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Removable singularities for the Yang-Mills-Higgs equations in two dimensions

by

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ABSTRACT. — We prove that under a holonomy decay condition; with L^1 growth of curvature and integral growth bounds on the Higgs field (depending on the sign of the coupling constant) that isolated point singularities of the Yang-Mills-Higgs equations on a vector bundle over a 2-dimensional manifold are removable by a smooth gauge transformation.

Key words : Yang-Mills-Higgs, removable singularities, elliptic regularity, holonomy.

RÉSUMÉ. — Sujet à la condition d'une holonomie asymptotiquement triviale, avec croissance de courbure dans L^1 et des bornes sur 1 a croissance du champ de Higgs (qui dépendent du signe de la constante de couplage), données par des intégrales, nous prouvons que les points singuliers isolés des équations de Yang-Mills-Higgs d'un espace fibré vectoriel sur une variété de dimension 2 peuvent être enlevés par une transformation de jauge lisse.

Classification A.M.S. : 53 C, 35 J.

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1. INTRODUCTION

In this paper we prove a removable singularities theorem for the coupled Yang-Mills-Higgs equations over a two dimensional base manifold M. This problem is local so at no loss of generality we assume that $M = B_4^2 - \{0\}$, where $B_4^2 - \{0\}$ is the punctured 2-ball of radius 4 centered at the origin. We also assume that every connection has some gauge in which it is C^1 over the *punctured* ball.

Let M be a domain in R^2 and η be a vector bundle over M with compact structure group $G \subset O(n)$ and Lie algebra $\mathfrak{G}$. Let the metric on G be induced by the trace inner product on $O(n)$ and let η have a metric compatible with the action of G. Let d be exterior differentiation, δ its adjoint, and let [,] denote the Lie bracket in $\mathfrak{G}$.

A connection determines a covariant derivative D which within a local trivialization defines a Lie algebra valued 1-form A by $D = d + A$. On p-forms we have locally $D\omega = d\omega + [A, \omega]$, $D^*\omega = \delta\omega + {}^*[A, {}^*\omega]$, where D^* is the adjoint of D. We denote the curvature 2-form by F and have $F = dA + \frac{1}{2}[A, A]$ in this local trivialization.

Gauge transformations are sections of Aut η which act on connections and curvature forms according to the transformations:

$$A^g = g^{-1} A g + g^{-1} dg$$
$$F^g = g^{-1} F g.$$

The pair (A, F) is gauge equivalent to $(\bar{A}, \bar{F})$ iff there is a gauge transformation g such that $\bar{A} = A^g$ and $F = F^g$.

We now follow [Sb2] *exactly* and define the Higgs field φ using the determinant bundle. We denote by L the determinant bundle of η raised to the $\frac{1}{2}$-power. Sections of this bundle are constant in a fixed co-ordinate system but have weight 1 under scale transformations.

The Higgs field φ is a section of $\eta \otimes L$. Therefore, in a fixed co-ordinate system φ may be regarded as a matrix-valued function. Under scale charges $y = rx$, $\varphi(y) = \frac{\varphi(x)}{r}$ (*cf.*: [P], [SB2]).

The Yang-Mills-Higgs equations are:

(YMH 1) $$D^* F = [D\varphi, \varphi]$$

(YMH 2) $$D^* D\varphi = \frac{\lambda}{2}(|\varphi|^2 - m^2)\varphi;$$

where λ is a fixed real constant and where m is a section of L constant in a fixed co-ordinate system but having weight 1 under scale changes. Thus under the transformation $y=rx$ we have $m'=m/r$. The equations (YMH1, 2) are thus invariant under the scale transformation $y=rx$.

Certain norms are invariant under scale transformations. For example $\|\varphi\|_{L^2}$ is invariant and if ψ is any p-form $\|\psi\|_{L^{2/p}}$ is invariant. We also have an important fact used in [U1].

Fact [*U*1]

Let ψ be a p-form and suppose $\psi\in L^{2/p}$ with $\|\psi\|_{L^{2/p}}$ invariant. Then, given a domain D in R^2 and $\gamma>0$ there is a metric g_0 conformally equivalent to the Euclidean metric in which on bounded sets in R^2; $\int_D |\psi|^{2/p}\,dx<\gamma$.

