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Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent

by

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ABSTRACT. — We study the asymptotic behavior of positive solutions of semilinear equations with nearly critical nonlinearity. The solutions are shown to blow up at exactly one point. The exact rate and location of blowing up are studied.

RÉSUMÉ. — On étudie le comportement asymptotique de solutions positives d'équations elliptiques semi-linéaires présentant une non-linéarité presque critique. Les solutions explosent en un seul point, et on étudie la localisation de ce point et le rythme d'explosion.

1. INTRODUCTION

Let Ω be a (smooth) bounded domain in R^N with $N \geqq 3$. Consider the problem

$$\begin{aligned} -\Delta u &= N(N-2)\, u^{p-\varepsilon} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

where $p=(N+2)/(N-2)$ and $\varepsilon \geqq 0$. It is well known that when $\varepsilon > 0$, problem (1) has a solution u_ε. On the other hand, when $\varepsilon = 0$, problem (1)

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becomes delicate. In [P], Pohozaev discovered that (1) does not have a solution if Ω is starshaped. Recently Bahri and Coron [BC] showed that (1) has a solution when Ω has non-trivial topology in the sense that $H_k(\Omega; Z_2) \neq 0$ for some positive integer k, where $H_k(\Omega; Z_2)$ is the kth homology group of Ω in Z_2 coefficients. While Ding [D] showed later that even if Ω is contractible, (1) can still have a solution when the geometry of Ω is non-trivial in a certain sense. $p=(N+2)/(N-2)$ is often called the critical exponent for (1).

It is interesting to study the asymptotic behavior of the subcritical solutions u_ε of (1) as $\varepsilon \to 0$. In [AP], Atkinson and Peletier made the first study when Ω is the unit ball in R^3. They showed, using ODE argument, that

$$\lim_{\varepsilon \to 0} \varepsilon u_\varepsilon^2(0) = \frac{32}{\pi}$$

and at any $x \in \Omega \setminus \{0\}$:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/2} u_\varepsilon(x) = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left(\frac{1}{|x|} - 1\right).$$

In [BP], Brezis and Peletier returned to this problem. They used PDE methods to give another proof of the above result still for the spherical domains, along with some other interesting results. They conjectured that similar behavior occurs also for non-spherical domains but left the problem open. We solve this problem for non-spherical domains:

THEOREM 1. — *Let u_ε be a solution of problem* (1), *assume*

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{L^{p+1-\varepsilon}(\Omega)}} = S_N + o(1) \quad as\ \varepsilon \to 0, \tag{2}$$

where S_N is the best Sobolev constant in R^N:

$$S_N = \pi N(N-2)\left[\frac{\Gamma(N/2)}{\Gamma(N)}\right]$$

Then we have (*after passing to a subsequence*)*:*

(i) *there exists $x_0 \in \Omega$ such that as $\varepsilon \to 0$,*

$$u_\varepsilon \to 0 \quad in\ C^1(\Omega \setminus \{x_0\})$$

and

$$|\nabla u_\varepsilon|^2 \to N(N-2)\left[\frac{S_N}{N(N-2)}\right]^{N/2} \delta_{x_0}$$

in the sense of distributions;

(ii) *the* x_0 *above is a critical point of* φ, i. e.,

$$\varphi'(x_0)=0$$

where $\varphi(x)=g(x, x)$, $x\in\Omega$, *and* $g(x, y)$ *is the regular part of the Green's function* $G(x, y)$, i. e.,

$$g(x, y)=G(x, y)-\frac{1}{(N-2)\sigma_N |w-y|^{N-2}}$$

where σ_N *is the area of the unit sphere in* R^N.

(iii)

$$\lim_{\varepsilon\to 0} \varepsilon \|u_\varepsilon\|^2_{L^\infty(\Omega)}=2\sigma_N^2\left[\frac{N(N-2)}{S_N}\right]^{N/2}|g|$$

where $g=\varphi(x_0)$ *with* x_0 *the same as in* (i);

(iv) *for any* $x\in\Omega\setminus\{x_0\}$, *we have:*

$$\frac{u_\varepsilon(x)}{\sqrt{\varepsilon}}\to\left[\frac{N(N-2)}{S_N}\right]^{N/4}\frac{(N-2)G(x, x_0)}{\sqrt{2|g|}}$$

with the same $|g|$ *as in* (iii).

Our proof, along the lines in [BP], exploits Pohozaev identity and finds a good approximation for u_ε. It is easy to see from our hypotheses that $\|u_\varepsilon\|_{L^\infty(\Omega)}\to+\infty$ as $\varepsilon\to 0$. The usual blowing up technique gives us a rough idea how the solutions blow up, but we need finer control over the blowing up. We show that $\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon\to(N-2)\sigma_N G(x, x_0)$ for some $x_0\in\Omega$ in appropriate norms. The proof of this fact uses the blowing up technique and a certain crucial estimate, Lemma 3 below, which is not in [BP].

