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Prolongation of classical solutions and singularities of generalized solutions

by

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ABSTRACT. – We consider the Cauchy problem for general partial differential equations of first order. It is well known that it admits locally a smooth solution. When we extend a solution of class C^1, what kinds of phenomena may appear? The aim of this paper is to see what may happen in this extension. Our method depends principally on the analysis of characteristic curves.

Key words : Classical solutions, life span, generalized solutions, singularities.

RÉSUMÉ. – Nous considérons le problème de Cauchy pour les équations aux dérivées partielles du premier ordre. Nous savons bien qu'il admet localement une solution régulière. Qu'est ce qui se passe quand on prolonge la solution de classe C^1 ? Le but de cet article est d'étudier les phénomènes qui apparaissent dans cette extension. Notre méthode dépend principalement de l'analyse des courbes caractéristiques.

Mots clés : Solutions classiques, le temps de vie, solutions généralisées, singularités.

Classification A.M.S. : 35 F 25, 35 L 67, 35 B 60.

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1. INTRODUCTION

We consider the Cauchy problem for general partial differential equations of first order as follows:

$$\frac{\partial u}{\partial t}+f\left(t, x, u, \frac{\partial u}{\partial x}\right)=0 \quad \text{in } \{t>0,\ x\in \mathbf{R}^n\}, \tag{1.1}$$

$$u(0,x)=\varphi(x) \quad \text{on } \{t=0,\ x\in \mathbf{R}^n\}, \tag{1.2}$$

where $f(t, x, u, p)\in C^2(\mathbf{R}^1\times\mathbf{R}^n\times\mathbf{R}^1\times\mathbf{R}^n)$ and $\varphi(x)\in C^2(\mathbf{R}^n)$. A vector $p=(p_1, \ldots, p_n)$ is corresponding to $(\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$. We are interested in the global theory for the Cauchy problem (1.1)-(1.2). It is well known that it admits uniquely a solution of class C^2 in a neighborhood of the origin. A "classical solution" of (1.1) means a solution of class C^1 which may not be of class C^2. From now on, we write a solution of class C^k as C^k-solution. Our aim is to see what kinds of phenomena may appear when we extend the classical solutions. In this extension, we need the uniqueness of classical solutions for the Cauchy problem (1.1)-(1.2) which will be treated in 2. In 3, we will discuss the life span of classical solutions. When we can not extend the classical solutions, we must introduce the notion of generalized solutions which are the solutions with singularities. The properties of singularities depend on the behavior of characteristic curves which are determined by the type of equations. For example, generalized solutions of Hamilton-Jacobi equations are continuous, though they are not differentiable. On the other hand, weak solutions of equations of conservation law are not continuous. The principal aim of 4 and 5 is to discuss the reason why the property of singularities may depend on the type of equations. The central problem of 5 is to solve Rankine-Hugoniot's equation (5.7). Though this has been well studied, there remains the problem of the uniqueness of solutions of the equation (5.7), because the right hand side of (5.7) is not Lipschitz continuous. One of the aims of 5 is to solve this problem which is necessary to construct shocks.

2. UNIQUENESS OF CLASSICAL SOLUTIONS

It is well known that the Cauchy problem (1.1)-(1.2) has a C^2-solution in a neighborhood of $t=0$, and that the solution is unique in C^2-space. C^k-space is a set of functions of class C^k. But, as (1.1) is the partial differential equation of first order, we should consider C^1-solutions, *i. e.*, classical solutions. Our first question is whether or not it admits another C^1-solution which is not of class C^2. Moreover, when we prolong the classical solutions of (1.1), we need the uniqueness of solutions in C^1-space. Concerning this problem, we have nice results obtained by

A. Haar [5] and T. Wazewski [16]. As it seems to us that their results are not so familiar today, we present them here without proofs. Let $T=\{(t, x)\in R^1\times R^n;\quad 0\leqq t<a,\quad c_i+L_i t\leqq x_i\leqq d_i-L_i t\ (i=1,2,\ldots,n)\}$ where $L_i\geqq 0$, $c_i<d_i$ and $a\leqq(d_i-c_i)/2L_i$ $(i=1,2,\ldots,n)$, and K be any compact set in $\{(u, p);\ u\in R^1,\ p\in R^n\}$.

THEOREM 1. – *Suppose that $f(t, x, u, p)$ is in $C^0(T\times K)$, and that it satisfies a Lipschitz condition as follows:*

$$|f(t, x, u, p)-f(t, x, v, q)|\leqq\sum_{i=1}^{n} L_i|p_i-q_i|+M|u-v|$$

where (t, x, u, p) and (t, x, v, q) both are in $T\times K$. Let $u_i(t, x)\,(i=1,2)$ be in $C^1(T)$ and $(u_i(t, x), (\partial u_i/\partial x)(t, x))\in K$ for any $(t, x)\in T$. If $u_i(t, x)\,(i=1,2)$ satisfy the equation (1.1) in the domain T with $u_1(0,x)=u_2(0,x)$, then $u_1(t, x)=u_2(t, x)$ in T.

This theorem was first proved by A. Haar [5] for the case $n=1$, and next by T. Wazewski [16] for the general case. This theorem can be easily rewritten in the following form:

THEOREM 2. – *Suppose that $f(t, x, u, p)$ is continuous with respect to (t, x, u, p) and locally Lipschitz continuous with respect to (u, p). If $u(t, x)$ and $v(t, x)$ are C^1-solutions of (1.1) in a neighborhood of the origin with $u(0,x)=v(0,x)$, then there exists Ω, a neighborhood of the origin, such that $u(t, x)=v(t, x)$ in Ω.*

3. LIFE SPANS OF CLASSICAL SOLUTIONS

We consider at first a quasi-linear partial differential equation of first order as follows:

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n} a_i(t, x, u)\frac{\partial u}{\partial x_i}=a_0(t, x, u)\quad\text{in } \{t>0, x\in R^n\},\tag{3.1}$$

$$u(0,x)=\varphi(x)\quad\text{on } \{t=0, x\in R^n\},\tag{3.2}$$

where $a_i(t, x, u)\in C^1(R^1\times R^n\times R^1)\,(i=0,1,2,\ldots,n)$ and $\varphi(x)\in C^1(R^n)$. The characteristic equations for (3.1)-(3.2) are written in the following form:

$$\left.\begin{aligned}\frac{dx_i}{dt}&=a_i(t,x,v)\qquad(i=1,2,\ldots,n)\\ \frac{dv}{dt}&=a_0(t,x,v)\end{aligned}\right\}\tag{3.3}$$

$$x_i(0)=y_i\,(i=1,2,\ldots,n),\qquad v(0)=\varphi(y).\tag{3.4}$$

We write the solution of (3.3)-(3.4) as $x=x(t, y)$ and $v=v(t, y)$. In this case $v=v(t, y)$ is the value of the solution of (3.1) restricted on the curve $x=x(t, y)$, *i. e.*, $v(t, y)=u(t, x(t, y))$. Here we assume the following condition:

(A) The Cauchy problem (3.3)-(3.4) has a unique global solution $x=x(t, y)$ and $v=v(t, y)$ on $\{t \geqq 0\}$ for any $y \in \mathbf{R}^n$.

