n} - \Delta$ by setting

$$H(k^p z) = k^{2p} H(z) \qquad \text{for} \quad z \in k\,D - \Delta, \quad p \in \mathbb{N}^*. \tag{1.11}$$

We finally set $H(z) = \lambda(1-\varepsilon)$ for $z \in \Delta$. It is now easy to check properties (i) to (iv):

(i) is obvious,

(ii) follows from (1.11), for this implies

$$H''(k^p z) = H''(z) \qquad \text{for} \quad z \in k\,D - \Delta, \quad p \in \mathbb{N}^*$$

since H'' is continuous, it is bounded on $k\,D - \Delta$, hence on $\mathbb{R}^{2n}$.

(iii) follows also from (1.11) for it implies

$$H(z) \geqq \lambda(1-\varepsilon) \frac{|z|^2}{k^2 R^2}$$

and we can assume $\lambda(1-\varepsilon) \geqq \frac{a}{2} k^2 R^2$.

(iv) Consider a non-constant solution, its trajectory has to be contained in $\bigcup_{p \geqq 1} k^p U$, thus yielding a periodic solution of $(\mathcal{N})$. ⊙

2. THE DUAL ACTION FUNCTIONAL AND ITS FINITE DIMENSIONAL REDUCTION

Let H be a function on $\mathbb{R}^{2n}$ such that H'' is bounded. Then we can find some positive real number, K, such that $H'' + KI$ is everywhere greater than εI for some positive ε, thus $H_K(z) = H(z) + \frac{K}{2}|z|^2$ is strictly convex, and we can consider its dual function in the sense of Fenchel (*see* [E-T])

$$H_K^*(y) = \sup_{z \in \mathbb{R}^{2n}} [(z, y) - H_K(z)]$$

which main virtue is to satisfy

$$\nabla H_K^*(\nabla H_K(z)) = \nabla H_K(\nabla H_K^*(z)) = z. \tag{2.1}$$

For $x \in X = W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$, we define

$$F_K(x) = \int_0^T \left[\frac{1}{2}(J\dot{x} - Kx, x) + H_K^*(-J\dot{x} + Kx)\right] ds \tag{2.2}$$

Assume $\frac{KT}{2\pi} \notin \mathbb{Z}$, then $x \to -J\dot{x} + Kx$ is an Hilbert space isomorphism from $W^{1,2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$ to $L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$, whose inverse we denote by M_K. Then critical points of F_K are precisely the solutions of $(\mathscr{H})$.

Our goal is to find critical points of F_K, but as we cannot prove that it satisfies condition (C) of Palais and Smale (to prove that this condition is satisfied, one usually needs some condition like $\nabla H(x).x \geqq \gamma H(x) > 0$, hence the level hypersurfaces of H are starshaped; (*cf.* [B-L-M-R] or [R]) we shall use a finite dimensional reduction of F_K that we shall now describe.

Let us first set for $u \in E = L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$, $\psi_K(u) = F_K(M_K u)$, that is

$$\psi_K(u) = \int_0^T \left[\frac{1}{2}(-M_K u, u) + H_K^*(u)\right] ds$$

and since M_K is an Hilbert space isomorphism, we can as well look for the critical points of ψ_K, and build a finite dimensional reduction of ψ_K.

The main point is to remark that ψ_K is convex in the direction orthogonal to some finite dimensional vector space: consider

$$\langle \psi_K'(u) - \psi_K'(v), u - v \rangle = \langle -M_K(u - v) + \nabla H_K^*(u) - \nabla H_K^*(v), u - v \rangle$$

since H_K^* is strictly convex

$$\langle \nabla H_K^*(u) - \nabla H_K^*(v), u - v \rangle \geqq \varepsilon \| u - v \|^2.$$

Let us mention that what we here denoted by M_K is in fact the composition of M_K and the Sobolev compact inclusion from $W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^{2n})$ into $L^2(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$ so that, as an endomorphism of L^2, M_K is self adjoint and compact. Thus if G is the finite dimensional subspace of E generated by the eigenvectors of M_K, the eigenvalues of which are greater than $\frac{\varepsilon}{2}$, we get for $u - v \in G^\perp$

$$\langle \psi_K'(u) - \psi_K'(v), u - v \rangle \geqq \frac{\varepsilon}{2} \| u - v \|^2 \tag{2.3}$$

and ψ_K is strictly convex in the direction of $G^\perp$ (i.e. for any $g \in G$, $h \to \psi_K(g+h)$, defined on $G^\perp$, is strictly convex).