This fact follows from conformal invariance and the continuity of the L^p-norms. *See* [UF] for details.

1.b. Statement of the Main Theorem

Now we state our Main Theorem:

THEOREM M. – *Let* $M=B_4^2-\{0\}$ *and let* η *be as above. Let* Λ *be a connection on* η *that satisfies condition* H, *defined in section* 1.c. *Let* F *be the curvature form of* Λ *and let* F *be* C^∞ *over* M. *Let* (F, φ) *satisfy* (YMH1) *and* (YMH2) *over* M. *Let* $F\in L^1(B_4^2)$.

If $\lambda\geqq 0$ *let* $\varphi\in H_2^1(B_4^2)$. *If* $\lambda<0$ *let* $\varphi\in L^{2+\varepsilon}(B_4^2)$ *and* $\overline{\lim_{t\to 0}}\int_{B_1/B_t}\frac{|\varphi|^2}{|x|^2\log^2(1/t)}=0$. *Then, there exists a continuous gauge transformation such that* (F, φ) *is gauge equivalent to a* C^∞*-pair over* B_4^2 *and the bundle extends continuously to a bundle over* B_4^2.

A theorem of this type was first proved by K. Uhlenbeck for the pure Yang-Mills equations over R^4 in [U1]. Later Parker [P] extended the result to the coupled Yang-Mills-Higgs equations over R^4. Papers of L. M. and R. J. Sibner [SB1], [SB2], [SB3] proved similar theorems for dimension 3 and for all higher dimensions. This paper fills the two-dimensional gap in the literature.

We would like to thank L. M. Sibner for suggesting this problem and C. Taubes for a useful abelian example suggesting that holonomy would be important.

1.c. Auxiliary Gauges

Condition H

We wish to introduce a condition on the connection Λ that insures that the bundle is trivial over the punctured disk M above. This condition is a "holonomy" condition. We call it condition H.

We use the conventions of [KN1], Vol. 1, pp. 71-72. We first define some useful paths.

DEFINITION. — Let $l_R : [0, 1] \to S^1_R$ be given by $l_R : t \mapsto (R \cos 2\pi t, R \sin 2\pi t)$ with $S^1_R = \{ x \in R^2 \mid |x| = R \}$. We say that l_R is the standard loop for S^1_R. Let $L_\theta : [0,1] \to R$ be given by $L_\theta : t \mapsto (R t, \theta)$. We call L_θ the standard ray.

For each R, let $g(R)$ be the holonomy of Λ around the loop l_R.

DEFINITION 1.1. — The map $C_R : (0,4] \to G$ given by $R \to g(R)$ is a path denoted by C_R.

Now, we define condition H.

DEFINITION 1.2 (condition H). — If as $R \downarrow 0$ the elements $g(R)$ considered as points on the carrier of the path C_R approach the identity element we say the connection satisfies *condition* H.

THEOREM 1.1. — *The following is equivalent to condition* H: *There exists a trivialization over a small ball* $B_{R_0} - \{0\}$, $\exists_{R_0}$, $0 < R_0 \leqq 4$ *centered at the origin, in which the connection defines a local co-variant derivative*

$$D = d + A, \qquad A = A_r(r, \theta)\, dr + A_\theta(r, \theta)\, d\theta$$

with

$$A_r(r, \theta),\ A_\theta(r, \theta) \in \Gamma(\mathfrak{G} \otimes T^*(B_{R_0} - \{0\}))$$

and with $\lim_{r \to 0} A_\theta(r, \theta) = 0$, with the limit taken in the sup-norm topology on $\mathfrak{G}$.