There exist several contributions which give the sufficient conditions to guarantee the above condition (A), for example B. Doubnov [3]. The seminal idea of the results which assures the condition (A) comes from the following examples: The Cauchy problem $(d/dt)x(t)=x(t)^\alpha$ with $x(0)=y$ is globally solvable for any $y \in \mathbf{R}^1$ if and only if $\alpha \leqq 1$.

When we assume the condition (A), we get a smooth mapping $x=x(t, y)$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ for each $t \geqq 0$. Here $(\mathrm{D}x/\mathrm{D}y)(t, y)$ means the Jacobian of the mapping $x=x(t, y)$. Concerning the life spans of classical solutions of (3.1)-(3.2), we get the following

THEOREM 3. — *Under the condition* (A), *suppose that* $(\mathrm{D}x/\mathrm{D}y)(t^0, y^0)=0$ *and* $(\mathrm{D}x/\mathrm{D}y)(t, y^0) \neq 0$ *for* $t<t^0$. *Then the first derivative* $(\partial u/\partial x)(t, x)$ *of the solution* $u=u(t, x)$ *tends to infinity when* t *goes to* t^0-0 *along the curve* $x=x(t, y^0)$.

Proof. — This is almost a corollary of Theorem 4 in Tsuji-Li [14]. Let us put $\mathrm{L}=\{(t, y^0); 0 \leqq t<t^0\}$ and $\mathrm{C}_0=\{(t, x); x=x(t, y^0), 0 \leqq t<t^0\}$. By the assumption, we can get an open neighborhood $\tilde{\mathrm{V}}$ of L so that the Jacobian $(\mathrm{D}x/\mathrm{D}y)(t, y)$ does not vanish on $\mathrm{V} \underset{\text{def}}{=} \tilde{\mathrm{V}} \cap \{0 \leqq t<t^0\}$, and we write $\mathrm{U}=\{(t, x);\ x=x(t, y).\ (t, y) \in \mathrm{V}\}$. By the theorem of inverses functions, we can uniquely solve the equation $x=x(t, y)$ with respect to y for any $(t, x) \in \mathrm{U}$, and write it as $y=y(t, x)$. If we define $u(t, x)=v(t, y(t, x))$, $u(t, x)$ is a C^1-solution of (3.1)-(3.2) in the domain U. Moreover by Theorem 2, $u(t, x)$ is the unique C^1-solution of (3.1)-(3.2) in U. Here we have

$$\frac{\partial u}{\partial x_j}=\sum_{i=1}^{n} \frac{\partial v}{\partial y_i} \frac{\partial y_i}{\partial x_j} \quad \text{in U}, \tag{3.5}$$

$$\frac{\partial v}{\partial y_i}=\sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial y_i} \quad \text{in V}. \tag{3.5$'$}$$

On the other hand, from (3.3), $\partial x_j/\partial y_i$ and $\partial v/\partial y_j$ $(i, j=1, 2, \ldots, n)$ satisfy the following system of linear ordinary differential equations:

$$\left.\begin{aligned}
\frac{d}{dt}\left(\frac{\partial x_i}{\partial y_j}\right) &= \sum_{k=1}^{n} \frac{\partial a_i}{\partial x_k} \frac{\partial x_k}{\partial y_j}+\frac{\partial a_i}{\partial v} \frac{\partial v}{\partial y_j} \\
\frac{d}{dt}\left(\frac{\partial v}{\partial y_j}\right) &= \sum_{k=1}^{n} \frac{\partial a_0}{\partial x_k} \frac{\partial x_k}{\partial y_j}+\frac{\partial a_0}{\partial v} \frac{\partial v}{\partial y_j}
\end{aligned}\right\} \tag{3.6}$$

with the initial data

$$\frac{\partial x_i}{\partial y_j}(0)=\begin{cases}1 & (i=j)\\ 0 & (i\neq j)\end{cases} \quad \text{and} \quad \frac{\partial v}{\partial y_j}(0)=\frac{\partial \varphi}{\partial y_j}(y).$$

As the system (3.6) is linear with respect to $\partial x_i/\partial y_j$ and $\partial v/\partial y_j$ $(i,j=1,2,\ldots,n)$, we get

$$\operatorname{rank}\begin{bmatrix}\dfrac{\partial x_1}{\partial y}(t)\\ \vdots \\ \dfrac{\partial x_n}{\partial y}(t)\\ \dfrac{\partial v}{\partial y}(t)\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\dfrac{\partial x_1}{\partial y}(0)\\ \vdots \\ \dfrac{\partial x_n}{\partial y}(0)\\ \dfrac{\partial v}{\partial y}(0)\end{bmatrix}=n \quad \text{for any } t\geqq 0, \tag{3.7}$$

where $\dfrac{\partial x_i}{\partial y}=\left(\dfrac{\partial x_i}{\partial y_1},\ldots,\dfrac{\partial x_i}{\partial y_n}\right)$ and $\dfrac{\partial v}{\partial y}=\left(\dfrac{\partial v}{\partial y_1},\ldots,\dfrac{\partial v}{\partial y_n}\right)$.

Assume that all the components of $\partial u/\partial x$ remain bounded in U, then we can pick up a sequence $\{t^n\}$ such that

(i) $$\lim_{n\to\infty} t^n=t^0,\ t^n<t^0$$

and

(ii) $$\lim_{n\to\infty}\frac{\partial u}{\partial x}(t^n, x(t^n, y^0))=c=(c_1,\ldots,c_n)\in \mathrm{R}^n.$$

From (3.5)′, we have

$$\frac{\partial v}{\partial y}(t^0, y^0)=\sum_{i=1}^{n} c_i\frac{\partial x_i}{\partial y}(t^0, y^0). \tag{3.8}$$

As $(\mathrm{D}x/\mathrm{D}y)(t^0, y^0)=0$, (3.8) contradicts to (3.7). This means that at least one component of $(\partial u/\partial x)(t, x(t, y^0))$ tends to infinity when $t\to t^0$.

Q.E.D.

Next we consider general partial differential equation of first order as follows:

$$\frac{\partial u}{\partial t}+f\left(t,x,u,\frac{\partial u}{\partial x}\right)=0, \quad \text{in } \{t>0, x\in \mathrm{R}^n\}, \tag{3.9}$$

$$u(0,x)=\varphi(x) \quad \text{on } \{t=0, x\in\mathrm{R}^n\}, \tag{3.10}$$

where $f(t, x, u, p)\in C^2(\mathrm{R}^1\times\mathrm{R}^n\times\mathrm{R}^1\times\mathrm{R}^n)$ and $\varphi(x)\in C^2(\mathrm{R}^n)$. The characteristic equations for (3.9)-(3.10) are written down as

$$\left.\begin{aligned}\frac{dx_i}{dt}&=\frac{\partial f}{\partial p_i}(t, x, v, p) \qquad (i=1,2,\ldots,n)\\ \frac{dv}{dt}&=\sum_{j=1}^{n}\frac{\partial f}{\partial p_j}(t, x, v, p)p_j-f(t, x, v, p)\\ \frac{dp_i}{dt}&=-\frac{\partial f}{\partial x_i}(t, x, v, p)-\frac{\partial f}{\partial v}(t, x, v, p)p_i\,(i=1,2,\ldots,n).\end{aligned}\right\} \tag{3.11}$$

$$x_i(0)=y_i, \qquad v(0)=\varphi(y), \qquad p_i(0)=\frac{\partial\varphi}{\partial y_i}(y) \qquad (i=1,2,\ldots,n). \quad (3.12)$$

Here we assume the condition (A)':

(A)' The Cauchy problem (3.11)-(3.12) has a unique global solution $x=x(t, y)$, $v=v(t, y)$ and $p=p(t, y)$ on $\{ t\geqq 0 \}$ for any $y\in \mathbf{R}^n$.