We now prove.

PROPOSITION 2.4. — *For any $g \in G$, the function $h \to \psi_K(g+h)$ defined on $G^\perp$ has a unique minimum: $h(g)$. The map from G to $G^\perp$ given by $g \to h(g)$ has its image in $G^\perp \subset W^{1,2}$, and is continuous as a map from G to $G^\perp \subset W^{1,2}$.*

Set $\psi_K(g) = \psi_K(g+h(g))$. Then ψ_K is a C^∞ function on G, whose critical points are in a one to one correspondance with those of ψ_K. □

Remark. — The main feature of ψ_K is that it satisfies condition (C), that we shall prove in Chapter 3.

Proof. — As $h \to \psi_K(g+h)$ is strictly convex, it has a unique minimum $h(g)$ satisfying $\psi'_K(g+h(g)).h = 0$ for any $h \in G^\perp$. Let us first prove that, as a map from G to $G^\perp$, h is Lipschitz.

Take $g_1, g_2 \in G$ and set $h_1 = h(g_1)$, $h_2 = h(g_2)$, then

$$\langle \psi'_K(g_1+h_1) - \psi'_K(g_2+h_2),\ h_1 - h_2 \rangle = 0 \tag{2.5}$$

since $h_1 - h_2 \in G^\perp$.

But

$$\langle \psi'_K(g_1+h_2) - \psi'_K(g_1+h_1),\ h_2 - h_1 \rangle \geqq \frac{\varepsilon}{2} \| h_1 - h_2 \|^2 \tag{2.6}$$

by (2.4), and

$$\| \psi'_K(g_2+h_2) - \psi'_K(g_1+h_2) \| \leqq C \| g_1 - g_2 \| \tag{2.7}$$

because M is linear (hence Lipschitz) and ∇H_K^* is Lipschitz (because

$$H_K^{*\prime\prime}(\nabla H_K(z)) = [H''_K(z)]^{-1}$$

and H'' is bounded).

From (2.7) we get,

$$\langle \psi'_K(g_1+h_2) - \psi'_K(g_2+h_2),\ h_2 - h_1 \rangle \geqq C \| g_1 - g_2 \| \| h_1 - h_2 \|$$

which compared to (2.5) and (2.6) yields

$$\frac{\varepsilon}{2} \| h_1 - h_2 \|^2 \leqq C \| g_1 - g_2 \| \| h_1 - h_2 \|$$

hence h is Lipschitz of ratio $\dfrac{2C}{\varepsilon}$.

Recall now that $h(g)$ is defined by

$$\frac{\partial}{\partial h}\psi_{K}(g+h(g))=0 \tag{2.8}$$

that we can write

$$-M_K h=Q\nabla H_K^*(g+h) \qquad \text{with} \quad h=h(g) \tag{2.9}$$

and Q the orthogonal projection on $G^{\perp}$.

We wish to prove that $h\in W^{1,2}$.

First $M_K h$ is in $W^{1,2}$, hence $Q\nabla H_K^*(g+h)$ also. Now for $z\in L^2$, $z-Qz\in G\subset W^{1,2}$, so if $Qz\in W^{1,2}$ then $z\in W^{1,2}$, whence we see that $\nabla H_K^*(g+h)\in W^{1,2}$.

Since $x\to\nabla H_K(x)$ has a bounded differential, it maps $W^{1,2}$ in $W^{1,2}$, hence $\nabla H_K(\nabla H_K^*(g+h)=g+h$ is in $W^{1,2}$, and eventually h is in $W^{1,2}$.

We finally prove that ψ is C^1 and that $d\psi_K(g)=d\Psi_K(g+h(g))$. We shall not prove here that ψ is C^∞, since using a pseudo gradient vector field, C^1 is sufficient in order to perform min-max theory.