We also prove another related conjecture of Brezis and Peletier:

THEOREM 2. — *Let* $\Omega\subset R^N$, $N\geqq 4$, *be a bounded domain with smooth boundary. Let* u_ε *be a solution of*

$$\begin{aligned} -\Delta u&=N(N-2)u^p+\varepsilon u \quad in\ \Omega,\\ u&>0 \quad in\ \Omega,\\ u&=0 \quad on\ \partial\Omega. \end{aligned} \tag{3}$$

where $p=(N+2)/(N-2)$. *Assume* u_ε *is a minimizing sequence for the Sobolev inequality. Then* (i), (ii) *of Theorem* 1 *hold,* (iii) *and* (iv) *are modified as:*

(iii)

$$\lim_{\varepsilon\to 0} \varepsilon \|u_\varepsilon\|^{2(N-4)/(N-2)}_{L^\infty(\Omega)}=\frac{(N-2)^3\sigma N}{2a_N}|g| \qquad if \quad N>4$$

$$\lim_{\varepsilon\to 0} \varepsilon \log\|u_\varepsilon\|_{L^\infty}(\Omega)=4\sigma_4 \qquad if \quad N=4.$$

where $g=\varphi(x_0)$ *and*

$$a_N=\int_0^\infty \frac{r^{N-1}\,dr}{(1+r^2)^{N-2}}.$$

(iv) *for any* $x\in\Omega\setminus\{x_0\}$, *we have :*

$$\|u_\varepsilon\|_{L^\infty(\Omega)}\,u_\varepsilon(x)\to(N-2)\,\sigma_N\,G(x,x_0)\quad as\ \varepsilon\to 0.$$

Remark 1. — *It is easy to see from the maximum principle that* $\varphi(x)<0$, $\forall x\in\Omega$ *and* $\varphi(x)\to-\infty$ *as* $x\to\partial\Omega$, *so* φ *at least has a maximum point. Generally the number* n *of critical points of* φ *depends on the geometry of* Ω, *but we have:*

$$n\geqq\begin{cases}\sum\beta_i, & \text{if } \varphi \text{ is a Morse function, with } \beta_i \text{ the ith Betti number of } H_*(\Omega;Z);\\ \operatorname{cat}(\Omega), & \text{in general, where } \operatorname{cat}(\Omega) \text{ is the category number of } \Omega.\end{cases}$$

Remark 2. — *When* Ω *is strictly starshaped, we have an easy proof of* (ii).

We learned that O. Rey independently proved results similar to those of this paper. He uses different methods [R3].

2. PROOF OF THEOREM 1 AND THEOREM 2

Since the proofs of the two theorems are very similar, we will give a detailed proof of Theorem 1, and indicate the necessary changes when proving Theorem 2. From now on we will concentrate on Theorem 1.

Let u_ε be a solution to (1). Multiplying (1) by u_ε and integrating by parts, we obtain:

$$\int_\Omega|\nabla u_\varepsilon|^2=N(N-2)\int_\Omega u_\varepsilon^{p+1-\varepsilon}.$$

Together with the assumption (2), we arrive at:

$$[S_N+o(1)]\,\|u_\varepsilon\|_{p+1-\varepsilon}^2=N(N-2)\,\|u_\varepsilon\|_{p+1-\varepsilon}^{p+1-\varepsilon},$$

hence

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon^{p+1-\varepsilon}=\left[\frac{S_N}{N(N-2)}\right]^{N/2}. \tag{4}$$

Next we state the Pohozaev identity for u_ε.

LEMMA 1. — *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, and u be a classical solution of*

$$\begin{cases} -\Delta u = f(x, u) & in\ \Omega, \\ u=0 & on\ \partial\Omega. \end{cases}$$

Then the following identily holds:

$$\int_\Omega \left[NF(x, u) - \frac{N-2}{2} uf(x, u) + (x-y).F_x(x, u) \right] dx$$
$$= \int_{\partial\Omega} \left[(x-y, \nabla u) \frac{\partial u}{\partial n} - (x-y, n) \frac{|\nabla u|^2}{2} \right.$$
$$\left. + (x-y, n) F(x, u) + \frac{N-2}{2} u \frac{\partial u}{\partial n} \right] dS_x \quad for\ any\ y \in \mathbb{R}^N \quad (5)$$

where $F(x, u) = \int_0^u f(x, t)\, dt$, F_x *is the gradient of* F *with respect to* x, dS_x *is the volume element of* $\partial\Omega$, *and* n *is the unit outward normal of* $\partial\Omega$.

The proof is by now standard. Applying Lemma 1 to (1), we have:

$$\frac{N(N-2)^3}{2N-\varepsilon(N-2)} . \varepsilon . \int_\Omega u_\varepsilon^{p+1-\varepsilon} = \int_{\partial\Omega} (x-y, n) \left(\frac{\partial u_\varepsilon}{\partial n} \right)^2 dS_x \quad (6)$$

for any $y \in \mathbb{R}^N$.

Next we will study the blowing up behavior of u_ε. It is easy to see that $\|u_\varepsilon\|_{L^\infty(\Omega)} \to +\infty$ as $\varepsilon \to 0$. For suppose, on the contrary, that $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)}$ remains bounded for a sequence $\varepsilon_n \to 0$ as $n \to \infty$. Then, in view of the elliptic regularity theory, u_{ε_n} remains bounded in $C^1(\bar{\Omega})$. So we can extract a subsequence, still denoted as u_{ε_n}, which converges uniformly to a limit v. By (4), $v \neq 0$, hence by taking limit in (2) we find that v achieves the best Sobolev constant, a contradiction to the well known fact that the best Sobolev constant is never achieved on a bounded domain.