As $(\mathrm{D}x/\mathrm{D}y)(0, y)=1$ for all $y\in \mathbf{R}^n$, the Jacobian does not vanish in a neighborhood of $t=0$. Therefore we can uniquely solve the equation $x=x(t, y)$ with respect to y, and write it as $y=y(t, x)$. Define $u(t, x)=v(t, y(t, x))$, then $u=u(t, x)$ is a C^2-solution of (3.9)-(3.10) in a neighborhood of $t=0$. Moreover, we can see by Theorem 2 that there does not exist another C^1-solution of (3.9)-(3.10) in a neighborhood of $t=0$. When we extend this solution for large t, we get the following

THEOREM 4. – *Under the condition* (A)', *suppose that* $(\mathrm{D}x/\mathrm{D}y)(t^0, y^0)=0$ *and* $(\mathrm{D}x/\mathrm{D}y)(t, y^0)\neq 0$ *for* $t<t^0$. *Then* $\sum_{i,j=1}^{n} |\partial^2 u/\partial x_i \partial x_j)(t, x)|$ *tends to infinity when* t *goes to* t^0-0 *along the curve* $x=x(t, y^0)$.

This is almost a corollary of Theorem 5 in Tsuji-Li [14]. Moreover, as the idea of the proof was already developed in the above Theorem 3, we omit the proof of this theorem. Theorem 4 says that, if the Jacobian vanishes at a point (t^0, y^0), then the second derivatives of the solution blow up at the point (t^0, y^0). But this does not prevent the existence of C^1-solutions even if the Jacobian may vanish. To consider this situation, we need to know the behavior of characteristic curves in a neighborhood of the point where the Jacobian vanishes. We define smooth mappings H_t and H by $x=x(t, y)=\mathrm{H}_t(y)$ and $\mathrm{H}(t, y)=(t, \mathrm{H}_t(y))$. From the results of [15], we get the following two cases:

(C.1) Though the Jacobian of the mapping $x=\mathrm{H}_t(y)$ vanishes at a point $(t, y)=(t^0, y^0)$, the mapping H is bijective from an open neighborhood of (t^0, y^0) to the one of (t^0, x^0) where $x^0=x(t^0, y^0)$.

(C.2) For $t>t^0$ where $(\mathrm{D}x/\mathrm{D}y)(t^0, y^0)=0$, the characteristic curves meet in a neighborhood of the point (t^0, x^0).

THEOREM 5. – *Under the hypothesis* (A)', *suppose the condition* (C.1). *Then the solution* $u(t, x)=v(t, y(t, x))$ *remains a* C^1*-solution of* (3.9) *in a neighborhood of* (t^0, x^0), *though it is not of class* C^2.

Proof. – Let V and U be open neighborhoods of (t^0, y^0) and (t^0, x^0) respectively such that the mapping H is bijective from V to U. As H is bijective, we can uniquely solve the equation $x=x(t, y)$ with respect to y and denote it by $y=y(t, x)$ which is continuous in the domain U. As usual, we write $u(t, x)=v(t, y(t, x))$. Put $\mathrm{S}=\{ (t, y)\in \mathrm{V}\,;\, (\mathrm{D}x/\mathrm{D}y)(t, y)=0 \}$ and $\mathrm{H}(\mathrm{S})=\{ \mathrm{H}(t, y)\,;\, (t, y)\in \mathrm{S} \}$. By Sard's theorem, Lebesgue measure of

H(S) is zero. Hence U − H(S) is dense in U. For any $(\tilde{t}, \tilde{x}) \in U - H(S)$, there exists a unique point $(\tilde{t}, \tilde{y}) \in V$ satisfying $(Dx/Dy)(\tilde{t}, \tilde{y}) \neq 0$ where $\tilde{x} = x(\tilde{t}, \tilde{y})$. Therefore H is a diffeomorphism from certain open neighborhhod of $(\tilde{t}, \tilde{y})$ to the one of $(\tilde{t}, \tilde{x})$. This guarantees that $u = u(t, x)$ is of class C^2 in a neighborhood of $(\tilde{t}, \tilde{x})$. It is obvious that it satisfies the equation (3.9). Next we show that $u(t, x)$ is of class C^1 in the domain U. We pick up arbitrarily a point $(\tilde{t}^0, \tilde{x}^0)$ in H(S). Then we can choose a sequence of points $\{ (t^m, x^m) \}_{m=1, 2, \ldots}$ in U − H(S) such that, when m goes to infinity, (t^m, x^m) is convergent to $(\tilde{t}^0, \tilde{x}^0)$. As the mapping H is bijective from V to U, there exists uniquely a point (t^m, y^m) satisfying $H(t^m, y^m) = (t^m, x^m)$ for each m. As $u(t, x)$ is continuously differentiable at (t^m, x^m), we have

$$p(t^m, y^m) = \frac{\partial u}{\partial x}(t^m, x^m).$$

As $p = p(t, y)$ is continuously differentiable for all (t, y), we get

$$\lim_{m \to \infty} \frac{\partial u}{\partial x}(t^m, x^m) = p(\tilde{t}^0, \tilde{y}^0) \qquad \text{where} \quad \tilde{x}^0 = x(\tilde{t}^0, \tilde{y}^0).$$

Therefore, if we define the derivative of $u(t, x)$ at the point $(\tilde{t}^0, \tilde{x}^0)$ by

$$\frac{\partial u}{\partial x}(\tilde{t}^0, \tilde{x}^0) = p(\tilde{t}^0, \tilde{y}^0),$$

then $(\partial u/\partial x)(t, x) = p(t, y(t, x))$ is continuous in U, that is to say, $u(t, x)$ is continuously differentiable in U. Moreover Theorem 2 guarantees that $u(t, x)$ is the unique classical solution of (3.9)-(3.10) in U.

Q.E.D.