Proof (1 → 2). — Choose an orthonormal framing $\{v_i(r, \theta)\}$ of η over the ray $\{(r, 0 \mid 0 \leqq \varepsilon\}$. Extend this to a framing $\{v_i(r, \theta)\}$ by parallel translation around the circles l_R. Then, $\nabla_\theta v_i = 0$, and holonomy appears. Thus, $v_i(r, 2\pi) = v_i(r, 0).g(r)$ for some $g(r) = g(r, 2\pi) \in G$. The hypothesis implies that for small ε, the element $g(r)$ is close to the identity so that $g(r) = \exp(h(r))$ for some $h(r) \in \mathfrak{G}$. Let $\varphi : [0,2\pi] \to [0,1]$ be a smooth function which vanishes near 0 and is 1 near 2π. Then $w_i(r, \theta) = v_i(r, \theta).\exp(-\varphi(\theta) h(r))$ is a smooth orthonormal framing of η over $B_2 - \{0\}$. In this framing the connection form is:

$$(A_\theta)^i_j = \langle \nabla_\theta w_i, w_j \rangle = \langle [\nabla_\theta (v_i.\exp(-\varphi(\theta) h(r))], w_j \rangle = -\varphi'(\theta) h(r) \delta_{ij}.$$

Hence $|A_\theta| \leqq c|h(r)| \downarrow 0$ as $r \downarrow 0$. $(2 \rightarrow 1)$. This follows from standard O.D.E. estimates on integrating the parallel transport equation for each horizontal lift of l_R.

Q.E.D.

Remark 1.1. – Thus condition H implies that the bundle η is trivial over $B^2_{R_0} - \{0\}$.

Remark 1.2. – Condition H is not implied by any L^p-condition on F, but, in Corollary 1.1 it is shown that an L_{1+r} bound on F for any $r>0$ implies the existence of a unique limit for $\{g(R)\}''_{R \rightarrow 0}$. We give an example of a bundle, trivial over the punctured ball with zero curvature form F.

We consider the rank-2 bundle over $B_{R_0} - \{0\}$ with $G = SO(2, R)$, $\mathcal{G} = \mathcal{SO}(2, R)$, $A = A_r(r, \theta)\,dr + A_\theta(r, \theta)\,d\theta$; where $A_r(r, \theta) = \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}$ and

$A_\theta(r, \theta) = \begin{bmatrix} 0, & \frac{1}{3} \\ -\frac{1}{3}, & 0 \end{bmatrix}$. Note that $F = dA + \frac{1}{2}[A, A] = 0$ which is in all L^P, $1 \leqq P \leqq \infty$.

Assume that condition H holds, then there exists, by Theorem 1.1, a new trivialization in which the connection form becomes

$$\tilde{A} = \tilde{A}_r, \theta)\,dr + \tilde{A}_\theta(r, \theta)\,d\theta$$

where $\lim_{r \rightarrow 0} A_\theta(r, \theta) = 0$ with the limit taken in the supremum topology on $\mathcal{G}$. The gauge transformation equation gives:

$$g^{-1}(r, \theta)\frac{\partial g(r, \theta)}{\partial\theta} + g^{-1}(r, \theta)\,A_\theta(r, \theta)\,g(r, \theta) = \tilde{A}_\theta(r, \theta)$$

for some $g(r, \theta) \in SO(2, R)$. Let

$$g(r, \theta) = \begin{bmatrix} \cos f(r, \theta) \sin f(r, \theta) \\ -\sin f(r, \theta) \cos f(r, \theta) \end{bmatrix}$$

since $g(r, \theta)$ and $A_\theta(r, \theta)$ commute we have

$$\frac{dg(r, \theta)}{d\theta} = g(r, \theta)[\tilde{A}_\theta - A_\theta]$$

Let $\tilde{A}_\theta(r, \theta) = \begin{bmatrix} 0\,b(r, \theta) \\ -b(r, \theta)\,0 \end{bmatrix} \in \mathcal{SO}(2, R)$. Direct computation now gives

$$\begin{bmatrix} \frac{\partial f(r, \theta)}{\partial\theta} & 0 \\ 0 & \frac{\partial f(r, \theta)}{\partial\theta} \end{bmatrix} = \begin{bmatrix} b(r, \theta) - \frac{1}{3} & 0 \\ 0 & b(r, \theta) - \frac{1}{3} \end{bmatrix}.$$

Thus: $\frac{\partial f(r, \theta)}{\partial \theta} = b(r, \theta) - \frac{1}{3}$

$$f(r, \theta) = \int_0^\theta b(r, \zeta)\, d\zeta - \frac{1}{3}\theta + \mathrm{P}(r); \ \exists\, \mathrm{P}(r) : \mathrm{R}^1 \to \mathrm{R}^1.$$

Let $\theta = 0$, we see that $\mathrm{P}(r) = f(r, 0)$.

