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# Travelling fronts in cylinders

by

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ABSTRACT. — This work is concerned with travelling front solutions of semilinear parabolic equations in an infinite cylindrical domain $\Sigma = \mathrm{R} \times \omega$ where $\omega \subset \mathrm{R}^{n-1}$ is a bounded domain. We write for $x$ in $\Sigma$, $x = (x_1, y)$, with $y \in \omega$. The problems we consider are of the following type

$$\begin{cases} \Delta u - (c + \alpha(y))\, \partial_1 u + f(u) = 0 & \text{in } \Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma \\ u(-\infty, .) = 0, \qquad u(+\infty, .) = 1 \end{cases}$$

where the unknowns are the parameter $c \in \mathrm{R}$ and the function $u$. The function $\alpha \in \mathrm{C}^0(\bar{\omega})$ and the nonlinear term $f: [0, 1] \to \mathrm{R}$ are given. (More general coefficients than $c + \alpha(y)$, $\beta(y, c)$ are also treated.)

We obtain a fairly general resolution of this problem. The results depend on the type of nonlinearity considered. When $f$ is of the bistable type, or of the "ignition temperature" type in combustion, we show the existence and uniqueness of $c$ and of the profile $u$. In the case that $f > 0$ in $(0, 1)$, we show the existence of $c^* \in \mathrm{R}$ such that a solution exists if and only if $c \geqq c^*$. These results extend to higher dimensions various classical results. In particular they extend the result of Kolmogorov, Petrovsky and Piskounov.

RÉSUMÉ. — Cet article a pour objet les solutions de type *front progressif* pour des équations paraboliques semi-linéaires dans un domaine cylindrique infini $\Sigma = \mathrm{R} \times \omega$ où $\omega \subset \mathrm{R}^{n-1}$ est un domaine borné. Pour $x$ dans $\Sigma$, nous noterons : $x = (x_1, y)$, avec $y \in \omega$. Les problèmes que nous envisageons sont de la forme

$$\begin{cases} \Delta u-(c+\alpha(y))\,\partial_1 u+f(u)=0 \quad \text{dans } \Sigma \\ \dfrac{\partial u}{\partial \nu}=0 \quad \text{sur } \partial\Sigma \\ u(-\infty,.)=0, \qquad u(+\infty,.)=1 \end{cases}$$

où l'on cherche à déterminer le paramètre $c \in \mathrm{R}$ et la fonction $u$. Le coefficient $\alpha \in C^0(\bar{\omega})$ ainsi que le terme non linéaire $f:[0,1] \to \mathrm{R}$ sont donnés. (Des coefficients plus généraux que $c+\alpha(y)$, du type $\beta(y,c)$ sont également traités.)

Nous obtenons une résolution essentiellement complète de ce problème. Les résultats varient avec le type de nonlinéarité $f$. Lorsque $f$ est du type « bistable » ou bien du type avec température d'ignition dans les modèles de combustion, nous montrons l'existence et l'unicité de $c$ et du profil $u$. Dans le cas où $f>0$ sur $(0,1)$, nous montrons l'existence de $c^* \in \mathrm{R}$ tel qu'une solution existe si et seulement si $c \geq c^*$. Ces résultats étendent aux dimensions supérieures divers travaux classiques en dimension un. En particulier, ils généralisent le résultat de Kolmogorov, Petrovsky et Piskounov.

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## 1. INTRODUCTION

This paper is devoted to the study of travelling front solutions of semilinear parabolic equations in several space dimensions.

These equations have the form

$$\frac{\partial u}{\partial t}-\Delta u+\alpha(y)\,\partial_1 u=f(u).$$

They are set for $t \in \mathrm{R}$, and spatial variables in an infinite cylinder $\Sigma=\mathrm{R}\times\omega$, where $\omega \subset \mathrm{R}^{N-1}$ is a bounded smooth domain. We will systematically represent $x \in \Sigma$ as $x=(x_1, y)$ with $x_1 \in \mathrm{R}$ and $y \in \omega$. We denote $\dfrac{\partial u}{\partial x_1}$ by $\partial_1 u$ or $u_1$. The outward unit normal to $\partial\omega$ or to $\partial\Sigma$ is denoted by $\nu$.

Here, $\alpha(y)\dfrac{\partial u}{\partial x_1}$ is a given (predetermined) transport term, or a driving flow, in the $x_1$-direction, along the direction of the cylinder. In some sense, the flow is *driven* by some exogeneously given flow represented by $\alpha(y)$ which is independent of $x_1$. We will always assume that $\alpha \in C^0(\bar{\omega})$ and $\alpha>0$. The term $f(u)$ represents a source term and we always assume $f(0)=f(1)=0$.

Travelling front solutions are defined as solutions of the form $u=u(x_1+ct, y)$, with $u:\bar{\Sigma} \to \mathrm{R}$, and $c$ a real parameter (the speed of the front), which is to be determined. We require that $u \to 0$ or $1$ as $x_1$ approaches $-\infty$ or $+\infty$ respectively, and impose Neumann boundary conditions on $\partial\Sigma$. One is thus led to the following semilinear elliptic problem in $\Sigma$

$$\Delta u-(c+\alpha(y))\,\partial_1 u+f(u)=0 \quad \text{in } \Sigma \tag{1.1}$$

$$\frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Sigma \tag{1.2}$$

$$u(-\infty, y)=0, \qquad u(+\infty, y)=1. \tag{1.3}$$

The limits in (1.3) are understood to be uniform in $y \in \bar{\omega}$.

The boundary condition (1.2) means that there is no flux of $u$ across the boundary $\partial\Sigma$. We remark that the methods in this paper are quite general. They apply to more general boundary conditions than (1.2) – such as Dirichlet conditions or conditions of mixed type. In addition, most of the results extend to a nonlinearity $f(y, u)$ in place of $f(u)$ – with suitable modifications of the assumptions. These extensions are briefly discussed at theend of the Introduction.

The results depend very critically on the behaviour of $f(u)$ near 0 and 1. Throughout the paper we assume

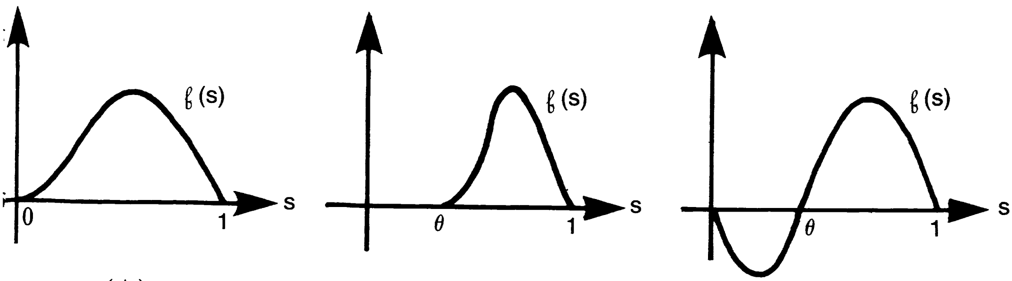
$$\left.\begin{array}{c} f \text{ is Lipschitz continuous on } [0,1], \text{ and } f(0)=f(1)=0, \\ f'(0) \text{ and } f'(1) \text{ exist}, f'(1)<0 \end{array}\right\} \tag{1.4}$$

and,

$$\left.\begin{array}{c} \text{for some constants } \mathrm{M} \geqq 0,\ s_0, \delta>0, \\ |f(s)-f'(0)s|,\ |f(1-s)+f'(1)s| \leqq \mathrm{M}\, s^{1+\delta} \text{ for } 0<s \leqq s_0. \end{array}\right\} \tag{1.5}$$

The last condition in (1.5) is satisfied in case $f \in \mathrm{C}^1$ near $s=0$ and $s=1$, and $f'$ is Hölder continuous at $s=0$ and $s=1$. For some results, further conditions will be imposed on $f$.

We will distinguish between three classes of nonlinearities. These will be called types A, B or C and are represented in the diagram below.




(A) (B) (C)

More precisely, we say that $f$ is of the *type A* if

$$f>0 \quad \text{in } (0,1); \tag{1.6}$$

$f$ is of *type B* if, for some $\theta \in (0,1)$

$$f=0 \quad \text{on } (0,\theta) \qquad \text{and} \qquad f>0 \quad \text{on } (\theta,1). \tag{1.7}$$

Last, $f$ is *of type C* if, for some $\theta \in (0,1)$,

$$f<0 \quad \text{on } (0,\theta) \qquad \text{and} \qquad f>0 \quad \text{on } (\theta,1). \tag{1.8}$$

As we know from the one-dimensional problems (and as we will see more generally) it is indeed necessary to distinguish between the various types of nonlinearities.

Before stating our results we decribe some of the history of problems of the form (1.1)-(1.3). They arise in various applications – including the three types of nonlinearities. For instance, cases A and C arise in some problems of biology (population dynamics, gene developments, epidemiology...). A general reference here is the book of P. Fife [Fi]. See also [FM], [AW1, 2] and the references therein. For instance when $f(s)=s(1-s)$ (case A) and $\alpha \equiv 1$, (1.1) is known as the Fisher equation. This one-dimensional situation has been studied in the well known 1937 paper of Kolmogorov, Petrovsky and Piskounov [KPP]. Case C is usually referred to as the "bistable nonlinearity" – since $s=0$ and $s=1$ are then two stable rest points.

Cases A and B also arise in combustion: in the study of flame propagation in a tube. In this model, after normalization of the variables, $u$ represents the temperature and $1-u$ the concentration of (premixed) reactant. A flame is defined as a travelling front solution of (1.1)-(1.3). The term $f(u)$ is then the reaction rate at which the reaction takes place and is given by the Arrhenius law (involving an exponential factor in $u$) and the law of mass action ($f(s)$ is proportional to $1-s$). The boundary condition (1.2) means that the wall of the tube is adiabatic. Both cases A and B occur in this model. In fact, case B is perhaps the most natural choice. It corresponds to an *ignition temperature* assumption. The reaction does not occur until the temperature passes the threshold $\theta$. A detailed description of this model, and derivation of equations (1.1)-(1.3), can be found in [BL2, Chapter 1].

In the combustion model, $f(s)$ is flat, almost zero, except for values of $s<1$ very close to 1. It is customary to consider the limit as $f(s)$ converges to a Dirac mass concentrated at $s=1$. The mathematical analysis of this singular limit, as well as the study of the resulting free boundary problem, are carried out in the joint paper with L. Caffarelli [BCN]. Also, in this context, the stability of the travelling front has recently been established in [BLR] (linearized stability), [R1] (nonlinear stability) and [R2] (global stability).

Case C also arises in the context of combustion for some reversible reactions or two step, not purely exothermic reaction systems.

For one space dimension there is much literature on problems of this kind. This corresponds to $\alpha \equiv 0$ and $u = u(x_1)$. The problem (1.1)-(1.3) then reduces to the following problem for a nonlinear ODE:

$$\left.\begin{array}{l} \ddot{u} - c\dot{u} + f(u) = 0 \quad \text{on } \mathbb{R} \\ u(-\infty) = 0, \qquad u(+\infty) = 1. \end{array}\right\} \tag{1.11}$$

When considering (1.11), the paper by Kolmogorov, Petrovsky and Piskounov [KPP], treats essentially a particular class in case A, and under the additional condition:

$$f(s)/s \leqq f'(0) \qquad \forall s \in (0, 1]. \tag{1.12}$$

This is referred to as the KPP case. In particular, KP and P have shown that (1.11) has a solution if and only if

$$c \geqq \sqrt{2f'(0)}. \tag{1.13}$$

They further analyzed the *stability* of these travelling fronts and derived several important properties of the evolution problem. In particular, they showed that the travelling front corresponding to the speed $c = \sqrt{2f'(0)}$ is stable and that in some sense, $\sqrt{2f'(0)}$ is the "preferred" or natural, speed at which solutions travel.

Problem (1.11) was also studied in case A, for $f$ close to a Dirac mass at $s=1$, in the work by Zeldovich and Frank-Kamenetskii [ZFK]. Their work was motivated by the combustion model and focussed on the singular limit when $f$ approaches the Dirac mass at $s=1$. In this direction, see also Kanel [K1, 2], Johnson and Nachbar [JN] and Berestycki, Nicolaenko and Scheurer [BNS].

The most general study of the ordinary differential equation (1.11) was then carried out by Fife and McLeod [FM]. For case C they established stability properties of the unique travelling front solution. We also refer to [BL2, chapter 2] for a complete study of equation (1.11). Solutions of (1.11) also yield "planar front" solutions $u(t, x) = u(x \cdot \xi + ct)$ of the equation

$$\frac{\partial u}{\partial t} - \Delta u = f(u) \quad \text{in } x \in \mathbb{R}^n \tag{1.14}$$

for some unit vector $\xi \in \mathbb{R}^n$. The stability of solutions of (1.11) with respect to (1.14) is studied in the work of Aronson and Weinberger [AW1] which also treats models in population genetics leading to (1.14).

The preceding list is by no means complete. There is a vast literature devoted to this subject, including generalizations and applications – see the surveys of A. J. Volpert [Vo 1, 2] and the bibliographies there and [VV1-3].

In higher dimensions there are not many studies of nonplanar travelling fronts. In an interesting paper, R. Gardner [G] studied problems of type (1.1) in a cylinder in $\mathbf{R}^2$, with Dirichlet boundary conditions, for a particular nonlinearity of type C. He used the Conley index. Another interesting related work in this context is that of S. Heinze [H]; for $\alpha \equiv 0$, and Dirichlet boundary condition, he presented a variational characterization of $c$. Probabilistic methods have been used in such problems. See H. P. McKean [McK] and recent work of M. Freidlin [Fr1, 2]; other references may be found there.

In this work, we present a complete investigation of problem (1.1)-(1.3) in higher dimensions. In particular, we will present some higher dimensional versions of the KPP theory.

In recent interesting papers, J.M. Vega [Ve1], [Ve2], treats equation (1.1) in the cylinder $\Sigma$, with $f=f(y,u)$, and for *Dirichlet boundary conditions* $u=0$ in $\partial\Sigma$ — but only for the case $\alpha(y)\equiv 0$. In place of (1.3) he considers the problem $u(x_1,y)\to v_\pm(y)$ as $x_1\to\pm\infty$, where $v_\pm(y)$ are solutions of

$$\begin{aligned} \Delta_y v+f(y,v)=0 \quad &\text{in } \omega \\ v=0 \quad &\text{on } \partial\omega, \end{aligned}$$

with $v_-(y)<v_+(y)\,\forall\, y\in\omega$. In [Ve1] he obtains rather precise information on the asymptotic behaviour of $u$ as $x_1\to\pm\infty$. His method is different from ours of [BN1], or of this paper (we use results of [AN] and [P]). It appears to us that his method cannot work in case $\alpha\not\equiv$ constant. Using the sliding method of [BN1], he proves monotonicity and uniqueness. In [Ve2], under various conditions he then proves several existence theorems, and uniqueness of $u$, modulo translation. In addition he makes use of sub and supersolutions to solve the corresponding problems in finite cylinders.

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We now describe our main results.

In some applications, the dependence on $c$ in (1.1) may be slightly different. For instance, in some combustion models, (1.1) is replaced by

$$\Delta u-c\,\alpha(y)+f(u)=0 \quad \text{in } \Sigma, \tag{1.15}$$

here $\alpha$ is positive continuous function in $\bar\omega$.

Since no additional work is required, we shall treat an equation with a more general dependence on $c$, in which $c+\alpha(y)$ of (1.1) or $c\,\alpha(y)$ of (1.15) are replaced by a function $\beta(y,c)$ of the form

$$\Delta u-\beta(y,c)\frac{\partial u}{\partial x_1}+f(u)=0 \quad \text{in } \Sigma. \tag{1.16}$$

We always assume that

$$\left.\begin{array}{c}\beta=\beta(y,c) \text{ is continuous in } \bar{\omega}\times \mathrm{R} \text{ and}\\ \text{strictly increasing in } c,\ \forall y\in\bar{\omega}.\end{array}\right\} \qquad (1.17)$$

$$\left.\begin{array}{c}\beta(y,c)\to\infty \text{ as } c\to\infty \text{ uniformly in } y,\\ \beta(y,c)\to-\infty \text{ as } c\to-\infty \text{ uniformly in } y.\end{array}\right\} \qquad (1.18)$$

We now describe our main results, beginning with case B (*see* (1.7)).

THEOREM 1.1. — *Assume that $f$ is of type B then, there exists a solution $(c,u)$ of problem* (1.16), (1.2), (1.3) *with $u_1>0$ in $\bar{\Sigma}$.*

THEOREM 1.1'.. — *Assume that $f$ is of type B and satisfies — in addition to conditions* (1.4), (1.5) — *the condition*

$$\left.\begin{array}{c}|f(1-s)-f(1-s')+f'(1)(s-s')|\leqq \mathrm{M}(s+s')^{\delta}|s-s'|\\ \text{for } 0<s,s'\leqq s_0.\end{array}\right\} \qquad (1.19)$$

*Then the solution $(c,u)$ is unique, i.e., if $(c',u')$ is also a solution; then $c=c'$ and $u'(x_1,y)=u(x_1+\tau,y)$ for some real $\tau$.*

Conditions (1.5) and (1.19) are satisfied in case $f\in C^1$ near $s=0$ and $s=1$ and $f'$ is Hölder continuous at $s=0$ and $s=1$.

For $\beta=c\,\alpha(y)$, $\alpha(y)>0$, Theorem 1.1 was already proved in [BL], and for $\beta=c+\alpha(y)$, in [BLL]. For both cases, Theorem 1.1' was proved in [BN1]. For completeness, we include here the proofs of Theorems 1.1 and 1.1' for general $\beta(y,c)$ satisfying (1.17) and (1.18).

We turn next to case C: $f$ satisfies (1.8). It is similar to case B but new difficulties occur. Indeed, in case C, there may exist nonconstant solutions of the "stationary problem" in $\omega$:

$$\left.\begin{array}{l}\Delta_y\psi+f(\psi)=0 \quad \text{in } \omega\\ \dfrac{\partial\psi}{\partial\nu}=0 \quad \text{on } \partial\omega\end{array}\right\} \qquad (1.20)$$

($\Delta_y$ is taken only with respect to the $n-1$ variables $y$).

For a solution $\psi$ of (1.20), let $\mu_1(\psi)$ denote the principal (or least) eigenvalue of the operator $-\Delta_y-f'(\psi(y))$ in $\omega$, with Neumann conditions. That is, $\mu_1(\psi)$ is characterized by the existence of $\Phi>0$ in $\bar{\omega}$ such that

$$\left.\begin{array}{l}-\Delta_y\Phi-f'(\psi(y))\Phi=\mu_1(\psi)\Phi \quad \text{in } \omega\\ \dfrac{\partial\Phi}{\partial\nu}=0 \quad \text{on } \partial\omega.\end{array}\right\} \qquad (1.21)$$

THEOREM 1.2. — *Assume that $f$ is of type C. Then there exists a solution $(c,u)$ of* (1.1)-(1.2) *which satisfies*

$$u(-\infty,.)=0, \qquad u(+\infty,y)=\psi(y) \qquad (1.22)$$

*for some nonconstant solution* $\psi=\psi(y)$ *of* (1.20), *with* $\psi\equiv 1$, *or* $0<\psi<1$; *also* $u_1>0$ *in* $\bar{\Sigma}$.

Here the function $\psi$ is unknown. The corresponding uniqueness result is

THEOREM 1.2′. — *Assume that* $f$ *satisfies condition* (1.19). *Suppose that* $\psi$ *is stable in the sense that* $\mu_1(\psi)>0$ (*this includes the case* $\psi\equiv 1$). *Then the solution* $(c, u)$ *of* (1.16), (1.2) *and* (1.22) *is unique* (*modulo translation*) *in each of the following cases:*

(*a*) $f'(0)<0$

(*b*) $f'(0)=0$, $\beta(y, c)=c\,\alpha(y)$, $\alpha>0$ *on* $\bar{\omega}$, *and* $\int_0^1 f(s)\,ds>0$.

*Remark.* — One can also reverse the roles of the conditions at $+\infty$ and $-\infty$. Indeed, one can prove that there exists a solution $(c, u)$ of (1.16), (1.2), (1.3) such that

$$u(-\infty, y)=\psi(y), \qquad u(+\infty, y)=1 \tag{1.23}$$

with $\psi\equiv 0$ or $\psi$ some nonconstant solution of (1.20), $0<\psi<1$.

Theorem 1.2 does not yield a solution of problem (1.16), (1.2), (1.3), since the function $\psi$ cannot be prescribed *a priori*, and thus (1.3) is not necessarily verified. With an extra assumption, though, we can obtain the desired solution.

THEOREM 1.3. — *Assume the conditions in Theorem* 1.2, *and suppose that* $\omega$ *is convex, and that* $f\in C^{1,\delta}$ *on* $[0, 1]$, $0<\delta<1$. *Then, there exists a solution* $(c, u)$ *of* (1.16), (1.2), (1.3).

The corresponding statement for a nonconvex domain $\omega$ remains open.

Last, we turn to case A which is quite different from the two others.

THEOREM 1.4. — (i) *Suppose* $f$ *is of type A. Then there exists* $c^*\in\mathbb{R}$ *such that there exists a solution* $u$ *of* (1.16), (1.2), (1.3) *if and only if* $c\geqq c^*$. *For every* $c\geqq c^*$, *there is a solution with* $u_1>0$.

(ii) *Furthermore, if* $f'(0)>0$, *then the solution* $u$ *is unique* (*modulo translation*).

Our last result is a higher dimensional extension of KPP theory.

THEOREM 1.5. — *In case A, suppose further that*

$$\frac{f(s)}{s}\leqq f'(0) \quad \text{for} \quad 0<s<1. \tag{1.24}$$

*Then,* $c^*$ *is "explicitly" determined from* $\omega$, $\alpha(y)$, *and the value of* $f'(0)$.

The "explicit" construction of $c^*$ is described in detail in Section 10. It agrees of course with the value $c^*=\sqrt{2f'(0)}$ in case $\alpha\equiv 0$.

The results described above may appear rather simple but some of the proofs are truly intricate. In addition, we were not able to prove monotonicity and uniqueness in all cases. Open problems remain:

*Problem* 1. – Does Theorem 1.2′ hold in general, *i.e.*, if condition (*b*) is replaced by the simpler condition $f'(0)=0$?

*Problem* 2. – In case $f'(0)=0$ in Theorem 1.4, does every solution satisfy $u_1>0$? Is there uniqueness?

The methods we develop in the paper allow one to consider somewhat more general problems of the type

$$\Delta u-\beta(y,c)\partial_1 u+f(y,u)=0 \quad \text{in } \Sigma \tag{1.25}$$

$$\partial_\nu u=0 \quad \text{on } \partial\Sigma \tag{1.26}$$

$$u(-\infty,.)=\psi_-(y), \qquad u(+\infty,.)=\psi_+(y). \tag{1.27}$$

Here, $\beta(y,c)$ is as before (in particular it satisfies (1.17)-(1.18)). The function $f$ is a $C^{1,\delta}$, $\delta>0$, function $f:\bar{\omega}\times[a,b]\to \mathbb{R}$ where $a=\min\psi_-$, $b=\max\psi_+$. The functions $\psi_-$ and $\psi_+$ are stationary solutions of

$$\left.\begin{array}{c}\Delta_y\psi+f(y,\psi)=0 \quad \text{in } \omega\\ \partial\psi/\partial\nu=0 \quad \text{on } \partial\omega\end{array}\right\} \tag{1.28}$$

satisfying $\psi_-<\psi_+$, and we are looking for solutions $(c,u)$ of (1.25) with $\psi_-(y)<u(x_1,y)<\psi_+(y)$.

Instead of formulating our assumptions on the nonlinearity $f(y,u)$ explicitly, we state them as conditions on the stationary problem (1.28). Indeed in the paper, the stated conditions on $f$ are only used to derive such properties.

For a stationary solution $\psi$, we denote by $\mu_1(\psi)$ the principal eigenvalue of the linearized problem:

$$\left.\begin{array}{c}-\Delta\chi-f'_\psi(y,\psi)\chi=\mu_1(\psi)\chi \quad \text{in } \omega\\ \dfrac{\partial\chi}{\partial\nu}=0 \quad \text{on } \partial\omega.\end{array}\right\} \tag{1.29}$$

In the following, we say that $\psi$ is a *stable* solution of (1.28) if $\mu_1(\psi)>0$, and *unstable* if $\mu_1(\psi)<0$.

We can now state our results for the more general situation (analogues of Theorems 1.1-1.2′ respectively).

THEOREM 1.6. – *Suppose $\psi_-$ and $\psi_+$ are two stable solutions of* (1.28). *There exists a connection* $(c,u)$, *i.e. a solution of* (1.25)-(1.26), *such that*

$$u(-\infty,y)=\psi(y), \qquad u(+\infty,y)=\psi_+(y)$$

*where $\psi$ is some stationary solution of* (1.28), *with* $\psi_-\leqq\psi<\psi_+$.

THEOREM 1.7. – *Under the assumption above, if the solution* $(c,u)$ *of* (1.25)-(1.27) *exists, then it is unique* (*up to translation for* $u$).

These two statements corresponds to Case C above (Theorems 1.2 and 1.2'). The more general statement corresponding to Theorem 1.3 is the following.

THEOREM 1.8. — *Suppose furthemore that the two stable solutions of* 1.28), $\psi_-$ *and* $\psi_+$ *have the property that every stationary solution* $\psi$ *of* (1.28) *satisfying* $\psi_- < \psi < \psi_+$ *is unstable. Then, there exists a connection from* $\psi_-$ *to* $\psi_+$, *that is, a solution* $(c, u)$ *of* (1.25)-(1.26), *such that*

$$u(-\infty, y) = \psi_-(y) \qquad u(+\infty, y) = \psi_+(y)$$

The next statement is the analogue of case A (Theorem 1.4). (Note however that since, here, $\mu_1(\psi_-) \neq 0$. The conditions are not exactly the same as above.)

THEOREM 1.9. — *Suppose* $\psi_-$ *and* $\psi_+$ *are stationary solutions of* (1.28) *such that* $\psi_-$ *is unstable and* $\psi_+$ *is stable. Assume, furthermore, that there is no stationary solution* $\psi$ *of* (1.28) *with* $\psi_- < \psi < \psi_+$. *Then, there exists* $c^* \in \mathbb{R}$ *such that* (1.25)-(1.27) *admits a solution if and only if* $c \geqq c^*$.

From our assertions, when $f(y, u) = f(u)$, $\psi_- = 0$, and $\psi_+ = 1$, it is clear that the case $\mu_1(\psi_-) = 0$ is more difficult to handle. Indeed, there, the cases $f(s) > 0$ or $f(s) \leqq 0$ in a right neighbourhood of $s = 0$ differ radically. Therefore, this case is omitted in our more general statement.

In this paper we only discuss in detail the problem (1.16), (1.2)-(1.3). The proofs readily extend to the more general problem (1.25)-(1.27). Furthermore, many of the results and methods presented here extend to other situations, in particular to different boundary conditions. But such extensions are not explored here.

We now describe the structure of the paper. An essential step in proving uniqueness and monotonicity (as already in [BN1]) is to establish rather precise exponential behaviour of solutions $u$ as $x_1 \to -\infty$ and of $1-u$ as $x_1 \to +\infty$. For instance we will show that $u = a e^{\lambda x_1} \varphi(y) + o(e^{\lambda x_1})$ as $x_1 \to -\infty$ where $a > 0$ is a constant and $\lambda > 0$ is some eigenvalue with eigenfunction $\varphi = \varphi(y)$. The next three sections are devoted to this question.