Exemple 1. — This is the example which satisfies the condition (C.1). We consider the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + f\left(t, u, \frac{\partial u}{\partial x}\right) = 0 \quad \text{in } \{ t > 0, x \in R^1 \}, \\ &u(0, x) = \frac{1}{2} x^2 \quad \text{on } \{ t = 0, x \in R^1 \}, \end{aligned} \tag{3.13}$$

where $f(t, u, p) = \alpha'(t) e^{-t} p^2 + \beta'(t) e^{-3t} p^4 - u$ and the functions $\{ \alpha(t), \beta(t) \}$ have the following properties:

1. $\alpha(t)$ and $\beta(t)$ are in $C^1(R^1)$,
2. $\alpha(0) = 1/2$, $\alpha(t) \geqq 0$ and $\alpha(t) = 0$ for any $t \geqq K > 0$,
3. $\beta(0) = 0$, $\beta(t) \geqq 0$ and $\beta(t) \neq 0$ for any $t \geqq K$,
4. $\alpha(t) + \beta(t) \neq 0$ for any $t \geqq 0$.

Then the characteristic curves are written as

$$x(t, y) = 2\alpha(t) y + 4\beta(t) y^3 \qquad \text{and} \qquad v(t, y) = \alpha(t) e^t y^2 + 3\beta(t) e^t y^4.$$

Therefore the Jacobian $(\partial x/\partial y)(t, y)=2\alpha(t)+12\beta(t)y^2$ vanishes on $L=\{(t, y); t\geqq K \text{ and } y=0\}$. But $x=x(t, y)$ is a bijective mapping defined in a neighborhood of $y=0$ for $t\geqq K$, *i. e.*, the condition (C. 1) is satisfied. In this case the solution $u(t, x)=v(t, y(t, x))=\text{Const. } \beta(t)^{-1/3}e^t x^{4/3}$ in a neighborhood of L. This means that $u(t, x)$ is of class C^1, but not of class C^2, in a neighborhood of L.

4. EQUATIONS OF HAMILTON-JACOBI TYPE

In this section we will consider the Cauchy problem for general partial differential equations of first order (3.9)-(3.10) which satisfies the condition (A)′. We have seen in 3 that, even if the Jacobian of the mapping H may vanish at some points, the bijectivity of H guarantees that the Cauchy problem (3.9)-(3.10) can keep the C^1-solution in the neighborhoods of them. Therefore we will discuss here the case (C.2) in which the characteristic curves meet after the Jacobian vanishes. In this case, as the Cauchy problem (3.9)-(3.10) can not have the classical solutions, we must introduce generalized solutions which contain singularities. Our interests are focused on the properties of their singularities.

One of the examples satisfying the condition (A)′ is classical Hamilton-Jacobi equations. Convex Hamilton-Jacobi equations have been extensively studied, and the global existence and uniqueness of generalized solutions have been proved. For detailed bibliography, refer to S. Benton [1] and P.-L. Lions [9]. Recently, M. G. Grandall, L. C. Evans and P.-L. Lions ([2], [9], . . .) have considered general Hamilton-Jacobi equations without convexity condition, and they have established the notion of viscosity solutions. The generalized solutions of convex Hamilton-Jacobi equations are Lipschitz continuous, and the viscosity solutions are continuous. On the other hand, though the equations of conservation law are also of first order, their weak solutions are generally not continuous. One of our aims of the following discussions is to consider the reason.

We consider the Cauchy problem (3.9)-(3.10) in one space dimension, *i. e.*,

$$\frac{\partial u}{\partial t}+f\left(t, x, u, \frac{\partial u}{\partial x}\right)=0 \quad \text{in } \{t>0, x\in R^1\}, \tag{4.1}$$

$$u(0, x)=\varphi(x) \quad \text{on } \{t=0, x\in R^1\}, \tag{4.2}$$

where $f(t, x, u, p)\in C^\infty(R^1\times R^1\times R^1\times R^1)$ and $\varphi(x)\in C^\infty(R^1)$. For the construction of singularities of generalized solutions, we need a little strong assumptions on the regularity of $f(t, x, u, p)$ and $\varphi(x)$. Then the characteristic curves of (4.1)-(4.2) are defined as the solution curves of

(3.11)-(3.12) for $n=1$. We write them by $x=x(t, y)$, $v=v(t, y)$ and $p=p(t, y)$.

Suppose the condition (A)′ and consider (4.1)-(4.2) under the following situation: (I) $(\partial x/\partial y)(t^0, y^0)=0$ and (II) $(\partial x/\partial y)(t, y)\neq 0$ for $t<t^0$ and $y\in \mathrm{I}$ where I is an open neighborhood of $y=y^0$. As written in the above, we will consider the case (C.2) where the characteristic curves meet after the time t^0. Here we assume the following condition (4.3) which guarantees the situation (C.2) (*see* Tsuji-Li [14]):

$$\frac{\partial^2 f}{\partial p^2}(t^0, x^0, v^0, p^0)\neq 0 \tag{4.3}$$

where $x^0=x(t^0, y^0)$, $v^0=v(t^0, y^0)$ and $p^0=p(t^0, y^0)$. If the condition (4.3) is not satisfied, we can construct an example in which the situation (C.2) does not happen. *See* Example 1 given after Theorem 5 in 3.

From now on, we will construct the singularities of generalized solutions for $t>t^0$ where $t-t^0$ is small. First we solve the equation $(\partial x/\partial y)(t, y)=0$ with respect to t. As $(\partial x/\partial y)(t, y)$ is positive for $t<t^0$ and $y\in \mathrm{I}$, it holds $(\partial^2 x/\partial y^2)(t^0, y^0)\leqq 0$. If $(\partial^2 x/\partial y^2)(t^0, y^0)<0$, we have $(\partial x/\partial y)(t^0, y)<0$ for $y>y^0$. As $x=x(t^0, y)$ is monotonously increasing with respect to y, this is a contradiction. Therefore we get $(\partial^2 x/\partial y^2)(t^0, y^0)=0$. We assume here $(\partial^3 x/\partial y^3)(t^0, y^0)\neq 0$. This assumption is natural from the generic point of view. In this case, it turns to be

$$\frac{\partial^3 x}{\partial y^3}(t^0, y^0)>0. \tag{4.4}$$

If (4.4) is negative, we have $(\partial^2 x/\partial y^2)(t^0, y)<0$ for $y>y^0$, *i. e.*, $(\partial x/\partial y)(t^0, y)<0$ for $y>y^0$ which also contradicts the monotony of $x=x(t, y)$ with respect to y. Let us draw a figure of the curve $x=x(t, y)\,(t>t^0)$.