Let $x_\varepsilon \in \Omega$, $\mu_\varepsilon \in \mathbb{R}^+$ such that

$$u_\varepsilon(x_\varepsilon) = \mu_\varepsilon^{-((N-2)/2)} = \|u_\varepsilon\|_{L^\infty(\Omega)}.$$

We first claim that x_ε will stay away from the boundary $\partial\Omega$ of Ω. This is a consequence of the moving planes method as in [GNN] and an interior integral estimate of the solutions [DLN]. Let $\varphi_1 > 0$ be the principal eigenfunction of $-\Delta$. Multiply (1) by φ_1 and integrate by parts:

$$\lambda_1 \int_\Omega u_\varepsilon \varphi_1 = N(N-2) \int_\Omega u_\varepsilon^{p-\varepsilon} \varphi_1$$

where λ_1 is the principal eigenvalue of $-\Delta$. Choose $\delta>0$ such that $N(N-2)\,\delta^{p-1-\varepsilon}\geqq 2\lambda_1$, we have

$$\begin{aligned}\lambda_1\int_\Omega u_\varepsilon\varphi_1 &= N(N-2)\int_{(u_\varepsilon\geqq\delta)} u_\varepsilon^{p-\varepsilon}\varphi_1+N(N-2)\int_{(u_\varepsilon\leqq\delta)} u_\varepsilon^{p-\varepsilon}\varphi_1\\ &\geqq N(N-2)\,\delta^{p-1-\varepsilon}\int_{(u_\varepsilon\geqq\delta} u_\varepsilon\varphi_1 - C'\\ &\geqq 2\lambda_1\int_\Omega u_\varepsilon\varphi_1 - C\end{aligned}$$

for some constants C', C. Therefore $\int_\Omega u_\varepsilon\varphi_1\leqq C/\lambda_1$, which further implies that $\int_{\Omega'} u_\varepsilon\leqq C(\Omega')$ for any $\Omega'\subset\subset\Omega$. If the domain Ω is strictly convex, applying the moving planes method in [GNN], there exist $t_0\alpha>0$ depending on the domain Ω only, ruch that $u(x-t\nu)$ *is nondecreasing for* $t\in[0, t_0]$, $\nu\in\mathbb{R}^N$ *satisfying* $|\nu|=1$ *and* $(\nu, n(x))\geqq\alpha$ *and* $x\in\partial\Omega$. Therefore we can find γ, $\delta>0$ such that *for any* $x\in\{z\in\bar\Omega: d(z, \partial\Omega)<\delta\}$ *there exists a measurable set* Γ_x *with* (i) $\operatorname{meas}(\Gamma_x)\geqq\gamma$, (ii) $\Gamma_x\subset\{z\in\Omega: d(z, \partial\Omega)>\delta/2\}$, *and* (iii) $u(y)\geqq u(x)$ *for any* $y\in\Gamma_x$. Actually, Γ_x can be taken to a piece of cone with vertex at x. Let $\Omega'=\{z\in\Omega: d(z, \partial\Omega)>\delta/2\}$, then for any

$$x\in\{z\in\bar\Omega: d(z, \partial\Omega)<\delta\}$$
$$u(x)\leqq\frac{1}{\operatorname{meas}(\Gamma_x)}\int_{\Gamma_x} u(y)\,dy\leqq\gamma^{-1}\int_{\Omega'} u,$$

hence back to our argument, since $u_\varepsilon(x_\varepsilon)\to\infty$ as $\varepsilon\to 0$, x_ε will stay out of the region $\{z\in\Omega: d(z, \partial\Omega)<\delta\}$. For a general domain, one can first use a Kelvin transform near each boundary point, and then apply the method in [GNN]. Pick any point $P\in\partial\Omega$, for instance. Since we assume the boundary of the domain Ω is smooth, we may assume, without loss of generality, that the ball B(O, 1) contacts P from the exterior of Ω. Let w_ε be the Kelvin transform of u_ε:

$$w_\varepsilon(x)=|x|^{2-N}u_\varepsilon\left(\frac{x}{|x|^2}\right)$$

then

$$\begin{cases}-\Delta w_\varepsilon(x)=N(N-2)\,|x|^{-(N-2)\varepsilon}\,w_\varepsilon^{p-\varepsilon} & \text{in } \Omega_P\\ w_\varepsilon\geqq 0\end{cases}$$

where Ω_P is the image of Ω under the Kelvin transform. The conditions in [GNN] are obviously satisfied along the PO direction, *see* Corollary 1 on p. 227 of [GNN], therefore w_ε is nondecreasing along the PO direction in a neighborhood of P. The same argument as in the last paragraph shows the interior integral estimate.