Let us compute

$$\begin{aligned}\psi_K(g+\delta g)-\psi_K(g)&=\Psi_K(g+\delta g+h(g+\delta g))-\Psi_K(g+h(g))\\&=\frac{\partial}{\partial g}\Psi_K(g+h(g))\,\delta g+\frac{\partial}{\partial h}\Psi_K(g+h(g))\,(h(g+\delta g)-h(g))\\&\qquad\qquad+o(\|\delta g\|+\|h(g+\delta g)-h(g)\|).\end{aligned}$$

Since $g\to h(g)$ is Lipschitz,

$$o(\|h(g+\delta g)-h(g)\|)=o(\|\delta g\|)$$

and because $\frac{\partial}{\partial h}\Psi_K(g+h(g))=0$, we see that

$$d\psi_K(g)=\frac{\partial}{\partial g}\Psi_K(g+h(g)).$$

Since Ψ_K is C^1 and $g\to h(g)$ is continuous, this implies that ψ_K is C^1. ⊙

Remark. – We write ψ_K for the finite dimensional reduction of ψ_K, and f_K for the corresponding reduction of F_K: since M_K preserves G and $G^{\perp}$ we set $\psi_K(M_K y)=f_K(y)$.

3. ψ_K satisfies condition (C)

The aim of this chapter is to prove

PROPOSITION 3.1. — *ψ_K satisfies condition* (C) *of Palais and Smale.*

Proof. — Let $g_n \in G$ be a sequence such that $\psi'_K(g_n) \to 0$ and $\psi_K(g_n)$ is bounded. Then if g_n is bounded it has a converging subsequence and there is nothing to prove, so we assume that $|g_n|$ goes to infinity.

Set $u_n = g_n + h(g_n)$, then $|u_n|_{C^0} \to +\infty$ since

$$\int_0^T \exp\left(\frac{2\pi}{T} k J s\right) u_n(s)\, ds \to +\infty$$

for some k, and by the same argument, if we set $u_n = M_K z_n$ $|z_n|_{C^0} \to +\infty$.

Now by assumption

$$-M_K u_n + \nabla H_K^*(u_n) = \varepsilon_n \qquad \text{where} \quad \varepsilon_n \in G \text{ goes to zero.} \tag{3.2}$$

In terms of z_n, (3.2) is equivalent to

$$z_n - \varepsilon_n = \nabla H_K^*(-J\dot{z}_n + K z_n)$$

hence

$$\nabla H(z_n - \varepsilon_n) + K(z_n - \varepsilon_n) = -J\dot{z}_n + K z_n$$

yielding

$$J\dot{z}_n + \nabla H(z_n - \varepsilon_n) = K\varepsilon_n. \tag{3.3}$$

Assume that for large values of n, there exists $t_0 \in \mathbb{R}/T\mathbb{Z}$ such that

$$z_n(t_0) \notin \bigcup_{p \geq 1} k^p . U \quad (cf. \text{ section } 1). \tag{3.4}$$

Let $W \subset\subset U$ defined by

$$W = \{ x \in U / |\nabla H(x)| > 0 \}$$

$$V = \{ x \in U / d(x, W) < \varepsilon \}.$$

By modifying our choice of the function λ, we can take W to be contained in an arbitrarily small neighborhood of $\Sigma \times \{1\}$.

We now prove that (3.4) implies, for a good choice of ε, that

$$z_n(t) \notin \bigcup_{p \geqq 1} k^p . \mathrm{V}, \qquad \forall t \in \mathbb{R}/\mathrm{T}\mathbb{Z}. \tag{3.5}$$

Let us argue by contradiction, and assume that t_1 is the smallest value of t larger than t_0 such that $z_n(t_1) \in \bigcup_{p \geqq 1} k^p . \mathrm{V}$.

For n large enough, $|\varepsilon_n(t)| < \varepsilon/2\mathrm{KT}$, so if

$$z_n(t) \notin \bigcup_{p \geqq 1} k^p . \mathrm{V}, \qquad z_n(t) - \varepsilon_n(t) \notin \bigcup_{p \geqq 1} k^p . \mathrm{W}.$$

Hence $\nabla \mathrm{H}(z_n(t) - \varepsilon_n(t)) = 0$, so by (3.3)

$$|\dot{z}_n(t)| < \varepsilon/2\mathrm{T} \tag{3.6}$$

and by the mean value theorem

$$|z_n(t_1) - z_n(t_0)| < \varepsilon/2$$

since we should have $|t_1 - t_0| < \mathrm{T}$.