We will explain the reason why it is drawn as Figure 1. Taking the derivative of (3.11) with respect to y, we get a system of ordinary differential equations concerning $\partial x/\partial y$, $\partial v/\partial y$ and $\partial p/\partial y$, just like (3.6). For example, the equation concerning $\partial x/\partial y$ is written as

$$\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial p}(t, x, v, p)\frac{\partial x}{\partial y} + \frac{\partial^2 f}{\partial v\,\partial p}(t, x, v, p)\frac{\partial v}{\partial y}+\frac{\partial^2 f}{\partial p^2}(t, x, v, p)\frac{\partial p}{\partial y}. \tag{4.5}$$

Then this system of equations is linear concerning $\partial x/\partial y$, $\partial v/\partial y$ and $\partial p/\partial y$. As $(\partial x/\partial y(0, y), \partial v/\partial y(0, y), \partial p/\partial y(0, y))=(1, \varphi'(y), \varphi''(y))\neq(0,0,0)$, we get $(\partial x/\partial y(t, y), \partial v/\partial y(t, y), \partial p/\partial y(t, y))\neq(0,0,0)$ for any $(t, y)\in \mathrm{R}^2$. We

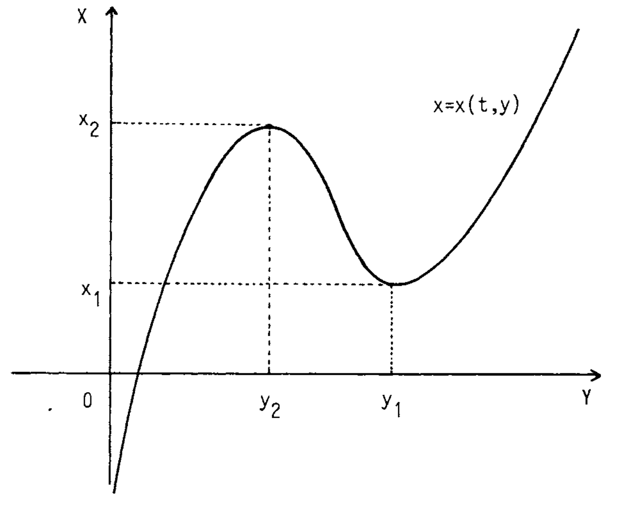

FIG. 1

recall here

$$p(t, y)\frac{\partial x}{\partial y}(t, y)=\frac{\partial v}{\partial y}(t, y) \quad \text{for any } (t, y)\in \mathbf{R}^2, \tag{4.6}$$

because, putting $z(t, y)=p(t, y)(\partial x/\partial y)(t, y)-(\partial v/\partial y)(t, y)$, we have

$$\frac{d}{dt}z(t, y)=-\frac{\partial f}{\partial v}(t, x(t, y), v(t, y), p(t, y))\, z(t, y),$$
$$z(0, y)=0.$$

As $(\partial x/\partial y)(t^0, y^0)=0$, it holds $(\partial v/\partial y)(t^0, y^0)=0$. Therefore we have $(\partial p/\partial y)(t^0, y^0)\neq 0$. On the other hand, as $(\partial x/\partial y)(t^0, y^0)=0$ and $(\partial x/\partial y)(t, y)>0$ for $t<t^0$ and $y\in I$, the left hand side of (4.5) must be non-positive at (t^0, y^0). Using these results in (4.5), we get

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial y}\right)\bigg|_{(t, y)=(t^0, y^0)}=\frac{\partial^2 f}{\partial p^2}(t^0, x^0, v^0, p^0)\frac{\partial p}{\partial y}(t^0, y^0)<0. \tag{4.7}$$

Therefore we can uniquely solve the equation $(\partial x/\partial y)(t, y)=0$ with respect to t in a neighborhood of (t^0, y^0), and write it by $t=\rho(y)$ which is of class C^∞. Then we have

$$\frac{d}{dy}\left(\frac{\partial x}{\partial y}(\rho(y), y)\right)=\frac{\partial^2 x}{\partial t\,\partial y}(\rho(y), y)\,\rho'(y)+\frac{\partial^2 x}{\partial y^2}(\rho(y), y)=0.$$

As $(\partial^2 x/\partial y^2)(t^0, y^0)=0$ and $(\partial^2 x/\partial t\,\partial y)(t^0, y^0)<0$, it follows $\rho'(y^0)=0$. Similarly we have

$$\frac{d^2}{dy^2}\left(\frac{\partial x}{\partial y}(\rho(y), y)\right)\bigg|_{y=y^0}=\frac{\partial^2 x}{\partial t\,\partial y}(t^0, y^0)\,\rho''(y^0)+\frac{\partial^3 x}{\partial y^3}(t^0, y^0)=0$$

which induces immediately $\rho''(y^0)>0$. As $\rho'(y^0)=0$, it holds $\rho'(y)>0$ for $y>y^0$ and $\rho'(y)<0$ for $y<y^0$. Therefore, as $\rho(y)$ is strictly increasing for $y>y^0$ and strictly decreasing for $y<y^0$, the equation $t=\rho(y)$ has two solutions $y=y_1(t)$ and $y_2(t)\,(y_2<y^0<y_1)$ for $t>t^0$ where $t-t^0$ is small. Summing up these results, we get the following.

LEMMA 6. – *The equation* $(\partial x/\partial y)(t, y)=0$ $(t>t^0)$ *has two solutions* $y=y_1(t)>y_2(t)$ *in a neighborhood of* $y=y^0$. *The solutions* $y=y_i(t)$ $(i=1,2)$ *are continuous on* $\{t\geqq t^0\}$, *and of class* C^∞ *for* $t>t^0$.

The proof will be obvious by the above discussions. Here we put $x_i(t)=x(t, y_i(t))$ $(i=1,2)$, then $x_1(t)<x_2(t)$. Next we solve the equation $x=x(t, y)$ with respect to y for $x\in(x_1(t), x_2(t))$. Then we get three solutions $y=g_1(t, x)<g_2(t, x)<g_3(t, x)$, and define $u_i(t, x)=v(t, g_i(t, x))$ $(i=1,2,3)$. This means that the solution constructed by the characteristic method takes three values in the interval (x_1, x_2). By the assumption (4.3), we assume in this section

$$\frac{\partial^2 f}{\partial p^2}(t^0, x^0, v^0, p^0)>0. \tag{4.8}$$

Then we get the following

LEMMA 7. – (i) *For any* $x\in(x_1, x_2)$, *we have*

$$u_1(t, x)-u_2(t, x)<0 \qquad \textit{and} \qquad u_2(t, x)-u_3(t, x)>0.$$

(ii) *There exists uniquely* $x=\gamma(t)\in(x_1, x_2)$ *satisfying*

$$u_1(t, \gamma(t))=u_3(t, \gamma(t)).$$

Proof. – (i) By (4.7) and (4.8), it holds $(\partial p/\partial y)(t^0, y^0)<0$, *i. e.*, $(\partial p/\partial y)(t, y)<0$ in a small neighborhood of (t^0, y^0).

As $g_1(t, x)<g_2(t, x)<g_3(t, x)$, it follows

$$p(t, g_1(t, x))>p(t, g_2(t, x))>p(t, g_3(t, x)).$$

Using these inequalities, we have

$$\frac{\partial}{\partial x}\{u_1(t, x)-u_2(t, x)\}=p(t, g_1(t, x))-p(t, g_2(t, x))>0,$$

$$\{u_1(t, x)-u_2(t, x)\}\,|_{x=x_2}=0.$$

Hence we get $u_1(t, x)-u_2(t, x)<0$ for $x\in(x_1, x_2)$. We obtain similarly the another inequality.