Note that since $\lim_{r \to 0} \tilde{\mathrm{A}}_\theta(r, \theta) = 0$ we have $\lim_{r \to 0} b(r, \theta) = 0$ and thus $f(r, 0) \neq f(r, 2\pi)$ (Mod 2π).

Thus, $g(r, \theta) = \begin{bmatrix} \cos f(r, \theta) & \sin f(r, \theta) \\ -\sin f(r, \theta) & \cos f(r, \theta) \end{bmatrix}$ is not well-defined as an element of SO(2, R). We have obtained a contradiction that shows that condition H is not satisfied although $\mathrm{F} = 0$.

Q.E.D.

The Auxiliary Gauge

We will give in Section 3 a gauge-independant proof that under the conditions of Theorem M, the curvature F is actually in $\mathrm{L}_p(\mathrm{B}_\mathrm{R})$ for $1 \leq p < \infty$ if R is small enough, in any smooth gauge over $\mathrm{B}_\mathrm{R} - \{0\}$. This estimate, coupled with the existence of an "auxillary" gauge in which the connection form A is L_p-norm close to zero (flat connection), will enable us to use a new gauge-fixing argument [U3] of Uhlenbeck to build a Coulomb gauge over $\mathrm{B}_\mathrm{R} - \{0\}$, bypassing the original broken Hodge gauge argument of [U1]. Thus this paper is *much simplified* compared to the author's Max-Planck preprint [S] which preceded it.

In this section we construct the "auxillary" gauge and show that the L_p-norm of the induced connection form is small.

LEMMA 1.1. — *Under the conditions of Theorem* 1.1, *let the connexion satisfy condition* H. *Then, there exists a local trivialization in which the connection induces the local co-variant derivative* $\mathrm{D} = d + \mathrm{A}$, $\mathrm{A} := \mathrm{A}_r(r, \theta)\, dr + \mathrm{A}_\theta(r, \theta)\, d\theta$ *and we have:*

$$\lim_{r \to 0} \mathrm{A}_r(r, 0) = 0, \qquad \lim_{r \to 0} \mathrm{A}_\theta(r, \theta) = 0, \qquad \lim_{r \to 0} \frac{d}{dr}(\mathrm{A}_\theta(r, \theta)) = 0.$$

Proof. — We start with the orthonormal framing v_i over the standard ray L_θ used in the beginning of the proof or Theorem 1.1. We use this framing to give a local trivialization for the bundle restricted to have the standard ray as a base space. The connection restricts and we denote the restricted connection by ∇_r. This connection defines

$$\{\tilde{\mathrm{A}}_r(r, 0)\}^i_j := \langle \nabla_r v_i(r, 0), v_j(r, 0) \rangle.$$

Now we define $\hat{s}(r) \in G$ as the solution to:

$$\frac{d\hat{s}(r)}{dr} = -\tilde{A}_r(r, 0)\hat{s}(r), \ \hat{s}(R_0) = I, \qquad \exists R_0,\ 0 < R_0 < 1.$$

Now define $\bar{v}_i(r, 0) :\ = v_i(r, 0) . \bar{s}(r)$.

Note that

$$\{\bar{A}_r(r, 0)\}^i_j := \langle \nabla_r \bar{v}_i(r, 0), \bar{v}_j(r, 0) \rangle = \hat{s}^{-1}(r)\tilde{A}_r(r, 0)\hat{s}(r) + \hat{s}^{-1}(r)\frac{d\hat{s}(r)}{dr}$$

and thus; $\lim_{r \to 0} \{\bar{A}_r(r, 0)\}^i_j = 0 = \lim_{r \to 0} \langle \nabla_r \bar{v}_i(r, 0), \bar{v}_j(r, 0) \rangle$.