But if ε is small enough, $d(\mathrm{V}, (\mathrm{U}) > \varepsilon$, thus

$$z_n(t_0) \notin \bigcup_{p \geqq 1} k^p . \mathrm{U}$$

implies

$$z_n(t_1) \notin \bigcup_{p \geqq 1} k^p . \bar{\mathrm{V}}$$

which contradicts our assumption.

Now we prove that if z_n is such that $z_n(t) \notin \bigcup_{p \geqq 1} k^p . \bar{\mathrm{V}}$ for all t's, and $|z_n|_{\mathrm{C}^0} \to +\infty$, then $\mathrm{F}_{\mathrm{K}}(z_n)$ is unbounded.

Let us first compute

$$\int_0^{\mathrm{T}} \frac{1}{2}(\mathrm{J}\dot{z}_n - \mathrm{K} z_n, z_n)\, dt = -\frac{K}{2}\|z_n\|^2 + \frac{1}{2}\int_0^{\mathrm{T}} (\mathrm{J}\dot{z}_n, z_n)\, dt.$$

Using (3.6)

$$\left|\int_0^{\mathrm{T}} (\mathrm{J}\dot{z}_n, z_n)\, dt\right| \leqq \varepsilon \|z_n\|$$

thus

$$\left|\frac{1}{2}\int_0^T (J\dot{z}_n - K z_n, z_n)\,dt + \frac{K}{2}\|z_n\|^2\right| \leqq \varepsilon \|z_n\|. \tag{3.7}$$

Also, since

$$\frac{(K+a)}{2}|z|^2 - C \leqq H_K(z) \leqq \frac{(K+a')}{2}|z|^2 + C$$

for some $a' > a$. (The right hand side inequality follows from the boundedness of H''.)

$$\frac{1}{2(K+a')}|z|^2 - C \leqq H_K^*(z) \leqq \frac{1}{2(K+a)}|z|^2 + C$$

so

$$H_K^*(-J\dot{z}_n + K z_n) \leqq \frac{1}{2(K+a)}|-J\dot{z}_n + K z_n|^2 + C,$$

and using (3.6) again

$$\leqq \frac{1}{2(K+a)}(\varepsilon + K)^2 |z_n|^2 + C.$$

Eventually

$$F_K(z_n) \leqq -\frac{K}{2}\|z_n\|^2 + \frac{(\varepsilon+K)^2}{2(K+a)}\|z_n\|^2 + \varepsilon\|z_n\| + C. \tag{3.7}$$

Consider now

$$-\frac{K}{2} + \frac{(\varepsilon+K)^2}{2(K+a)}$$

for $\varepsilon = 0$ this equals $-\frac{K}{2}\left(1 - \frac{K}{K+a}\right) = -\frac{Ka}{2(K+a)}$, so for ε small enough, this quantity is smaller than $-\frac{Ka}{4(K+a)}$. So the right hand side of (3.6) goes to minus infinity with $\|z_n\|$, and $F_K(z_n) = \psi_K(u_n)$ is not bounded.

We now see that the only possibility is that $z_n(t) \in \bigcup_{p \geq 1} k^p . \mathrm{U}$ for all t, hence by a trivial connectedness argument

$$z_n(t) \in k^{p_n} . \mathrm{U} \qquad \text{for all} \quad t's.$$

Set $w_n = \frac{1}{k^{p_n}} z_n \in \mathrm{U}$. From (3.3) and $|\nabla \mathrm{H}(x)| \leqq C|x|$ we infer

$$|\dot{z}_n| \leqq \mathrm{C} |z_n|$$

so w_n is bounded in $\mathrm{W}^{1,2}$, hence there is a converging subsequence, still denoted w_n, such that $w_n \to w$ in the C^0 topology.

Let us remark that since $w_n(t) \in \mathrm{U}$, $w(t) \in \mathrm{U}$ for all t's.