Let $x_\varepsilon \to x_0 \in \Omega$. We define a family of rescaled functions

$$v_\varepsilon(x) = \mu_\varepsilon^{(N-2)/2}\, u_\varepsilon(\mu^{1-((N-2)/4)}\, x + x_\varepsilon),$$

then

$$\begin{aligned} -\Delta v_\varepsilon(x) &= \mu^{(N-2)/2}\,\mu^{2-((N-2)/2)}\,(-\Delta u_\varepsilon)\,(\mu_\varepsilon^{1-((N-2)/4)}\, x + x_\varepsilon) \\ &= N(N-2)\,[\mu_\varepsilon^{(N-2)/2}]^{((N+2)/(N-2)-\varepsilon}\, u_\varepsilon^{((N+2)/(N-2)-\varepsilon} \\ &\quad \times (\mu_\varepsilon^{1-((N-2)/4)\,\varepsilon}\, x + x_\varepsilon) \quad [\text{by}\,(1)] \\ &= N(N-2)\, v_\varepsilon^{p-\varepsilon} \quad \text{in } \Omega_\varepsilon = \frac{\Omega - x_\varepsilon}{\mu^{1-((N-2)/4)\,\varepsilon}}. \end{aligned} \tag{7}$$

Notice that $v_\varepsilon(0)=1$, $0 \leqq v_\varepsilon \leqq 1$ for $x \in \Omega_\varepsilon$, and that Ω_ε converges to R^N. Thus elliptic theory implies $\{v_\varepsilon\}$ is equicontinuous on every compact subset of R^N, hence by Arzela-Ascoli theorem, there exists a subsequence converging to some V uniformly on every compact set, and

$$\begin{aligned} -\Delta V &= N(N-2)\, V^p, \quad && x \in R^N \\ V(O) &= 1, && \\ 0 \leqq V &\leqslant 1, && x \in R^N. \end{aligned} \tag{8}$$

The solution of (8) is unique [CGS], and

$$V(x) = U_1(x)$$

where

$$U_\mu(x) = \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{(N-2)/2}.$$

Back to the convergence argument, we remark that as $\varepsilon \to 0$, $v_\varepsilon \to V$ in $H^1(R^N)$. This follows from assumption (2) and [S].

COROLLARY 1. — *There exists* $\delta > 0$ *such that*

$$\delta \leqq \mu_\varepsilon^\varepsilon \leqq 1.$$

Proof. — Since $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$, it follows that $\mu_\varepsilon^\varepsilon \leqq 1$. By the above convergence argument, $v_\varepsilon \to V$ uniformly over B_1 we have, for some $C > 0$, that:

$$\int_{B_1} v_\varepsilon^{p+1-\varepsilon} \geqq C$$

But

$$\begin{aligned}\int_{B_1} v_\varepsilon^{p+1-\varepsilon}\,dx &= \int_{|x|\leq 1} \mu^{(N-2)/2\,((2N/(N-2))-\varepsilon)}\, u_\varepsilon^{(2N/(N-2)-\varepsilon}(\mu_\varepsilon^{1-((N-2)/4)\varepsilon} x + x_\varepsilon)\,dx \\ &= \int_{|y-x_\varepsilon|\leq \mu_\varepsilon^{1-((N-2)/4)\varepsilon}} \mu^{N-((N-2)/2)\varepsilon}{}_\varepsilon (\mu_\varepsilon^{1-((N-2)/4)\varepsilon})^{-N} u_\varepsilon^{p+1-\varepsilon}(y)\,dy \\ &\leq \mu_\varepsilon^{((N-2)/2)^2\varepsilon} \int_\Omega u_\varepsilon^{p+1-\varepsilon}\end{aligned}$$

Thus with (4), $\delta \leq \mu_\varepsilon^\varepsilon \leq 1$ as $\varepsilon \to 0$, for some $\delta > 0$.

To proceed further, we need the following lemma, adapted from [BP].

LEMMA 2. — *Let u solve*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

ω *is a neighborhood of* $\partial\Omega$. *Then*

$$\| u \|_{W^{1,q}(\Omega)} + \| \nabla u \|_{C^{0,\alpha}(\omega')} \leq C(\| f \|_{L^1(\Omega)} + \| f \|_{C^\infty(\omega)})$$

for $q < \dfrac{N}{N-1}$, $\alpha \in (0, 1)$, *and* $\omega' \subset\subset \omega$ *is a strict subdomain of* ω.

Proof. — We first claim that for any $q < N/(N-1)$,

$$\| u \|_{W^{1,q}(\Omega)} \leq C \| f \|_{L^1(\Omega)}$$

This follows easily, by duality, from the fact that if v satisfies

$$\begin{cases} -\Delta v = f_0 + \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} f_i & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where f_0, f_i, $i = 1, \ldots, N$, $\in L^p(\Omega)$, then

$$\| v \|_{L^\infty(\Omega)} \leq C \sum_{i=0}^{N} \| f_i \|_{L^p(\Omega)}$$

for any $p > N$; in other words $(-\Delta)^{-1}$ maps $W^{-1,p}(\Omega)$ into $L^\infty(\Omega)$ and by duality, it also maps $L^1(\Omega)$ into $W_0^{1,q}(\Omega)$ with $1/q + 1/p = 1$.

Next we claim that

$$\| \nabla u \|_{C^{0,\alpha}(\omega')} \leq C(\| f \|_{L^1(\Omega)} + \| f \|_{L^\infty(\omega)})$$

for any neighborhoods $\omega' \subset\subset \omega$ of $\partial\Omega$. Let χ denote the characteristic function of ω and write $f = f_1 + f_2$ with $f_1 = \chi f$ and $f_2 = (1-\chi) f$. For $i = 1, 2$, let u_i be the solutions of the problems

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases}$$

so that $u=u_1+u_2$.