In Section 2 we analyze in detail the linear "eigenvalue" problems that arise in this context. Then, in Section 3, we show, under certain conditions, that solutions which tend to 0 as $x_1 \to -\infty$ necessarily decay exponentially. In addition, we prove a result of independent interest, Theorem 3.2, for positive solutions $u$ of a general linear second order elliptic equation in $\Sigma^- = \Sigma \cap \{x_1 < 0\}$, with $u_\nu = 0$ on the curved side of the cylinder. Without imposing any condition on $u$ as $x_1 \to -\infty$ we show that $u$ can grow at most exponentially there. This result should prove useful in other problems.

Incidentally, in connection with problem 2 above, at the beginning of Section 3 we describe a simple example in which the solution decays precisely like a negative power of $-x_1$ as $x_1 \to -\infty$. In addition, in Lemma 7.2, we show that if there is a solution $u$ satisfying $u \geq c|x_1|^{-k}$ for $x_1 < -1$, with $c$, $k$ positive constants, then there cannot exist a solution of the problem which decays exponentially at $-\infty$. This shows that the conditions of Section 3 for exponential decay are sharp. For $f$ of type A, and satisfying $f'(0)=0$, we do not know what further conditions to impose on $f$ in order to ensure exponential decay of solutions at $-\infty$, or to ensure uniqueness, or monotonicity in $x_1$.

In Section 4, we finally derive the exact asymptotic behaviour of solutions $u$ as $x_1 \to -\infty$.

The existence Theorems 1.1 and 1.2 are proved in Section 5 – and Theorem 1.3, in Section 6. Theorems 1.1′, 1.2′ are proved in Section 7 where uniqueness and monotonicity properties are studied. Last, case A is treated in Sections 8 and 9. Section 10 contains the proof of Theorem 1.5.

Most of the results of this paper have been announced and described in [BN2]. Our earlier paper [BN1] on this subject, treated only case B. There, we established the uniqueness of the solution $(c, u)$, and proved that $u_1 > 0$. We also derived precise exponential asymptotic behaviour of $u$ at $-\infty$ and $+\infty$. This involved the "eigenvalue problems" for solutions of the linearized equation near, say, $-\infty$, of the form $e^{\lambda x_1} \varphi(y)$. However our results in this paper include, and are more general than, those of [BN1].

## 2. LINEARIZATION AT INFINITY AND SOME ASSOCIATED LINEAR EIGENVALUE PROBLEMS

As we will see in the next section, the behaviour at infinity of solutions of semi-linear elliptic equations of the kind (1.1) in infinite cylindrical domains is governed by some special solutions, the *exponential solutions*, of the linearized equation at infinity. The latter is an equation in a half cylinder, say $[-\infty, 0] \times \omega$, of the form

$$\Delta w - \beta(y) w_1 - a(y) w = 0. \tag{2.1}$$

Here $\beta(y)$ and $a(y)$ are given functions on $\omega$. Exponential solutions are the particular solutions of (2.1) which are of the following type

$$u = e^{\lambda x_1} \psi(x_1, y), \qquad \psi_\nu = 0 \quad \text{for } y \in \partial\omega, \tag{2.2}$$

where $\psi$ is a polynomial in $x_1$:

$$\psi = x_1^k \psi_k(y) + x_1^{k-1} \psi_{k-1}(y) + \ldots + \psi_0(y), \qquad \psi_k \neq 0. \tag{2.3}$$

This yields the following system of $k+1$ linear elliptic problems for the functions $\psi_0, \ldots, \psi_k$ involving the unknown parameter $\lambda$

$$\left.\begin{array}{c}(\Delta+\lambda^2-\lambda\beta-a)\psi_k=0\\ (\Delta+\lambda^2-\lambda\beta-a)\psi_{k-1}+(2\lambda-\beta)k\psi_k=0\\ (\Delta+\lambda^2-\lambda\beta-a)\psi_{k-2}+(2\lambda-\beta)(k-1)\psi_{k-1}+k(k-1)\psi_k=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ (\Delta+\lambda^2-\lambda\beta-a)\psi_0+(2\lambda-\beta)\psi_1+2\psi_2=0\\ \dfrac{\partial\psi_k}{\partial\nu}=\dfrac{\partial\psi_{k-1}}{\partial\nu}=\ldots=\dfrac{\partial\psi_0}{\partial\nu}=0 \quad \text{on } \partial\omega.\end{array}\right\} \qquad (2.4)$$

One is thus led to a kind of eigenvalue problem in $\lambda$ for a system of equations. In particular, when $k=0$, *i.e.* for solutions of the type $w=e^{\lambda x_1}\varphi(y)$, the system reduces to one equation

$$\left.\begin{array}{c}-\Delta\varphi+a(y)\varphi=(\lambda^2-\lambda\beta(y))\varphi \quad \text{in } \omega,\\ \varphi_\nu=0 \quad \text{on } \partial\omega.\end{array}\right\} \qquad (2.5)$$

Note that in the general case, *i.e.* for arbitrary $k$, the leading order term of $\psi$ in (2.3), *i.e.* $\varphi=\psi_k(y)$ is also determined by the same problem (2.5). As we will see, this problem determines the "eigenvalue" $\lambda$.

In this section we derive some spectral properties for this class of problems. We first investigate the generalized eigenvalue problem (2.5). Afterwards, we will completely describe the set of positive exponential solutions of (2.1). We will state all our results in the case of the half cylinder

$$\Sigma^-=(-\infty,0]\times\omega.$$

Obviously, analogous properties hold in the right half cylinder as well. Since our solutions are positive and tend to zero, we will only be concerned here with solutions with positive $\lambda$ and for which the leading order term $\psi_k(y)$ does not change sign. Hence, of primary concern to us are the existence, uniqueness and characterization of a positive (or negative) *principal eigenvalue* of (2.5). Such an eigenvalue is defined as a $\lambda$ for which there exists an eigenfunction $\varphi$ of the corresponding problem (2.5) which does not change sign.

It may be worth to emphasize the fact that although this is a linear self-adjoint problem with respect to $\varphi$, the dependance on $\lambda$ does not allow one to use the usual spectral results for these operators. Most of our results can be readily adapted to other boundary conditions and to more general settings.

Throughout this paper, we will use the following notation. We denote by $\mu_1$ the first (least) eigenvalue, and by $\sigma(y)$ the associated eigenfunction of the linear eigenvalue problem with Neumann conditions

$$\left.\begin{aligned}(-\Delta+a)\sigma=\mu_1\sigma \quad \text{in } \omega,\\ \sigma_\nu=0 \quad \text{on } \partial\omega.\end{aligned}\right\} \tag{2.6}$$

Here, $\sigma(y)>0$ in $\bar{\omega}$. In our earlier work [BN1], we had studied the eigenvalue problem (2.5) under the additional condition that $\mu_1 \geqq 0$. For the more general type of equations we study here, this condition in general is not satisfied. Actually, as we will see, the results are somewhat different when $\mu_1<0$. We will consider here the general setting, so that the ones in [BN1] will be obtained as particular cases.

The main existence and multiplicity result for (2.1) is the following.

THEOREM 2.1. — *We assume that* $\beta:\bar{\omega}\to\mathbb{R}$ *is an arbitrary continuous function. Then* (2.5) *possesses* 0, 1 *or* 2 *principal real eigenvalues. Furthermore, we have the following characterization* (*recall that* $\mu_1$ *and* $\sigma(y)$ *are the principal eigenvalue and associated eigenfunction defined from* (2.6)).

(*a*) *If* $\mu_1>0$, *then* (2.5) *admits exactly one positive and one negative principal eigenvalue.*

(*b*) *If* $\mu_1=0$, *then, aside from* $\lambda=0$, (2.5) *admits exactly one principal eigenvalue which is positive* (*resp. negative*) *if* $\int_\omega \beta(y)\sigma(y)^2\,dy>0$ (*resp.* $<0$). *If* $\int_\omega \beta(y)\sigma(y)^2\,dy=0$, *then* (2.5) *admits no principal eigenvalues aside from* $\lambda=0$.

(*c*) *If* $\mu_1<0$, (2.5) *may admit* 0, 1 *or* 2 *principal eigenvalues. When two exist, they always have the same sign, namely that of* $\int_\omega \beta(y)\sigma(y)^2\,dy$. *If* $\int_\omega \beta(y)\sigma(y)^2\,dy=0$, *then no principal eigenvalue exists.*

(*d*) *Furthermore, in cases* (*a*) *and* (*b*), *i.e.,* $\mu_1\geqq 0$, *the positive principal eigenvalue is the smallest positive eigenvalue of* (2.5) *and, likewise, the negative one is the largest negative eigenvalue. In all cases, the eigenspace associated with a principal eigenvalue is always one-dimensional.*

(*e*) *Suppose* $\mu_1\geqq 0$, *and suppose* $\beta \lneqq \bar{\beta}$. *If the principal positive eigenvalue* $\lambda$ *corresponding to* $\beta$ *exists, then so does the principal positive eigenvalue* $\bar{\lambda}$ *for* $\bar{\beta}$, *and* $\lambda<\bar{\lambda}$. *Similarly, if the principal negative eigenvalue* $-\bar{\tau}$ *corresponding to* $\bar{\beta}$ *exists, so does the one* $-\tau$ *for* $\beta$, *and* $-\tau<-\bar{\tau}$.

*Remark* 2.1. — Case (*c*) will be discussed further in Theorem 2.2. Also, it should be noted that, contrary to the case $\mu_1\geqq 0$, it is not necessarily true when $\mu_1<0$ that the smallest positive *principal* eigenvalue, if it exists, is the smallest positive eigenvalue. This fact – and examples – will be clear from our proof (compare Remark 2.3 below).

*Proof of Theorem* 2.1. – The proof relies on some properties of the principal eigenvalue of a linear problem and, in particular, its dependence on the zero order coefficient. We first recall these properties.

Consider the following eigenvalue problem

$$\left.\begin{aligned} &(-\Delta+a(y)+q(y))\varphi=\lambda\varphi \quad \text{in } \omega \\ &\frac{\partial\varphi}{\partial\nu}=0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{2.7}$$

With $a$ fixed, denote by $\rho(q)$ the principal eigenvalue of (2.7). That is, $\rho(q)\in\mathbb{R}$ is uniquely determined by the existence of an associated eigenfunction $\varphi>0$ in $\bar{\omega}$. We also know that $\rho(q)$ is the smallest eigenvalue of (2.7) and it is a simple eigenvalue.

PROPOSITION 2.1. – *The mapping $q\mapsto\rho(q)$ defined on $C^0(\bar{\omega})$ into $\mathbb{R}$ has the following properties.*

(i) *$\rho(q)$ is continuous with respect to $q$, in fact 1-Lipschitz continuous in the $L^\infty(\omega)$, norm:*

$$|\rho(q)-\rho(q')|\leqq\|q-q'\|_{L^\infty(\omega)}$$

(ii) *$q\mapsto\rho(q)$ is monotone: if $q_1\leqq q_2$ and $q_1\not\equiv q_2$ on $\bar{\omega}$, then $\rho(q_1)<\rho(q_2)$.*

(iii) *The mapping $q\mapsto\rho(q)$ is concave: if $q_1, q_2\in C^0(\bar{\omega})$, $s\in[0, 1]$, then*

$$\rho[sq_1+(1-s)q_2]\geqq s\,\rho(q_1)+(1-s)\,\rho(q_2).$$

*Proof.* – These classical properties are obvious consequences of the variational characterization of the principal eigenvalue

$$\rho(q)=\inf_{\varphi\in H^1(\omega)} J_q(\varphi) \tag{2.8}$$

with

$$J_q(\varphi)=\frac{\displaystyle\int_\omega |\nabla\varphi|^2+a|\varphi|^2+q|\varphi|^2}{\displaystyle\int_\omega |\varphi|^2}.$$

Indeed, $|J_{q_1}(\varphi)-J_{q_2}(\varphi)|\leqq\|q_1-q_2\|_{L^\infty(\omega)}$ for each fixed $\varphi\in H^1(\omega)$. The mapping $q\mapsto J_q(\varphi)$ being affine and monotone, $\rho(q)$ is concave and monotone (in the sense of (ii) and (iii)).

*Remarks* 2.2. – (*a*) These properties remain true in a more general setting. For instance, the very same statement holds for the principal eigenvalue of the problem

$$\left.\begin{aligned} &\mathrm{L}\varphi+q\varphi=\lambda\varphi \quad \text{in } \omega \\ &\frac{\partial\varphi}{\partial\nu}=0 \quad \text{on } \partial\omega \end{aligned}\right\} \tag{2.9}$$

where $\mathrm{L}=-a_{ij}(x)\partial_{ij}+b_i(x)\partial_i+c(x)$ is an elliptic operator, not necessarily in divergence form, with $a_{ij}(x)\xi_i\xi_j\geqq c_0|\xi|^2$, $\forall\xi\in\mathrm{R}^m$, $\forall x\in\bar{\omega}$, with $c_0>0$; $a_{ij}, b_i, c\in\mathrm{C}^0(\bar{\omega})$.

(*b*) Part (i) can be improved. For instance, it is clear from the variational formulation (2.8) that $\rho(q)$ is continuous with respect to $q$ in the $\mathrm{L}^{m/2}(\omega)$ topology, where $m=n-1$. ■

We can now complete the proof of Theorem 2.1. For each fixed $t\in\mathrm{R}$, we let $\mu_1(t)$ denote the principal eigenvalue of the operator $-\Delta+a(y)+t\beta$ in $\mathrm{H}^2(\omega)$ with Neumann data on $\partial\omega$.

That is, $\mu_1(t)$ is characterized by the existence of a unique $\varphi=\varphi(t)\in\mathrm{H}^1(\omega)$ such that $\varphi(t)(y)>0$, $\forall y\in\bar{\omega}$, normalized by $\|\varphi(t)\|_{\mathrm{L}^\infty(\omega)}=1$ such that

$$\left.\begin{aligned} &-\Delta\varphi+a(y)\varphi+t\beta(y)\varphi=\mu_1(t)\varphi \quad \text{in } \omega \\ &\frac{\partial\varphi}{\partial\nu}=0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{2.10}$$

In other terms, $\mu_1$ is defined by

$$\mu_1(t)=\rho(t\beta).$$

Likewise, we denote by $\mu_k(t)$ the $k$-th eigenvalue of this problem (counting multiplicity). With this notation, we see that $\lambda\in\mathrm{R}$ is an eigenvalue of (2.5) if and only if

$$\lambda^2=\mu_k(\lambda)$$

for some $k\in\mathbb{N}^*$. In particular, since eigenfunctions associated with $\mu_k(t)$ are only positive when $k=1$, it follows that $\lambda$ is a principal eigenvalue of (2.5) if and only if

$$\mu_1(\lambda)=\lambda^2. \tag{2.11}$$

The proof of Theorem 2.1 rests on the following assertions.

Proposition 2.2. – *The function* $t\mapsto\mu_1(t)$ *is continuous and concave on* R. *At* $t=0$, $\mu_1(0)=\mu_1$ *and* $\mu_1(t)$ *is differentiable with*

$$\mu_1'(0)=\frac{\displaystyle\int_\omega\beta(y)\sigma^2(y)\,dy}{\displaystyle\int_\omega\sigma^2(y)\,dy.}$$

*Proof of Proposition* 2.2. – Continuity and concavity of $\mu_1$ will follow from Proposition 2.1. That $\mu_1(0)=\mu_1$ is the definition (2.6) of $\mu_1$. We just need to show that $\mu_1$ is differentiable at 0 and to compute $\mu_1'(0)$.

First, since our eigenfunction $\varphi=\varphi(0)$ of (2.10) associated with $\mu_1(t)$ is unique, it is an obvious consequence of the continuity of $\mu_1(t)$ that $t\mapsto\varphi(t)$ is continuous from R into $W^{2,p}(\omega)$ for all $1\leqq p<\infty$. In particular, $\varphi(t)(y)\to\sigma(y)$ in $W^{2,p}(\omega)$ as $t\to 0$, where $\sigma$ is the eigenfunction of (2.6) normalized by $\|\sigma\|_{L^\infty}=1$, $\sigma>0$ in $\bar{\omega}$.

Decompose $\varphi(t)$ along $R\{\sigma\}\oplus\{\sigma\}^\perp$, in the $L^2(\omega)$ sense. That is, write

$$\varphi(t)=s(t)\sigma+h(t) \tag{2.12}$$

where $s(t)\in R$, and $h(t)$ satisfies

$$\int_\omega \sigma(y)h(t)(y)\,dy=0 \qquad \forall t\in R. \tag{2.13}$$

From (2.10) we obtain the following equation for $h(t)$:

$$\left.\begin{aligned}-\Delta h+ah+t\beta h+ts(t)\beta(y)\sigma\\ =\mu_1(t)h(t)+(\mu_1(t)-\mu_1(0))s(t)\sigma \quad \text{in } \omega\\ \frac{\partial}{\partial\nu}h(t)=0 \quad \text{on } \partial\omega.\end{aligned}\right\} \tag{2.14}$$

Multiplying (2.14) by $\sigma$ and using (2.6), which implies,

$$\int_\omega(-\Delta h+ah)\sigma\,dy=0,$$

we obtain

$$s(t)\frac{\mu_1(t)-\mu_1(0)}{t}\int_\omega\sigma^2=\int_\omega\beta h(t)\sigma\,dy+s(t)\int_\omega\beta\sigma^2.$$

Since $s(t)\to 1$, and $h(t)\to 0$ as $t\to 0$, we see that $\mu_1'(0)$ exists and has the value

$$\mu_1'(0)=\frac{\displaystyle\int_\omega\beta(y)\sigma(y)^2\,dy}{\displaystyle\int_\omega\sigma(y)^2\,dy}. \quad \blacksquare$$

*Conclusion of the proof of Theorem* 2.1. – Since $\mu_1(t)$ is concave, the equation $\mu_1(t)=t^2$ admits at most two roots. The existence and signs of these roots are then immediately determined from knowledge of the position of $\mu_1(0)=\mu_1$ and of the sign of $\mu_1'(0)$ which is that of $\displaystyle\int_\omega\beta(y)\sigma(y)^2\,dy$.

Note that when $\mu_1 \geqq 0$ (cases (*a*) and (*b*)), then since any eigenvalue $\lambda'$ of (2.5) must correspond to a solution of $\mu_k(\lambda') = \lambda'^2$ for some $k \geqq 1$, and since $\mu_k(t) > \mu_1(t)$, $\forall t \geqq 0$, $\forall k \geqq 2$, we see that the positive principal eigenvalue is the smallest positive eigenvalue of (2.5). The last assertion of (*d*) of the theorem is an obvious consequence of the fact that if $\lambda$ is a principal eigenvalue of (2.5), then $\mu_1(\lambda) = \lambda^2$ is the principal eigenvalue of (2.10) which is well known to be simple.

Part (*e*) is Lemma 3.2 of [BN1] and its proof is simple; we just indicate the proof of the first assertion: recall that $\lambda$ is the positive number satisfying $\lambda^2 = \mu_1(\lambda)$ where for $t > 0$, $\mu_1(t)$ is the first eigenvalue of (2.10). The corresponding first eigenvalue $\bar{\mu}_1(t)$ of (2.10), with $\bar{\beta} \gneqq \beta$ in place of $\beta$, satisfies $\bar{\mu}_1(t) > \mu_1(t)$. It follows that $\bar{\lambda} > \lambda$.

The proof of Theorem 2.1 is complete.

*Remark* 2.3. — In case (*c*), when $\mu_1 < 0$, and also $\mu_2(0) < 0$, the smallest positive eigenvalue of (2.5) is no longer a *principal* eigenvalue as seen from the previous construction. Now, this circumstance only depends on the choice of $a(y)$ and not on $\beta$. We will see that for any $a(y)$, there exists $\beta(y)$ for which (2.5) admits positive principal eigenvalues. Thus, if $\mu_2(0) < 0$, there is a $\beta$ for which the principal positive eigenvalue is not the smallest positive eigenvalue.

Next, some further discussion for case (*c*), when $\mu_1 < 0$. The structure of this problem is better understood if we let the term $\beta$ depend on some parameter $\tau$ — compare with Section 1. Thus, we consider the more general family of problems

$$\left.\begin{aligned} -\Delta\varphi + a(y)\,\varphi = (\lambda^2 - \lambda\beta_\tau(y))\,\varphi \quad \text{in } \omega, \\ \varphi_\nu = 0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{2.15}$$

Here $\beta_\tau(y) = \beta(y, \tau)$ is a family of functions satisfying (1.17) and (1.18).

The typical examples one should keep in mind (and which are the relevant ones for the applications in this work) are

$$\beta_\tau(y) = \alpha(y) + \tau \tag{2.16}$$

or

$$\beta_\tau(y) = \tau\alpha(y) \tag{2.17}$$

for some continuous positive function $\alpha$ on $\bar{\omega}$.

THEOREM 2.2. — *Suppose that the principal eigenvalue* $\mu_1$ *of* (2.6) *is negative. Suppose furthermore that the family* $\beta_\tau$ *satisfies* (1.17), (1.18). *Then, there exists a critical value of* $\tau$, $\tau^*$, *such that if* $\tau < \tau^*$, (2.15) *has no principal eigenvalue; for* $\tau = \tau^*$, (2.15) *possesses exactly one principal eigenvalue which is positive, while for* $\tau > \tau^*$, (2.15) *admits exactly two distinct principal eigenvalues which are both positive. Likewise, there exists* *a critical value* $\tau_-^* < \tau^*$ *such that* (2.15) *possesses respectively exactly one principal eigenvalue which is negative, two negative principal eigenvalues, or no negative principal eigenvalue, according as* $\lambda = \tau_-^*, \lambda < \tau_-^*$ *or* $\lambda > \tau_-^*$.

*Proof.* — Consider again our previous construction but with $\beta$ replaced by $\beta_\tau$. The corresponding principal eigenvalue of (2.10) will be denoted by $\mu_1^\tau(t)$. From Proposition 2.1, we see that $\mu_1^\tau(t)$ depends continuously on $\tau$. Furthermore, if $\tau < \tau'$, then $\mu_1^\tau(t) < \mu_1^{\tau'}(t)$, $\forall t \geqq 0$ (with reversed inequality if $t < 0$). Since

$$(\mu_1^\tau)'(0) = \frac{\int_\omega \beta_\tau(y)\,\sigma^2}{\int_\omega \sigma^2}$$

it is clear that $\lim_{\tau \to \pm\infty} (\mu_1^\tau)'(0) = \pm\infty$. Hence, there exists $\tau_0 \in \mathrm{R}$ such that $(\mu_1^{\tau_0})'(0) = 0$. Furthermore, by concavity, $\mu_1^\tau(t) \leqq \mu_1 + t(\mu_1^\tau)'(0)$, and so for $|\tau - \tau_0|$ small, no principal eigenvalue exists.

Using these facts, Theorem 2.2 now follows easily with the aid of:

$$\lim_{\tau \to +\infty} \mu_1^\tau(1) = +\infty, \qquad \lim_{\tau \to -\infty} \mu_1^\tau(-1) = +\infty \tag{2.18}$$

which we leave to the reader. ■

We now consider once more the problem (2.5) or system (2.4) and investigate some consequences of the existence of a principal eigenvalue of (2.5).

Since the problem is not a self adjoint eigenvalue problem, there may exist complex eigenvalues of (2.5). For instance, if $\beta \equiv 0$, $a(y) \equiv -1$, then $\lambda = \pm i$ is a (principal) eigenvalue of (2.5), associated with constant eigenfunctions.

This, however, cannot happen as soon as there exists a real principal eigenvalue.

THEOREM 2.3. — *If there exists a real principal eigenvalue* $\lambda$ *of* (2.5), *then* *all* *eigenvalues of* (2.5) *are real.*

*Proof.* — Let $\lambda$ be a real principal eigenvalue and let $\varphi > 0$ be an eigenfunction associated with $\lambda$, that is (2.5) holds:

$$-\Delta\varphi + a(y)\,\varphi = (\lambda^2 - \lambda\beta(y))\,\varphi \quad \text{in } \omega$$
$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\omega.$$

We can also read (2.5) as saying (since $\varphi>0$) that the first eigenvalue of $\mathscr{L}=-\Delta+a(y)-(\lambda^2-\lambda\beta(y))$ in $H^2(\omega)$ with Neumann boundary conditions is 0, $\varphi$ being the associated eigenfunction. The variational characterization of the first eigenvalue of $\mathscr{L}$ therefore shows that

$$\int_\omega \{|\nabla w|^2+a(y)|w|^2-(\lambda^2-\lambda\beta(y))|w|^2\}\geqq 0 \qquad \forall w\in H^1(\omega) \quad (2.19)$$

Now suppose that $\zeta\in\mathbb{C}$ is another eigenvalue of (2.5) associated with an eigenfunction $\psi\not\equiv 0$ in $\omega$. Write $\zeta=\mu+i\nu$ and assume by way of contradiction that $\nu\neq 0$.

We can write

$$\left.\begin{aligned} -\Delta\psi+a\psi+(\mu^2+\nu^2)\psi=\zeta p(y)\psi \quad \text{in } \omega\\ \frac{\partial\psi}{\partial\nu}=0 \quad \text{on } \partial\omega \end{aligned}\right\} \quad (2.20)$$

with $p(y)=2\mu-\beta(y)$. Multiply this equation by $\bar\psi$ to get

$$\int_\omega |\nabla\psi|^2+a|\psi|^2+(\mu^2+\nu^2)|\psi|^2=\zeta\int_\omega p(y)|\psi|^2. \quad (2.21)$$

The left hand side in (2.21) being real, as well as $\int_\omega p(y)|\psi|^2$, we infer that if $\nu=\operatorname{Im}\zeta\neq 0$, we must have $\int_\omega p|\psi|^2\,dy=0$, *i.e.*,

$$\int_\omega \beta(y)|\psi|^2=2\mu\int_\omega|\psi|^2. \quad (2.22)$$

Furthermore, the left hand side in (2.21) is zero:

$$\int_\omega |\nabla\psi|^2+a|\psi|^2+(\mu^2+\nu^2)|\psi|^2=0.$$

From the variational characterization (2.19), with $w=\psi$, we then find

$$\int_\omega (\mu^2+\nu^2)|\psi|^2+[\lambda^2-\lambda\beta(y)]|\psi|^2\leqq 0.$$

Combining with (2.22) we obtain

$$[\mu^2+\nu^2+\lambda^2-2\mu\lambda]\int|\psi|^2\leqq 0. \quad (2.23)$$

Since $\psi\not\equiv 0$, this is obviously impossible if $\nu\neq 0$. Contradiction; the theorem is proved.

*Remark* 2.4. — This is an extension of our earlier Theorem 3.1 in [BN2]. There we had assumed that the first eigenvalue $\mu_1$ of (2.6) is nonnegative and we proved that if $\mu_1>0$, there is always a positive principal eigenvalue. In case $\mu_1=0$, then $\lambda=0$ is a real principal eigenvalue. Therefore in both cases the result there follows from Theorem 2.3 here.

*Remark* 2.5. — The theorem holds under more general conditions than (1.17), (1.18). It suffices, namely, to assume that $\beta(y,\tau)$ is continuous on $\bar{\omega}\times R$, satisfies

$$\text{if } \tau<\tau' \text{ then } \beta(y,\tau) \underset{\not\equiv}{\geq} \beta(y,\tau'); \tag{2.24}$$

and that there exist nonempty open sets $\omega', \omega'' \subset \omega$ such that

$$\left.\begin{aligned} \lim_{\tau\to+\infty} \{\inf_{\omega'} \beta(y,\tau)\} = +\infty \\ \lim_{\tau\to-\infty} \{\sup_{\omega''} \beta(y,\tau)\} = -\infty. \end{aligned}\right\} \tag{2.25}$$

*Remark* 2.6. — Does the converse to Theorem 2.3 hold? Namely, if (2.5) does not have a real principal eigenvalue is there necessarily a complex (non real) eigenvalue? This is the case for instance if $\beta(y)$ is a constant.