(ii) Using the results of (i), we have

$$\frac{\partial}{\partial x}\{u_1(t, x)-u_3(t, x)\}=p(t, g_1(t, x))-p(t, g_3(t, x))>0,$$
$$\{u_1(t, x)-u_3(t, x)\}|_{x=x_2}=\{u_2(t, x)-u_3(t, x)\}|_{x=x_2}>0,$$
$$\{u_1(t, x)-u_3(t, x)\}|_{x=x_1}=\{u_1(t, x)-u_2(t, x)\}|_{x=x_1}<0.$$

Therefore there exists uniquely $x=\gamma(t)\in(x_1, x_2)$ satisfying

$$u_1(t, \gamma(t))=u_3(t, \gamma(t)).$$

Q.E.D.

As we are looking for single-valued and continuous solution, we define the solution $u=u(t, x)$ in the interval (x_1, x_2) as follows:

$$u(t, x)=\begin{cases} u_1(t, x), & x\leqq\gamma(t), \\ u_3(t, x), & x>\gamma(t). \end{cases}$$

Moreover we can easily prove that, when (4.8) is satisfied, this solution $u(t, x)$ is semi-concave in a neighborhood U of (t^0, x^0), *i. e.*, there exists a constant K such that

$$u(t, x+y)+u(t, x-y)-2u(t, x)\leqq K|y|^2$$

for any $(t, x+y)$ and $(t, x-y)$ in U. Therefore the solution $u(t, x)$ constructed as above is reasonable.

Remark. – The above construction of singularities is local. In some cases we can extend the solution for large t by the same method. But, if $(\partial^2 f/\partial p^2)$ changes the sign, the solution may generally lose the property of simi-concavity or semi-convexity.

5. QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

In this section we consider the Cauchy problem for quasi-linear equations of first order as follows:

$$\frac{\partial u}{\partial t}+a_1(t, x, u)\frac{\partial u}{\partial x}=a_0(t, x, u) \quad \text{in } \{t>0, x\in R^1\}, \tag{5.1}$$
$$u(0, x)=\varphi(x) \quad \text{on } \{t=0, x\in R^1\}, \tag{5.2}$$

where $a_1(t, x, u)\in C^\infty(R^1\times R^1\times R^1)$ $(i=0, 1)$ and $\varphi(x)\in C^\infty(R^1)$. Then the characteristic curves for (5.1)-(5.2) are the solution curves $x=x(t, y)$ and $v=v(t, y)$ of (3.3)-(3.4) for $n=1$. As in 4, we assume the condition (A), stated in 3, which assures the global existence of characteristic curves. We have seen by Theorem 3 that, if the Jacobian of the mapping $x=x(t, y)$ vanishes at a point (t^0, y^0), then the classical solution blows up at a point

(t^0, x^0) where $x^0 = x(t^0, y^0)$. Therefore we must introduce weak solutions for the equation (5.1). To define it, we rewrite the equation (5.1) as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(t, x, u) = g(t, x, u) \tag{5.1$'$}$$

where

$$(\partial f/\partial u)(t, x, u) = a_1(t, x, u)$$

and

$$g(t, x, u) - (\partial f/\partial x)(t, x, u) = a_0(t, x, u).$$

If $g(t, x, u) = 0$, then the equation (5.1) is of conservation law. Let $w(t, x)$ be locally integrable in R^2. The function $w(t, x)$ is called to be a weak solution of (5.1) if it satisfies (5.1) in distribution sense, *i. e.*,

$$\iint \left\{ w \frac{\partial \kappa}{\partial t} + f(t, x, w) \frac{\partial \kappa}{\partial x} + g(t, x, w) \kappa \right\} dt\, dx = 0$$

for all $\kappa(t, x) \in C_0^\infty(\mathrm{R}^2)$.

From now on, we will consider the Cauchy problem (5.1)-(5.2) in a neighborhood of (t^0, x^0) under the following assumptions (I) and (II): (I) $(\partial x/\partial y)(t^0, y^0) = 0$ and (II) $(\partial x/\partial y)(t, y) \neq 0$ for $t < t^0$ and $y \in \mathrm{I}$ where I is an open neighborhood of $y = y^0$. Our problem is to see what kinds of phenomena may appear for $t > t^0$. Before we proceed to this problem, we give an important property of the equation (5.1) which is the point different from the equations satisfying the condition (A)$'$.

Consider the Cauchy problem for an ordinary differential equation concerning $p = p(t, y)$ as follows:

$$\frac{dp}{dt} = -\frac{\partial a_1}{\partial u}(t, x, v) p^2 + \left\{ \frac{\partial a_0}{\partial u}(t, x, v) - \frac{\partial a_1}{\partial x}(t, x, v) \right\} p + \frac{\partial a_0}{\partial x}(t, x, v), \tag{5.3}$$

$$p(0, y) = \varphi'(y), \tag{5.4}$$

where $x = x(t, y)$ and $v = v(t, y)$ are the solutions of (3.3)-(3.4) for $n = 1$. The equation (5.3) corresponds to the last one of (3.11). More concretely, when $u = u(t, x)$ is a C^2-solution of (5.1), we get

$$p(t, y) = (\partial u/\partial x)(t, x(t, y)).$$

LEMMA 8. — *Assume* $(\partial x/\partial y)(t^0, y^0) = 0$ *and* $(\partial x/\partial y)(t, y^0) \neq 0$ *for* $t < t^0$. *Then the solution* $p = p(t, y^0)$ *of* (5.3)-(5.4) *tends to infinity when* t *goes to* $t^0 - 0$.

Proof. — As $\partial x/\partial y$ and $\partial v/\partial y$ satisfy the system of linear differential equations (3.6), we get $(\partial x/\partial y, \partial v/\partial y) \neq (0, 0)$ for all $(t, y) \in \mathrm{R}^2$. Therefore, if $(\partial x/\partial y)(t^0, y^0) = 0$, then $(\partial v/\partial y)(t^0, y^0) \neq 0$. As (4.6) is also true for the above $x=x(t, y)$, $v=v(t, y)$ and $p=p(t, y)$, we have

$$p(t, y) \frac{\partial x}{\partial y}(t, y)=\frac{\partial v}{\partial y}(t, y)$$

Hence we can easily get the conclusion of this lemma.

Q.E.D.

This lemma says that, for the quasi-linear equations, the condition (A)′ is not compatible with the property that the Jacobian vanishes somewhere.

In the case (C.1) where the mapping $\mathrm{H}(t, y)=(t, x(t, y))$ is the bijective one defined in a neighborhood of (t^0, y^0), we can get the similar result like Theorem 5. That is to say, though the solution of (5.1) is continuous in a neighborhood of (t^0, x^0), it is not Lipschitz continuous. Because Lemma 8 means that, if the Jacobian of $x=x(t, y)$ vanishes somewhere, then $(\partial u/\partial x)(t, x)=p(t, y(t, x))$ can not remain bounded where $y=y(t, x)$ is the inverse function of $x=x(t, y)$.