By the L^p regularity theory we have

$$\|u_1\|_{W^{2,q}(\Omega)} \leqq C\|f_1\|_{L^\infty(\Omega)} = C\|f\|_{L^\infty(\omega)}$$

for any $q<\infty$, and consequently

$$\|u_1\|_{C^{1,\alpha}(\bar{\Omega})} \leqq C\|f\|_{L^\infty(\omega)}$$

for any $\alpha<1$. On the other hand, as above,

$$\|u_2\|_{W^{1,q}(\Omega)} \leqq C\|f_2\|_{L^1(\Omega)} \leqq C\|f\|_{L^1(\Omega)}$$

for any $q<N/(N-1)$. Finally we note that u_2 satisfies

$$\begin{cases} -\Delta u_2 = 0 & \text{in } \omega \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from the standard elliptic regularity theory that

$$\|u_2\|_{C^{1,\alpha}(\bar{\omega}')} \leqq C\|u_2\|_{W^{1,q}(\omega)}$$

for any neihgborhood ω' of $\partial\Omega$, strictly smaller than ω. Combine the above estimates, we prove the lemma.

We also need the following crucial estimate, the proof of which will be delayed to the next section.

LEMMA 3. – *There exists* $\Lambda>0$, *such that*

$$u_\varepsilon(x) \leqq \Lambda\, U_{\mu_\varepsilon}(x-x_\varepsilon), \qquad \forall x \in \Omega.$$

An immediate consequence of this lemma is a lower bound for μ_ε.

COROLLARY 2. – *There exists a constant* $C>0$, *such that*

$$\varepsilon \leqq C\,\mu_\varepsilon^{(N-2)(1-\varepsilon)} \tag{9}$$

Proof. – First we will use Lemmas 2 and 3 to establish the estimate:

$$\int_{\partial\Omega} (x, n)\left(\frac{\partial u}{\partial n}\right)^2 \leqq C\,\mu_\varepsilon^{(N-2)(1+\varepsilon)}$$

Then (9) follows from (6) and (4). According to Lemma 2, it is sufficient to estimate the right hand side of (1) in $L^1(\Omega)$ and $L^\infty(\omega)$. Lemma 3 implies

$$\begin{aligned}\int_\Omega u_\varepsilon^{p-\varepsilon} &\leqq \Lambda^{p-\varepsilon}\int_\Omega U_{\mu_\varepsilon}^{p-\varepsilon}(x-x_\varepsilon)\,dx \\ &= \Lambda^{p-\varepsilon}\int_\Omega\left[\frac{\mu_\varepsilon}{\mu_\varepsilon^2+|x-x_\varepsilon|^2}\right]^{(N-2)/2\,[((N+2)/(N-2)-\varepsilon]} dx \\ &\leqq C\,\mu_\varepsilon^{(N-2)/2\,(1+\varepsilon)}\end{aligned}$$

for some $C>0$. On the other hand, for $x \neq x_0$:

$$u_\varepsilon^{p-\varepsilon}(x) \leqq \Lambda^{p-\varepsilon} U_{\mu_\varepsilon}^{p-\varepsilon}(x-x_\varepsilon) \leqq \frac{C}{|x-x_0|^{N+2-(N-2)\varepsilon}} \mu_\varepsilon^{((N+2)/2)-((N-2)/2\varepsilon}.$$

These estimates together with Lemma 2 finishes the proof.

COROLLARY 3 :

$$|\mu_\varepsilon^\varepsilon - 1| = O(\mu_\varepsilon^{N-2} \log \mu_\varepsilon) \quad as\ \varepsilon \to 0$$

Proof. — By the theorem of the mean,

$$|\mu_\varepsilon^\varepsilon - 1| = |\mu_\varepsilon^{t\varepsilon} \varepsilon \log \mu_\varepsilon|$$

for some $0<t<1$. Therefore (9) gives the result.

Next we claim

PROPOSITION 1 :

$$\| u_\varepsilon \|_{L^\infty(\Omega)} u_\varepsilon(x) \to (N-2)\sigma_N G(x, x_0)$$

in $C^{1,\alpha}(\omega)$ *for any neighborhood* ω *of* $\partial\Omega$, *not containing* x_0.

Proof. — The proof again uses Lemmas 2 and 3. The equation for $\| u_\varepsilon \|_{L^\infty(\Omega)} u_\varepsilon$ is

$$-\Delta(\| u_\varepsilon \|_{L^\infty(\Omega)} u_\varepsilon) = N(N-2)\mu^{-2)/2} u_\varepsilon^{p-\varepsilon}$$

Now by Lemma 3 and Corollary 3, the $L^1(\Omega)$ norm of the right hand side is

$$\mu_2^{-(N-2)/2} \int_\Omega u_\varepsilon^{p-\varepsilon} = \mu^{-((N-2)/2)^2\varepsilon} \int_{\Omega_\varepsilon} v_\varepsilon^{(N+2)/(N-2)}(y)\, dy$$

$$\to \int_{R^N} V^{(N+2)/(N-2)}(y)\, dy$$

$$= \frac{\sigma_N}{N}$$

Also for any $x \neq x_0$:

$$\mu_\varepsilon^{-(N-2)/2} u_\varepsilon^{p-\varepsilon}(x) \leqq \Lambda^{p-\varepsilon} \mu_\varepsilon^{-(N-2)/2} U_{\mu_\varepsilon}^{p-\varepsilon}(x-x_\varepsilon) \leqq \Lambda^{p-\varepsilon} \frac{C}{|x-x_0|^{N+2-(N-2)\varepsilon}} \mu_\varepsilon^{2-((N-2)/2)\varepsilon}.$$

Therefore Lemma 2 gives the conclusion of Proposition 1.