We now turn to the analysis of the system (2.4). Our goal is to describe the set of bounded and positive exponential solutions on $R^-$ of (2.1). We will prove that they are either of the form $w=e^{\lambda x_1}\Phi(y)$ or $w=e^{\lambda x_1}(-x_1\Phi(y)+\Phi_0(y))$ with $\lambda$ a principal eigenvalue and $\Phi$ an associated eigenfunction of (2.5).

Recall that an exponential solution is defined as

$$u=e^{\lambda x_1}\psi(x_1,y)$$
$$\frac{\partial\psi}{\partial\nu}(x_1,y)=0 \qquad \text{for} \quad y\in\partial\omega$$

with

$$\psi(x_1,y)=(-1)^k x_1^k\psi_k(y)+x_1^{k-1}\psi_{k-1}(y)+\ldots+\psi_0(y)$$
$$\psi_k\not\equiv 0 \quad \text{in } \omega.$$

If $u$ is to be bounded, then for $k>0$, we require $\lambda$ to be positive. Furthermore, if $u>0$, then necessarily $\psi_k>0$ in $\omega$. We recall that $\psi_0, \psi_1, \ldots, \psi_k$ satisfy the system (2.4).

In particular, $\psi_k=\varphi$ is then an eigenfunction of (2.5) associated with $\lambda$ which is therefore a principal positive eigenvalue of (2.5). Thus, we now assume that (2.5) possesses such a principal positive eigenvalue.

THEOREM 2.4. — *Suppose that* $\lambda>0$ *is a principal eigenvalue of* (2.5). *Suppose that u defined in* (2.2)-(2.3) *is an exponential solution of* (2.1) *with* $\psi_k=\varphi>0$ *being an eigenfunction of* (2.5) *associated with* $\lambda$. *Then* $k\leq 1$.

*Furthermore, $k=0$ and $u$ is of the form $u=e^{\lambda x_1}\varphi(y)$ in each of the following cases:*

(*a*) $\mu_1 \geqq 0$

*or*

(*b*) $\mu_1 < 0$ *and* (2.5) *has two distinct positive principal eigenvalues.*

This result was proved in the case $\mu_1 \geqq 0$ (where $k=0$) in our earlier work [BN1] (compare Theorem 3.2 there). Therefore, we will only consider here the case $\mu_1 < 0$ which is somewhat more delicate.

For $\tau \in \mathrm{R}$, set

$$\beta_\tau(y) = \beta(y) + \tau.$$

The family $\beta_\tau$ clearly satisfies our assumptions (1.17)-(1.18). Let $\tau^*$ be the critical value of $\tau$ for this family, given by Theorem 2.2. We recall that $\tau^*$ is such that (2.15) (that is the problem (2.5) with $\beta_\tau$ instead of $\beta$) has a principal positive eigenvalue if and only if $\tau \geqq \tau^*$. For $\tau = \tau^*$ there is one and only one such eigenvalue while for $\tau > \tau^*$ there are exactly two positive principal eigenvalues.

Problem (2.5) corresponds to $\tau = 0$. The assumption that there exists a principal eigenvalue $\lambda > 0$ thus means that $\tau^* \leqq 0$. In this setting, the assumption that (2.5) has two distinct positive eigenvalues is equivalent to $\tau^* < 0$. Therefore we will distinguish between the case $\tau^* < 0$ and the case $\tau^* = 0$. In the former we will prove that necessarily $k=0$ while in the latter we will show $k \leqq 1$. Actually, in the latter, as we will see, there is always a bounded positive exponential solution with $k=1$ and $\psi_1 > 0$ in (2.3).

We use the same notation as before. For $t \in \mathrm{R}$, we denote by $\mu_1(t)$ the principal eigenvalue of (2.10), and by $\varphi(t)$ our associated eigenfunction normalized by $\varphi(t) > 0$ on $\bar{\omega}$, $\|\varphi(t)\|_{L^\infty(\omega)} = 1$. They satisfy

$$\left.\begin{aligned} & L_t \varphi := -\Delta\varphi + a(y)\varphi + t\beta(y)\varphi = \mu_1(t)\varphi \quad \text{in } \omega \\ & \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\omega; \qquad \varphi = \varphi(t) > 0 \quad \text{on } \bar{\omega} \\ & \|\varphi(t)\|_{L^\infty(\omega)} = 1. \end{aligned}\right\} \tag{2.26}$$

First, a consequence of Proposition 2.2.

PROPOSITION 2.3. – *The function $t \mapsto \mu_1(t)$ is $C^1$ on* R *and*

$$\mu_1'(t) = \frac{\displaystyle\int_\omega \beta(y)\,|\varphi(t)(y)|^2\,dy}{\displaystyle\int_\omega |\varphi(t)(y)|^2\,dy}. \tag{2.27}$$

*Proof.* – (2.27) follows from Proposition 2.2 by taking

$$\hat{a}(y) = a(y) + t_0\,\beta(y) \qquad \text{and} \qquad \hat{\mu}_1(s) = \mu_1(t_0 + s).$$

We now prove Theorem 2.4.

A. PROOF OF THEOREM 2.4 IN CASE $\tau^*<0$. — We prove that $k=0$. As $\tau^*<0$, there are exactly two positive roots of $\mu_1(t)=t^2$. We denote these roots by $\lambda_-$ and $\lambda_+$ such that $0<\lambda_-<\lambda_+$. Our $\lambda$ must be one of these roots. From the concavity of $\mu_1(t)$ it follows that $\mu_1'(\lambda_-)>2\lambda_-$ and $\mu_1'(\lambda_+)<2\lambda_+$. In any case,

$$\mu_1'(\lambda)\neq 2\lambda. \tag{2.28}$$

To prove that $k=0$, we argue by contradiction and assume $k\geqq 1$. The first two equations in system (2.4) read

$$\left.\begin{array}{c}\left.\begin{array}{r}L\varphi_k:=(\Delta+\lambda^2-\lambda\beta-a)\varphi_k=0\\ (\Delta+\lambda^2-\lambda\beta-a)\varphi_{k-1}+(2\lambda-\beta)k\varphi_k=0\end{array}\right\}\text{ in }\omega\\ \dfrac{\partial\varphi_k}{\partial\nu}=\dfrac{\partial\varphi_{k-1}}{\partial\nu}=0\quad\text{in }\partial\omega.\end{array}\right\} \tag{2.29}$$

Multiply the first equation by $\varphi_{k-1}$ and the second by $\varphi_k$. Use of Green's formula yields

$$\int_\omega (2\lambda-\beta(y))\,|\varphi_k(y)|^2\,dy=0. \tag{2.30}$$

The first equation shows that up to a normalization, $\varphi_k$ is the eigenfunction of (2.5) associated with $\lambda$. From (2.27) (in Proposition 2.3) it follows that

$$\int_\omega \beta(y)\,|\varphi_k(y)|^2\,dy=\mu_1'(\lambda)\int_\omega |\varphi_k(y)|^2\,dy. \tag{2.31}$$

Whence, by (2.30)

$$(2\lambda-\mu_1'(\lambda))\int_\omega |\varphi_k(y)|^2\,dy=0. \tag{2.32}$$

This contradicts (2.28) since $\varphi_k\not\equiv 0$.

Therefore, one has $k=0$, and $u$ is of the form $e^{\lambda x_1}\varphi(y)$.

B. PROOF OF THEOREM 2.4 IN CASE $\tau^*=0$. — Our goal here is to show that $k\leqq 1$. By way of contradiction assume $k\geqq 2$. Hence, (2.29)-(2.32) hold as well as (from (2.4))

$$L\varphi_{k-2}+(k-1)(2\lambda-\beta(y))\varphi_{k-1}+k(k-1)\varphi_k=0. \tag{2.33}$$

Multiplying (2.33) by $\varphi_k$ and integrating we find, since $k>1$,

$$\int_\omega (2\lambda-\beta)\varphi_{k-1}\varphi_k+k\int_\omega \varphi_k^2=0$$

so that

$$\int_\omega (2\lambda-\beta)\,\varphi_{k-1}\,\varphi_k<0.$$

On the other hand, if we multiply the second equation in (2.29) by $\varphi_{k-1}$ and integrate, we obtain

$$k\int_\omega (2\lambda-\beta)\,\varphi_k\,\varphi_{k-1}=-\int_\omega \varphi_{k-1}\,\mathrm{L}\,\varphi_{k-1}\geqq 0$$

since 0 is the lowest eigenvalue of $-\mathrm{L}$ (for $\varphi_k>0$). With this contradiction, Theorem 2.4 is proved.

To complete the section, we show that in case $\tau^*=0$ there is an exponential solution of the form (2.2)-(2.3), with $k=1$.

Let $\varphi(t)$ be the eigenfunction of (2.10) associated with $\mu_1(t)$ and normalized by

$$\varphi(t)>0 \qquad \text{and} \qquad \|\varphi(t)\|_{\mathrm{L}^2(\omega)}=1. \tag{2.34}$$

As before (when the normalization involved the $\mathrm{L}^\infty(\omega)$ norm rather than $\mathrm{L}^2(\omega)$), $\varphi(t)$ is a continuous function of $t$ in $\mathrm{L}^2(\omega)$. Since $\tau^*=0$, there is only one principal eigenvalue $\lambda$ of (2.5). Thus $\lambda$ is the only root of the equation $t^2=\mu_1(t)$. Since $t^2-\mu_1(t)\geqq 0$ for all $t$, we see that

$$\mu_1'(\lambda)=2\lambda.$$

We will require a differentiability property of $\varphi(t)$:

Lemma 2.1. – *For $t>0$, $\varphi(t)$ is differentiable at $t$ and $\varphi'(t)=\varphi'(t)(y)$ satisfies*

$$\left.\begin{aligned}&[-\Delta+a(y)+t\,\beta(y)-\mu_1(t)]\,\varphi'(t)=(\mu_1'(t)-\beta)\,\varphi(t) \quad \textit{in } \omega,\\ &\frac{\partial}{\partial\nu}\varphi(t)=0 \quad \textit{on } \partial\omega.\end{aligned}\right\} \tag{2.35}$$

Postponing the proof for a moment, we get

$$e^{\lambda x_1}(-x_1\,\varphi(\lambda)(y)-\varphi'(\lambda)(y))$$

as an exponential solution of (2.2) of the form (2.3) with $k=1$, $\psi_k=\varphi(\lambda)$ and $\psi_{k-1}=\varphi'(\lambda)\,(=\varphi'(\lambda)(y))$.

*Proof of Lemma* 2.1. – $\mu_1(t)$ and $\varphi(t)$ satisfy (2.10). Take the orthogonal decomposition

$$\varphi(t+h)-\varphi(t)=w(t,h)+\alpha(t,h)\,\varphi(t)$$

where $w(t,h)$ is orthogonal to $\varphi(t)$ in $\mathrm{L}^2(\omega)$. By continuity,

$$\|w(t,h)\|_{\mathrm{L}^2(\omega)}\to 0 \qquad \text{and} \qquad \alpha(t,h)\to 0 \quad \text{as } h\to 0.$$

For $L=-\Delta+a+t\beta-\mu_1(t)$, we find, using (2.10), that

$$L\frac{w(t,h)}{h}=\left(\frac{\mu_1(t+h)-\mu_1(t)}{h}-\beta\right)\varphi(t+h). \tag{2.36}$$

Since the right hand side is in the range of L, it is orthogonal in $L^2(\omega)$ to $\varphi(t)$. If N denotes the space of functions orthogonal in $L^2(\omega)$ to $\varphi(t)$, $L^{-1}:N\to N$ is a bounded map. The right hand side converges in $L^2$ to $(\mu_1'(t)-\beta)\varphi(t)$. Consequently

$$\frac{w(t,h)}{h}\to v(t) \quad \text{in } L^2$$

with the function $v(t)$ in N, satisfying

$$Lv(t)=(\mu_1'(t)-\beta)\varphi(t).$$

Next, using the relation

$$\int_\omega(\varphi(t+h)-\varphi(t))(\varphi(t+h)+\varphi(t))\,dy=0$$

we find

$$\|w(t,h)\|_{L^2}^2+\alpha(t,h)\int(\varphi(t+h)+\varphi(t))\varphi(t)\,dy=0.$$

Since $\|w(t,h)\|_{L^2}=O(h)$, as we have shown, and since the coefficient of $\alpha(0,h)$ in the preceding equation is close to 0, it follows that

$$\alpha(t,h)=O(h^2).$$

Consequently

$$\frac{\varphi(t+h)-\varphi(t)}{h}\overset{L^2}{\to}v(t).$$

Thus $\varphi_1'(t)=v(t)$ and it satisfies (2.35). Clearly $\varphi'$ is smooth.

## 3. EXPONENTIAL DECAY OF SOLUTIONS

In this section we study the decay of positive solutions $u<1$ in the infinite cylinder $\Sigma=(-\infty,\infty)\times\omega$ of the problem

$$\left.\begin{aligned}&\Delta u-\beta(y)u_1+f(y,u)=0 \quad \text{in } \Sigma\\ &u_\nu(x,y)=0 \qquad \text{for} \quad y\in\partial\omega,\end{aligned}\right\} \tag{3.1}$$

with $u(x_1,y)\to 0$, and 1, respectively, as $x_1$ tends to $-\infty$ and $+\infty$, uniformly for $y\in\bar{\omega}$. Here $f$ is a continuous function which is also uniformly

Lipschitz in $u$, and $f(y,0)=f(y,1)=0\ \forall y\in\bar{\omega}$. Further regularity of $f$ will be assumed at $u=0$ and $u=1$.

In our study of $u$ near $+\infty$, we always asume $f_u(y,1)\lneqq 0$; this is the simplest case. The corresponding assumption was made in [BN1] and the asymptotic behaviour of $1-u$ was described there in detail. It is also described in Theorem 4.5 in the next section.

In fact, here, we will concentrate our attention on the behaviour of $u$ as $x_1\to-\infty$ (we will use the expressions: at $-\infty$, or near $-\infty$). As will be seen, we need to distinguish several cases.

Here we will prove, under various conditions, that the solution $u$, and also $|\nabla u|$, decay exponentially at $-\infty$. In the next section, arguing as in [BN1]—using the results of Section 2, and results of [AN] and [P]—we will derive more precise asymptotic behaviour. Thus in this and the next section, we treat (3.1) in a half cylinder $\Sigma^-=(-\infty,0]\times\omega$. Throughout, $u>0$ in $\bar{\Sigma}^-$, and tends to zero as $x_1\to-\infty$, uniformly in $y$.

Before stating our positive results we point out that solutions of (3.1) need not always decay exponentially at $-\infty$. For example on $R^1$, the function $u=-1/x$, near $-\infty$, satisfies, for $c>0$,

$$\ddot{u}-c\dot{u}+cu^2-2u^3=0. \tag{3.2}$$

In fact this example may easily be extended to a solution of (3.1) on all of $R^1$—for large $c$. Namely, let $u$ be a smooth, strictly increasing, function of $x$, $0<u<1$, which equals $-1/x$ for $-x$ large, and equals $1-e^{-x}$ for $x$ large. Choose $c$ so large that $\ddot{u}-c\dot{u}<0$ for all $x$. Since $u$ is strictly increasing, for a suitable smooth function $f$,

$$\ddot{u}-c\dot{u}+f(u)=0$$

and $f>0$ in $(0,1)$, $f(0)=f(1)=0$. Furthermore, as above, for $u$ small, $f=cu^2-2u^3$ and for $u$ close to 1 (*i.e.*, $x$ large), $f=(1+c)(1-u)$.

On the other hand, the function

$$u=\frac{1}{1+e^{-x}}$$

on $R^1$ satisfies

$$\ddot{u}-\dot{u}+2u^2-2u^3=0. \tag{3.3}$$

This equation is almost the same as (3.2) and yet the solutions behave very differently near $-\infty$. In both of these cases, $f(0)=f'(0)=0$ and $f''(0)>0$. In such a situation we do not know how to tell when the solutions decay exponentially and when not. For (3.1) on all of $\Sigma$, in case $f=f(u)>0$ on $(0,1)$, $f'(1)<0$ and $f'(0)=0$, if there is a solution $u$ which decays no faster than a power of $|x_1|^{-1}$ near $-\infty$, then we do not know, for $n>1$, if $u$ is necessarily monotonic in $x_1$, or if it is the unique solution of (3.1) (tending to zero at $-\infty$ and to 1 at $+\infty$).

The aim of this section is to establish exponential decay at $-\infty$ in a number of cases – indeed, in most cases. In the proofs we will use the fact that a solution $u$ of (3.1) satisfies the linear equation

$$\Delta u - \beta(y) u_1 + c(x_1, y) u = 0 \quad \text{in } \Sigma^- \tag{3.4}$$

and

$$u_\nu = 0 \qquad \text{for} \quad y \in \partial\omega \tag{3.4'}$$

where $c = c(x_1, y) = f(y, u)/u$; by our assumption, $c \in L^\infty$. We will make use of several results which old for more general elliptic operators.

We always assume that the coefficients, $\beta$, $c$ in (3.4) are in $L^\infty$, and

$$|\beta|, |c| \leqq b. \tag{3.5}$$

In addition we always assume that our functions $u$ are in $W^{2,p}_{\text{loc}}(\bar{\Sigma}^-)$ for some fixed $p > n$.

In the conditions below, $s_0$ is a fixed positive number, and the conditions holds for all $y \in \bar{\omega}$.

THEOREM 3.1. – *For some positive constants* $\varepsilon$, $C_1$, *our solution of* (3.1) *in* $\Sigma^-$ *satisfies*

$$u(x) + |\nabla u(x)| \leqq C_1 e^{\varepsilon x_1} \quad \textit{in } \Sigma^- \tag{3.6}$$

*in each of the following cases:*

CASE 1. $f(y, s) \leqq -a_0(y) s$ *for* $0 \leqq s \leqq s_0$, *and the first eigenvalue* $\mu_1$ *of* $-\Delta_y + a_0(y)$, *i.e., of* (2.6), *is positive;*

CASE 1'. $f(y, s) \leqq -a_0(y)\, s$ *for* $0 \leqq s \leqq s_0$, *the first eigenvalue* $\mu_1$ *of* $-\Delta_y + a_0(y)$ *is zero, and, if* $\sigma$ *is the corresponding positive eigenfunction* (*of* (2.6)),

$$\int_\omega \beta(y) \sigma^2(y)\, dy > 0;$$

CASE 2. $f(y, s) \geqq -a_0(y) s$ *for* $0 \leqq s \leqq s_0$ *and the first eigenvalue* $\mu_1$ *of* $-\Delta_y + a_0(y)$ *is negative.*

*Remark* 3.1. – If $f(y, 0) = 0$ and $f(y, s) = -a(y) s + o(s)$ then the only case which is not covered is the case: the first eigenvalue $\mu_1$ of $-\Delta + a$, in $\omega$, is $\mu_1 = 0$, and assumption of case 1' is not satisfied. The example (3.2) falls into this case.

*Proof of Theorem* 3.1 *in Cases* 1 *and* 1'. – These cases are proved easily with the aid of the maximum principle and Theorem 2.1. For both cases, according to (*a*) and (*b*) of Theorem 2.1, there is a positive principal eigenvalue $\lambda$ and also a positive principal eigenfunction $\varphi(y)$ in $\bar{\omega}$ of (2.5):

$$(-\Delta+a_0(y)-\lambda^2+\lambda\beta(y))\varphi=0 \quad \text{in } \omega$$
$$\varphi_\nu=0 \quad \text{on } \partial\omega.$$

Thus if

$$L_0=\Delta-\beta(y)\partial_1-a_0(y),$$

then $L_0(e^{\lambda x_1}\varphi)=0$. By our hypotheses, $L_0 u \geqq 0$. For some positive constant C, the function

$$w=u-Ce^{\lambda x_1}\varphi$$

is $\leqq 0$ for $x_1=0$, it satisfies $w_\nu=0$ for $y\in\partial\omega$, and the elliptic inequality

$$L_0\omega\geqq 0. \tag{3.7}$$

We claim that $w\leqq 0$ in $\Sigma^-$. This is easily verified. By our hypotheses, there is a nonnegative eigenvalue $\mu_1$, with a positive eigenfunction $\sigma(y)$ in $\bar{\omega}$ of (2.6):

$$(-\Delta_y+a_0)\sigma=\mu_1\sigma \quad \text{in } \omega$$
$$\sigma_\nu=0 \quad \text{on } \partial\omega.$$

Set $w=\sigma z$; then $z_\nu=0$ for $y\in\partial\omega$. By (3.7),

$$0\leqq\frac{L_0 w}{\sigma}=\Delta z+\frac{2}{\sigma}\sigma_i z_i-\beta z_1-\mu_1 z.$$

Since $\mu_1\geqq 0$, we may apply the maximum principle and the Hopf lemma, recalling that $z_\nu=0$ for $y\in\partial\omega$, and conclude that $z\leqq 0$ in $\Sigma^-$; consequently $w\leqq 0$ in $\Sigma^-$.

We have proved that

$$u\leqq Ce^{\lambda x_1}\varphi(y) \quad \text{in } \Sigma^-. \tag{3.8}$$

Since $u$ satisfies (3.4) in $\Sigma^-$, and (3.4)′, we may apply local estimates, up to the boundary: for $x_1\leqq -1$,

$$|\nabla u(x_1,y)|\leqq \text{Const.} \max_{\substack{|\xi-x_1|\leqq 1/2\\ \eta\in\bar{\omega}}} u(\xi,\eta). \tag{3.9}$$

Using (3.8) we obtain the desired conclusion (3.6).

To prove Case 2 of the theorem we will make use of the following

LEMMA 3.1. — *Let u be a positive function in* $\bar{\Sigma}^-$ *which tends to zero uniformly at* $-\infty$ *and which satisfies*

$$\Delta u-\beta(y)u_1-a_0(y)u\leqq 0 \quad \textit{in } \Sigma^- \tag{3.10}$$
$$u_\nu=0 \quad \textit{for } y\in\partial\omega. \tag{3.11}$$

*Assume that the principal eigenvalue* $\mu_1$ *of* $-\Delta_y+a_0(y)$ *is negative, with* $\sigma(y)$ *as positive eigenfunction in* $\bar{\omega}$:

$$\left.\begin{aligned}(-\Delta_y+a_0)\sigma=\mu_1\sigma \quad \textit{in } \omega, \quad \mu_1<0,\\ \sigma_\nu=0 \quad \textit{on } \partial\omega.\end{aligned}\right\} \tag{3.12}$$

*Then, for some positive constants* $\varepsilon$, C,

$$\int_{-\infty}^{-N}\int_{\omega} u \leqq C e^{-\varepsilon N}, \qquad \textit{for} \quad N \geqq 0, \tag{3.13}$$

*where* $\varepsilon$ *depends only on* $b$, $\mu_1$, $\min \sigma$, $\max \sigma$, *and* C *depends only on these numbers and on*

$$\tilde{C} = \max_{\substack{-1 \leqq x_1 \leqq 0 \\ y \in \bar{\omega}}} (u + |\nabla u|).$$

*Proof of Lemma* 3.1. – With positive integers $N < R$, let $\zeta = \zeta_{N, R}$, be a $C^\infty$ function on $R^1$, $0 \leqq \zeta \leqq 1$, satisfying $\zeta \equiv 0$ for $x_1 \leqq -R-1$, $\zeta \equiv 1$ for $-R \leqq x_1 \leqq -N$, and

$$\zeta_{N, R} \equiv 0 \qquad \text{for} \quad x_1 \geqq -N+1.$$

We may choose $\zeta$ in such a way that

$$C_1 := \| \zeta_{N, R} \|_{C^2}$$

is independent of N and R.

Multiply (3.10) by $\sigma\zeta(x_1)$ (where $\zeta = \zeta_{N, R}$) and integrate over $\Sigma^-$:

$$0 \geqq \int_{\Sigma^-} \sigma\zeta(\Delta u - \beta u_1 - a_0 u) = \int u[\sigma\zeta'' + \zeta(a_0 - \mu_1)\sigma + \beta\sigma\zeta' - a_0\zeta\sigma],$$

by Green's theorem and (3.11) and (3.12). Thus

$$0 \geqq \int u(-\mu_1\zeta\sigma + \sigma\zeta'' + \beta\sigma\zeta').$$

Consequently

$$-\mu_1 \int_{-R}^{-N}\int_{\omega} \sigma u \leqq (1+b) C_1 \left[\int_{-R-1}^{-R}\int_{\omega} \sigma u + \int_{-N}^{-N+1}\int_{\omega} \sigma u\right].$$

Since $u \to 0$ uniformly as $x_1 \to -\infty$, the first $n$ dimensional integral on the right hand side tends to zero as $R \to \infty$. Letting, then, $R \to \infty$, and setting $a = -\mu_1 [(1+b) C_1]^{-1}$, we conclude that $\sigma u \in L^1(\Sigma^-)$ and

$$g(N) := \int_{-\infty}^{-N}\int_{\omega} \sigma u \leqq \frac{1}{a}\int_{-N}^{-N+1}\int_{\omega} \sigma u,$$

or

$$a g(N) \leqq g(N-1) - g(N)$$

*i.e.*

$$g(N) \leqq \frac{g(N-1)}{1+a}.$$

This implies that

$$g(\mathrm{N}) \leqq e^{-\varepsilon(\mathrm{N}-1)} g(1) \text{ for any integer } \mathrm{N} \geqq 1,$$

which yields (3.13).

*Proof of Theorem* 3.1 *case* 2. — Inequality (3.6) follows from Lemma 3.1, as before, with the aid of a standard local estimate (up to the boundary) which holds for solutions of (3.4) satisfying (3.4)′: for $x_1 < -1$

$$u(x_1, y) + |\nabla u(x_1, y)| \leqq \mathrm{C}_2 \int_{x_1-1}^{x_1+1} \int_{\omega} u. \tag{3.14}$$

Theorem 3.1 is proved.

Though we will not use it, we mention a sharper form of Lemma 3.1 in case

$$0 < \beta_0 \leqq \beta(y) \leqq b \tag{3.15}$$

with $\beta_0$ a constant:

Lemma 3.2. — *Let $u$ be a positive function in $\bar{\Sigma}^-$ satisfying* (3.10), (3.11) *and with $a_0$ satisfying the conditions of Lemma* 3.1. *Suppose, in addition, that* (3.15) *holds. Then for some positive constant $\varepsilon$ depending only on $b$, $\mu_1$,* $\min \sigma$, $\max \sigma$ *and* $\beta_0$,

$$\int_{\Sigma^-} e^{-\varepsilon x_1} u \leqq \mathrm{C}_3. \tag{3.16}$$

*Here $\mathrm{C}_3$ is a constant depending only on $b, \ldots, \beta_0$ and on*

$$\tilde{\tilde{\mathrm{C}}} = \max_{y \in \bar{\omega}} (u(0, y) + |\nabla u(0, y)|).$$

Note that nothing is assumed about the behaviour of $u$ near $-\infty$. In the proof we denote by C any constant depending on $b$, $\tilde{\tilde{\mathrm{C}}}$ and $\max \sigma$.