Next we consider the case (C.2). The suffisient condition which guarantees the situation (C.2) is written as

$$\frac{\partial a_1}{\partial v}(t^0, x^0, v^0)\neq 0. \tag{5.5}$$

If (5.5) is not satisfied, we can construct an example in which the characteristic curves do not meet though the Jacobian vanishes. See the example in Tsuji-Li [15]. Therefore we suppose the condition (5.5). Since $(\partial^2 x/\partial y^2)(t^0, y^0)=0$, we assume, as in 4, $(\partial^3 x/\partial y^3)(t^0, y^0)\neq 0$. This assumption is natural from the generic point of view. To represent the solution $v=v(t, y)$ as a function of (t, x), we solve the equation $x=x(t, y)$ with respect to y in a neighborhood of $y=y^0$ for $t>t^0$. This calculation is almost same to the one developed in 4. Especially, the graph of the curve $x=x(t, y)$ is drawn just as Figure 1 in 4. Therefore we use the same notations introduced in 4. The functions $y=y_1(t)$ and $y_2(t)$ $(y_2<y_1)$ are the solutions of $(\partial x/\partial y)(t, y)=0$ and we put $x_i(t)=x(t, y_i(t))$ $(i=1, 2)$. When we solve the equation $x=x(t, y)$ with respect to y for $x\in(x_1, x_2)$, we get three solutions $y=g_1(t, x)<g_2(t, x)<g_3(t, x)$ and define $u_i(t, x)=v(t, g_i(t, x))$ $(i=1, 2, 3)$. As we are looking for single-valued solution, we must choose only one velue from $\{u_i(t, x);\ i=1, 2, 3\}$ so that it is weak solution of (5.1). In this case we can not get the result as Lemma 7 in 4. Therefore we try to obtain the weak solution which is piecewise smooth. If $w(t, x)$ is a weak solution of (5.1) which has jump discontinuity along a curve $x=\gamma(t)$, we get Rankine-Hugoniot's jump condition which is familiar for equations of conservation law:

$$\frac{d\gamma}{dt}=\frac{f(t, \gamma(t), w_+(t))-f(t, \gamma(t), w_-(t))}{w_+(t)-w_-(t)} \tag{5.6}$$

where $w_+(t)=w(t, \gamma(t)+0)$ and $w_-(t)=w(t, \gamma(t)-0)$. This suggests us that we jump from a branch $u=u_1(t, x)$ to the other $u=u_3(t, x)$ along a curve $x=\gamma(t)$ on which the condition (5.6) is satisfied. Therefore our problem is to get the curve which is a solution of the Cauchy problem of the following ordinary differential equation:

$$\frac{dx}{dt}=\frac{f(t, x, u_1(t, x))-f(t, x, u_3(t, x))}{u_1(t, x)-u_3(t, x)} \underset{\text{def}}{=} j(t, x), \qquad t>t^0, \tag{5.7}$$
$$x(t^0)=x^0.$$

The function $j(t, x)$ is differentiable in

$$U=\{(t, x);\ t>t^0 \text{ and } x_2(t)>x>x_1(t)\}$$

and continuous on $\bar{U}$. But it is not Lipschitz continuous at the point (t^0, x^0), because $(\partial j/\partial x)(t, x)$ tends to infinity when t goes to t^0+0. The proof will be given in Lemma 10. As $j(t, x)$ is continuous in $\bar{U}$, there is no problem on the existence of solutions. But we can not get the uniqueness of solutions. In the works done up to now on the construction of shocks, for example in J. Guckenheimer [6] and G. Jennings [7], they did not pay attention to this point. This is the problem which we would like to consider in this section.

As we have put the hypothesis (5.5), we assume here more concretely

$$\frac{\partial a_1}{\partial v}(t^0, x^0, v^0)>0. \tag{5.8}$$

As $j(t, x)$ is in $C^1(U)$ where U does not contain the point (t^0, x^0), we will restrict our discussions in a small neighborhood of (t^0, x^0).

LEMMA 9. — *(i)* $(\partial v/\partial y)(t^0, y^0)<0$.

(ii) *For* $(t, x)\in U$, *we get*

$$u_1(t, x)>u_2(t, x)>u_3(t, x)$$

and

$$\frac{\partial u_1}{\partial x}(t, x)<0, \qquad \frac{\partial u_2}{\partial x}(t, x)>0, \qquad \frac{\partial u_3}{\partial x}(t, x)<0.$$

(iii) *When* (t, x) *goes to* (t^0, x^0) *in* U, *then* $(\partial u_i/\partial x)(t, x)$ $(i=1, 2, 3)$ *tend to infinity.*

Proof. — (i) As $(\partial x/\partial y)(t^0, y^0)=0$ and $(\partial x/\partial y)(t, y^0)>0$ for $t<t^0$, we have

$$\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)\Bigg|_{(t, y)=(t^0, y^0)}=\frac{\partial a_1}{\partial v}(t^0, x^0, y^0)\frac{\partial v}{\partial y}(t^0, y^0)\leqq 0.$$

Since $(\partial x/\partial y(t, y),\ \partial v/\partial y(t, y))\neq(0, 0)$ for all (t, y), it holds $(\partial v/\partial y)(t^0, y^0)\neq 0$. Hence we get (i) by (5.8).

(ii) By the definition, we have $g_1(t, x) < g_2(t, x) < g_3(t, x)$ and $u_i(t, x) = v(t, g_i(t, x))$ $(i = 1, 2, 3)$. Using the property (i), we get the first half of (ii). As $g_1(t, x) < y_2(t) < g_2(t, x) < y_1(t) < g_3(t, x)$ where $x_1(t) < x < x_2(t)$, we have

$$\frac{\partial x}{\partial y}(t, g_1(t, x)) > 0, \qquad \frac{\partial x}{\partial y}(t, g_2(t, x)) < 0, \qquad \frac{\partial x}{\partial y}(t, g_3(t, x)) > 0.$$

As $(\partial u_i/\partial x)(t, x) = \{(\partial v/\partial y)(t, y)/(\partial x/\partial y)(t, y)\}|_{y = g_i(t, x)}$, we get the second part of (ii), and also (iii).

Q.E.D.

LEMMA 10. – *(i) Though $j(t, x))$ is in $C^1(U) \cap C^0(\bar{U})$ where*

$$U = \{(t, x);\ t > t^0,\ x_2(t) > x > x_1(t)\},$$

it is not Lipschitz continuous at a point (t^0, x^0).

(ii) For $t > t^0$, $j(t, x)$ is decreasing with respect to x.