Now carry out the proof of Theorem 1.1 with $\{v_i\}$ replaced by $\{\bar{v}_j\}$. Note that in the gauge constructed for which $\lim_{r \to 0} A_\theta(r, \theta) = 0$ we have

$$\lim_{r \to 0} \{A_r(r, 0)\}^i_j = \lim_{r \to 0} \langle \nabla_r w_i(r, 0), w_j(r, 0) \rangle$$

$$= \lim_{r \to 0} \langle \nabla_r \bar{v}_i(r, 0), \bar{v}_j(r, 0) \rangle = 0$$

Now we show that, in fact Condition H and $F \in L^1$ imply

$$\lim_{r \to 0} \frac{d}{dr}(A_\theta(r, \theta)) = 0.$$

We apply the argument [Karch] p. 92, line 12-15; p. 93, line 8-15 to the bundle η with our fiber norm as above and to the curves given in Figure 1. On page 93, line 9, we do *not* pull the norm of curvature through the integral sign. *See* our Figure 1.

The homotopy between $c_1(S)$ and $c_2(S)$ is given by: $(0 \leqq S \leqq 2\pi,\ r_1 \leqq t \leqq r_2)$ for any $\bar{\varepsilon}$ with $0 \leqq \bar{\varepsilon} \leqq 2\pi$.

$$c_t(S) = \left\{ \begin{array}{ll} \left(0, \left(\frac{t - r_1}{\bar{\varepsilon}}\right) S + r_1\right); & 0 \leqq S \leqq \bar{\varepsilon} \\ \left(\left(\frac{2\pi}{2\pi - 2\bar{\varepsilon}}\right) S + \frac{2\pi\bar{\varepsilon}}{2\bar{\varepsilon} - 2\pi}, t\right); & \bar{\varepsilon} \leqq S \leqq 2\pi - \bar{\varepsilon} \\ \left(0, \left(\frac{r_1 - t}{\bar{\varepsilon}}\right) S + \frac{2\pi(t - r_1)}{\bar{\varepsilon}} + r_1\right); & 2\pi - \varepsilon \leqq S \leqq 2\pi; \end{array} \right\}$$

where $(-,\ -)$ is the polar coordinate of the image in R^2.

Geometrically c_t goes linearly in S from p to μ along $\overline{p\mu}$ when $0 \leqq S \leqq \bar{\varepsilon}$, linearly in S clockwise around the circle of radius t when $\bar{\varepsilon} \leqq S \leqq 2\pi - \bar{\varepsilon}$, then linearly in $\bar{S}$ from, μ to p along $\overline{\mu p}$ when $2\pi - \varepsilon \leqq S \leqq 2\pi$.

Remark. — Note that when applying the argument of [Karch] to this homotopy, in *Karcher's Notation*, we have [Karch, p. 93, line 15];

$$\int_0^{2\pi}\int_{r_1}^{r_2} \mathrm{R}\left(\frac{\partial}{\partial \mathrm{S}}c, \frac{\partial}{\partial t}c\right)\mathrm{X}(s, t)\,ds\,dt \leqq \int_0^{2\pi}\int_{r_1}^{r_2} \|\mathrm{R}\| \left|\frac{\partial}{\partial \mathrm{S}}c \wedge \frac{\partial}{\partial t}c\right|$$

$$ds\,dt = \int_{\bar{\varepsilon}}^{2\pi-\bar{\varepsilon}}\int_{r_1}^{r_2} \|\mathrm{R}\| \left|\frac{\partial}{\partial \mathrm{S}}c \wedge \frac{\partial}{\partial t}c\right| r\,dr\,d\theta \leqq c \,.\int_0^{2\pi}\int_{r_1}^{r_2} \|\mathrm{R}\|\, r\,dr\,d\theta.$$

(Here in Karcher's notation $\frac{\partial}{\partial \mathrm{S}}c$ and $\frac{\partial}{\partial t}$ are tangent vector fields to the homotopy, *because the wedge product vanishes when either* $0 \leqq \mathrm{S} \leqq \bar{\varepsilon}$ *or* $2\pi - \varepsilon \leqq \mathrm{S} \leqq 2\pi$.)

Thus, since R is Karcher's notation for vector bundle curvature we obtain:

$$|v^1 - v^2|_{\text{fiber}} \leqq \mathrm{C}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})} \qquad \text{where } v^1 \text{ is any parallel}$$

transport of a vector W, in our lifted frame above p, around c_1 and v^2 is any parallel transport of the same vector W around the path c_2. Here $r_1 \leqq r \leqq r_2 < \tau < \mathrm{R}_0$ with R_0 as in Theorem 1.1.