*Proof.* — For R large, and $0 < \varepsilon$ small (to be fixed), we multiply (3.10) by $e^{-\varepsilon x_1} \sigma(y) \zeta(x_1)$ where $\zeta = \cos(x_1/\mathrm{R})$, and integrate over $\left(-\dfrac{\pi}{2}\mathrm{R}, 0\right) \times \omega$. Applying Green's theorem, and using (3.11) and (3.12), we find

$$\begin{aligned} 0 \geqq & \int_{-\pi/2\,\mathrm{R}}^{0} \int_{\omega} e^{-\varepsilon x_1} \sigma\zeta(\Delta u - \beta u_1 - a_0 u) \\ & = \int_{\omega} \sigma(u_1(0, y) - \beta(y) u(0, y))\, dy + \iint e^{-\varepsilon x_1} u \zeta(a_0 - \mu_1) \sigma \\ & \qquad + \iint - \sigma u_1 (e^{-\varepsilon x_1} \zeta)_1 + \sigma\beta u (e^{-\varepsilon x_1} \zeta)_1 - e^{-\varepsilon x_1} a_0 \sigma \zeta u. \end{aligned}$$

(Here double integrals refer to the measure $dx_1\,dy$ while single integrals involve only $dy$.) Hence

$$\iint -\mu_1 e^{-\varepsilon x_1}\zeta\sigma u+\sigma\beta u(e^{-\varepsilon x_1}\zeta)_1-\sigma u_1(e^{-\varepsilon x_1}\zeta)_1\leqq C.$$

Integrating by parts once more, we find

$$\iint \sigma u[-\mu_1 e^{-\varepsilon x_1}\zeta+\beta(e^{-\varepsilon x_1}\zeta)_1 + (e^{-\varepsilon x_1}\zeta)_{11}]+\int_\omega (\sigma u e^{-\varepsilon x_1}\zeta_1)\Big|_{x_1=-\pi R/2} dy\leqq C.$$

Hence

$$\iint e^{-\varepsilon x_1}\sigma u[(-\mu_1-\varepsilon\beta+\varepsilon^2)\zeta+(\beta-2\varepsilon)\zeta_1+\zeta_{11}]\leqq C.$$

Now set

$$\varepsilon=\min\left\{\frac{\beta_0}{2},\ -\frac{\mu_1}{2b}\right\}.$$

Since $\zeta_1\geqq 0$ and $\zeta_{11}=-\dfrac{1}{R^2}\zeta$, we find

$$\int_{-(\pi/2)R}^{0}\int_\omega e^{-\varepsilon x_1}\sigma\zeta u\left(-\frac{\mu_1}{2}-\frac{1}{R^2}\right)\leqq C.$$

We may now let $R\to\infty$ and conclude that

$$\int_{-\infty}^{0}\int_\omega e^{-\varepsilon x_1}\sigma u\leqq\frac{2C}{-\mu_1},$$

which yields (3.16). ■

Our original proof of Lemma 3.2 used a slightly different function $\zeta$; the one used here was suggested by H. Brezis.

*Remark* 3.2. — In case 3.15 holds, with $c(x)\leqq -a_0(y)$, one can strengthen the result in Theorem 3.1: the constants $C_1$ and $\varepsilon$ in (3.6) can be chosen to depend only on $b$, $\mu_1<0$, $\min\sigma$, $\max\sigma$, $\beta_0$, $\tilde{\tilde{C}}$ and $\omega$.

Turn back to the general case. Our next result shows that any positive solution of (3.4) satisfying (3.4)′, cannot decay more rapidly near $-\infty$ than some fixed exponential, nor grow more rapidly there than some exponential. Since it may prove useful on other occasions we present it for a general uniformly elliptic operator

$$L=a_{ij}(x)\,\partial_{ij}+b_i(x)\,\partial_i+c(x)$$

in $\Sigma^-$. Here we assume the $a_{ij} \in C(\bar{\Sigma}^-)$ and satisfy, for some $c_0 > 0$

$$c_0 |\xi|^2 \leqq a_{ij} \xi_i \xi_j \leqq c_0^{-1} |\xi|^2 \qquad \forall x \in \Sigma^-, \quad \forall \xi \in \mathbf{R}^n. \tag{3.17}$$

The coefficients $b_i$, $c$ are in $L^\infty(\Sigma^-)$, and

$$\sqrt{\Sigma b_i^2}, \qquad |c| \leqq b. \tag{3.18}$$

THEOREM 3.2. — *Let $u \in W_{loc}^{2,p}(\bar{\Sigma}^-)$ be a positive solution of*

$$\left.\begin{aligned} Lu &= 0 \quad \text{in } \Sigma^- \\ u_\nu &= 0 \quad \text{on } y \in \partial\omega. \end{aligned}\right\} \tag{3.19}$$

*There is a constant $a > 0$ depending only on $c_0$, $b$ and $\omega$, such that*

$$e^{ax_1} \min_{\bar{\omega}} u(-2, y) \leqq u(x_1, y) \leqq e^{ax_1} \max_{\bar{\omega}} u(-2, y). \tag{3.20}$$

It will be clear from the proof that any negative number could be used in place of $-2$.

The proof is short but it makes use of a deep result; the Krylov-Safonov Harnack inequality for positive solutions. In fact we will use a form of this inequality which is valid up to the boundary, as presented in [BCN] (*see* Theorem 2.1 as well as other references there). It asserts that there is a constant $C_0$ depending only on $c_0$, $b$, and $\omega$, such that on any slice $S = [\alpha - 1, \alpha] \times \bar{\omega}$, with $\alpha < -1$,

$$\max_S u \leqq C_0 \min_S u. \tag{3.21}$$

*Proof of Theorem* 3.2. — By standard local estimates up to the boundary, the solution $u$ of (3.19) satisfies: for $x_1 \leqq -2$,

$$|u_1(x_1, y)| \leqq C_5 \max_{\substack{|\xi - x_1| \leqq 1/2 \\ \eta \in \bar{\omega}}} u(\xi, \eta).$$

By (3.21), we conclude that for $x_1 \leqq -2$,

$$|u_1(x_1, y)| \leqq a u(x_1, y) \qquad \forall y \in \bar{\omega}$$

with $a = C_5 C_0$. Hence, for $x_1 \leqq -2$,

$$-a u(x_1, y) \leqq u_1(x_1, y) \leqq a u(x_1, y),$$

and consequently

$$e^{ax_1} u(x_1, y) \leqq \max_{\bar{\omega}} u(-2, y)$$
$$e^{-ax_1} u(x_1, y) \geqq \min_{\bar{\omega}} u(-2, y).$$

A number of results are known concerning exponential decay near infinity of solutions — not necessarily positive — of elliptic equations in a cylinder. See primarily Lax [L] and [AN]. Recent results were also obtained by O. A. Oleinik, to appear.

For example, if Theorems 3.4 and 5.9 of [AN] are applied to our linearized operator $\Delta-\beta\partial_1+f_u(y,0)$, then they imply that if $u(x_1,y)$ satisfies

$$\begin{aligned}(\Delta-\beta\partial_1+f_u(y,0))\,u=r \quad &\text{in } \Sigma^-\\ u_\nu=0 \qquad &\text{for } \partial\omega\end{aligned}$$

with

$$|r|\leqq C(1-x_1)^{-3}$$

and if $u\in L^2(\Sigma^-)$ then

$$\int_{\Sigma^-}|e^{-\varepsilon x_1}u|^2<\infty$$

for some $\varepsilon>0$. In case $f$ satisfies: for some $s$, $\delta$, $M>0$

$$|f(y,s)-f_u(y,0)s|\leqq M s^{1+\delta} \qquad \text{for} \quad 0<s\leqq s_0,$$

then we could apply the result above to obtain exponential decay for our solution of (3.1) provided we could show that

$$u(x,y)\leqq\frac{C}{(1-x_1)^k} \quad \text{in } \Sigma^-$$

for $k>\dfrac{3}{\delta+1}$.

## 4. ASYMPTOTIC EXPONENTIAL BEHAVIOUR

In this section we improve Theorem 3.1 by obtaining precise exponential asymptotic behaviour near $-\infty$, of solutions of (3.1) in $\Sigma^-$, in cases analogous to 1, 1′ and 2 of the theorem. These will be obtained using the preceding section, and applying the general theory of Agmon and Nirenberg [AN] about exponential behaviour of solutions of linear PDE's, as well as the extension of Pazy [P]. In [BN4] we carried out a direct approach to the exponential behaviour for the problems we study here – semilinear second order elliptic equations in an infinite cylinder. There, we gave a direct proof relying essentially on versions of the Harnack inequality and its extensions – as well as various generalizations of these results.

Here we will assume that $f(y,s)$ is differentiable in $s$ at $s=0$, uniformly in $y$, and that

$$a(y):=-f_s(y,0) \tag{4.1}$$

is in $L^\infty$. Set

$$h(y,s)=f(y,s)+a(y)s. \tag{4.2}$$

We assume that for some constants $s_0$, $\delta>0$, $M\geqq 0$,

$$|h(y,s)|=|f(y,s)+a(y)s|\leqq M s^{1+\delta} \quad \text{for} \quad 0<s\leqq s_0. \tag{4.3}$$

This assumption is satisfied, for instance, if $f$ is of class $C^{1,\delta}$ in a neighbourhood of $s=0$.

In various cases we will show that as $x_1 \to -\infty$,

$$\left.\begin{aligned} u(x_1,y)&=e^{\lambda x_1}\psi(x_1,y)+o(e^{\lambda x_1})\\ \nabla u(x_1,y)&=\nabla(e^{\lambda x_1}\psi(x_1,y))+o(e^{\lambda x_1}).\end{aligned}\right\} \tag{4.4}$$

Here $w=e^{\lambda x_1}\psi$ is an exponential solution, with $\lambda>0$, of the linear problem (2.1):

$$\left.\begin{aligned} \Delta w-\beta(y)w_1-a(y)w=0 \quad &\text{in } \Sigma^-\\ w_\nu=0 \qquad \text{for} \quad &y\in\partial\omega.\end{aligned}\right\} \tag{4.5}$$

That is, as in (2.3),

$$\psi=(-x_1)^k\psi_k(y)+(-x_1)^{k-1}\psi_{k-1}(y)+\ldots+\psi_0(y). \tag{4.6}$$

We will show, furthermore, that

$$\psi_k>0 \quad \text{in } \bar{\omega}. \tag{4.7}$$

This means, as in Section 2, that $\psi_k$ is a principal eigenfunction, and $\lambda$ is a positive principal eigenvalue: $\lambda$ and $\psi_k=:\varphi$ satisfy (2.5).

Our results will be derived with the aid of the following

THEOREM 4.1. – *Let $u$ be a positive function in $\bar{\Sigma}^-$ which is a solution of* (3.1). *Assume that $f$ satisfies* (4.1) *and* (4.3). *Assume also that for some $0<\varepsilon<a$, and $c_0, c_1>0$,*

$$c_0 e^{a x_1}\leqq u(x_1,y)\leqq c_1 e^{\varepsilon x_1} \quad \textit{in } \Sigma^-. \tag{4.8}$$

*Then there is a positive principal eigenvalue $\lambda$, $\varepsilon\leqq\lambda\leqq a$ and a corresponding exponential solution $w=e^{\lambda x_1}\psi$ of* (4.5) *so that* (4.4) *holds. Furthermore, $\psi$ is given by* (4.6), *and $\psi_k$ is a positive principal eigenfunction, i.e. $\psi_k=\varphi$ satisfies* (2.5).

The proof of Theorem 4.1 is based on results in [AN] (*see* Theorem 5.6 and the remark on the next page there) and in [P] (Lemma 5.2 there). In Theorems 2.1 and 2.2 of [BN1] we have described these results for general elliptic operators of any orders. We now present special cases of these results – and for second order operators only – which will suffice for our purposes. First the result from [AN]:

THEOREM 4.2. – *Let $u^0$ be a solution of* (4.5) *in $\Sigma^-$ satisfying* (4.8). *Then there is an exponential solution $w=e^{\lambda x_1}\psi_k$, with $\varepsilon\leqq\lambda\leqq a$, such that* (4.4) *holds for $u^0$.*

The result in [P] refers to the inhomogeneous form of (4.5) (*see* (2.16) in [BN1]):

THEOREM 4.3. — *Let u be a solution of*

$$\left.\begin{aligned} \Delta u-\beta(y)\,u_1-a(y)\,u=r(x) \quad &in\ \Sigma^- \\ u_\nu=0 \qquad for \quad &y\in\partial\omega, \end{aligned}\right\} \tag{4.9}$$

*with r satisfying, for some positive* $\alpha$, C,

$$|r(x_1,y)|\leqq C\,e^{\alpha x_1} \quad in\ \Sigma^-.$$

*Then* $u=u^0+u^*$ *where* $u^0$ *is a solution of* (4.5) *and* $u^*$ *is a solution of* (4.9). *Furthermore,* $\forall\,\varepsilon'>0$, $\exists\,C_{\varepsilon'}$, *such that*

$$|u^*(x_1,y)|+|\nabla u^*(x_1,y)|\leqq C_{\varepsilon'}\,e^{(\alpha-\varepsilon')x_1} \quad in\ \Sigma^-.$$

Using the last two theorems we now give the

*Proof of Theorem* 4.1. — Let

$$\tau_0=\sup\{\tau;\,u(x_1,y)\leqq C_\tau e^{\tau x_1}\text{ in }\Sigma^-\text{ for some constant }C_\tau\}.$$

Clearly $\varepsilon\leqq\tau_0\leqq a$.

Now $u$ satisfies

$$\Delta u-\beta u_1-a(y)\,u=-h(y,u(x_1,y))=:r(x_1,y). \tag{4.10}$$

It follows from (4.3) that for any $\tau<\tau_0$,

$$|r|\leqq MC_\tau^{1+\delta}\,e^{\tau(1+\delta)x_1} \quad \text{in } \Sigma^-. \tag{4.11}$$

Applying Theorem 4.3 to (4.10) we find that $u=u^0+u^*$ where $u^0$ is a solution of (4.5); furthermore, for any $\tau<\tau_0$, there is a constant $D_\tau$ such that

$$|u^*(x_1,y)|+|\nabla u^*(x_1,y)|\leqq D_\tau\,e^{\tau(1+\delta)x_1}.$$

For $\tau$ close to $\tau_0$, we have $\tau(1+\delta)>\tau_0$. Thus $u^0$ satisfies $\forall\,\tau<\tau_0$

$$|u^0(x_1,y)|\leqq E_\tau\,e^{\tau x_1}$$

with $E_\tau=C_\tau+D_\tau$.

If we now apply Theorem 4.2 to $u^0$ we see that (4.4) holds for a suitable exponential solution $w=e^{\lambda x_1}\,\psi$. Necessarily, then, $\lambda=\tau_0$ and, since $u>0$, we must have, in (4.6), $\psi_k>0$ in $\bar\omega$. ■

We now apply Theorem 4.1 to treat our positive solutions of (3.1) $-u$ tending to zero at $-\infty-$ under various assumptions on $f$. We always assume here that $f$ satisfies (4.1) and (4.3); $\mu_1$ will always represent the first eigenvalue of $(-\Delta_y+a(y))$ with Neumann conditions. That is, $\mu_1$ is the principal eigenvalue, of (2.6) and $\sigma$ is the corresponding positive eigenfunction.

THEOREM 4.4. — *The conclusion* (4.4) *of Theorem* 4.1 *holds in each of the following cases:*

*Case I.* $\mu_1 > 0$.

*Case I'.* $\mu_1 = 0$, $h(y, s) \leqq 0$, *for* $0 < s \leqq s_0$ *and*

$$\int_{\omega} \beta \sigma^2 \, dy > 0.$$

*Case II.* $\mu_1 < 0$.

*Furthermore, in each of these cases we have* $k=0$ *in* (4.6), *with one possible exception, where* $k$ *may equal* 1: *namely, Case II and when, in addition, there is exactly one positive principal eigenvalue* $\lambda$ *of* (2.5).

*Proof.* — In every case, $u$ satisfies (3.4)' and an equation of the form (3.4). By Theorem 3.2, it follows that for some $c_0$, $a > 0$,

$$c_0 e^{a x_1} \leqq u(x_1, y) \quad \text{in } \Sigma^-.$$

To complete the proof of Theorem 4.4 we have only to establish the other inequality in (4.8):

$$u(x_1, y) \leqq c_1 e^{\varepsilon x_1} \quad \text{in } \Sigma^- \tag{4.12}$$

for some $c_1$, $\varepsilon > 0$.

Consider first case I. For $a_0(y) = a(y) - \varepsilon'$, and $\varepsilon' > 0$ sufficiently small, the first eigenvalue $\mu_1'$ of $-\Delta_y + a_0$ is also positive. Fix such an $\varepsilon'$. By (4.3), $|h(y, u(x_1, y))| \leqq \varepsilon' u$ for $-x_1$ large, say $x_1 \leqq -\mathrm{N}$. There $u$ satisfies

$$\Delta u - \beta u_1 - a_0(y) u \geqq 0.$$

We are then in case 1 of Theorem 3.1 and we may apply the theorem in the semi-infinite cylinder $(-\infty, \mathrm{N}] \times \omega$ to conclude that (4.12) holds.

The other cases are similar. In case II, for $a_0 = a + \varepsilon'$, $0 < \varepsilon'$ small, the first eigenvalue $\mu_1'$ of $-\Delta_y + a_0(y)$ is negative. Then for $-x_1$ large, $x_1 \leqq -\mathrm{N}$,

$$\Delta u - \beta u_1 - a_0(y) u \leqq 0.$$

Inequality (4.12) then follows from Case 2 of Theorem 3.1.

Finally in Case I', Case 1' of Theorem 3.1 applies directly, and yields (4.12).

The last assertion of Theorem 4.4 follows from Theorem 2.4. ■

So far, we have concentrated on the behaviour of the solution $u$ of (3.1) near $-\infty$. Clearly this also yields the behaviour near $+\infty$ by a change of variables. We describe it here in the simplest case — the one we will actually need later. This case was also done earlier in Proposition 4.3 in [BN1]. (The reader may state the various other cases.)

We always assume that for $0<v\leqq s_0$,

$$0<f(y,1-v)=a(y)v+H(y,v), \qquad a(y)\underset{\not\equiv}{\geqq}0, \quad a\in L^\infty, \tag{4.13}$$

and, as in (4.3),

$$|H(y,s)|\leqq M s^{1+\delta} \qquad \text{for} \quad 0\leqq s\leqq s_0. \tag{4.14}$$

We also assume $1-u(x_1,y)$ tends to zero, uniformly in $y$, as $x_1\to\infty$.

THEOREM 4.5. – *Under the conditions above, there exist a principal eigenvalue* $-\tau<0$, *and an eigenfunction* $\varphi'(y)>0$, *as* $x_1\to\infty$,

$$1-u(x_1,y)=\gamma e^{-\tau x_1}\varphi'(y)+o(e^{-\tau x_1}) \tag{4.15}$$

$$-\nabla u(x_1,y)=\gamma\nabla(e^{-\tau x_1}\varphi'(y))+o(e^{-\tau x_1}), \tag{4.16}$$

*for some constant* $\gamma>0$. *Here* $w=e^{-\tau x_1}\varphi'(y)$ *is an exponential solution of*

$$\left.\begin{aligned}(\Delta-\beta\partial_1-a(y))w=0 \quad &\text{in } \omega\\ w_\nu=0 \quad &\text{in } \partial\omega.\end{aligned}\right\} \tag{4.17}$$

*Proof.* – After replacing $x_1$ by $-x_1$ the theorem follows easily from Case I of Theorem 4.4.

Recall that $-\tau$ is the principal negative eigenvalue for (2.5).

For use in Section 9 we present here a result which was included in the proof of Theorem 7.1 of [BN1].

LEMMA 4.1. – *Assume that f satisfies* (4.13), (4.14), *and in addition,*

$$|H(y,s)-H(y,s')|\leqq M(s+s')^\delta|s-s'| \qquad \textit{for} \quad 0<s,s'\leqq s_0. \tag{4.18}$$

*Let* $v$ *and* $v'$ *be positive* $C^1$ *functions in the closure of the half cylinder* $\Sigma^+=\Sigma\cap\{x_1>0\}$ *satisfying* (3.1) *in* $\Sigma^+$, *with* $v\geqq v'$, *and assume that* (4.15) *holds for both* $v$ *and* $v'$ – *with the same constant* $\gamma$. *Then* $v\equiv v'$.

In the proof we will make use of the following result, Lemma 4.3 of [BN1]. Here $\tau$ and $\varphi'$ are as in Lemma 4.1.

LEMMA 4.2. – *In* $\bar\Sigma^+$ *consider a nonnegative solution* $z\in C(\bar\Sigma^+)$ *of a linear problem*

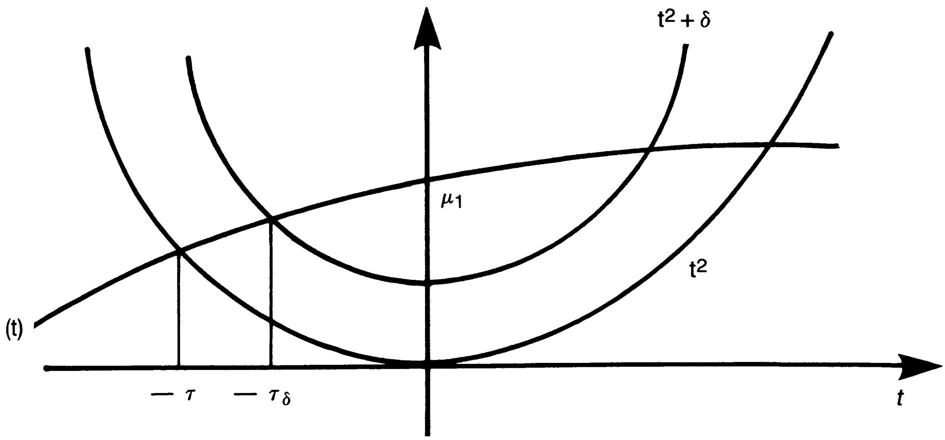
$$\left.\begin{aligned}Lu\equiv(\Delta-\beta(y)\partial_1-a(y)+d(x_1,y))z=0 \quad &\text{in } \Sigma^+\\ z_\nu=0 \qquad \text{for} \quad &y\in\partial\omega\end{aligned}\right\} \tag{4.19}$$

*with* $z$ *and* $d$ *tending to zero uniformly as* $x_1\to+\infty$. *Here we assume that the first eigenvalue* $\mu_1$ *of* $(-\Delta_y+a)$, *under the usual Neumann boundary condition, is positive. Then for any* $\varepsilon>0$, $\exists$ *constants* A, $B>0$ *such that*

$$A e^{-(\tau+\varepsilon)x_1}\leqq z(x_1,y)\leqq B e^{(-\tau+\varepsilon)x_1} \quad \text{in } \bar\Sigma^+. \tag{4.20}$$

For the convenience of the reader we include here proofs of both lemmas. The proof of Lemma 4.2 differs slightly from that in [BN1].

*Proof of Lemma* 4.2. — It suffices to prove the lemma for $\varepsilon$ small. For $0<\delta$ small, let $-\tau_\delta$ be the principal negative eigenvalue of (2.5) with $a(y)$ replaced by $a(y)-\delta$. Let $\varphi_\delta$ be the associated positive eigenfunction. Recall that $-\tau_\delta$ is obtained via the following diagram (*see* Section 2)




Thus $-\tau_\delta > -\tau$, and we now fix $\delta$ so that $\tau-\tau_\delta<\varepsilon$. We have

$$\left.\begin{aligned}(-\Delta_y+a-\delta)\,\varphi_\delta=[\tau_\delta^2-\tau_\delta\,\beta]\,\varphi_\delta \quad \text{in } \omega\\ \partial_\nu\,\varphi_\delta=0 \quad \text{on } \partial\omega.\end{aligned}\right\} \tag{4.21}$$

Set

$$w_\delta=e^{-\tau_\delta x_1}\,\varphi_\delta(y);$$

then

$$\mathrm{L}\,w_\delta=(\Delta-\beta\partial_1-a+d)\,w_\delta=(d-\delta)\,w_\delta<0 \quad \text{in } \Sigma^{\mathrm{R}}=\Sigma\cap\{x_1>\mathrm{R}\}$$

for R large.

Choose $\mathrm{B}_1>0$ so that

$$\zeta:=\mathrm{B}_1\,w-z$$

is positive when $x_1=\mathrm{R}$. The function $\zeta$ tends to zero at $+\infty$, $\mathrm{L}\zeta<0$ in $\Sigma^{\mathrm{R}}$, and $\zeta_\nu=0$ if $y\in\partial\omega$. We claim that $\zeta\geq 0$ in $\Sigma^{\mathrm{R}}$ for R large. To see this we write

$$\zeta=\xi\,\sigma$$

where $\sigma$ is the positive eigenfunction of $-\Delta_y+a$ with eigenvalue $\mu_1$. Then in $\Sigma^{\mathrm{R}}$

$$0>\mathrm{L}\zeta=\sigma\left\{\Delta\xi-\beta\xi_1+\frac{2}{\sigma}\Sigma\,\sigma_{y_i}\xi_{y_i}+(d-\mu_1)\,\xi\right\}.$$

Since $d \to 0$ as $x_1 \to \infty$, for R large, $d - \mu_1 < 0$. We may therefore apply the maximum principle and the Hopf lemma, and conclude that $\xi \geqq 0$ in $\Sigma^R$. Hence $\zeta \geqq 0$ in $\Sigma^R$.

Thus $z \leqq B_2 w$ in $\Sigma^+$ for a suitable constant $B_2$. Since $-\tau_\delta < -\tau + \varepsilon$, we obtain the right hand inequality of (4.20).

To derive the left inequality of (4.20) we consider $\varphi_\delta$, the positive solution of (4.21), but with $\delta < 0$. The rest of the argument proceeds as before. Lemma 4.2 is proved.

*Proof of Lemma* 4.1. — In $\Sigma^+$, the function

$$z = v - v' \geqq 0$$

satisfies $z_\nu = 0$ for $y \in \partial\omega$ and the equation

$$\Delta z - \beta z_1 - az + H(y, 1-v) - H(y, 1-v') = 0. \tag{4.22}$$

By (4.22), (4.15) and (4.14),

$$\Delta z - \beta z_1 - az = O(e^{-\tau(1+\delta)x_1}) \quad \text{as } x_1 \to \infty.$$

Arguing as in the proof of Theorem 4.4, using Theorems 4.2 and 4.3, we see that for some constant $\rho$,

$$z = \rho e^{-\tau x_1} \varphi'(y) + O(e^{-\varkappa x_1}) \quad \text{as } x_1 \to \infty \tag{4.23}$$

for some $\kappa > \tau$.