Now we give the proof of Theorem 1.

Proof of Theorem 1. — Rewrite the Pohozaev identity (6) as

$$\frac{N(N-2)^3}{2N-\varepsilon(N-2)} \varepsilon \| u_\varepsilon \|_{L^\infty(\Omega)}^2 \int_\Omega u_\varepsilon^{p+1-\varepsilon} = \int_{\partial\Omega} (x-x_0, n) \left(\frac{\partial}{\partial n} (\| u_\varepsilon \|_{L^\infty(\Omega)} u_\varepsilon) \right)^2$$

Proposition 1 allows us to take limit:

$$\lim_{\varepsilon \to 0} \varepsilon \| u_\varepsilon \|^2_{L^\infty(\Omega)} = \frac{2\sigma_N^2}{N-2}\left[\frac{N(N-2)}{S_N}\right]^{N/2} \int_{\partial\Omega} (x-x_0, n)\left(\frac{\partial G(x, x_0)}{\partial n}\right)^2$$

where we also used (4). Then (iii) of Theorem 1 follows from the following lemma:

LEMMA 4 [BP]. — *For every* $x_0 \in \Omega$

$$\int_{\partial\Omega} (x-x_0, n)\left(\frac{\partial G(x, x_0)}{\partial n}\right)^2 dS_x = -(N-2)\, g(x_0, x_0).$$

A consequence of (6) is

$$\int \left(\frac{\partial u_\varepsilon}{\partial n}\right)^2 n(x)\, dS_x = 0. \tag{10}$$

Pass to the limit in (10), we obtain

$$\int_{\partial\Omega} \left(\frac{\partial G(x, x_0)}{\partial n}\right)^2 n(x)\, dS_x = 0.$$

Using the following lemma, we obtain (ii) of Theorem 1.

LEMMA 5 [BP]. — *For every* $x_0 \in \Omega$

$$\int_{\partial\Omega} \left(\frac{\partial G(x, x_0)}{\partial n}\right)^2 n(x)\, dS_x = -\nabla \varphi(x_0).$$

(iv) of Theorem 1 follows from Proposition 1 and (ii). (i) is more or less standard, see also Remark 3.

Proof of Lemma 4. — Without loss of generality we may assume $x_0 = 0$. Let $u = G(x, O)$, we apply Lemma 1 to u on $\Omega \setminus B(O, r)$ for r small:

$$\int_{\partial\Omega} \left[(x, \nabla u) - (x, n)\frac{|\nabla u|^2}{2} + \frac{N-2}{2} u \frac{\partial u}{\partial n}\right]$$
$$= \int_{\partial B(O, r)} \left[(x, \nabla u) - (x, n)\frac{|\nabla u|^2}{2} + \frac{N-2}{2} u \frac{\partial u}{\partial n}\right]$$

Since $u = 0$ on $\partial\Omega$,

$$\text{LHS} = \frac{1}{2}\int_{\partial\Omega} (x, n)\left(\frac{\partial u}{\partial n}\right)^2.$$

While

$$u = [x]^{2-N}/(N-2)\,\sigma_N + g(x, 0),$$

therefore

$$\nabla u = -\sigma_N^{-1} |x|^{-N} x + \nabla g(x, O).$$

Hence

$$\text{RHS}=\int_{\partial B(0,r)}\Big\{[-\sigma_N^{-1}|x|^{2-N}+x.\nabla g(x,0)]^2/r$$
$$-\frac{r}{2}[\sigma_N^{-2}|x|^{2-2N}+|\nabla g(x,O)|^2-2\sigma_N^{-1}|x|^N x.\nabla g(x,O)]$$
$$+\frac{N-2}{2}(|x|^{2-N}/(N-2)\sigma_N+g(x,0))$$
$$\times(-\sigma_N^{-1}|x|^{1-N}+n.\nabla g(x,0))\Big\}$$

Let $r\to 0$, the limit of PHS is easily seen to be $-(N-2)/2g(0,0)$, therefore we have :

$$\int_{\partial\Omega}(x,n)\left(\frac{\partial u}{\partial n}\right)^2=-(N-2)g(0,0)$$

The proof of Lemma 5 follows similarly.