Since the action of G carries parallel transport in η we have: (with slight change of notation to conform to the usual conventions for matrix representations of leftwise multiplication). Thus

$$v^1 = [g(r_1)]\,.\,\mathrm{W} \qquad \text{and} \qquad v^2 = [\tilde{g}]^{-1}[g(r_2)][\tilde{g}]\,.\,\mathrm{W}$$

and we have:

$$|[g(r_1)]\,.\,\mathrm{W} - [\tilde{g}]^{-1}[g(r_2)][\tilde{g}]\,.\,\mathrm{W}|_{\text{fiber}} \leqq \mathrm{C}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})}$$

(here $[\tilde{g}]$ carries parallel transport over the line segment $p\sigma$ (This makes sense since h is trivial over $p\sigma$ and our framing gives a canonical meaning to this construction.); and [ ] refers to a matrix representation of the group action. Thus:

$$|([g(r_1)] - [\tilde{g}]^{-1}[g(r_2)][\tilde{g}])\,.\,\mathrm{W}|_{\text{fiber}} \leqq \mathrm{C}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})}$$

Thus:

$$\langle ([g(r_1)] - [\tilde{g}]^{-1}[g(r_2)][\tilde{g}])^{\mathrm{T}} \times ([g(r_1) - [\tilde{g}][g(r_2)][\tilde{g}])\,.\,\mathrm{W}, \mathrm{W}\rangle^{1/2} \leqq \tilde{\mathrm{C}}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})}.$$

Now using theorem (19) p. 73, line 8-10 [Bell] (the largest eigenvalue λ_1 of a real symmetric matrix A is given by: $\lambda_1 = \max_{(x,\, x)=1} \langle x, \mathrm{A}x\rangle$ we obtain:

$$\|[g(r_1)] - [\tilde{g}]^{-1}[g(r_2)][\tilde{g}]\|_{\text{trace inner product norm}} \leqq \tilde{\mathrm{C}}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})}.$$

Now, we will estimate $\| [\tilde{g}] \|_M$ from above and below.

We denote the group element that carries parallel transport over the line segment $\overline{p, (r, 0)}$ by $g(r)$, here $p=(r_1, 0)$ and $0<r_1<r<r_2<\tau<R_0$. This makes sense because η is *trivial* over this line segment (even over the *punctured* disk), hence our framing gives a canonical meaning to this construction. Using the parallel transport equation:

$$\frac{d[\tilde{g}(r)]}{dr} = -A_r(r, 0)[\tilde{g}(r)], \qquad [\tilde{g}(r_1)]=I$$

and since $\lim_{r\to 0} A_r(r, 0)=0$.

we have: $[\tilde{g}(r)]=[\tilde{g}(r_1]+\int_{r_1}^{r} -A_r(S, 0)[\tilde{g}(S)]\,dS$

$$\| [\tilde{g}(r)] \|_M \leqq \| [\tilde{g}(r_1)] \|_M + C\int_{r_1}^{r} \| A_r(S, 0 \|_{\text{trace inner product norm}} \| [\tilde{g}(S)] \|_M \, dS$$

(We have used equivalence of finite dimensional norms.) Here $\| \;\; \|_M$ is the matrix norm given by the largest absolute value of an entry of the matrix. *Cf.* [H] lemma 4.1, p. 54 with $f=0$.

Now by (standard) Gronwalls inequality we have:

$$\| [\tilde{g}(r)] \|_M \leqq 1\, e^{\tilde{C}\int_{r_1}^{r} \| A_r(S, O) \| dS_{\text{trace inner product norm}}} \leqq e^{\tilde{\beta}\cdot(r-r_1)}$$

where $0<r_1\leqq r\leqq r_2<\tau<R_0$ and $\tilde{\beta}\downarrow 0$ as $\tau\downarrow 0$.

Thus, we have:

upper bound:

$$\| [\tilde{g}(r)] \|_M \leqq 1+\tilde{\beta}(r-r_1)+0(\tilde{\beta}.(r-r_1))$$

where $0<r_1\leqq r\leqq r_2<\tau<R_0$ and $\tilde{\beta}\downarrow 0$ as $\tau\downarrow 0$.