On the other hand, since $v$ and $v' \to 0$ at $+\infty$, it follows from (4.18) and (4.22) that $z$ satisfies a linear equation of the form (4.19), with $d$ tending uniformly to zero at $+\infty$. By the maximum principle, if $z \not\equiv 0$, then $z > 0$. By Lemma 4.2, (4.20) holds—in which we may take $\varepsilon < \kappa - \tau$. This implies that $\rho > 0$ in (4.23). But (4.23) then contradicts our original assumption on $v$ and $v'$, according to which $v - v' = o(e^{-\tau x_1})$ as $x_1 \to \infty$. Consequently we must have $z \equiv 0$. ■

## 5. PROOFS OF THEOREMS 1.1 AND 1.2

In this section we are concerned with both cases B and C described in the Introduction. Therefore we assume that for some $\theta$, $0 < \theta < 1$, the function $f$ satisfies

$$\left.\begin{aligned} f(u) &\leqq 0 \quad \text{for } 0 < u < \theta \\ f(u) &> 0 \quad \text{for } \theta < u < 1. \end{aligned}\right\} \tag{5.1}$$

Theorems 1.1 and 1.2 will be proved by solving corresponding problems in finite cylinders $\Sigma_a = (-a, a) \times \omega$ and then letting $a \to \infty$. We will make repeated use of Theorem 5.1 below, which is a special case of Theorem 7.2 in [BN3], though our conditions here are slightly weaker than those there. Here we permit $f$ to depend on $y$, $f$ is continuous and uniformly Lipschitz continuous in $u$. We set

$$\Gamma=\{(x_1,y);\, x_1=\pm a,\, y\in\partial\omega\} \qquad \text{and} \qquad \tilde{\Sigma}_a=\bar{\Sigma}_a\setminus\Gamma.$$

Theorem 5.1 is concerned with the following problem in $\Sigma_a$: find a function $u$ such that

$$u\in C(\bar{\Sigma}_a)\cap W^{2,p}_{\mathrm{loc}}(\tilde{\Sigma}_a) \quad \text{for some fixed } p>n \tag{5.2}$$

$$\Delta u-\beta(y)\,u_1+f(y,u)=0 \quad \text{in } \Sigma_a \tag{5.3}$$

$$u_\nu(x_1,y)=0 \quad \text{in } (-a,a)\times\partial\omega \tag{5.4}$$

$$\left.\begin{array}{l}\psi_1(y)=u(-a,y)<u(x_1,y)<u(a,y)=\psi_2(y)\\ \text{for}\\ \qquad -a<x_1<a, \qquad y\in\bar{\omega}.\end{array}\right\} \tag{5.5}$$

Here the functions $\psi_1$, $\psi_2$ are assumed to belong to $W^{2,\infty}(\omega)$ and to satisfy

$$\partial_\nu\psi_j=0 \quad \text{on } \partial\omega, \quad j=1,2. \tag{5.6}$$

THEOREM 5.1. – *Let $\underline{u}\leqq\bar{u}$ be sub and super solutions of the problem* (5.2)-(5.4), *belonging to $C(\bar{\Sigma}_a)\cap W^{2,p}_{\mathrm{loc}}(\tilde{\Sigma}_a)$, i.e. they satisfy*

$$\Delta\underline{u}-\beta(y)\,\underline{u}_1+f(y,u)\geqq\Delta\bar{u}-\beta(y)\,\bar{u}_1+f(y,\bar{u}) \quad \textit{in } \Sigma_a \tag{5.7}$$

*and*

$$\underline{u}_\nu(x_1,y)\leqq 0\leqq\bar{u}_\nu(x_1,y) \qquad \textit{for} \quad -a<x_1<a, \quad \forall y\in\partial\omega. \tag{5.8}$$

*and satisfying also the conditions*

$$\left.\begin{array}{l}\psi_1(y)=\underline{u}(-a,y)\leqq\underline{u}(x_1,y)\leqq\psi_2(y)\\ \psi_1(y)\leqq\bar{u}(x_1,y)\leqq\bar{u}(a,y)=\psi_2(y)\end{array}\right\} \quad \textit{for} \quad -a<x_1<a, \quad \forall y\in\omega \tag{5.9}$$

*Both $\underline{u}$ and $\bar{u}$ are not solutions of* (5.3), (5.4). (5.10)

*Then there exists a unique solution $u\in W^{2,p}_{\mathrm{loc}}(\tilde{\Sigma}_a)\cap C(\bar{\Sigma}_a)$ of* (5.2)-(5.5). *Furthermore*

$$u_{x_1}(x_1,y)>0 \qquad \textit{for} \quad -a<x_1<a, \quad y\in\bar{\omega}. \tag{5.11}$$

The more general Theorem 7.2 in [BN3] was proved using sub and super solutions and monotone iteration – in slightly smaller regions in which the corners have been rounded off. Then one does a limiting argument. In addition, the sliding method was used to prove uniqueness and (5.11).

Since Theorem 5.1 here is a particular case of Theorem 7.2 of [BN3], we will not include its proof. A somewhat modified argument also yields the proof of Lemma 1 of [BN2].

We will make use of the following addition to Theorem 5.1.

LEMMA 5.1. — *Let $v$ and $z$, belonging to $W^{2,p}_{loc}(\bar{\Sigma}_a) \cap C(\bar{\Sigma}_a)$, be sub and super solutions of* (5.1), *i.e.*

$$\left.\begin{aligned} \Delta v-\beta(y)v_1+f(y,v)\geqq 0\geqq \Delta z-\beta(y)z_1+f(y,z) \quad &in\ \Sigma_a, \\ v_\nu\leqq 0\leqq z_\nu \quad for\quad y\in\partial\omega, & \end{aligned}\right\} \tag{5.12}$$

*and*

$$\begin{aligned} v(-a,y)<z(x_1,y) \quad & for \quad -a<x_1\leqq a, \quad \forall y\in\bar{\omega} \\ v(x_1,y)<z(a,y) \quad & for \quad -a\leqq x_1<a, \quad \forall y\in\bar{\omega}. \end{aligned}$$

*Then $v\leqq z$ in $\Sigma_a$.*

*Proof.* — Shift $z$ to the left, *i.e.*, for $0\leqq r\leqq 2a$, consider

$$z^r(x_1,y)=z(x_1+r,y) \quad \text{in } \Sigma_a^r=(-a-r,a-r)\times\omega.$$

Here $\Sigma_a^0=\Sigma_a$. For $r=2a$, $\bar{\Sigma}_a^r\cap\bar{\Sigma}_a=\{(-a,y);y\in\bar{\omega}\}$, and on this set, $z^r>v$. Now decrease $r$. For any $r$ in $0<r<2a$, $z^r>v$ on the lateral, *i.e.* left and right, boundaries $\{x_1=-a,\ y\in\bar{\omega}\}$ and $\{x_1=a-r,\ y\in\bar{\omega}\}$ of $\Sigma_a^r\cap\Sigma_a$, while on the curved part, where $y\in\partial\omega$, $(z^r-v)_\nu\geqq 0$. On decreasing $r$ suppose there is a first value in $0<r<2a$ such that $z^r\geqq v$ in $\bar{\Sigma}_a^r\cap\bar{\Sigma}_a$ with equality holding somewhere. Then, since, from (5.12), $z^r-v$ satisfies a linear elliptic inequality for some $c\in L^\infty$,

$$(\Delta-\beta\partial_1+c(x_1,y))(z^r-v)\leqq 0,$$

we find by the maximum principle and the Hopf lemma that $z^r-v\equiv 0$, which is impossible. Hence no such $r$ exists; letting $r\to 0$ we find therefore that $z\geqq v$ in $\Sigma_a$. ■

COROLLARY 5.1. — *The unique solution $u$ in Theorem* 5.1 *satisfies:*

(*a*) *$u$ is strictly decreasing in its dependence on the function $\beta$. That is, if $\beta'\leqq\beta$ and $\beta'\not\equiv\beta$, then*

$$u'>u \quad in\ \Sigma_a$$

*where $u'$ is the solution of* (5.2)-(5.5) *for $\beta'$;*

(*b*) *$u$ is strictly decreasing in its dependence on $f$. That is, if $f'$ satisfies the conditions of the theorem, and $f' \underset{\not\equiv}{\geqq} f$ then $u'>u$ in $\Sigma_a$ where $u'$ is the solution of* (5.2-5) *for $f'$.*

*Proof.* — We only prove (*a*). The proof of (*b*) is similar. Since $u'_1>0$ we see that

$$\Delta u'-\beta u'_1+f(y,u')\underset{\not\equiv}{\leqq}0 \quad \text{in } \Sigma_a.$$

Also since $u_1>0$ we see that $v=u$ and $z=u'$ satisfy the conditions of Lemma 5.1. Thus $u\leqq u'$. Clearly $u\not\equiv u'$. Since $u'-u$ satisfies an elliptic inequality

$$(\Delta-\beta\partial_1+c(x_1,y))(u'-u)\leqq 0$$

it follows that $u'>u>0$. ∎

Theorem 5.1 will be used to prove the following basic preliminary result.

THEOREM 5.2. – *Assume f is Lipschitz continuous on* $[0, 1]$, $f(0)=f(1)=0$, *and f satisfies* (5.1). *Assume* $\beta(y,c)$ *satisfies conditions* (1.17), (1.18). *Then for a large, there is a unique* $\tilde{c}$ *such that for* $\beta(y)=\beta(y,\tilde{c})$, *the system* (5.2)-(5.5), *with* $\psi_1=0$, $\psi_2=1$ *has a solution u satisfying, in addition,*

$$\max_{y\in\bar{\omega}} u(0,y)=\theta. \tag{5.13}$$

*Here* (5.5) *takes the simple form*

$$\left.\begin{array}{c} 0=u(-a,y)<u(x_1,y)<u(a,y)=1 \\ \text{for} \\ -a<x_1<a, \qquad y\in\bar{\omega}. \end{array}\right\} \tag{5.5)'$$

*Furthermore,* $\exists\, c_0<c_1$ *independent of a, such that* $c_0\leqq\tilde{c}\leqq c_1$.

Note that $\underline{u}\equiv 0$ and $\bar{u}\equiv 1$ are sub and super solutions in the sense of Theorem 5.1. Consequently by that theorem, for any fixed $c$, the problem (5.2)-(5.4), (5.5)′, with $\beta=\beta(y,c)$, has a unique solution; call it $u^c$. Theorem 5.2 asserts that there exists a unique $c=\tilde{c}$ so that the additional condition (5.13) holds.

We will first prove Theorem 5.2 in the one dimensional case.

*Remark* 5.1. – Note that by the uniqueness part of Theorem 5.1, if $\beta(y,c)$ is independent of $y$, then the solution $u^c$ must agree with the solution (assuming it exists) of the one dimensional problem.

The one dimensional problem (5.2-4), (5.5)′, for a function $v(x)$ in $\Sigma_a=(-a,a)$, reads

$$\left.\begin{array}{ll} \ddot{v}-k\dot{v}+f(v)=0 \quad \text{and} & 0<v<1 \quad \text{in } \Sigma_a \\ v(-a)=0, & v(a)=1. \end{array}\right\} \tag{5.14}$$

By Theorem 5.1, for any real $k$, (5.14) has a unique solution $v^k$ and it satisfies $\dot{v}>0$ on $\Sigma_a$. We will make use of the following observations.

LEMMA 5.2. – (i) *There is a unique* $k^*=k^*(a)$, *such that*

$$v^{k^*}(0)=\theta. \tag{5.15}$$

*Furthermore, there exists a constant* K *such that for all* $a\geqq 1$, *the following estimate holds:*

$$-\mathrm{K}\leqq k^*(a)\leqq \mathrm{K}. \tag{5.16}$$

Various constants depending only on $\theta$, $M=\max|f|$, and $F(1)$ will be denoted by C. We will always suppose $a \geqq 1$.

The proof of the lemma is elementary.

*Proof of Lemma* 5.2. – Since $v^k$ is unique, it follows easily that it depends continuously on $k$. By Corollary 5.1, $v^k(x)$ is strictly decreasing in $k$, $\forall x \in \Sigma_a$. Part (i) of the theorem will then follow from the following assertions. For fixed $a \geqq 1$

$$\lim_{k \to -\infty} v^k(0)=1 \quad \text{and} \quad \lim_{k \to +\infty} v^k(0)=0.$$

This is easily seen by direct computations on comparison functions.

It now remains to prove the estimate (5.16). We follow [BNS]. In this part of the proof we denote $k^*$ by $k=k(a)$.

First we will derive the upper bound K for $k$. We may suppose in this step that $k \geqq 1$.

Denoting $v^k$ by $v$ we see that if $\chi_a$ is the characteristic function of $(0, a)$ then

$$\ddot{v} - k\dot{v} \geqq -M\chi_a$$

where $M=\max|f|$. By the maximum principle it follows that if $z$ is the solution of

$$\left.\begin{aligned} \ddot{z} - k\dot{z} = -M\chi \quad &\text{on } (-a, a) \\ z(-a)=0, \quad &z(a)=1, \end{aligned}\right\} \tag{5.17}$$

then $v \leqq z$. Consequently $\theta \leqq z(0) =: \tau$.

Direct computation yields

$$z = \tau \frac{e^{kx} - e^{-ka}}{1 - e^{-ka}} \quad \text{for} \quad x<0$$

$$z = \frac{Mx}{k} + \tau + \alpha(e^{kx}-1) \quad \text{for} \quad x>0.$$

We know further that $\alpha$ is chosen so that $z(a)=1$; since $z \in C^1$, at $x=0$,

$$\dot{z}(0) = \frac{\tau k}{1-e^{-ka}} = \frac{M}{k} + \alpha k.$$

This yields the identity

$$\tau\left(\frac{e^{ka}-1}{1-e^{-ka}}\right) = \frac{M}{k^2}(e^{ka}-1) + 1 - \frac{Ma}{k} - \tau.$$

Hence, since $a \geqq 1$

$$\theta \leqq \tau \leqq \frac{M}{k^2} + \frac{1}{e^k - 1}. \tag{5.18}$$

This clearly yields an upper bound $k(a) \leqq K_1$.

The previous proof also yields a lower bound on $k(a)$. Indeed, consider the function $w$, solution of

$$\left.\begin{array}{l} \ddot{w} - k\dot{w} - M\chi_a(-x) = 0 \quad \text{on } (-a, a) \\ w(-a) = 0, \qquad w(a) = 1. \end{array}\right\} \tag{5.19}$$

This time, $w \leqq v$ and consequently $\theta \geqq w(0)$. Now $\zeta(x) = 1 - w(-x)$ solves the same equation as $z$ does except that $k$ is substituted by $-k$ in (5.17). Using that $1 - \theta \leqq \zeta(0)$, assuming that $k \leqq -1$, we may repeat the previous computation and obtain as in (5.18), an upper bound for $-k$. Lemma 5.2 is proved.

*Proof of Theorem* 5.2. — Since for each $c$ the solution $u^c$ of (5.2)-(5.4), (5.5)' is unique, it is easily seen that $u^c$ depends continuously on $c$. Let $v$ be the solution of (5.14) with $k = k^*$.

Using (5.16), let $c_0$ be such that $\beta(y, c_0) < -K \leqq k$, $\forall y \in \bar{\omega}$. Then by Corollary 5.1 and Remark 5.1,

$$\max_{\bar{\omega}} u^{c_0}(0, y) > \theta.$$

Similarly, if $c_1$ is such that $\beta(y, c_1) \geqq K$, $\forall y \in \bar{\omega}$, we find

$$\max_{\bar{\omega}} u^{c_1}(0, y) < \theta.$$

By the continuity and strict monotonicity of $u^c$ with respect to $c$, it follows that for a unique $\tilde{c}$ in $c_0 < \tilde{c} < c_1$,

$$\max_{\bar{\omega}} u^{\tilde{c}}(0, y) = \theta$$

Theorem 5.2 is proved.

An immediate consequence of Corollary 5.1 is

LEMMA 5.3. — *If $f'$ satisfies the conditions of Theorem* 5.2, *and* $f' \underset{\neq}{\geq} f$ *then the corresponding unique $\tilde{c}'$ satisfies $\tilde{c}' > c'$.*

Now we will let $a \to \infty$ and obtain a solution satisfying almost all the conditions of Theorems 1.1 and 1.2—which we summarize in

PROPOSITION 5.1. — *Under the assumptions of Theorem* 5.2, *let $(u_a, c_a)$ be the solution of* (5.2)-(5.4), (5,5)' *satisfying* (5.12). *There exists a sequence $a = a_n \to \infty$ such that $c_a \to c$ in* R *and $u_a \to u$ locally in $\bar{\Sigma}$. Furthermore, $0 < u < 1$ is a solution in the closure of the infinite cylinder $\Sigma$ of*

$$\left.\begin{array}{l} \Delta u - \beta(y, c)\, u_1 + f(u) = 0 \quad \text{in } \Sigma \\ u_\nu = 0 \quad \text{if } y \in \partial\omega \end{array}\right\} \tag{5.20}$$

*with* $\max_{\bar{\omega}} u(0, y)=\theta$. *The solution u satisfies* $u_1 \geqq 0$,

$$u(x_1, y) \to \text{a constant } \gamma \leqq \theta \text{ as } x_1 \to -\infty, \text{ uniformly in } y \tag{5.21}$$

$$u(x_1, y) \to \psi(y) \text{ uniformly as } x_1 \to +\infty \tag{5.22}$$

*where* $\psi$ *is some solution, with* $0<\psi \leqq 1$ *of*

$$\left.\begin{aligned} \Delta_y \psi + f(\psi) = 0 \quad &\text{in } \omega \\ \psi_\nu = 0 \quad &\text{on } \partial\omega. \end{aligned}\right\} \tag{5.23}$$

*Furthermore* $\nabla u \to \nabla \psi$ *uniformly as* $x_1 \to \infty$.

*Proof.* — For $a$ large, let $(\tilde{c}_a, u_a)$ be the unique solution of (5.2-4), (5.12), (5.5)′ given by Theorem 5.2. We let $a \to \infty$ through successive integers. Since $c_0 \leqq \tilde{c}_a \leqq c_1$, we may choose a subsequence of $a$'s such that $\tilde{c}_a \to$ some number $c$ with $c_0 \leqq c \leqq c_1$. It is straightforward from standard local estimates up to the boundary, to see, taking a further subsequence, that the functions $u_a$ tend to some function $u$, uniformly on compact sets. $u \in W^{2,p}_{loc}(\bar{\Sigma})$, and it satisfies (5.20). Furthermore, $\max_{\bar{\omega}} u(0, y)=\theta$ and $u_1 \geqq 0$.

It follows that as $x_1 \to -\infty$ and $+\infty$, $u(x_1, y)$ tends, respectively to functions $\psi_1(y)$ and $\psi(y)$, uniformly in $y$; these functions are in $W^{2,p}(\omega)$ and each is a solution of (5.23). Also $u_1$ tends uniformly to zero as $|x_1| \to \infty$. Now $\psi_1 \leqq \theta$ and hence $f(\psi_1) \leqq 0$. Integrating the corresponding equation in (5.23) we find $\int_\omega f(\psi_1)=0$, which implies $f(\psi_1(y)) \equiv 0$. But then $\Delta\psi_1=0$ and so $\psi_1=\text{constant}$.

Since $\max \psi \geqq \theta$ and $\psi$ satisfies (5.23) we see with the aid of the maximum principle and the Hopf lemma that $\psi>0$ in $\bar{\omega}$. The last assertion in the theorem follows easily with the aid of standard elliptic estimates. ■

We are now in a position to give the

*Proof of Theorem* 1.1. — We recall that we are concerned here with case B, namely $f$ satisfies (5.1) and $f \equiv 0$ on $[0, \theta]$. In (5.20), we denote $\beta(y, c)$ by $\beta$. By Theorem 5.3, the only things left to prove are:

$$\gamma := \lim_{x_1 \to -\infty} u(x_1, y) \quad \text{is zero} \tag{5.24}$$

and

$$\psi(y) \equiv 1. \tag{5.25}$$

Recall that in this case, $f=0$ on $(0, \theta)$ and $f>0$ on $(\theta, 1)$. The proof proceeds in several steps — taken from [BLL].

STEP 1. — We show that

$$\int_{\Sigma} f(u) < \infty, \qquad \int_{\Sigma} |\nabla u|^2 < \infty. \tag{5.26}$$

To obtain the first inequality we integrate the equation for $u$ over $\Sigma_a$. This leads to

$$-\int_{\Sigma_a} f(u) = \int_{\Sigma_a} \Delta u - \beta u_1$$
$$= \int_{\omega} [u_1(a, y) - u_1(-a, y) - \beta u(a, y) + \beta u(-a, y)]\, dy.$$

Letting $a \to \infty$ we find

$$\int_{\Sigma} f(u) = \int_{\omega} \beta(y)(\psi(y) - \gamma)\, dy. \tag{5.27}$$

The second inequality in (5.26) is obtained in a similar way by multiplying the equation for $u$ by $u$, integrating over $\Sigma_a$, and letting $a \to \infty$. Namely

$$0 = \int_{\Sigma_a} u(\Delta u - \beta u_1 + f(u))$$
$$= \int_{\Sigma_a} [-|\nabla u|^2 + uf(u)] - \int_{\omega} \frac{\beta}{2}(u^2(a, y) - u^2(-a, y)) + o(1) \text{ as } a \to \infty.$$

Letting $a \to \infty$ we find

$$\int_{\Sigma} |\nabla u|^2 = \int_{\Sigma} uf(u) - \int_{\omega} \frac{\beta}{2}(\psi^2 - \gamma^2) \tag{5.28}$$

and (5.26) is proved.

STEP 2. — Since $\nabla_y u(x_1, y) \to \nabla_y \psi(y)$ as $x_1 \to \infty$ and $\int_{\Sigma} |\nabla u|^2 < \infty$, necessarily $\psi$ is a constant. But then $\psi \equiv \theta$ or $\psi \equiv 1$.

If $\psi = \theta$ we claim $\gamma = \theta$ and hence $u \equiv \theta$. This is easily seen. Indeed, if $\psi \equiv \theta$, since $u \leqq \theta$, $f(u) = 0$. If $\theta > \gamma$, then, by (5.27),

$$\int_{-\omega} \beta\, dy = 0.$$

But then, by (5.28), $\int |\nabla u|^2 = 0$, *i.e.* $u$ is a constant. Contradiction. Thus $\gamma = \theta$ and hence $u \equiv \theta$.

STEP 3. — Recall that the solution $u$ was obtained as a limit of a subsequence of the $u_a$. We now show that there is a constant $\tau > 0$ such that for $a>1$,

$$\int_{\Sigma_a} f(u_a) \geqq \tau. \tag{5.29}$$

*Proof.* — Fix a number $\lambda$ in $(\theta, 1)$. Now $u_a(x_0, y_0)=\lambda$ for some $x_0$ in $(0, a)$ and some $y_0 \in \omega$. Recalling that $|\nabla u_a| \leqq K$ independent of $a$, we see that

$$\frac{1-\lambda}{K} \leqq a-x_0 \quad \text{and} \quad \frac{\lambda-\theta}{K} \leqq x_0$$

hence for some $r$, $\sigma$, $\varepsilon>0$ independent of $a$,

$$f(u_a(x_1, y)) \geqq \sigma \quad \text{for all } (x_1, y) \text{ in } \Sigma \cap B_r(x_0, y_0)$$

and $|\Sigma \cap B_r(x_0, y_0)| \geqq \varepsilon$. Consequently (5.29) holds with $\tau=\varepsilon\sigma$.

STEP 4. — We now show that

$$\int_\omega \beta(y, c)\, dy>0. \tag{5.30}$$

In case $\psi=1$, (5.30) follows from (5.27). So we need only consider the case $\psi=\theta$; then, according to Step 2, $u \equiv \theta$. Integrating the equation for $u_a$ over $\Sigma_a^+=\Sigma_a \cap \{x_1>0\}$, we find

$$-\int_{\Sigma_a} f(u_a)=\int_{\Sigma_a} \Delta u_a-\beta(y, c_a)\, \partial_1 u_a$$

$$=\int_\omega \{\partial_1 u_a(a, y)-\partial_1 u_a(0, y)-\beta(y, c_a)(1-u_a(0, y))\}\, dy.$$

With $\tau$ as in Step 3, using $\partial_1 u_a(a, y) \geqq 0$, we get

$$\tau \leqq \int_{\Sigma_a^+} f(u_a) \leqq \int_\omega \beta(1-u_a(0, y))\, dy+\int_\omega \partial_1 u_a(0, y)\, dy. \tag{5.31}$$

Since $u_a(0, y) \rightarrow \theta$ and $\partial_1 u_a(0, y) \rightarrow 0$ uniformly on $\bar{\omega}$, we derive from (5.31)

$$\tau(1-\theta)^{-1} \leqq \int_\omega \beta(y, c)\, dy$$

and this yields (5.30).

STEP 5. — Fix $c'<c$, but close to $c$, so that

$$\int_\omega \beta(y, c')\, dy>0.$$

We may then use Theorem 2.1 (*b*), with $a(y)\equiv 0$, *i.e.* $\mu_1=0$ there. Indeed in this case, $\sigma\equiv 1$, and the previous sign condition is precisely that of Theorem 2.1. (*b*). According to that result, there is an exponential solution $w=e^{\lambda x_1}\varphi(y)$ of (2.1):

$$(\Delta-\beta(y,c')\partial_1)w=0 \quad \text{in } \Sigma$$
$$w_\nu=0 \qquad \text{if} \quad y\in\partial\omega.$$

Returning to our sequence $c_a\to c$ and $u_a\to u$, for $a$ large, $c_a>c'$ and hence $\beta(y,c_a)>\beta(y,c')$. We will use $w$ as a comparison function, as we have done earlier. For some constant C, $C\varphi(y)\geqq\theta\ \forall y\in\bar\omega$. Now for $x_1<0$ in $\Sigma_a$,

$$0=(\Delta-\beta(y,c_a)\partial_1)u_a\leqq(\Delta-\beta(y,c')\partial_1)u_a.$$

Using the maximum principle in the region $\Sigma_a^-=(-a,0)\times\omega$, we find

$$Ce^{\lambda x_1}\varphi(y)\geqq u_a \quad \text{in } \Sigma_a^-.$$

Letting $a\to\infty$ we infer that

$$Ce^{\lambda x_1}\varphi(y)\geqq u \qquad \text{for} \quad x_1<0. \tag{5.31}$$

But this means that $\gamma=0$ and consequently $\psi=1$.

It is easily seen that $u_1$ satisfies $(\Delta-\beta\partial_1+f'(u))u_1=0$, $u_{1\,\nu}=0$ on $\partial\Sigma$. Using the maximum principle and the Hopf lemma one sees that $u_1>0$ in $\bar\Sigma$.

Theorem 1.1 is proved.

*Remark* 5.2. – In Theorem 5.2, Proposition 5.1 and Theorem 1.1, the function $\beta$ satisfies (1.17), (1.18). If we were to replace $\beta(y,c)$ by $-\beta(y,c)$ the results of the theorems would still hold. One simply changes $c$ to $-c$.

From Corollary 5.1 we immediately infer the following

LEMMA 5.4. – *In Theorem* 1.1 *if we have two functions* $f'\geqq f$ *satisfying the conditions of the theorem i.e.* $f=0$ *on* $(0,\theta)$, $f'=0$ *on* $(0,\theta')$ *etc. then we can find corresponding solutions* $(c',u')$ *and* $(c,u)$, *with* $c'\geqq c$.

Finally we prove Theorem 1.2. The proof is tricky; it relies on Theorem 1.1 which we have already proved.

*Proof of Theorem* 1.2. – Let $(c,u)$ be the solution obtained in Proposition 5.3. The only thing we have to prove is that $\gamma=0$. It then follows as before that $u_1>0$ in $\bar\Sigma$. Since $f<0$ on $(0,\theta)$, the only other possible value of $\gamma$ is $\gamma=\theta$. In that case, since $\max_{\bar\omega} u(\theta,y)=\theta$ we find by the maximum principle that $u_1\equiv 0$. Then $u\equiv\theta$. We have only to show that this case is impossible.