The proof of Theorem 2 is almost identical to that of Theorem 1, except that the Pohozaev identity appears differently:

$$\varepsilon\int_\Omega u_\varepsilon^2=\frac{1}{2}\int_{\partial\Omega}(x-y,n)\left(\frac{\partial u}{\partial n}\right)^2,\qquad \forall y\in R^N.$$

3. Proof of Lemma 3

We have seen that

$$v_\varepsilon\to V$$

uniformly on compact sets of R^N, which, in terms of u_ε, says $u_\varepsilon\to U_{\mu_\varepsilon}$ in a certain sens. Lemma 3 makes this precise. We first need:

LEMMA 6. — *Let Ω be a domain in R^N and $u\in H_0^1(\Omega)$ be a positive smooth solution of*

$$-\Delta u=a(x)u^{q-1}\tag{11}$$

where $a\in L^\infty(\Omega)$, $2<q_0\leq q\leq p+1=\dfrac{2N}{N-2}$. *Then there exist* $\varepsilon_0>0$ *and* $r_0>0$ *depending on* N, $\|a\|_{L^\infty(\Omega)}$, q_0 and $1<\beta\leqslant p$ *such that for any* $Q\in R^N$ *with* $\int_{\Omega(Q,2r)}u^q\leq\varepsilon_0$, $r\leq r_0$, *we have*

$$\|u\|_{L^{(\beta+1)((p+1)/2)}(\Omega(Q,r))}\leq\frac{C}{r^{2/(\beta+1)}}\|u\|_{L^{\beta+1}(\Omega(Q,2r))}$$

with C *depending on* N *only, where* $\Omega(Q, r)=\Omega \cap B(Q, r)$.

Proof. — Let η be a smooth cut-off function on R^N such that

$$\eta|_{B(Q, r)}=1$$
$$\eta|_{R^N \setminus B(Q, 2r)}=0$$
$$0 \leqq \eta \leqq 1$$
$$|\nabla \eta| \leqq \frac{2}{r}$$

Multiply (11) by $\eta^2 u^\beta (\beta>1)$ and integrate by parts

$$\int_\Omega \nabla(\eta^2 u^\beta)\nabla u=\int_\Omega a(x) u^{q+\beta-1}\eta^2.$$

The integral on the left

$$\int_\Omega \nabla(\eta^2 u^\beta)\nabla u=\int_\Omega \eta^2 \beta u^{\beta-1}|\nabla u|^2+2\eta u^\beta \nabla \eta \nabla u$$
$$\geqq \beta \int_\Omega \eta^2 u^{\beta-1}|\nabla u|^2-\frac{\beta}{2}\int_\Omega \eta^2 u^{\beta-1}|\nabla u|^2-\frac{2}{\beta}\int_\Omega |\nabla \eta|^2 u^{\beta+1}$$
$$=\frac{\beta}{2}\int_\Omega \eta^2 u^{\beta-1}|\nabla u|^2-\frac{2}{\beta}\int_\Omega |\nabla \eta|^2 u^{\beta+1}.$$

By the Sobolev inequality, we have:

$$S_N\left\{\int_{\Omega(Q, 2r)} |\eta u^{(\beta+1)/2}|^{p+1}\right\}^{2/(p+1)}$$
$$\leqq \int_{\Omega(Q, 2r)} |\nabla(\eta u^{(\beta+1)/2})|^2$$
$$\leqq 2\int_{\Omega(Q, 2r)} \eta^2\left(\frac{\beta+1}{2}\right)^2 u^{\beta-1}|\nabla u|^2+2\int_{\Omega(Q, 2r)} |\nabla \eta|^2 u^{\beta+1}$$
$$\leqq \frac{(\beta+1)^2}{\beta}\,\frac{\beta}{2}\int_{\Omega(Q, 2r)} \eta^2 u^{\beta-1}|\nabla u|^2+2\int_{\Omega(Q, 2r)} |\nabla \eta|^2 u^{\beta+1}$$
$$\leqq \frac{(\beta+1)^2}{\beta}\left\{\int_{\Omega(Q, 2r)} a(x) u^{q+\beta-1}\eta^2+\frac{2}{\beta}\int_{\Omega(Q, 2r)} |\nabla \eta|^2 u^{\beta+1}\right\}$$
$$+2\int_{\Omega(Q, 2r)} |\nabla \eta|^2 u^{\beta+1}$$
$$\leqq 4\|a\|_{L^\infty(\Omega)}\beta\int_{\Omega(Q, 2r)} \eta^2 u^{q+\beta-1}+10\int_{\Omega(Q, 2r)} |\nabla \eta| u^{\beta+1}$$
$$\leqq 4\|a\|_{L^\infty(\Omega)}\beta\left\{\int_{\Omega(Q, 2r)} (\eta u^{(\beta+1)/2})^{p+1}\right\}^{2/(p+1)}$$

$$\left\{ \int_{\Omega(Q,\,2r)} u^{(q-2)(N/2)} \right\}^{2/N} + 10 \int_{\Omega(Q,\,2r)} |\nabla \eta|^2 u^{\beta+1}.$$