Rewriting (3.3) and using the equation $\nabla \mathrm{H}(k^p z) = k^p \nabla \mathrm{H}(z)$ for $z \in \mathrm{U}$, yields

$$\mathrm{J} \dot{w}_n + \nabla \mathrm{H}\left(w_n - \frac{\varepsilon_n}{k^{p_n}}\right) = \mathrm{K} \frac{\varepsilon_n}{k^{p_n}}. \tag{3.8}$$

Let n go to infinity, we thus obtain

$$\mathrm{J} \dot{w} + \nabla \mathrm{H}(w) = 0 \tag{3.9}$$

so w is a solution of $(\mathscr{H})$ such that $w(t) \in \mathrm{U}$ for all t's.

Now, let us show that $\mathrm{F_K}(z_n) \sim k^{2p_n} \mathrm{F_K}(w)$. Since if w is a constant in U, $\mathrm{F_K}(w)$ is non zero, this will imply that w is a nontrivial solution of $(\mathscr{H})$.

Obviously,

$$\int_0^{\mathrm{T}} \frac{1}{2}(\mathrm{J} \dot{z}_n - \mathrm{K} z_n, z_n)\, dt = k^{2p_n} \int_0^{\mathrm{T}} \frac{1}{2}(\mathrm{J} \dot{w}_n - \mathrm{K} w_n, w_n)\, dt$$

On the other hand, by (3.3)

$$-\mathrm{J} \dot{z}_n + \mathrm{K} z_n = \nabla \mathrm{H_K}(z_n - \varepsilon_n),$$

therefore

$$\mathrm{H_K^*}(-\mathrm{J} \dot{z}_n + \mathrm{K} z_n) = (z_n - \varepsilon_n) . \nabla \mathrm{H_K}(z_n - \varepsilon_n) - \mathrm{H_K}(z_n - \varepsilon_n). \tag{3.10}$$

As before $z_n - \varepsilon_n = k^{p_n}\left(w_n - \frac{\varepsilon_n}{k^{p_n}}\right)$, and for n large enough, $w_n - \frac{\varepsilon_n}{k^{p_n}}$ is in U since $|\varepsilon_n|_{\mathrm{C}^0}$ goes to zero.

Thus, using (1.11),

$$H_K^*(-J\dot{z}_n+Kz_n)=k^{2p_n}\left[\left(w_n-\frac{\varepsilon_n}{k^{p_n}}\right).\nabla H_K\left(w_n-\frac{\varepsilon_n}{k_{p_n}}\right)-H_K\left(w_n-\frac{\varepsilon_n}{k_{p_n}}\right)\right]$$
$$=k^{2p_n}H_K^*(-J\dot{w}_n+Kw_n)$$

the last equality follows from (3.8).

This proves $F_K(z_n)=k^{2p_n}F_K(w_n)$, since w_n converges strongly to w, we indeed get $F_K(z_n)\sim k^{2p_n}F_K(w)$.

Finally, if w is a constant in U, solution of ($\mathscr{H}$), then $\nabla H(w)=0$, hence $\nabla H_K(w)=Kw$, so

$$H_K^*(Kw)=w.Kw-H_K(w)=\frac{K}{2}|w|^2-H(w)$$

and

$$F_K(w)=T\left[-\frac{K}{2}|w|^2+\frac{K}{2}|w|^2-H(w)\right]<-TH(0).$$

So if $F_K(z_n)$ is bounded w is a non constant solution of ($\mathscr{H}$). ⊙

Remark. – It is easy to see that in order that (P.S.) holds, we only need that $F_K(w)\neq 0$ for all solutions w of ($\mathscr{H}$). Now, a computation yields

$$F_K(w)=w.\nabla H(w)-H(w)$$

and it can be shown that if the set of periods of closed characteristics of Σ is discrete, we can choose λ so that 0 is not a critical value of F_K.

4. PROOF OF THE THEOREM

Now that we proved that f_K satisfies condition (C) we must prove that it has a non trivial critical point.

Let us first remark that if we let $S^1=\mathbb{R}/T\mathbb{Z}$ act on $X=W^{1,2}(\mathbb{R}/T\mathbb{Z},\mathbb{R}^{2n})$ by

$$\theta.z(.)=z(\theta+.)$$

then F_K is equivariant, as well as f_K since $g\to h(g)$ is equivariant.