With the aid of Theorem 1.1 we first construct several auxiliary functions which will be used in our argument. On [0, 1] we define

$$\underline{f}=\begin{cases} f & \text{on } [0,\theta] \\ 0 & \text{on } [\theta,1] \end{cases} \qquad \overline{f}=\begin{cases} 0 & \text{on } [0,\theta] \\ f & \text{on } [\theta,1]. \end{cases} \tag{5.32}$$

Clearly

$$\underline{f}\leqq f\leqq \overline{f}.$$

Now fix $\varepsilon$ satisfying $0<\varepsilon<\theta<1-\varepsilon$.

By Theorem 1.1 there exists a solution $(c_\varepsilon, u_\varepsilon)$, with $\theta-\varepsilon<u_\varepsilon<1$, and $\partial_1 u_\varepsilon>0$ in $\bar{\Sigma}$, of

$$\left.\begin{aligned} \Delta u_\varepsilon-\beta(y,c_\varepsilon)\,\partial_1 u_\varepsilon+\overline{f}(u_\varepsilon)=0 \quad \text{in } \Sigma \\ \partial_\nu u_\varepsilon=0 \quad \text{on } \partial\Sigma \end{aligned}\right\} \tag{5.33}$$

$$\lim_{x_1\to-\infty} u_\varepsilon(x_1,y)=\theta-\varepsilon, \qquad \lim_{x_1\to+\infty} u_\varepsilon(x_1,y)=1 \text{ uniformly in } y.$$

This result is obtained directly from Theorem 1.1 by considering $v_\varepsilon=\dfrac{u_\varepsilon-\theta+\varepsilon}{1-\theta+\varepsilon}$.

Similarly, there is a solution $(c^\varepsilon, u^\varepsilon)$, with $0<u^\varepsilon<\theta+\varepsilon$, and $\partial_1 u^\varepsilon>0$ in $\bar{\Sigma}$, of

$$\left.\begin{aligned} \Delta u^\varepsilon-\beta(y,c^\varepsilon)\,\partial_1 u^\varepsilon+\underline{f}(u^\varepsilon)=0 \quad \text{in } \Sigma \\ \partial_\nu u^\varepsilon=0 \quad \text{on } \partial\Sigma \end{aligned}\right\} \tag{5.34}$$

$$\lim_{x_1\to-\infty} u^\varepsilon(x_1,y)=0, \qquad \lim_{x_1\to+\infty} u^\varepsilon(x_1,y)=\theta+\varepsilon \text{ uniformly in } y.$$

This is obtained by setting

$$v^\varepsilon(x_1,y)=1-\frac{u^\varepsilon(-x_1,y)}{\theta+\varepsilon}.$$

Then (5.34) is equivalent to

$$\left.\begin{aligned} \Delta v^\varepsilon+\beta(y,c^\varepsilon)\,v_1^\varepsilon-\frac{1}{\theta+\varepsilon}\,\underline{f}[(\theta+\varepsilon)(1-v^\varepsilon)]=0 \quad \text{in } \Sigma \\ \partial_\nu v^\varepsilon=0 \quad \text{on } \partial\Sigma \\ \lim_{x_1\to-\infty} v^\varepsilon(x_1,y)=0, \qquad \lim_{x_1\to+\infty} v^\varepsilon(x_1,y)=1 \text{ uniformly in } y \end{aligned}\right\} \tag{5.35}$$

to which one applies Theorem 1.1 and Remark 5.2.

LEMMA 5.5. — $c_\varepsilon>c^\varepsilon$.

*Proof.* — Clearly $u^\varepsilon<u_\varepsilon$ for $|x_1|$ large and this remains true for any $x_1$-translation of $u_\varepsilon$. We may translate $u_\varepsilon$ so far to the left so that it is $>u^\varepsilon$

everywhere. Then start translating $u_\varepsilon$ back to the right until its graph first touches $u^\varepsilon$. Then the resulting $u_\varepsilon$ satisfies $u_\varepsilon \geqq u^\varepsilon$ with equality holding somewhere. Now suppose $c_\varepsilon \leqq c^\varepsilon$; then $\beta(y, c_\varepsilon) \leqq \beta(y, c^\varepsilon)$. Since $\bar{f} \geqq \underline{f}$ and $\partial_1 u_\varepsilon > 0$, $u_\varepsilon$ would satisfy

$$\Delta u_\varepsilon - \beta(y, c^\varepsilon)\,\partial_1 u_\varepsilon + \underline{f}(u_\varepsilon) \leqq 0.$$

But then the function $z = u_\varepsilon - u^\varepsilon \geqq 0$ would satisfy a linear elliptic inequality: for some $c \in L^\infty(\Sigma^-)$

$$\Delta z - \beta(y, c^\varepsilon)\,\partial_1 z + c(x_1, y)\, z \leqq 0.$$

Since $z = 0$ somewhere, it follows from the maximum principle and the Hopf lemma, that $z \equiv 0$, which is impossible. The lemma is proved.

We will need several similar results.

LEMMA 5.6. — *Suppose there is a solution $v$, $0 < v < \theta$, of*

$$\left.\begin{aligned} &\Delta v - \beta(y, c)\, v_1 + f(v) = 0 \quad \textit{in } \Sigma^+ = (0, \infty) \times \omega \\ &v_\nu = 0 \quad \textit{on } (0, \infty) \times \partial\omega, \\ &v(0, y) = 0, \qquad \lim_{x_1 \to \infty} v(x_1, y) = \theta \textit{ uniformly in } y. \end{aligned}\right\} \tag{5.36}$$

*Then $c^\varepsilon > c$.*

*Proof.* — Clearly $u^\varepsilon > v$ if $x_1 = 0$ and if $x_1$ is large, and this is true for any translation of $u$. Arguing then exactly as in the previous lemma — note that $\underline{f} = f$ on $(0, \theta)$ — we obtain the desired result.

Similarly, since $\bar{f} = f$ on $[\theta, 1]$, we find

LEMMA 5.7. — *If $w$, with $\theta < w < 1$, is a solution of*

$$\left.\begin{aligned} &\Delta w - \beta(y, c)\, w_1 + f(w) = 0 \quad \textit{on } \Sigma^- = (-\infty, 0) \times \omega \\ &w_\nu = 0 \quad \textit{on } (-\infty, 0) \times \partial\omega \\ &\lim_{x_1 \to -\infty} w(x_1, y) = \theta \textit{ uniformly in } y; \qquad w(0, y) = 1, \end{aligned}\right\} \tag{5.37}$$

*then $c > c_\varepsilon$.*

*Completion of the Proof of Theorem* 1.2. — We complete our argument by contradiction. We have assumed that the solution $u$ of Theorem 5.3, which was obtained as a limit as $a \to \infty$ of solutions $u_a$ of (5.2-4), (5.5)′, is $u \equiv \theta$. The $u_a$ tend therefore to $\theta$ uniformly on every compact set. The functions $u_a$ satisfied $\max_{\bar{\omega}} u(0, y) = \theta$. Now shift these functions to the right so that the new functions — we still call them $u_a$ — satisfy

$$\max_{\bar{\omega}} u(0, y) = \frac{\theta}{2}.$$

Each new $u_a$ is defined on $(\tau_a, \tau_a + 2a) \times \omega$ with $\tau_a < 0$. Now let $a \to \infty$ so that (for a subsequence) the $u_a$ converge to a function $v$. There are two possibilities. If the $\tau_a \to -\infty$. Then $v$ is a solution of

$$\left.\begin{array}{c} \Delta v - \beta(y,c)\, v_1 + f(v) = 0 \quad \text{in } \Sigma \\ v_\nu = 0 \quad \text{on } \partial\Sigma, \\ \lim_{x_1 \to -\infty} v(x_1, y) = 0, \qquad \lim_{x_1 \to +\infty} v(x_1, y) = \theta \\ v_1 > 0, \qquad \max_{\bar{\omega}} v(0, y) = \theta/2. \end{array}\right\} \tag{5.38}$$

The other possibility then leads to a solution, after suitable $x_1$-translation, of (5.36). In this second case, according to Lemma 5.6,

$$c^\varepsilon \geqq c. \tag{5.39}$$

We claim that (5.39) also holds in the case (5.38). This is proved in the, by now, familiar way. Suppose to the contrary that $c^\varepsilon < c$. Clearly $u^\varepsilon > v$ for $x_1$ large. By Theorem 4.4, Case C′ and Theorem 2.1 (*e*), $u^\varepsilon > v$ also for $-x_1$ large. These assertions also hold for any translation of $u^\varepsilon$. Translating $u^\varepsilon$ to the left so that it is greater than $v$ everywhere, and then shifting back to the right until the graphs first touch, we find as before—recall that $\underline{f} = f$ on $(0, \theta)$—that the new $u^\varepsilon \equiv v$. This is impossible. Thus (5.39) holds.

By a similar argument we will show that

$$c \geqq c_\varepsilon. \tag{5.40}$$

But this contradicts (5.39) and Lemma 5.5, and so is impossible.

To establish (5.40) we consider our solutions $u_a$ as before, but now translate them to the left, so that

$$\max_{\bar{\omega}} u_a(0, y) = \frac{1+\theta}{2}.$$

Then let $a \to \infty$. Again two possibilities arise. Either we have a solution $w$, $\theta < w < 1$, of

$$\left.\begin{array}{c} \Delta w - \beta(y,c)\, w_1 + f(w) = 0 \quad \text{on } \Sigma \\ w_\nu = 0 \quad \text{on } \partial\Sigma \\ \lim_{x_1 \to -\infty} w(x_1, y) = \theta, \qquad \lim_{x_1 \to +\infty} w(x_1, y) = 1 \text{ uniformly in } y \\ w_1 > 0, \qquad \max w(0, y) = \dfrac{1+\theta}{2}, \end{array}\right\} \tag{5.41}$$

or else we obtain, after translation, a function $w$ in $\Sigma^+$ satisfying (5.37). If the second case occurs then according to Lemma 5.7, $c \geqq c_\varepsilon$. If the first case occurs, *i.e.* (5.41), assuming $c_\varepsilon > c$ we argue as before and obtain a contradiction.

The proof of Theorem 1.2 is complete.

## 6. PROOF OF THEOREM 1. EXISTENCE OF A SOLUTION FOR THE BISTABLE NONLINEARITY IN CASE $\omega$ IS CONVEX

We continue our study of the bistable case (type C) begun in Section 5, with $f$ of type C. Assuming

$$f \in C^{1,\delta}([0,1]) \quad \text{for some positive } \delta < 1, \tag{6.1}$$

and that $\omega$ is convex, we prove the existence of a solution of problem (1.16), (1.2), (1.3), *i.e.* we prove Theorem 1.3.

From the results in Section 5 (viz. Theorem 1.2), we know that there exists a solution $u$ of (1.16), (1.2) with $u(-\infty,.)=0$ and $u(+\infty,.)=\psi$, for some undetermined solution $\psi=\psi(y)$, $0<\psi\leqq 1$, $\psi\not\equiv\theta$, of

$$\left.\begin{aligned} &\Delta\psi+f(\psi)=0 \quad \text{in } \omega \\ &\frac{\partial\psi}{\partial\nu}=0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{6.2}$$

Using terminology from dynamical systems we will refer to solutions $\psi$ of (6.2) as stationary solutions, and a pair $(c, u(x_1, y))$ satisfying (1.16), (1.2), with

$$\lim_{x_1 \to -\infty} u(x_1, y)=\psi_1(y), \qquad \lim_{x_1 \to +\infty} u(x_1, y)=\psi_2(y),$$

as a connection between stationary solutions $\psi_1$, $\psi_2$. As always, the limits are understood to be uniform in $y$.

Theorem 5.3 asserts the existence of a connection from 0 to some $\psi$, $0<\psi\leqq 1$ where $\psi$ *is a priori* undetermined. Here, as we said, assuming $\omega$ to be convex, we prove that there is actually a connection from 0 to 1.

The proof relies on two further properties of equation (6.2) in case of convex $\omega$. The first one is the *instability* of nonconstant stationary solutions. This has been established independently by Matano [Ma] and Casten and Holland [CH]. To state this result precisely we use the following notation.

Let $\psi$ be a stationary solution (of (6.2)) and let $\mu_1(\psi)$ be the principal (least) eigenvalue of $-\Delta-f'(\psi)$ with Neumann conditions on $\partial\omega$. That is,

$$\mu_1(\psi)=\inf_{\substack{\varphi\in H^1(\omega)\\ \varphi\not\equiv 0}} \frac{\displaystyle\int_\omega |\nabla\varphi|^2-f'(\psi)|\varphi|^2}{\displaystyle\int_\omega |\varphi|^2}.$$

It is characterized by the existence of a positive eigenfunction $\varphi(y)$; $\varphi$ satisfies

$$\left.\begin{aligned} -\Delta\varphi - f'(\psi(y))\,\varphi = \mu_1(\psi)\,\varphi \quad \text{in } \omega \\ \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{6.3}$$

We make $\varphi$ unique by the normalization

$$\|\varphi\|_{L^2(\omega)} = 1.$$

Proposition 6.1 ([Ma], [CH]). — *Under the assumptions above, in particular that $\omega$ is convex, let $\psi$ be a nonconstant solution of* (6.2). *Then $\psi$ is unstable in the sense that $\mu_1(\psi) < 0$.*

From this property, we now derive the second — a known result; it concerns solutions of (6.2).

Proposition 6.2. — *Under the assumptions above* (*with $\omega$ convex*), *there does not exist a pair of distinct ordered nonconstant stationary solutions, i.e., if $\psi$ and $\tilde{\psi}$ are solutions of* (6.2) *with $0 \leqq \psi \leqq \tilde{\psi} \leqq 1$, and $\psi \not\equiv 0$, $\tilde{\psi} \not\equiv 1$, then $\psi \equiv \tilde{\psi}$.*

*Proof.* — Suppose two such different solutions $\psi$ and $\tilde{\psi}$ exist. By the strong maximum principle and the Hopf lemma, we see that $0 < \psi < \tilde{\psi} < 1$ in $\bar{\omega}$. If one of the two — say $\psi$ — is the constant $\theta$, we have an obvious contradiction, for then $f(\tilde{\psi}) \geqq 0$ in $\omega$. On the other hand, integrating (6.2) yields

$$\int_{\omega} f(\tilde{\psi}) = 0,$$

which is impossible.

Hence, we may assume that $\psi \not\equiv \theta$ and $\tilde{\psi} \not\equiv \theta$. The same argument shows that

$$\min_{\bar{\omega}} \tilde{\psi} < \theta < \max_{\bar{\omega}} \psi. \tag{6.4}$$

Let $\mu_1$ and $\varphi > 0$ be the principal eigenvalue and eigenfunction of (6.3) corresponding to $\psi$, and $\tilde{\mu}_1$ and $\tilde{\varphi}$ the corresponding ones for $\tilde{\psi}$. By Proposition 6.1, $\mu_1$, $\tilde{\mu}_1 < 0$.

From this we see that for $0 < \varepsilon$ small,

$$\Delta(\psi + \varepsilon\varphi) + f(\psi + \varepsilon\varphi) > 0 \quad \text{in } \omega. \tag{6.5}$$

Therefore, $\psi + \varepsilon\varphi$ is a *subsolution* of equation (6.2). Likewise, for $0 < \varepsilon$ small, $\tilde{\psi} - \varepsilon\tilde{\varphi}$ is a supersolution. For $\varepsilon$ small, $\psi + \varepsilon\varphi < \tilde{\psi} - \varepsilon\tilde{\varphi}$ in $\bar{\omega}$. By the standard theory of sub and supersolutions it follows that there is a maximal solution $\bar{\psi}$ satisfying $\psi + \varepsilon\varphi < \bar{\psi} < \tilde{\psi} - \varepsilon\tilde{\varphi}$. (This is obtained by successively solving $(\Delta-k)\psi_{j+1}+f(\psi_j)+k\psi_j=0$ in $\omega$, $\partial_\nu\psi_{j+1}=0$ on $\partial\omega$, with $\psi_1=\tilde{\varphi}-\varepsilon\tilde{\varphi}$. Here $k>0$ is a constant chosen so that $f(s)+ks$ is increasing on $[0,1]$. Then $\bar{\psi}=\lim\psi_j$ as $j\to\infty$.) But then if $\bar{\mu}$ and $\bar{\varphi}$ are the principal eigenvalue and eigenfunction for $-\Delta-f'(\bar{\psi})$ we see, as above that for $\delta$ small, $\bar{\psi}+\delta\bar{\varphi}$ is a subsolution of (6.2) and $\bar{\psi}+\delta\bar{\varphi}<\tilde{\psi}-\varepsilon\tilde{\varphi}$. The sub and super solution theory yields again a stationary solution $\hat{\psi}$ satisfying $\bar{\psi}+\delta\bar{\varphi}<\hat{\psi}<\tilde{\psi}-\varepsilon\tilde{\varphi}$ – contradicting the maximality of $\bar{\psi}$. ■

We now prove the existence of a connection from 0 to 1, that is a solution of (1.16), (1.2), (1.3). In Section 5, we constructed in $\Sigma_a=(-a,a)\times\omega$ a solution $(c_a,u_a)$ of

$$\left.\begin{array}{c}\Delta u_a-\beta(y,c_a)\,\partial_1 u_a+f(u_a)=0 \quad \text{in } \Sigma_a\\ \partial_\nu u_a=0 \quad \text{on } (-a,a)\times\partial\omega\\ u_a(-a,.)=0, \qquad u_a(+a,.)=1\end{array}\right\} \tag{6.6}$$

with the normalization

$$\max_{y\in\bar{\omega}} u_a(0,y)=\theta. \tag{6.7}$$

In analyzing the limit as $a\to\infty$, we showed – see the proof of Theorem 5.3 – that for a sequence $a=a_j\to\infty$, $c_a\to c$ and $u_a\to u$ locally in $C^{1,\mu}$ norm, $0<u<1$, and $u\in W^{2,p}_{\text{loc}}(\bar{\Sigma})$, $\forall p>n$. $(c,u)$ is a solution of (1.16)-(1.2) and $u(-\infty,.)=0$, $u(+\infty,.)=\psi$, with $\psi$ a stationary solution of (6.2) such that

$$0<\psi\leqq 1 \qquad \text{and} \qquad \psi\not\equiv\theta.$$

We will prove that $\psi\equiv 1$, arguing by contradiction. Suppose $\psi\not\equiv 1$. Since $\psi\leqq 1$, it follows from the maximum principle and the Hopf lemma that $\psi<1$ in $\bar{\omega}$.

Fix a real number $d$ satisfying

$$\max_{\bar{\omega}}\psi<d<1. \tag{6.8}$$

Since $\partial_1 u_a>0$ in $\bar{\Sigma}_a$, there is a unique $t_a\in(0,a)$ such that

$$\max_{y\in\bar{\omega}} u_a(t_a,y)=d. \tag{6.9}$$

Since $u_a\to u$ locally, and $u\leqq\psi$, we see that $t_a\to+\infty$ as $a\to\infty$. Shift the origin to $x_1=t_a$ setting

$$v_a(x_1,y)=u_a(t_a+x_1,y).$$

This function $v_a$ is defined on $[-a-t_a,a-t_a]\times\bar{\omega}$.

Choosing an appropriate further subsequence $a=a_j\to\infty$, we have $a-t_a\to b$ with $0\leqq b\leqq\infty$. Since the $v_a$ are bounded locally in $W^{2,p}$, we can further impose that $v_a\to v$ locally in $C^{1,\mu}$ as $a=a_j\to\infty$. Now $v$ is a solution of (1.16), (1.2) in the domain $\Sigma'_b=(-\infty,b)\times\omega$ (this is all of $\Sigma$

in case $b=\infty$). Note that *this solution $v$ is obtained for the same value of $c$ as our connection $(c,u)$ from 0 to $\psi$*. Moreover, $\partial_1 v \geqq 0$ in this domain, and

$$\max_{y \in \bar{\omega}} v(0,y)=d. \tag{6.10}$$

The argument used in Section 5, shows that $v$ has a limit as $x_1 \to -\infty$:

$$v(-\infty,y)=\psi_1(y)$$

and $\psi_1$ is a stationary solution.

For any real $x_1$, and $A>0$, we have $x_1+t_a>A$ for $a$ large. Hence $v_a(x_1,y) \geqq u_a(A,y)$ for $y \in \omega$. Letting $a \to \infty$ we see that $v(x_1,y) \geqq u(A,y)$ for any $A>0$ and therefore $v(x_1,y) \geqq \psi(y)$. Thus, $\psi_1(y) \geqq \psi(y)$. In view of condition (6.10), $\psi_1 \leqq d<1$. Since $\psi_1$ and $\psi$ are ordered stationary solutions with $0<\psi \leqq \psi_1<1$, it follows from Proposition 6.2 that $\psi=\psi_1$.

Using (6.8) and (6.10), we see that $v$ is not constant, and since $\partial_1 v \geqq 0$ in $\Sigma'_b$, we conclude by the Hopf lemma that $\partial_1 v>0$ on $(-\infty,b) \times \partial\omega$.

Thus we have, for the same value of $c$, a connection $u$ from 0 to $\psi$, and a solution $v$ of (1.1)-(1.2) in $\Sigma'_b$ with $v(-\infty,y)=\psi(y)$ and $\partial_1 v>0$ in $\Sigma'_b$.

By analyzing the asymptotic behaviour of these, we now show that this is impossible. To do this, we will rely on the results of Sections 3 and 4.

First we consider the behaviour of $u$ near $x_1=+\infty$. Let $w(x_1,y)=\psi(y)-u(-x_1,y)$. This function satisfies the equation

$$\Delta w+\beta(y,c)\,\partial_1 w+g(y,w)=0 \quad \text{in } \Sigma^- \tag{6.11}$$

where $g(y,w)=f(\psi(y))-f(\psi(y)-w)$. Note that $g_w(y,0)=f'(\psi(y))$ and that the first eigenvalue of $-\Delta_y-g_w(y,0)$ is precisely $\mu_1(\psi)$ which is negative. For $0<\varepsilon$ small, the first eigenvalue of $-\Delta_y-(g_w(y,0)-\varepsilon)$, namely $\mu_1(\psi)+\varepsilon$, is still negative. Therefore, we may apply the results of Theorem 3.1 case 2, and Theorem 4.1. (The fact that $f \in C^{1,\delta}[0,1]$ is used in satisfying the conditions of the theorems.)

From these, we infer that there exists a positive principal eigenvalue $\lambda_1>0$ and a corresponding eigenfunction $\varphi_1=\varphi_1(y)>0$ in $\bar{\omega}$ of the problem

$$\left.\begin{aligned} -\Delta_y\varphi-f'(\psi)\,\varphi&=[\lambda^2+\lambda\beta(y,c)]\,\varphi \quad \text{in } \omega\\ \frac{\partial\varphi}{\partial\nu}&=0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{6.12}$$

The behaviour of $u$ is then given by

$$u(x_1,y)=\psi(y)-\rho\, e^{-\lambda_1 x_1}\,\varphi(y)+o(e^{-\lambda x_1}) \tag{6.13}$$

as $x_1 \to +\infty$, with $\rho$ a positive constant.

Let us now consider the behaviour of $v$ as $x_1 \to -\infty$. Working with the function $v(x_1, y) - \psi(y)$ we find that we can again apply Theorems 3.1 (case 2) and 4.1. Therefore, the behaviour of $v$ as $x_1 \to -\infty$ is given by

$$v(x_1, y) = \psi(y) + \rho' e^{\lambda'_1 x_1} \varphi'(y) + o(e^{\lambda'_1 x_1}) \tag{6.14}$$

for some positive constant $\rho' > 0$. Here, $\lambda'_1 > 0$ and $\varphi' > 0$ are the principal positive eigenvalue and corresponding eigenfunction of

$$\left.\begin{aligned} -\Delta_y \varphi - f'(\psi)\varphi &= [\lambda^2 - \lambda\beta(y, c)]\varphi \quad \text{in } \omega \\ \frac{\partial\varphi}{\partial\nu} &= 0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{6.15}$$

We emphasize again that we have the same value of $c$ in both (6.12) and (6.15). Therefore, the same eigenvalue problem (6.15) admits both a positive principal eigenvalue $\lambda'_1$ and a negative one, $-\lambda_1$, from (6.12).

However, since $\mu_1(\psi) < 0$, the case ($c$) of Theorem 2.1 applies. It asserts that principal eigenvalues, when they exist, always have the same sign. We have reached a contradiction. This means that $u_a$ cannot break into two pieces—for example, into a connection from 0 to $\psi$ and one from $\psi$ to 1. Necessarily $u_a$ converges to a connection from 0 to 1. The proof of Theorem 1.3 is complete.

## 7. UNIQUENESS AND MONOTONICITY

In Section 7 of [BN1] the sliding method was used to prove uniqueness and monotonicity of the solution $(c, u)$ of the problem (1.1), with $f$ as in case B. In this section we will extend the argument there to other cases. In Theorem 7.1 below we will first deal with uniqueness and monotonicity of any solution $u$ if $c$ is fixed. Uniqueness of $c$, under various conditions, is then proved in Theorem 7.2.

We consider positive solutions $u < 1$, $u \in C^1(\bar{\Sigma})$, of

$$\left.\begin{aligned} \Delta u - \beta(y) u_1 + f(y, u) &= 0 \quad \text{in } \Sigma \\ u_\nu &= 0 \qquad \text{for } y \in \partial\omega, \end{aligned}\right\} \tag{7.1}$$

with $u \to 0$ and 1 respectively, at $-\infty$ and $+\infty$, uniformly in $y$. The function $f$ is assumed to be continuous in $\bar{\omega} \times [0, 1]$ and uniformly Lipschitz continuous in $u$. In addition, we assume, as in Section 4, that for $M \geqq 0$ and positive constants $s_0$, $\delta$, and for $L^\infty$ functions $a'(y) \gneqq 0$, and $a(y)$, the functions

$$h(y, s) := f(y, s) + a(y)s, \qquad H(y, s) := f(y, 1 - s) - a'(y)s \tag{7.2}$$

satisfy

$$|h(y,s)|,\ |H(y,s)|\leqq M s^{1+\delta} \qquad \text{for} \quad 0<s\leqq s_0, \tag{7.3}$$

and

$$|H(y,s)-H(y,s')|\leqq M(s+s')^{\delta}|s-s'| \qquad \text{for} \quad 0<s,s'\leqq s_0. \tag{7.4}$$

Note that unlike [BN1], no condition like (7.4) is required of $h(y,s)$.

We denote by $\mu_1$ the first eigenvalue of $-\Delta_y+a(y)$, and by $\sigma$, the corresponding positive eigenfunction.

THEOREM 7.1. — *Assume all the conditions above. Suppose furthermore that* $\mu_1\neq 0$, *then the solution u is unique, up to translation in the* $x_1$ *direction, and* $u_1>0$ *in* $\bar{\Sigma}$. *Moreover the same result holds if* $\mu_1=0$, $h(y,s)\leqq 0$, *for* $0<s\leqq s_0$ *and*

$$\int_{\omega} \beta(y)\,\sigma^2\,dy > 0. \tag{7.5}$$

*Proof.* — According to Theorem 4.4, any function $u$ satisfying the conditions of Theorem 7.1 satisfies (4.4), *i.e.*, for some $\alpha>0$,

$$u(x_1,y)=\alpha e^{\lambda x_1}\varphi(y)+o(e^{\lambda x_1}) \quad \text{as } x_1\to-\infty \tag{7.6}$$

or

$$u(x_1,y)=\alpha e^{\lambda x_1}(-x_1\varphi(y)+\varphi_0(y))+o(e^{\lambda x_1}) \quad \text{as } x_1\to-\infty. \tag{7.7}$$

Here $\lambda$ and $\varphi(y)$ are principal positive eigenvalues and eigenvectors of (2.5). Furthermore, the case (7.7) may occur only if $\mu_1<0$ and the principal positive eigenvalue $\lambda$ is unique. In that case,

$$w=e^{\lambda x_1}(-x_1\varphi(y)+\varphi_0(y))$$

satisfies (2.1).