Since $q \leqq \frac{2N}{N-2}$, we have $(q-2)\frac{N}{2} \leqq q$, choosing ε_0 and r_0 small enough such that

$$4\|a\|_{L^\infty(\Omega)} \beta \left\{ \int_{\Omega(Q,\,2r)} u^{(q-2)(N/2)} \right\}^{2/N}$$

$$\leqq 4\|a\|_{L^\infty(\Omega)} \beta \left\{ \int_{\Omega(Q,\,2r)} u^q \right\}^{(a-2)/q} |\Omega(Q,\,2r)|^{(2/N)-((q-2)/q)}$$

$$\leqq 4\|a\|_{L^\infty(\Omega)} \beta\, \varepsilon_0^{(a-2)/q} |\Omega(Q,\,2r)|^{(2/N)-((q-2)/q)} \leqq \frac{S_N}{2}$$

then

$$\frac{S_N}{2} \left\{ \int_{\Omega(Q,\,2r)} (\eta\, u^{(\beta+1)/2})^{p+1} \right\}^{2/(p+1)} \leqq 10 \int_{\Omega(Q,\,2r)} |\nabla \eta|^2 u^{\beta+1}$$

which implies:

$$\|u\|_{L^{(\beta+1)(p+1)/2}(\Omega(Q,\,r))} \leqq \frac{C}{r^{2/(\beta+1)}} \|u\|_{L^{\beta+1}(\Omega(Q,\,2r))}.$$

LEMMA 7. — *Let $u \in H_0^1(\Omega)$ be a positive smooth solution of*

$$-\Delta u = a(x)\, u$$

with $a \in L^\alpha(\Omega)$ for some $\alpha > \frac{N}{2}$. Then for any $Q \in R^N$

$$\sup_{\Omega(Q,\,r)} u \leqq C \left[\frac{1}{r^N} \int_{\Omega(Q,\,2r)} u^{p+1} \right]^{1/(p+1)}$$

where C depends on $\|a\|_{L^\infty(\Omega)}$, α, N.

This is more or less standard elliptic regularity, see, for instance, Theorem 8.17 in [GT].

Now we give the proof of Lemma 3.

Proof. — We first remark that Lemma 3 is equivalent to

$$v_\varepsilon(x) \leqq C\,V(x) \tag{12}$$

where $V(x) = \left(\frac{1}{1+|x|^2} \right)^{(N-2)/2}$.

Let's recall the equation for v_ε

$$-\Delta v_\varepsilon = N(N-2)\, v_\varepsilon^{p-\varepsilon} \quad \text{in } \Omega_\varepsilon = \frac{\Omega - x_\varepsilon}{\mu_\varepsilon^{1-((N-2)/4)\,\varepsilon}}$$
$$0 \leqq v_\varepsilon \leqq 1 \quad \text{in } \Omega_\varepsilon$$
$$v_\varepsilon(0) = 1$$

and recall that $v_\varepsilon \to V$ in H^1 and uniformly on compact sets. Let w_ε be the Kelvin transform of v_ε:

$$w_\varepsilon(x) = |x|^{2-N}\, v_\varepsilon\left(\frac{x}{|x|^2}\right)$$

then

$$\left.\begin{aligned} -\Delta w_\varepsilon(x) = N(N-2)\,|x|^{-(N-2)\,\varepsilon}\, w_\varepsilon^{p-\varepsilon} \quad \text{in } \Omega_\varepsilon^* \\ w_\varepsilon \geqq 0 \end{aligned}\right\} \tag{13}$$

where Ω_ε^* is the image of Ω_ε under Kelvin transform. Then it is easy to see that (12) is equivalent to

$$w_\varepsilon(x) \leqq B, \qquad x \in \Omega_\varepsilon^* \tag{14}$$

for some B. Notice that Ω_ε^* is the whole R^N except a small region near O and by definition of w_ε and $0 \leqq v_\varepsilon \leqq 1$, we have

$$w_\varepsilon(x) \leqq |x|^{2-N}, \qquad x \in \Omega_\varepsilon^* \tag{15}$$

so we only need to bound w_ε near O.

We apply Lemmas 6 and 7. First let $a(x) = N(N-2)\,|x|^{-(N-2)\,\varepsilon}$. By Corollary 1, we see $\|a\|_{L^\infty(\Omega_\varepsilon^*)}$ is bounded independent of ε. That $v_\varepsilon \to V$ in H^1 norm implies $v_\varepsilon \to V$ in L^{p+1} and also $w_\varepsilon \to V$ in L^{p+1}. So for the ε_0, r_0 given in Lemma 6, we can find sufficiently small $r > 0$ such that

$$\int_{B(O,\,2r)} w_\varepsilon^{p+1} \leqq \varepsilon_0 \tag{16}$$

Fix $r > 0$, then Lemma 6 with $\beta = p$ gives

$$\int_{B(O,\,r)} w_\varepsilon^{(p+1)^2/2} \leqq C, \quad \text{for some } C > 0.$$

Then we write (13) as

$$-\Delta w_\varepsilon = b(x)\, w_\varepsilon \tag{17}$$

with $b(x) = N(N-2)\,|x|^{-(N-2)\,\varepsilon}\, w_\varepsilon^{p-1-\varepsilon}$. Applying Lemma 7 with $\alpha = \dfrac{(p+1)^2}{2}\,\dfrac{1}{p-1-\varepsilon} > \dfrac{N}{N-2}\,\dfrac{N}{2} > \dfrac{N}{2}$ gives

$$\|w_\varepsilon|_{L^\infty(B(O,\,r/2))} \leqq C_1, \quad \text{for some } C_1 < 0. \tag{18}$$

(18) with (15) gives (14), which then gives (12), finishing the proof.

Remark 3. – *We can prove* (i) *of Theorem* 1 *by following Sacks-Uhlenbeck* [SU] *and using Lemmas* 6 *and* 7.

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