Now let F be the set of fixed points of X by the S^1 action, that is F is the set of constant paths. We now prove

PROPOSITION 4.1. — *There are two* S^1-*invariant vector subspaces of* G, V *and* W *such that* $V \overset{\neq}{\supset} W \supset F$ *and if we denote by* S *the unit sphere of* G,

$$\text{(i)} \qquad \alpha S \cap W^{\perp} \subset G - G^{-\gamma}$$

$$\text{(ii)} \qquad G - G^{C} \cap V = \varnothing$$

for α *small,* $0 < \gamma + TH(0) < \varepsilon$, *with* ε *small,* C *large. Moreover* $\dim V - \dim W \geqq 1$ *for a large enough* [*a was defined in* (1.8)]. □

Proof. — Set

$$Q_{\lambda, K}(x) = \frac{1}{2}\int_0^T \left[(J\dot{x} - Kx, x) + \frac{1}{(K+\lambda)} |J\dot{x} - Kx|^2 \right] ds$$

Then since

$$H_K^*(x) \leqq \frac{1}{2(K+a)} |x|^2 + C'$$

we have for $x \in G$

$$f_K(x) \leqq F_K(x) \leqq Q_{a, K}(x) + \frac{C'}{2} T$$

so for C large, $X - X^C$ does not meet the negative eigenspace of $Q_{a, K}$, that shall be our space V (remark that indeed $V \subset G$).

On the other hand, for x in a neighborhood of the origin, we assumed

$$H(x) = H(0)$$

so that

$$F_K(z) = Q_{a, K}(z) - H(0) \qquad \text{for} \qquad |z|_{C^0} \text{ small enough,}$$

hence since for small g, $|g + h(g)|_{C^0}$ is small, $f_K(g) = q_{0, K}(g) - H(0)$ near the origin, where $q_{0, K}$ is obtained from $Q_{0, K}$ in the same way as f_K is obtained from F_K.

It is easy to see that $q_{0, K}$ and $Q_{0, K}$ have the same index, and we take for W the non positive eigenspace of $Q_{0, K}$.

An easy computation shows that

$$\operatorname{index} Q_{\lambda,K} = 2n\left(\left[(K+\lambda)\frac{T}{2\pi}\right]+1\right):$$

Set $y=\sum_{k\in Z}\exp(2\pi k t J/T)\,y_k$, then

$$\begin{aligned}
Q_{\lambda,K}(y,y) &= \int_0^T\left[(J\dot{y}-Ky,y)+\frac{1}{K+\lambda}|J\dot{y}-Ky|^2\right]dt\\
&= \sum_{k\in Z} -\left(\frac{2\pi k}{T}+K\right)y_k^2+\frac{1}{K+\lambda}\left(\frac{2\pi k}{T}+K\right)^2 y_k^2\\
&= \sum_{k\in Z} -\left(\frac{2\pi k}{T}+K\right)\left[\frac{1}{K+\lambda}\left(\frac{2\pi k}{T}+K\right)-1\right]y_k^2\\
&= \sum_{k\in Z}\left(\frac{2\pi k}{T}+K\right)\frac{1}{(K+\lambda)}\left(\frac{2\pi k}{T}-\lambda\right)y_k^2
\end{aligned}$$

whose index is given by

$$2n.\left\{k\in\mathbb{Z}/-\frac{KT}{2\pi}<k<\frac{\lambda T}{2\pi}\right\}=2n\left(\left[(\lambda+K)\frac{T}{2\pi}\right]+1\right)$$

so that

$$\dim V = 2n\left(\left[(K+a)\frac{T}{2\pi}\right]\right)$$

and

$$\begin{aligned}
\dim W &= \operatorname{index} Q_{0,K}+\operatorname{nullity} Q_{0,K}\\
&= 2n\left[\frac{KT}{2\pi}\right]+2n
\end{aligned}$$

(because $\ker Q_{0,K}=F$ has dimension $2n$)

$$\text{so } \dim V-\dim W = 2n\left(\left[(K+a)\frac{T}{2\pi}\right]-\left[\frac{KT}{2\pi}\right]-1\right).\quad\odot$$