Proof. – The first part of (i) is obvious. Taking the derivative of $j(t, x)$ with respect to x, we have

$$\begin{aligned}\frac{\partial}{\partial x} j(t, x) = \frac{1}{u_1 - u_3} &\left\{\frac{\partial f}{\partial x}(t, x, u_1) - \frac{\partial f}{\partial x}(t, x, u_3)\right\} \\ &+ \frac{1}{u_1 - u_3}\left[\frac{\partial f}{\partial u}(t, x, u_1) \right.\\ &\left. - \frac{1}{u_1 - u_3}\left\{f(t, x, u_1) - f(t, x, u_3)\right\}\right]\frac{\partial u_1}{\partial x}(t, x) \\ &+ \frac{1}{u_1 - u_3}\left[\frac{1}{u_1 - u_3}\{f(t, x, u_1) - f(t, x, u_3)\}\right.\\ &\left. - \frac{\partial f}{\partial u}(t, x, u_3)\right]\frac{\partial u_3}{\partial x}(t, x). \qquad (5.9)\end{aligned}$$

When (t, x) goes to (t^0, x^0) in U, the first term of (5.9) is convergent to $(\partial^2 f/\partial x\, \partial u)(t^0, x^0, u^0)$ where $u^0 = u(t^0, x^0)$. Therefore it is bounded in a neighborhood of (t^0, x^0). When $(t, x) \to (t^0, x^0)$ in U, the coefficient of $(\partial u_1/\partial x)$ tends to

$$(\partial^2 f/\partial u^2)(t^0, x^0, u^0)/2 = (\partial a_1/\partial u)(t^0, x^0, u^0)/2 > 0.$$

The coefficient of $(\partial u_3/\partial x)(t, x)$ has the same property as $(\partial u_1/\partial x)(t, x)$. Here we use (ii) and (iii) of Lemma 9 for (5.9), then we see that, when (t, x) goes to (t^0, x^0) in U, $(\partial j/\partial x)(t, x)$ tends to $-\infty$. This means that

$j(t, x)$ is not Lipschitz continuous at (t^0, x^0), and that it is monotonously decreasing with respect to x in U.

Q.E.D.

LEMMA 11. – *The functions* $x_i(t) = x(t, y_i(t))$ $(i=1, 2)$ *satisfy the following properties:*

(i) $\dfrac{d}{dt} x_i(t) = a_1(t, x_i(t), v(t, y_i(t)))$ *for* $t > t^0$ $(i=1, 2)$,

(ii) $\dfrac{dx_1}{dt}(t) < j(t, x_1(t))$ *and* $\dfrac{dx_2}{dt}(t) > j(t, x_2(t))$ *for* $t > t^0$.

Proof. – (i) As the functions $y = y_i(t)$ $(y_2(t) < y_1(t))$ are the solutions of $(\partial x/\partial y)(t, y) = 0$ with respect to y for $t > t^0$, we have

$$\begin{aligned}\frac{d}{dt} x_i(t) &= \frac{\partial x}{\partial t}(t, y_i(t)) + \frac{\partial x}{\partial y}(t, y_i(t)) \frac{dy_i}{dt}(t) \\ &= \frac{\partial x}{\partial t}(t, y_i(t)) = a_1(t, x(t, y_i(t)), v(t, y_i(t))).\end{aligned}$$

(ii) By the definition of $j(t, x)$, we have

$$j(t, x_1(t)) = a_1(t, x_1(t), u_3(t, x_1(t)) + \theta(u_1(t, x_1) - u_3(t, x_1))), \qquad 1 > \theta > 0.$$

As $(\partial a_1/\partial u)(t^0, x^0, v^0) > 0$, $a_1(t, x, u)$ is strictly increasing with respect to u in a neighborhood of (t^0, x^0, u^0). Moreover we have $u_1(t, x) > u_3(t, x)$ for $(t, x) \in \mathrm{U}$ and $v(t, y_1(t)) = u_3(t, x_1(t))$. Hence we get

$$j(t, x_1(t)) > a_1(t, x_1(t), u_3(t, x_1(t))) = \frac{d}{dt} x_1(t).$$

The second inequality can be similarly proved.

Q.E.D.

Using the above lemmas, we can obtain the following

THEOREM 12. – *The Cauchy problem* (5.7) *has a unique solution in the domain* $\mathrm{U} = \{(t, x);\ t > t^0,\ x_2(t) > x > x_1(t)\}$.

Proof. – The existence of solution is obtained by Lemma 9. As this has been proved in G. Jennings [7], we omit the proof. Our aim is to show the uniqueness of solutions. Let $x = \gamma_1(t) < \gamma_2(t)$ be two solutions of (5.7).

Then we have by Lemma 10

$$\frac{d}{dt} \gamma_1(t) = j(t, \gamma_1(t)) > j(t, \gamma_2(t)) = \frac{d}{dt} \gamma_2(t) \qquad \text{for} \quad t > t^0.$$

Hence we get $\gamma_1(t) > \gamma_2(t)$ for $t > t^0$. This contradicts the hypothesis.

Q.E.D.

Now we can define the weak solution of (5.1) in the interval

$(x_1(t), x_2(t))$ for $t>t^0$ by

$$u(t, x)=\begin{cases} u_1(t, x), & x \leqq \gamma(t), \\ u_3(t, x), & x>\gamma(t). \end{cases}$$

Then we can prove that this weak solution satisfies locally the entropy condition. Next we extend the weak solution for large t. If $(\partial a_1/\partial u)(t, x, u)$ changes the sign, the solution may sometimes lose the entropy condition. Then we must introduce another singularities, for example "contact discontinuity". This subjet will be treated in a forthcoming paper.

From the above discussions, we may say that the essential difference between Hamilton-Jacobi equations and equations of conservation law is the condition (A)′ which means the global solvability of ordinary differential equation (5.3)-(5.4). Therefore we would like to call partial differential equations (1.1) with the condition (A)′ to be of Hamilton-Jacobi type.

We will give here some historical remarks on the subjects treated in this section. For the construction of shocks, we must solve the ordinary differential equation (5.7), though we do not need to do so for Hamilton-Jacobi equations. For the equations of conservation law in one space dimension, we can reduce the construction of shocks to Hamilton-Jacobi equations. Suppose that $u=u(t, x)$ satisfies the following conservation law

$$\frac{\partial}{\partial t} u+\frac{\partial}{\partial x} f(t, x, u)=0. \tag{5.10}$$

Put $u(t, x)=(\partial/\partial x)\, w(t, x)$, then we have

$$\frac{\partial}{\partial t} w+f\left(t, x, \frac{\partial w}{\partial x}\right)=0. \tag{5.11}$$

For Hamilton-Jacobi equation (5.11), we can construct the singularities of generalized solutions, as done in 4. Then $(\partial w/\partial x)(t, x)$ is the weak solution of (5.10) which has jump discontinuity satisfying locally the entropy condition. B. Rozhestvenskii had written this idea a little in [11]. But we can not apply it to quasi-linear equations of first order which are not of conservation law. Because the above transform $u(t, x)=(\partial/\partial x)\, w(t, x)$ does not work well to get Hamilton-Jacobi equations. Moreover the equations treated in [6] and [7] does not depend on (t, x), *i.e.*, $f=f(u)$. By these reasons, the discussions in 5 are necessary to construct the singularities of shock type for general quasi-linear partial differential equations of first order. Concerning the construction of singularities for Hamilton-Jacobi equations in two space dimensions, *see* M. Tsuji [13]. S. Nakane [10] has constructed the shocks for single conservation law in several space dimensions. But his discussions also lack a proof on the uniqueness of solutions of Rankine-Hugoniot's equation. This problem will be considered in our forthcoming paper by the method used here. The next problem is to consider the global behavior of singularities of generalized solutions. For single conservation law, *see* D. G. Schaeffer [12] for $n=1$ and B. Gaveau [4] for $n=2$ where n is the space dimensions.

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