According to Theorem 4.5, the function $u$ also satisfies: for some $\gamma>0$,

$$1-u=\gamma e^{-\tau x_1}\varphi'(y)+o(e^{-\tau x_1}) \quad \text{as } x_1\to+\infty. \tag{7.8}$$

Here $\tau>0$ and $\varphi'(y)>0$ are the principal positive eigenvalue and eigenfunction of (2.5) — with, however, $-\beta$ in place of $\beta$.

Suppose now that $u$ and $\underline{u}$ satisfy the conditions of Theorem 7.1. Suppose they satisfy (7.8) with respective constants $\gamma$, $\underline{\gamma}$, and (7.6) or (7.7) with respective constants $\alpha$, $\underline{\alpha}$. Without loss of generality we need only consider the following cases:

(1) $u$ and $\underline{u}$ satisfy (7.6) with respective constants $\alpha$ and $\underline{\alpha}$.

(2) $u$ and $\underline{u}$ both satisfy (7.7) with respective constants $\alpha$, $\underline{\alpha}$.

(3) $u$ satisfies (7.7) and $\underline{u}$ satisfies (7.6) with respective constants $\alpha$, $\underline{\alpha}$.

For any real $r$, $u^r(x_1,y)=u(x_1+r,y)$ is a solution of (7.1), and it satisfies (7.8) with $\gamma$ replaced by $\gamma e^{-\tau r}$. In each of the cases (1), (2) and

(3), the function $u^r$ satisfies (7.6) or (7.7) with $\alpha$ replaced by $\alpha e^{\lambda r}$ (and $\varphi_0$ replaced by $\varphi_0 - r\varphi$ in case of (7.7)).

We may assume, after shifting $u$ to the left if necessary, that $\gamma < \underline{\gamma}$ and $\alpha > \underline{\alpha}$. We shall make use of the following.

LEMMA 7.1. – *For some positive r large, $u^r > \underline{u}$ everywhere.*

*Proof.* – In any of the cases (1), (2), (3), using (7.6) or (7.7), and (7.9), we easily verify that for some $R > 0$,

$$u(x_1', y) > \underline{u}(x_1, y) \qquad \text{for} \quad x_1' \geqq x_1, y \in \bar{\omega}$$

if $x_1' \leqq -R$ or $x_1 \geqq R$. Suppose the assertion of the lemma is false, then $\exists$ sequences $r_j \to \infty$, and $(x_j, y_j) \in \Sigma$ such that

$$u(x_j + r_j, y_j) \leqq \underline{u}(x_j, y_j), \qquad j = 1, 2, \ldots \tag{7.9}$$

By the preceding remark, either $x_j < R$ or $x_j + r_j > -R$. Two cases are possible: (*a*) the $x_j$ are all bounded, *i.e.* $|x_j| \leqq M$, or (*b*) a subsequence of the $x_j$ tend to $-\infty$. In case (*a*), the right hand side of (7.9) is bounded above by $1 - \delta$ for some $\delta > 0$ while the left hand side tends to 1. Impossible. In case (*b*), for the subsequence $x_j \to -\infty$, the left hand side is $\geqq \delta$ for some $\delta > 0$ while the right hand side tends to zero – once more this is impossible. ■

Return to the proof of Theorem 7.1. With $r$ chosen according to Lemma 7.1, now start shifting $u^r$ back, *i.e.* decreasing $r$, until we reach a value $r = s$, for which one of the following *first* happens:

(i) $u^s = \underline{u}$ somewhere in $\bar{\Sigma}$,

(ii) $\gamma e^{-\tau s} = \underline{\gamma}$,

(iii) $\alpha e^{\lambda s} = \underline{\alpha}$.

This must occur for some finite $s$.

We will show that in cases (i) and (ii),

$$u^s \equiv \underline{u}, \tag{7.10}$$

under the assumptions of the theorem. For case (i), this is clear by the maximum principle and the Hopf lemma – since $u^s - \underline{u}$ is nonnegative, and satisfies a linear elliptic equation in $\Sigma$. For case (ii), (7.10) follows from Lemma 4.1. Consider, then, case (iii). In this case $u^s > \underline{u}$, and we may suppose $\gamma e^{-\tau s} > \underline{\gamma}$; otherwise we are in case (ii).

Now we treat the various cases listed above. Consider first the case when $\mu_1 = 0$. In this case, $u^s > \underline{u}$ for $-x_1$ large, and this remains true if we shift $u^s$ to the right by any amount. Thus we may decrease $r$ further, beyond $s$ until we reach a value $r = t < s$ such that $u^t \geqq \underline{u}$, and either (i)′ $u^t = \underline{u}$ somewhere in $\bar{\Sigma}$, or (ii)′ $\gamma e^{\tau t} = \underline{\gamma}$. But then we are back in cases (i) and (ii) and, as above, we conclude that $u^t \equiv \underline{u}_1$.

We are left, finally, with case (iii) and $\mu_1 \neq 0$. If we now shift $u^s$ to the right (by any amount) it will lie below $\underline{u}$ for $-x_1$ large. Arguing as in the proof of Lemma 7.1 we see that we may shift $u^s$ so far to the right, to $u^t$, so that $u^t < \underline{u}$ in $\bar{\Sigma}$ and $\gamma e^{-\tau t} < \underline{\gamma}$. At this point, start shifting it back to the left, *i.e.*, increase $t$. There will be a first value $d < s$, such that $u^d$ first touches $\underline{u}$, *i.e.* $u^d \leqq \underline{u}$, and either

$$u^d = \underline{u} \quad \text{somewhere in } \bar{\Sigma}$$

or

$$\gamma e^{-\tau d} = \underline{\gamma}.$$

Such a value $d < s$ must occur. But then we are back to case (i) or (ii) and we conclude, as before, that $u^d \equiv \underline{u}$.

We have proved the uniqueness of $u$ modulo shift. To prove monotonicity, we apply the same sliding argument with $\underline{u} = u$ itself. In this case, for any $r > 0$, $u^r > u$ for $|x_1|$ large. Applying Lemma 7.1 again we may take $r$ so large that $u^r > u$ everywhere. Then, shifting to the right to $u^s$, so that we are in case (i), (ii) or (iii), we find that necessarily $s \geqq 0$ and then, in fact, $s = 0$. This implies that $u$ is strictly increasing in $x_1$.

Applying $\partial_1$ to the equation for $u$ we see that $u_1 \geqq 0$ satisfies a linear elliptic equation with bounded coefficients. Since $u_1 \not\equiv 0$ it follows by the maximum principle and the Hopf Lemma, that $u_1 > 0$ in $\bar{\Sigma}$.

Theorem 7.1 is proved.

*Remark* 7.1. – Our proof of uniqueness in the theorem hinges on knowing that solutions decay exponentially as $x_1 \to -\infty$. At the beginning of Section 3 we presented some examples of type A, with $f'(0) = 0$, in which $u$ decays at $-\infty$ like a power of $|x_1|^{-1}$. Those examples were merely one-dimensional – and in one dimension, uniqueness and monotonicity are easily established. However in higher dimensions, in case $\mu_1 = 0$, we do not know whether solutions of (7.1) (which tend to 0 and 1 at $-\infty$ and $+\infty$) are monotonic, or unique.

The proof of Theorem 7.1 yields the following

LEMMA 7.2. – *Let u be a solution of* (7.1), $0 < u < 1$ *tending to* 0 *and* 1 *at* $-\infty$ *and* $+\infty$ *and satisfying*

$$u(x_1, y) \geqq \frac{C}{1 + |x_1|^k} \qquad \text{for} \quad x_1 < 0,$$

*for some positive constants* C, $k$. *Then no solution of the problem can decay at* $-\infty$ *faster than* $|x_1|^{-l}$ *for any* $l > k$.

We turn now to the case where $\beta$ depends on a parameter $c$, $\beta = \beta(y, c)$ – satisfying the conditions (1.17), (1.18). Having (1.16) in mind, we restrict ourselves to $f$ independent of $y$ and satisfying (7.2)-(7.4).

As before we consider functions $u$, $0<u<1$ in $\bar{\Sigma}$, tending to 0 and 1 as $x_1 \to -\infty$ and $+\infty$, uniformly in $y$, which satisfy

$$\left.\begin{array}{ll} \Delta u-\beta(y,c)\,u_1+f(u)=0 & \text{in } \Sigma \\ u_\nu=0 & \text{for } \; y\in\partial\omega. \end{array}\right\} \tag{7.11}$$

According to Theorem 7.1, with $c$ fixed, any such solution of (7.11) is unique modulo translation, and $u_1>0$. As we have remarked before, standard local elliptic estimates imply that $u_1 \to 0$ as $|x_1| \to \infty$.

Under further conditions on $f$ we will now prove that also $c$ is unique. We will treat the following cases (in which $0<\theta<1$):

CASE B. $f=0$ on $[0,\theta]$, $f>0$ on $(\theta, 1)$.

CASE C′. $f<0$ on $(0,\theta)$, $f>0$ on $(\theta, 1)$ and $f'(0)<0$.

CASE C″. $f<0$ on $(0,\theta)$, $f>0$ on $(\theta, 1)$, $f'(0)=0$.

(Indeed in case A, $c$ may be any value in an interval $[c^*, \infty)$ – compare Section 9 below.)

THEOREM 7.2. – *Let $u$ and $\beta(y,c)$ be as above and with $f$ satisfying (7.2)-(7.4). Then in either case B or C′, the constant $c$ is unique. In case C″, $c$ is also unique if $\beta(y,c)=c\,\alpha(y)$, $\alpha>0$ on $\bar{\omega}$, and* $\displaystyle\int_0^1 f(s)\,ds>0$.

This is essentially contained in Theorem 7.2 of [BN1]; for completeness we include the proof.

*Proof.* – (*a*) Consider a solution $(c,u)$ of (7.11). By Theorem 4.5, for some constant $\gamma>0$,

$$1-u(x_1,y)=\gamma e^{-\tau x_1}\varphi'(y)+o(e^{-\tau x_1}) \quad \text{as } x_1 \to +\infty. \tag{7.12}$$

Here $-\tau$ is the negative principal eigenvalue and $\varphi'$ the corresponding positive eigenfunction, for (2.5) – with $(-\beta(y,c))$ in place of $\beta$. Suppose we are in case B or C′. Then we claim that

$$u(x_1,y)=\alpha e^{\lambda x_1}\varphi(y)+o(e^{\lambda x_1}) \quad \text{as } x_1 \to -\infty \tag{7.13}$$

where $(\lambda,\varphi)$ are the positive principal eigenvalue and eigenvector of (2.5) with $\beta=\beta(y,c)$. In particular, we claim that these exist. Here we use Theorem 4.4. In case C′, we have $a=-f'(0)>0$; then case I of Theorem 4.4 yields the claim. If we are in case B, we integrate the equation in (7.11) over $\Sigma$ and find

$$\int_\omega \beta(y,c)\,dy=\int_\Sigma f(u)>0.$$

Then case I′ of Theorem 4.4 applies, and yields (7.13). Note indeed that here $\sigma\equiv 1$.

Suppose now that $\bar{u}$ is a solution of (7.11) with $\beta=\beta(y,\bar{c})$, $\bar{c}>c$; $\bar{u}$ is assumed to satisfy the same conditions as $u$. $\bar{u}$ then satisfies (7.13) and (7.14) with corresponding $(\bar{\lambda}, \bar{\varphi})$, $(\bar{\tau}, \bar{\varphi}')$, and $\bar{\alpha}$, $\bar{\gamma}>0$.

Since $\beta(y,\bar{c})>\beta(y,c)$ we may now make use of Theorem 2.1 (*e*), according to which,

$$\bar{\lambda}>\lambda>0>-\bar{\tau}>-\tau. \tag{7.14}$$

It follows that for $|x_1|$ large, $u>\bar{u}$. Consequently if we consider $u^r=u(x_1+r,t)$, as in the proof of Theorem 7.1, we see that for $r$ large,

$$u^r>\bar{u} \quad \text{in } \bar{\Sigma}. \tag{7.15}$$

In fact for every real $r$, $\exists\, \mathrm{R}(r)$ such that

$$u^r>\bar{u} \qquad \text{if} \quad |x_1|>\mathrm{R}(r), \tag{7.16}$$

and $\mathrm{R}(r)$ is bounded for $|r|$ bounded.

Now starting with large $r$, so that (7.15) holds, decrease it to a first value $s$, such that $u^s=\bar{u}$ somewhere in $\bar{\Sigma}$. Because of (7.16), such $s$ must exist. But then

$$z:=u^s-\bar{u}\geqq 0$$

satisfies

$$\Delta z-\beta(y,\bar{c})\,z_1+f(y,u^s)-f(y,\bar{u})=(\beta(y,c)-\beta(y,\bar{c}))\,u_1^s\leqq 0.$$

Thus for some bounded function $d$,

$$(\Delta-\beta(y,\bar{c})\,\partial_1+d)\,z\leqq 0.$$

Since $z\geqq 0$ and $z$ vanishes somewhere in $\bar{\Sigma}$ it follows from the maximum principle and the Hopf lemma that $z\equiv 0$. Contradiction. ■

(*b*) We now prove the last assertion of Theorem 7.2. Let $(c,u)$ be a solution of (7.11). If we multiply the equation in (7.17) by $u_1$ and integrate over $\Sigma$ we find

$$\int_\Sigma (u_1\,\Delta u-c\,\alpha(y)\,u_1^2)+\int_0^1 f(s)\,ds\cdot \mathrm{vol}(\omega)=0.$$

Applying Green's theorem to the first term we find that its integral vanishes (since $|\nabla u|\to 0$ as $|x_1|\to\infty$). Because of our hypotheses we see that $c>0$.

As in the preceding argument, $u$ satisfies (7.12). We claim that it also satisfies (7.13). This follows from Theorem 4.4, case I′ – because $\beta(y,c)=c\,\alpha(y)>0$ and because $f<0$ in $(0,\theta)$. Then the remainder of the argument in (*a*) carries over without change. Theorem 7.2 is proved.

Next, the

*Proof of Theorem* 1.1′. – The conditions in Theorem 1.1′ allow us to apply Theorem 7.1 case 3, and Theorem 7.2 case B – the desired conclusion follows.

*Proof of Theorem* 1.2'. — The proof proceeds exactly along the lines of the proofs of Theorem 7.1 (in case $\mu_1=0$, $\sigma\equiv 0$), and Theorem 7.2. We leave the details to the reader.

We conclude this section with a comparison principle for the unique solutions $(c, u)$, $0<u<1$, of

$$\left.\begin{array}{c} \Delta u-\beta(y, c) u_1+f(u)=0 \\ u_\nu=0 \\ u \to 0 \quad \text{at } -\infty, \qquad u \to 1 \quad \text{at } +\infty \end{array}\right\} \tag{7.17}$$

in case B.

If we extend $f$ as zero outside of [0, 1] then, for any $\varepsilon$, $0 \leqq \varepsilon \leqq 1$, we may consider the problem (7.18) but with the condition $u \to 0$ at $-\infty$ replaced by $u \to -\varepsilon$ at $-\infty$. It is clear that the proofs of existence and uniqueness (Theorems 7.1, 7.2) of solutions $(c_\varepsilon, u_\varepsilon)$ carry over as in the case $\varepsilon=0$.

Our next result shows how $c_\varepsilon$ changes with $\varepsilon$ and in addition, in case $\varepsilon=0$, how $c$ changes with $f$. We only consider case B.

PROPOSITION 7.1. — (i) *Consider the unique solution* $(c, u)$ *of Theorems* 7.1 *and* 7.2. *The unique* $c$ *is increasing in its dependence on* $f$, *i.e. if* $f'$ *also satisfies the conditions of the theorems, i.e.* $f'=0$ *on* $(0, \bar\theta)$, *etc., and if* $f' \geqq f$, *then the corresponding unique* $c' \geqq c$.

(ii) *In addition, for* $f$ *fixed, the unique* $c_\varepsilon$ *is nonincreasing in* $\varepsilon$.

*Proof.* — Assertion (i) follows from Lemma 5.4 and the uniqueness of $c$. To prove (ii), suppose to the contrary that for some $\varepsilon>\tilde\varepsilon$, with corresponding unique solutions $(c, u)$, $(\tilde c, \tilde u)$, $c>\tilde c$. Then

$$\beta := \beta(y, c) > \beta(y, \tilde c) =: \tilde\beta.$$

By Theorem 4.5, $u$ and $\tilde u$ satisfy at $+\infty$, for $\gamma$, $\tilde\gamma>0$,

$$\begin{aligned} 1-u &= \gamma(e^{-\tau x_1}\varphi')+o(e^{-\tau x_1}) \\ 1-\tilde u &= \tilde\gamma(e^{-\tilde\tau x_1}\tilde\varphi')+o(e^{-\tilde\tau x_1}). \end{aligned}$$

By Theorem 2.1 (*e*), $-\tilde\tau<-\tau$. Consequently $\tilde u>u$ for $x_1$ large and this remains true for any $x_1$-translation of $\tilde u$. Clearly $\tilde u>u$ near $-\infty$. As usual, translate $\tilde u$ to the left so that it is larger than $u$ everywhere. Then shift it back to the right until it is $\geqq u$ and equality holds somewhere. Since

$$\begin{aligned} \Delta\tilde u-\beta\tilde u_1+f(\tilde u) &< 0 \\ \Delta u-\beta u_1+f(u) &= 0 \end{aligned}$$

in $\Sigma$, we see that $\tilde u-u$ satisfies a suitable elliptic inequality and by the maximum principle and the Hopf lemma, we would have $\tilde u-u\equiv 0$. Impossible. ■

## 8. CASE A: THE POSITIVE SOURCE TERM. EXISTENCE OF $c^*$

In this and the next two sections we consider case A:

$$f>0 \quad \text{on } (0,1). \tag{8.1}$$

Here, as usual, $f(0)=f(1)=0$, $f$ is Lipschitz continuous $f'(0)\geqq 0$ and $f'(1)<0$ exist, and $f$ is assumed to satisfy (1.5) and (1.19): for some $\delta$, $s_0>0$ and some $M\geqq 0$,

$$\left.\begin{aligned} &|f(s)-f'(0)s|\leqq M s^{1+\delta}, \\ &|f(1-s)+f'(1)s|\leqq M s^{1+\delta} \quad \text{for } 0\leqq s\leqq s_0 \\ &|f(1-s)-f(1-s')+f'(1)(s-s')| \\ &\leqq M(s+s')^{\delta}|s-s'| \quad \text{for } 0\leqq s, s'\leqq s_0. \end{aligned}\right\} \tag{8.2}$$

In this and the next section we prove Theorem 1.4 in the general case A. In section 9 we consider further the case $f'(0)>0$, and in Section 10 we consider the KPP case.

In this section, we prove the existence of one value of $c$, $c^*$, for which there is a solution $u$, $0<u<1$, of

$$\left.\begin{aligned} &\Delta u-\beta(y,c^*)u_1+f(u)=0 \quad \text{in } \Sigma \\ &u_\nu=0 \quad \text{on } \partial\Sigma \\ &\lim_{x_1\to-\infty} u(x_1,y)=0, \qquad \lim_{x_1\to+\infty} u(x_1,y)=1 \quad \text{uniformly in } y. \end{aligned}\right\} \tag{8.3}$$

This will be achieved by truncating $f(s)$ near $s=0$, using case B, namely Theorems 1.1 and 1.2, and then a limiting procedure.

For $0<\theta\leqq\frac{1}{2}$, let $\chi_\theta$ be a cutoff function satisfying $\chi_\theta\in C_0^\infty(\mathbb{R})$, $0\leqq\chi_\theta\leqq 1$, $\chi_\theta(s)=0$, $\forall s\leqq\theta$, and $\chi_\theta(s)=1$, $\forall s\geqq 2\theta$. Furthermore we require: $\chi_\theta\leqq\chi_{\theta'}$ if $0<\theta'<\theta<\frac{1}{2}$. Set

$$f_\theta=f\chi_\theta$$

*i.e.* we cut off the source term $f(s)$ near $s=0$.

We may now apply Theorems 1.1 and 1.2, according to which there is a unique solution $(c_\theta, u_\theta)$ of

$$\left.\begin{aligned} &\Delta u-\beta(y,c)\partial_1 u+f_\theta(u)=0 \quad \text{in } \Sigma \\ &\partial_\nu u=0 \quad \text{on } \partial\Sigma \\ &\lim_{x_1\to-\infty} u(x_1,y)=0, \qquad \lim_{x_1\to+\infty} u(x_1,y)=1, \quad \text{uniformly in } y. \end{aligned}\right\} \tag{8.3$_\theta$}$$

$u_\theta$ is unique modulo translation. After translation, we may normalize $u_\theta$ so that

$$\max_{y \in \bar{\omega}} u_\theta(0, y) = \frac{1}{2}. \tag{8.4}$$

By construction,

$$f_\theta \leqq f_{\theta'} \qquad \text{if} \quad 0 < \theta' \leqq \theta \leqq \frac{1}{2}.$$

It follows from the comparison principle, Proposition 7.1, that

$$c_{1/2} \leqq c_\theta \leqq c_{\theta'} \qquad \text{if} \quad 0 < \theta' < \theta \leqq \frac{1}{2}. \tag{8.5}$$

Now we come to the main point, to obtain an upper bound for $c_\theta$ independent of $\theta$. This will be done again with the aid of a comparison function, and Lemma 5.1. Thus our derivation of the upper bound is not quite simple.

Let $(k_\theta, v_\theta)$ be the solution of the corresponding one dimensional problem: $0 < v < 1$

$$\left.\begin{aligned} &\ddot{v}_\theta - k_\theta \dot{v}_\theta + f_\theta(v_\theta) = 0 \quad \text{on } \mathrm{R} \\ &v_\theta(-\infty) = 0, \qquad v_\theta(\infty) = 1. \end{aligned}\right\} \tag{8.6}$$

Recall that our solution $(c_\theta, u_\theta)$ was obtained in the limit from solutions of the problem (5.2-4), (5.5)′ in the finite cylinder. It follows from the comparison principle that

$$\min_{y \in \bar{\omega}} \beta(y, c_\theta) \leqq k_\theta.$$

(Indeed one cannot have $\beta(y, c_\theta) > k_\theta$ everywhere; see the uniqueness argument above.) Using Proposition 7.1 (i) we see that to obtain an upper bound for $c_\theta$ it suffices to obtain one for $k_\theta$ independent of $\theta$.

To this end we are going to make use of a function $w$ similar to those in the examples near the beginning of Section 3. First we recall that by Theorems 4.4 and 4.6 – now just for the ordinary differential equation (8.6) –

$$\left.\begin{aligned} v(x) &= \alpha e^{\lambda x} \quad \text{as } x \to -\infty, \\ 1 - v(x) &= \gamma e^{-\tau x} + o(e^{-\tau x}) \quad \text{as } x \to \infty, \end{aligned}\right\} \tag{8.7}$$

for some positive constants $\alpha$, $\lambda$, $\gamma$, $\tau$, where $\lambda = k_\theta > 0$ and $\tau > 0$ is a positive root of $\tau^2 + \tau k_\theta + f'(1) = 0$.

Now we construct the comparison function $w$. Let $w$ be a smooth strictly increasing function of $x$, $0 < w < 1$ with $\dot{w} > 0$ and $w(0) = \frac{1}{2}$. Furthermore $w$ is to equal to $e^{\lambda' x}$ for $x \leqq -\mathrm{N}$, N large, and equals $1 - e^{-\tau' x}$ for $x \geqq \mathrm{N}$.

Here $\lambda'$, $\tau'$ are fixed, with $0<\lambda'<\lambda$ and $\tau'>\tau$. Choose $\kappa$ so large that $\ddot{w}-\kappa\dot{w}<0$ for all $x$. Since $\dot{w}>0$,

$$\ddot{w}-\kappa\dot{w}+g(w)=0 \tag{8.8}$$

for a suitable smooth function $g$ on $[0, 1]$, with $g>0$ on $(0, 1)$. We are going to choose $\kappa$ so large that $g \geqq f$ on $[0, 1]$. Observe that for $x \leqq -N$, *i.e.* for $w \leqq e^{-\lambda' N}$,

$$g(w)=\lambda'(\kappa-\lambda')w,$$

while for $w \geqq 1-e^{-\tau' N}$,

$$g(w)=\tau'(\kappa+\tau')(1-w).$$

By taking $\kappa$ large we may therefore achieve that

$$g(w) \geqq f(w) \quad \text{for} \quad 0 \leqq w \leqq e^{-\lambda' N}, \quad \text{and} \quad w \geqq 1-e^{-\tau' N}.$$

For the remaining values of $w$, we have $-N \leqq x \leqq N$. There, $\dot{w}$ is bounded away from zero, and we may therefore increase $\kappa$ further, if necessary, to achieve that $g \geqq f$ everywhere, since $g$ is defined by (8.8).

Having now chosen $\kappa$ so that $g \geqq f$, we observe by (8.7) and our choice of $w$, that

$$w \geqq v_\theta \quad \text{for } |x| \text{ large, say for } |x| \geqq a. \tag{8.9}$$

CLAIM: $\forall\,\theta$, $k_\theta \leqq \kappa$.

*Proof.* – Suppose $k_\theta>\kappa$ for some $\theta$; we will obtain a contradiction. By (8.9), for $a$ large,

$$w \geqq v_\theta \quad \text{at } x=\pm a.$$

Since $k_\theta>\kappa$ and $g \geqq f$,

$$\ddot{w}-k_\theta\dot{w}+f_\theta(w)<0, \quad \text{on } (-a, a).$$

We now apply Lemma 5.1 (just in R) and conclude that $w>v_\theta$ on $(-a, a)$. But this contradicts the normalization $w(0)=v_\theta(0)=\dfrac{1}{2}$. The claim is proved.

We now return to our solutions $(c_\theta, u_\theta)$. Since the $c_\theta$ are bounded above, there exists a limit

$$c^* := \lim_{\theta \searrow 0} \nearrow c_\theta.$$

We may then proceed as in Section 5. By local estimates, we find that for a sequence of $\theta \to 0$, the $u_\theta$ converge (uniformly on compact sets) to a solution $u$ of

$$\left.\begin{array}{c}\Delta u-\beta(y,c^*)\,u_1+f(u)=0 \quad \text{in } \Sigma\\ u_\nu=0 \quad \text{on } \partial\Sigma\\ \max\limits_{y\in\bar{\omega}} u(0,y)=\dfrac{1}{2}.\end{array}\right\} \tag{8.10}$$

Furthermore, since $\partial_1 u_\theta>0$, in the limit, $\partial_1 u\geqq 0$. Therefore, $u(x_1,y)$ has limits $\psi_0(y)$ and $\psi(y)$ as $x_1\to-\infty$ and $+\infty$ respectively; $\psi_0\leqq\psi$. As in Section 5, these are then solutions of the equation

$$\begin{array}{c}\Delta_y\psi+f(\psi)=0 \quad \text{in } \omega\\ \psi_\nu=0 \quad \text{on } \partial\omega.\end{array}$$

Integrating over $\omega$ we see that

$$\int_\omega f(\psi)=0,$$

and since $f>0$ on $(0,1)$ it follows that $\psi_0$ and $\psi$ are constants, either 0 or 1. In view of the last equation in (8.10), necessarily $\psi_0=0$ and $\psi=1$. Furthermore, as mentioned in Section 5, using local estimates, one finds that the convergence of $u$ to 0 and 1 as $x_1\to-\infty$ and $+\infty$ is uniform in $y$. As usual, we have $u_1>0$ in $\bar{\Sigma}$.