COROLLARY 4.2:

$$H^q_{S^1}(G-G^\gamma, G-G^C)\neq 0 \qquad \textit{for}\quad q\in]\dim V^\perp, \dim W^\perp]$$

thus f_k has a critical level in $[\gamma, C]$ *whence a non trivial critical circle.* □

Proof. – The idea is that if $i(A)$ is the Fadell-Rabinowitz cohomological index of A (*cf.* [F – R]), then by (ii) $i(G-G^C) \geqq \operatorname{codim}_G V$; on the other hand by (*i*) $i(G-G^\gamma) \geqq \operatorname{codim}_G W$ so that

$$H^q_{S^1}(G-G^\gamma, G-G^C) \neq 0 \quad \text{for all } q \text{ in }]\operatorname{codim}_G V, \operatorname{codim}_G W].$$

To be more precise, there are maps

$$\alpha S \cap W^\perp \to G-G^\gamma \to G-F,$$

since as proved at the end of chapter 3, $F \subset G^\gamma$

$$H^*_{S^1}(G-\{0\}) \to H^*_{S^1}(G-G^\gamma) \to H^*_{S^1}(\alpha S \cap W^\perp)$$

as the map $H^*_{S^1}(G-\{0\}) \to H^*_{S^1}(\alpha S \cap W^\perp)$ is surjective for $* \leqq \dim W^\perp$, $H^*_{S^1}(G-G^\gamma) \to H^*_{S^1}(\alpha S \cap W^\perp)$ will also be surjective.

On the other hand there is a homotopy commutative diagram

$$\begin{tikzcd} G-G^C \arrow[rr, "\pi"] \arrow[dr, hook] & & V^\perp-\{0\} \arrow[dl, hook] \\ & G-\{0\} & \end{tikzcd}$$

where π is the orthogonal projection on $V^\perp$ and "homotopy commutative" means that the inclusion of $G-G^C$ in $G-\{0\}$ is homotopic to π composed with the inclusion of $V^\perp-\{0\}$ in $G-\{0\}$; the homotopy being given by $\varphi_t(x)=(1-t)x+t\pi(x)$.

Thus there is a commutative diagram

$$\begin{tikzcd} H^*_{S^1}(G-G^C) & & H^*_{S^1}(V^\perp-\{0\}) \arrow[ll] \\ & H^*_{S^1}(G-\{0\}) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

as

$$H^*_{S^1}(V^\perp-\{0\}=0 \qquad \text{for} \quad * \geqq \dim V^\perp,$$

so

$$H^*_{S^1}(G-\{0\}) \to H^*_{S^1}(G-G^C)$$

is zero in these dimensions.

Finally let us write the cohomology sequences of the pairs $(G-G^\gamma, G-G^C)$; $(G-F, G-F)$ and the map between these sequences induced by the inclusion

$$\begin{array}{ccccc} H^*_{S^1}(G-G^\gamma, G-G^C) & \xrightarrow{\alpha} & H^*_{S^1}(G-G^\gamma) & \rightarrow & H^*_{S^1}(G-G^C) \\ \uparrow & & \uparrow \beta & & \uparrow \gamma \\ 0 & \rightarrow & H^*_{S^1}(G-F) & \xrightarrow{\text{id}} & H^*_{S^1}(G-F) \end{array}$$

γ is zero for $* \geqq \dim V^\perp$, β is non zero for any $* \leqq \dim W^\perp$. So let $y \in H^q_{S^1}(G-\{0\})$ where $\dim V^\perp \leqq q < \dim W^\perp$ such that $\beta(y) \neq 0$. As $\gamma(y)=0$, $\beta(y)$ is in the image of α hence

$$H^q_{S^1}(G-G^C) \neq 0. \odot$$

Remark. — An analogous statement is proved in [B−L−M−R] with some notion of genus instead of equivariant cohomology.

We can now conclude the proof of our theorem:

By corollary 4.2 f_K has at least one critical value in $[\gamma, C]$. Thus F_K ha a critical value, also in $[\gamma, C]$ since the critical values of f_K and F_K coïncide. Since $\gamma > -TH(0)$, the critical orbit thus found is non trivial. According to proposition 1.8, this yields a periodic orbit of $(\mathcal{N})$. $\odot$

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