We have established the existence of a solution $(c^*,u)$ of (8.3). Furthermore, $c^*$ is the limit of $c_\theta$ as $\theta\searrow 0$. ■

## 9. CASE A: THE RANGE OF VALUES OF $c$

With $c^*$ characterized as above, we now investigate problem (1.16) – case A – for other values of $c$. In this section, we complete the proof of Theorem 1.4. That is, we will show that (1.16) has no solution for any value of $c<c^*$ whereas there exists a solution of (1.16) for any $c\geqq c^*$. Part (ii) of Theorem 1.4 is an immediate consequence of Theorem 7.1.

### 9.1. Existence of solutions for $c\geqq c^*$

We will derive the existence of a solution of (1.16) for any $c\geqq c^*$ using Theorem 5.1 and our construction of $c^*$. Here, we denote by $u^*$ the solution of problem (1.16) associated with $c^*$.

$$\left.\begin{array}{c}\Delta u^*-\beta(y,c^*)\,u_1^*+f(u^*)=0 \quad \text{in } \Sigma\\ u_\nu^*=0 \quad \text{on } \partial\Sigma\\ u^*(-\infty,.)=0, \qquad u^*(+\infty,.)=1.\end{array}\right\} \tag{9.1}$$

By our construction,

$$\partial_1 u^* > 0 \quad \text{in } \Sigma. \tag{9.2}$$

Therefore, $u^*$ is a *supersolution* for the problem for any $c > c^*$.

Any constant $h$, $0 \leqq h \leqq 1$ is a *subsolution* of (1.1) (aside from the limiting condition near $x_1 \to -\infty$):

$$\Delta h - \beta(y, c) h_1 + f(h) \geqq 0 \quad \text{in } \Sigma.$$

We consider a fixed $c > c^*$. Let $a \geqq 1$ be fixed. Theorem 5.1 with $\bar{u} = u^*$, then yields the following: for any positive constant $h < \min_{y \in \bar{\omega}} u^*(-a, y)$ there exists a unique solution $v$ in $\Sigma_a = (-a, a) \times \omega$ of the problem

$$\left.\begin{array}{c} \Delta v - \beta(y, c) v_1 + f(v) = 0 \quad \text{in } \Sigma_a \\ v_\nu = 0 \quad \text{on } [-a, a] \times \partial\omega, \qquad v(-a, y) = h, \qquad v(a, y) = u^*(a, y), \\ h \leqq v \leqq u^* \quad \text{in } \Sigma_a. \end{array}\right\} \tag{9.3}$$

Moreover, for

$$\Gamma = \{(x_1, y); x_1 = \pm a, y \in \partial\omega\}$$

and $\tilde{\Sigma}_a = \bar{\Sigma}_a \setminus \Gamma$, $v \in W^{2,p}_{\text{loc}}(\tilde{\Sigma}_a) \cap C(\bar{\Sigma}_a)$, for any $p \geqq 1$, and

$$v_1 > 0 \quad \text{in } (-a, a) \times \bar{\omega}. \tag{9.4}$$

Let us now redo this construction but with $u^*$ shifted. More precisely, for $r \in R$, we let $u^{*r}(x_1, y) = u^*(x_1 + r, y)$, and

$$h^r = \min_{y \in \bar{\omega}} u^*(-a + r, y).$$

As in the previous construction, there exists a unique function $v^r \in W^{2,p}(\tilde{\Sigma}_a) \cap C(\bar{\Sigma}_a)$ $(\forall p \geqq 1)$ with $h^r < v^r < u^{*r}$ in $\Sigma_a$, satisfying

$$\left.\begin{array}{c} \Delta v^r - \beta(y, c) v_1^r + f(v^r) = 0 \quad \text{in } \Sigma_a \\ v_\nu^r = 0 \quad \text{on } [-a, a] \times \partial\omega \\ v^r(-a, y) = h^r, \qquad v^r(a, y) = u^*(a + r, y), \quad \forall y \in \bar{\omega}. \end{array}\right\} \tag{9.5}$$

Since $h^r$ and $u^*(a + r, y)$ vary continuously with $r \in \mathbb{R}$, the uniqueness of the solution of (9.5) implies that $v^r$ depends continuously—say in the $C^0(\bar{\Sigma}_a)$ topology—on $r$. Also, since $u^*$ is increasing in $x_1$, $v^r$ is a subsolution for the problem (9.5) corresponding to any $r' > r$. We may therefore apply Lemma 5.1 and conclude that $v^{r'} > v^r$. Obviously the behaviour of $u^*$ at $x_1 \to \pm\infty$ implies that $v^r \to 1$ if $r \to +\infty$ and $v^r \to 0$ as $r \to -\infty$, uniformly on $\bar{\Sigma}_a$.

From these facts we see that there exists one (and exactly one) value of $r$ such that $\max_{y \in \bar{\omega}} v^r(0, y) = \frac{1}{2}$. We denote by $u^a$ the corresponding solution $v^r$.

To sum up, $u^a$ satisfies

$$\left.\begin{array}{c}\Delta u^a-\beta(y,c)\,u_1^a+f(u^a)=0 \quad \text{in } \Sigma_a\\ u_\nu^a=0 \quad \text{on } (-a,a)\times\partial\omega\\ 0<u^a<1 \text{ and } \partial_1 u^a>0 \quad \text{in } (-a,a)\times\bar{\omega}\\ \max_{y\in\bar{\omega}} u^a(0,y)=\frac{1}{2}.\end{array}\right\} \tag{9.6}$$

For any $p>1$ the family $u^a$ is bounded in the $W^{2,p}$ norm on any compact set of $\bar{\Sigma}$ as $a\to\infty$. Hence, there exists a sequence $a_j\to\infty$ such that $u^{a_j}\to u$ uniformly on compact sets of $\bar{\Sigma}$, and $u$ satisfies

$$\left.\begin{array}{c}\Delta u-\beta(y,c)\,u_1+f(u)=0 \quad \text{in } \Sigma\\ u_\nu=0 \quad \text{on } \partial\Sigma\\ u_1\geqq 0, \qquad \max_{y\in\bar{\omega}} u(0,y)=\frac{1}{2}.\end{array}\right\} \tag{9.7}$$

The last condition in (9.7) (and the fact that $f>0$ in (0, 1)) show that $u$ is not constant. As in the previous section we may conclude that

$$u(-\infty,.)=0, \qquad u(+\infty,.)=1.$$

We have thus proved that for any $c\geqq c^*$, the problem (1.16), (1.2), (1.3) has a solution. ■

*Remark* 9.1. – Our construction of a solution when $c>c^*$ may appear intricate. In fact, this is a rather delicate point. For instance, one may think of directly trying the approximation method used in Section 5. Namely

$$\left.\begin{array}{c}\Delta w-\beta(y,c)\,w_1+f(w)=0 \quad \text{in } \Sigma_a\\ w_\nu=0 \quad \text{on } [-a,a]\times\partial\omega\\ w(-a,y)=0, \qquad w(a,y)=1, \quad \forall\, y\in\bar{\omega}\end{array}\right\} \tag{9.8}$$

However, in general, this does not produce a solution of (1.16) when $a\to\infty$ in the case $c>c^*$. Indeed, in the one dimensional case, where $\beta=c$, it is shown in [BL 2, chapter 2] that $w\to 0$ uniformly on compact sets when $a\to\infty$.

### 9.2. Nonexistence of solutions for $c<c^*$

To complete the proof of Theorem 1.4, it remains to be shown that, in our case, for any $c<c^*$, problem (1.16) has no solution.

In the proof, we will use some results about an auxiliary problem corresponding to a modification of the limiting condition as $x_1\to-\infty$. This problem was mentioned near the end of Section 7.

For $\varepsilon \geqq 0$, we consider the problem

$$\left.\begin{aligned} \Delta v - \beta(y, c)\, v_1 + f_\theta(v) = 0 \quad \text{in } \Sigma \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Sigma \\ v(-\infty, y) = -\varepsilon, \qquad v(+\infty, y) = 1, \quad y \in \bar{\omega}. \end{aligned}\right\} \tag{9.9$_\varepsilon$}$$

Here, $f_\theta$ is the truncation of $f$ defined in the previous section ($f_\theta \leqq f$). Recall that $f_\theta$ (as well as $f$) are always extended as zero outside (0, 1). The case $\varepsilon = 0$ corresponds to (8.3) with $f = f_\theta$.

As pointed out in Section 7, we know that there exists a unique solution $c = c_\theta^\varepsilon$ and $v = v_\theta^\varepsilon$ satisfying the normalization condition

$$\max_{y \in \bar{\omega}} v(0, y) = \theta. \tag{9.10}$$

By the comparison principle (Proposition 7.1), $c_0^\varepsilon$ is monotonic with respect to $\varepsilon$: for all $0 \leqq \varepsilon < \varepsilon'$, $c_\theta^{\varepsilon'} \leqq c_\theta^\varepsilon$.

Next we show

Lemma 9.1. — *The parameters $c_\theta^\varepsilon$ are continuous in $\varepsilon$.*

*Proof.* — Consider a monotonic sequence of values of $\varepsilon$ in $0 \leqq \varepsilon \leqq 1$ tending to some $\varepsilon^0$. We wish to prove that

$$\gamma := \lim_{\varepsilon \to \varepsilon^0} c_\theta^\varepsilon \text{ is equal to } c_\theta^{\varepsilon^0}.$$

By the usual limiting arguments—relying on local estimates—we know that a subsequence of $v_\theta^\varepsilon$ converges in $\bar{\Sigma}$ (uniformly on compact sets) to a function $v$ satisfying

$$\left.\begin{aligned} \Delta v - \beta(y, \gamma)\, v_1 + f_\theta(v) = 0 \quad \text{in } \Sigma \\ v_\nu = 0 \quad \text{on } \partial\Sigma, \end{aligned}\right\} \tag{9.11}$$

and the normalization (9.10). Furthermore $v_1 \geqq 0$ in $\bar{\Sigma}$. As in Section 8 we find that $v$ has uniform limits $v^-(y)$, $v^+(y)$ as $x_1 \to -\infty$ and $+\infty$ respectively; $v^- \geqq -\varepsilon^0$. Moreover, since $f_\theta \geqq 0$, these limits are constants, and $v^+ = \theta$ or 1. In order to show that $v^+ = 1$ it suffices to show that $v$ is not a constant. Concerning $v^-$, all we know at this point is that $-\varepsilon^0 \leqq v^- \leqq \theta$.

Therefore it suffices to show that $v^- = -\varepsilon^0$. Indeed, then, $v(-\infty, .) = -\varepsilon^0$, $v(+\infty, .) = 1$, which means that $v$ is a solution of $(8.3)_\theta$ with $c = \gamma$. By uniqueness, $\gamma = c_\theta^{\varepsilon^0}$.

To show that $v^- = -\varepsilon^0$, we use the exponential behaviour. For $\varepsilon \geqq 0$, let $\lambda^\varepsilon > 0$ be the principal eigenvalue and $\varphi_\varepsilon = \varphi_\varepsilon(y)$ the associated eigenfunction of

$$\left.\begin{aligned} -\Delta\varphi_\varepsilon &= [\lambda^2 - \lambda\beta(y, c_\theta^\varepsilon)]\,\varphi_\varepsilon \quad \text{in } \omega \\ \frac{\partial\varphi_\varepsilon}{\partial\nu} &= 0 \quad \text{on } \partial\omega, \end{aligned}\right\} \tag{9.12}$$

(*see* Section 2) with $\varphi_\varepsilon$ normalized by $\max_{\bar\omega} \varphi_\varepsilon = 1$.

By Theorem 2.1 (*e*) we see that $\lambda^\varepsilon$ is monotonic decreasing with respect to $\varepsilon \geqq 0$ – since $c_\theta^\varepsilon$ also is. Therefore $\lambda^0 \geqq \lambda^\varepsilon \geqq \lambda^1 > 0$ for all $0 \leqq \varepsilon \leqq 1$.

For a subsequence of $\varepsilon \to 0$, $\lambda^\varepsilon \to$ some $\lambda > 0$ and $\varphi_\varepsilon \to \varphi$ uniformly in $\bar\omega$; $\varphi \in W^{2,p}(\omega)$, $\forall p > 1$ and $\lambda$, $\varphi$ satisfy

$$\begin{aligned} -\Delta\varphi &= (\lambda^2 - \lambda\beta(y, \gamma))\,\varphi \quad \text{in } \omega \\ \varphi_\nu &= 0 \quad \text{on } \partial\omega. \end{aligned}$$

Also $\varphi \geqq 0$, $\max \varphi = 1$. By the maximum principle and the Hopf lemma, $\varphi \geqq \delta > 0$ for some constant $\delta$. In fact we must have, for our subsequence, $\varphi_\varepsilon \geqq \delta$ (with possibly a different $\delta$).

In view of (9.10), on $\Sigma \cap \{x_1 \leqq 0\}$,

$$\Delta v_\theta^\varepsilon - \beta(y, c_\theta^\varepsilon)\,\partial_1 v^\varepsilon = 0.$$

Using the maximum principle we find that

$$v_\theta^\varepsilon + \varepsilon \leqq \frac{(\theta + \varepsilon)}{\delta} e^{\lambda^\varepsilon x_1} \varphi_\varepsilon(y) \qquad \text{for} \quad x_1 \leqq 0.$$

Letting $\varepsilon \to \varepsilon^0$ through the subsequence we find that

$$v(x_1 y) + \varepsilon^0 \leqq \frac{\theta + \varepsilon_0}{\delta} e^{\lambda x_1} \varphi(y) \qquad \text{for} \quad x_1 < 0.$$

Hence $v^- = -\varepsilon^0$, which implies, as we observed, that $v^+ = 1$. Hence $\gamma = c_0^{\varepsilon^0}$. ■

We can now turn to the claim that there is no solution of (1.16) if $c < c^*$. We argue by contradiction and assume that for some $c < c^*$, there is a solution $u$. We do not assume *a priori* any further property of $u$. In particular, we do *not* assume that $u$ is monotonic in $x_1$.

By continuity, $c < c_\theta$ for some small enough $\theta > 0$, likewise, for $\varepsilon > 0$ small enough, $c < c_\theta^\varepsilon$. Let $\varepsilon > 0$ and $\theta > 0$ be chosen in this manner and let $(c^\varepsilon, v^\varepsilon)$ denote a solution of $(9.9)_\varepsilon$.

For $x_1 \to +\infty$, $u$ and $v^\varepsilon$ have exponential behaviour given by (4.15):

$$\left.\begin{aligned} u(x_1, y) &= 1 - \gamma e^{-\tau x_1} \varphi(y) + o(e^{-\tau x_1}) \\ v^\varepsilon(x_1, y) &= 1 - \gamma' e^{-\tau' x_1} \varphi'(y) + o(e^{-\tau' x_1}) \end{aligned}\right\} \tag{9.13}$$

where $\gamma$, $\gamma'$ are positive constants, $\tau$, $\tau' > 0$, $\varphi(y)$ and $\varphi'(y)$ are the principal elements of the corresponding eigenvalue problems.

By Theorem 2.1 (*e*), it follows that, since $c < c_\theta^\varepsilon$,

$$-\tau < -\tau' < 0. \tag{9.14}$$

Hence, near $x_1 = +\infty$, $u$ dominates $v_\theta^\varepsilon$. Since near $x_1 = -\infty$, $u \to 0$ and $v_\theta^\varepsilon \to -\varepsilon$, we may, after shifting $u$, assume that $u > v_\theta^\varepsilon$ everywhere in $\Sigma$. Now shift $u$ to the right until its graph first "touches" that of $v_\theta^\varepsilon$. In view of the exponential behaviour (9.13) and by (9.14), this occurs at a finite point. Therefore $u - v_\theta^\varepsilon \geqq 0$ in $\Sigma$ and $u - v_\theta^\varepsilon$ vanishes at some finite point in $\Sigma$. Let $z = u - v_\theta^\varepsilon$.

Since $c < c_\theta^\varepsilon$, we see that $u$ and $v = v_\theta^\varepsilon$ satisfy

$$\left.\begin{array}{l}\Delta u - \beta(y, c)\, u_1 + f_\theta(u) = 0\\ \Delta v - \beta(y, c)\, v_1 + f_\varepsilon(v) \geqq 0\end{array}\right\} \quad \text{in } \Sigma.$$

Hence, $z = u - v_\theta^\varepsilon$ satisfies some linear elliptic inequality

$$\Delta z - \beta(y, c)\, z_1 + qz \leqq 0 \quad \text{in } \Sigma$$

for some $q \in L^\infty(\Sigma)$. Since $z \geqq 0$ in $\Sigma$ and $z \not\equiv 0$ (indeed, $z(-\infty, .) = \varepsilon$), the maximum principle implies $z > 0$ in $\bar{\Sigma}$, and we have reached a contradiction.

Thus, for any $c < c^*$, there is no solution of problem (1.16). The proof of Theorem 1.4 is complete.

## 10. THE KPP CASE

The purpose of this section is to give a nearly explicit value of $c^*$ under the further assumption (1.12). The function $f$ satisfies the conditions of the previous section, in particular

$$f(s) > 0, \quad \forall s \in (0, 1); \qquad f(0) = f(1) = 0. \tag{10.1}$$

We recall that $f: [0, 1] \to \mathbb{R}$ is Lipschitz continuous. We also require $f$ to satisfy all the assumptions of the previous two sections. Namely, we assume conditions (8.2) which state in particular that $f$ has derivatives at 0 and 1 and $f'(1) < 0$; we also assume $f'(0) > 0$.

In this section, we further assume that $f$ satisfies condition (1.12):

$$f(s)/s \leqq f'(0) \qquad \forall s \in (0, 1). \tag{10.2}$$

The results of Sections 8 and 9 therefore apply.

We recall that there exists a real number $c^*$ such that (1.16), (1.2), (1.3) admit a solution $u$ if and only if $c \geqq c^*$. Moreover, we know that for each fixed $c \geqq c^*$, this solution is unique and satisfies $u_1 > 0$ in $\bar{\Sigma}$.

As before, we assume that $\beta(y, c)$ satisfies conditions (1.17), (1.18).

The characterization of $c^*$ will be in terms of an associated eigenvalue problem. We recall from Sections 2 and 4 that the exponential behaviour near $x_1 \to -\infty$ is governed by the eigenvalue problem

$$\left.\begin{aligned} -\Delta_y \varphi - f'(0)\varphi = [\lambda^2 - \lambda\beta(y,c)]\varphi \quad \text{in } \omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{10.3}$$

Since $f'(0)>0$, by Theorem 2.2 we know that there exists a unique value $\gamma$ such that (10.3) admits 0, 1 or 2 principal positive eigenvalues according to whether $c<\gamma$, $c=\gamma$ or $c>\gamma$ respectively. $\gamma$ depends only on $\omega$, $f'(0)$ and the function $\beta$.

THEOREM 10.1. – *Let $\gamma$ be the above critical number. Then,*

$$c^* = \gamma.$$

*Remark* 10.1. – Consider the one dimensional case – by uniqueness this is equivalent to the case $\beta(y,c)\equiv c$. In this case solutions $u$ depend only on $x_1$, the principal eigenfunctions of (10.3) are the constants, and the principal eigenvalues are determined from the quadratic equation

$$\lambda^2 - \lambda c + f'(0) = 0.$$

Hence, in this case,

$$c^* = \gamma = \sqrt{2f'(0)}. \tag{10.4}$$

This, of course, is the formula of [KPP] for the one dimensional problem. Theorem 10.1 is a higher dimensional version of this formula.

*Remark* 10.2. – In general, when (10.2) is not satisfied, $c^*$ and $\gamma$ do not necessarily coincide. In the proof below we will see that $c^* \geqq \gamma$ always holds. But it may happen that $c^*>\gamma$ if (10.2) is not satisfied. In fact, here is a simple one dimensional example. Suppose $\beta(y,c)=c$. Then $u=u(x_1)$ satisfies

$$\ddot{u} - c\dot{u} + f(u) = 0.$$

Let us derive a lower bound on $c$. It is easily seen that $\dot{u}$ converges to 0 as $x_1 \to \pm\infty$. Multiplying the equation successively by $\dot{u}$ and $(1-u)$ yields

$$c\int_{-\infty}^{+\infty} \dot{u}^2 = \int_0^1 f(s)\,ds$$

$$\int_{-\infty}^{+\infty} \dot{u}^2 - \frac{c}{2} \leqq 0.$$

Therefore,

$$c^2 \geqq 2\int_0^1 f(s)\,ds.$$

Now, $\gamma$ only depends on $f'(0)$. Thus for some fixed values of $f'(0)$ one may choose a function $f$ such that $\int_0^1 f(s)\,ds$ is arbitrarily large, thus making $c$ arbitrarily large, while keeping $\gamma$ bounded. Hence, we obtain

$$c^* > \gamma. \quad \blacksquare$$

*Proof of Theorem* 10.1. — First, assume that there exists a solution $u$ for some value $c$. Then, since $f'(0)>0$, $u$ has exponential behaviour near $x_1 \to -\infty$. (*See* Section 4.) In particular, since $u>0$ in $\Sigma$, it follows from Theorem 4.4 that there must exist at least one eigenfunction of (10.3) which has a constant sign. This means that there exists a principal eigenvalue of (10.3). Therefore, $c \geqq \gamma$ and we infer that $c^* \geqq \gamma$. Note that this is true in general as soon as $f'(0)>0$. We do not require (10.2) for this inequality.)

Assuming (10.2) let us now prove that $c^* = \gamma$. We argue by contradiction and assume that $c^* > \gamma$. Choose some $c$ satisfying

$$\gamma < c < c^*.$$

Consider, for $\theta > 0$ small, a truncation of $f$, $f_\theta$ satisfying all the conditions in Section 8. In particular, $f_\theta \leqq f$ on $[0, 1]$, for all $\theta$.

Since $c > \gamma$, we know that there exist two principal positive eigenvalues $\lambda$ of the problem

$$\left.\begin{aligned} -\Delta_y \varphi - f'(0)\varphi = (\lambda^2 - \lambda\beta(y, c))\varphi \quad \text{in } \omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{10.5}$$

Let $\lambda$ denote one of these two eigenvalues (it does not matter which) and let $\varphi$ be an associated eigenfunction; $\varphi > 0$ in $\bar{\omega}$. The function $z = e^{\lambda x_1}\varphi(y)$ defined on $\Sigma$ satisfies

$$\left.\begin{aligned} -\Delta z + \beta(y, c)\frac{\partial z}{\partial x_1} - f'(0) z = 0 \quad \text{on } \Sigma \\ \frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial\Sigma. \end{aligned}\right\} \tag{10.6}$$

Let $(c_\theta, u_\theta)$ be the unique solution of the problem with $f_\theta$:

$$\left.\begin{aligned} -\Delta u_\theta+\beta(y, c_\theta)\frac{\partial u_\theta}{\partial x_1} &= f_\theta(u_\theta) \quad \text{in } \Sigma \\ \frac{\partial u_\theta}{\partial \nu} &= 0 \quad \text{on } \partial\Sigma \\ u_\theta(-\infty, .)=0, &\qquad u_\theta(+\infty, .)=1 \\ \max_y u_\theta(\theta, y) &= \frac{1}{2}. \end{aligned}\right\} \tag{10.7}$$

Since $\lim_{\theta \searrow 0} c_\theta = c^*$ — as we have shown in Section 9 — we may choose $\theta > 0$ so that

$$\gamma < c < c_\theta \leqq c^*.$$

Because of condition (10.2) we see that $z$ is a *global supersolution* corresponding to the value $c$ and the nonlinearity $f$. That is, it satisfies, in *the whole cylinder*

$$\left.\begin{aligned} -\Delta z+\beta(y, c)\frac{\partial z}{\partial x_1} - f(z) &\geqq 0 \quad \text{in } \Sigma \\ \frac{\partial z}{\partial \nu} &= 0 \quad \text{on } \partial\Sigma \\ z(-\infty, .)=0, &\qquad z(+\infty, .)=+\infty. \end{aligned}\right\} \tag{10.8}$$

On the other hand, since $c_\theta > c$, $f \geqq f_\theta$ and $\dfrac{\partial u_\theta}{\partial x_1} > 0$ in $\Sigma$, we see that $u_\theta$ is a subsolution:

$$-\Delta u_\theta+\beta(y, c)\frac{\partial u_\theta}{\partial x_1} - f(u_\theta) \leqq 0 \quad \text{in } \Sigma.$$

Moreover, as $x_1 \to -\infty$, the behaviour of $u_\theta$ is governed by

$$u_\theta(x_1, y) = \alpha e^{\mu x_1} \psi(y) + o(e^{\mu x_1})$$

for some positive constant $\alpha > 0$, where $\mu > 0$ and $\psi > 0$ in $\bar{\omega}$ are determined from

$$\left.\begin{aligned} -\Delta_y \psi &= (\mu^2 - \mu\beta(y, c_\theta))\psi \quad \text{in } \omega \\ \frac{\partial \psi}{\partial \nu} &= 0 \quad \text{on } \partial\omega. \end{aligned}\right\} \tag{10.9}$$

It follows from Theorem 2.1 (*e*), that $\mu > \lambda$. This shows that $u_\theta$ is dominated by $z$ as $x_1 \to -\infty$. This is obviously also true as $x_1 \to +\infty$.

After, possibly, a finite translation of $u$, we may then assume that $u_\theta < z(x_1, y) = e^{\lambda x_1} \varphi(y)$ for all $(x_1, y) \in \Sigma$.

Fix $a$ so large that $z(a, y) \geqq 1$, $\forall y \in \bar{\omega}$. Theorem 5.1 yields the existence of a solution $u^a$ in the finite cylinder $\Sigma_a$ of

$$\left.\begin{aligned} &-\Delta u^a + \beta(y, c)\frac{\partial u^a}{\partial x_1} = f(u^a) \quad \text{in } \Sigma_a \\ &\frac{\partial u^a}{\partial \nu} = 0 \quad \text{on } (-a, a) \times \partial\omega \\ &u^a(-a, y) = u_\theta(-a, y), \qquad u^a(a, y) = 1 \\ &u_\theta(x_1, y) \leqq u^a(x_1, y) \leqq 1 \qquad \text{and} \qquad \partial_1 u^a \geqq 0 \quad \text{in } \Sigma_a. \end{aligned}\right\} \tag{10.10}$$

We may then apply Lemma 5.1 and conclude that, in addition,

$$u^a \leqq z \quad \text{in } \Sigma_a. \tag{10.11}$$

It is now straightforward to pass to the limit as $a \to \infty$. There is a sequence $a = a_j \to \infty$ such that the corresponding solution $u_a$ converges locally to $u$ satisfying

$$\left.\begin{aligned} &-\Delta u + \beta(y, c)\frac{\partial u}{\partial x_1} = f(u) \quad \text{in } \Sigma \\ &\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Sigma. \end{aligned}\right\} \tag{10.13}$$

In addition $u$ satisfies $u_1 \geqq 0$ and

$$u_\theta \leqq u \leqq \min\{z, 1\}.$$

Hence $u$ satisfies

$$\lim_{x_1 \to -\infty} u(x_1, y) = 0.$$

Since $u_\theta \leqq u \leqq 1$ we also see that

$$\lim_{x_1 \to +\infty} u(x_1, y) = 1.$$

We have constructed a solution $u$ of problem (1.1) corresponding to $c$. Since $c < c^*$ this is impossible — in view of Theorem 9.1.

We have proved that $c^* = \gamma$. ■

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