Now we obtain the lower bound. Since,

$$\frac{d[\tilde{g}(r)]}{dr} = -A_r(r, 0)[\tilde{g}(r)] \qquad \text{for} \quad r_1\leqq r\leqq r_2\leqq\tau$$

we have:

$$[\tilde{g}(r)]=[\tilde{g}(r_1)]+\int_{r_1}^{r} -A_r(S, 0)[\tilde{g}(S)]\,dS$$

$$\| [\tilde{g}(r)] \|_M = \| [\tilde{g}(r_1)] + \int_{r_1}^{r} A_r(S, 0)[\tilde{g}(S)]\,dS \|_M$$

$$\geqq \| [\tilde{g}(r_1)] \|_M - \left\| \int_{r_1}^{r} A_r(S, 0)[\tilde{g}(S]\,dS \right\|_M$$

(Since $\| x+y \| \geqq \| x \| - \| y \|$).

Thus

$$\|[\tilde{g}(r)]\|_M \geqq \|[\tilde{g}(r_1)]\|_M - \int_{r_1}^{r_2} \|A_r(S,0)\|_{\text{trace inner product norm}} \|[\tilde{g}(S)]\|_M \, dS$$

$$\geqq \|[\tilde{g}(r_1)]\|_M - \hat{\beta} \int_{r_1}^{r} \|[\tilde{g}(S)]\|_M \, dS$$

where $r_1 \leqq r \leqq r_2 \leqq \tau < R_0$ and $\hat{\beta} \downarrow 0$ as $\tau \downarrow 0$.

Now let

$$I(r) = \int_{r_1}^{r} \|[\tilde{g}(S)]\|_M \, dS$$

$$\frac{dI(r)}{dr} \geqq \|[\tilde{g}(r_1)]\|_M - \hat{\beta}\, I(r) = 1 - \hat{\beta}\, I(r)$$

since $[\tilde{g}(r_1)] = I$.

Integration gives:

$$\max_{[r_1, r]} \|[\tilde{g}(S)]\|_M \geqq \frac{1}{\hat{\beta}(r-r_1)} \cdot (1 - e^{-\hat{\beta}(r-r_1)})$$

$$\max_{[r_1, r]} \|[\tilde{g}(S)]\|_M \geqq 1 - \hat{\beta}(r-r_1) + o(\hat{\beta}(r-r_1))$$

since $A_r(S, 0)$ and $[\tilde{g}(S)]$ are continuous on $[r_1, r_2]$ so is $\dfrac{d[\tilde{g}(S)]}{dS}$. Thus $\left\| \dfrac{d[\tilde{g}(S)]}{dS} \right\|_M$ is bounded above on $[r_1, r_2]$. Thus $[\tilde{g}(S)]$ is Lipshitz continuous on $[r_1, r_2]$ in the $\| \;\; \|_M$ norm with Lipshitz constant L. L satisfies $L \downarrow 0$ as $\tau \downarrow 0 \; (r_1 \leqq r \leqq r_2 < \tau < R_0)$ since

$$L \cong \max_{[r_1, r_2]} (\|A_r(S,0)\|_{\text{trace inner product norm}} \cdot \|[\tilde{g}(S)]\|_M \leqq \beta . 2$$

by our upper Gronwall estimate on $\|[\tilde{g}(S)]\|_M$ above.

Thus, since

$$\max_{[r_1, r]} \|[\tilde{g}(S)]\|_M \leqq \max_{[r_1, r]} \|[\tilde{g}(S)] - [g(r)]\|_M + \|[g(r)]\|_M$$

$$\leqq L\,|\bar{S} - r_1| + \|g(r)\|_M \leqq 3\beta\,|r - r_1| + \|g(r)\|_M$$

(for some $\bar{S}$, $r_1 < \bar{S} < r_2$) we have:

lower bound

$$\|[\tilde{g}(r)]\|_M \geqq 1 - \gamma(r - r_1) \; (0 < r_1 \leqq r < r_2 < \tau < R_0)$$

where $\gamma \downarrow 0$ as $\tau \downarrow 0$ (if τ is small enough).

At no loss of generality we choose our matrix representation of $G \subset O(n)$ so that each $[g]$ in the component of the identity is given by a matrix in block form with 2×2 blocks given as rotations like $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$, and 1×1 blocks given as $[1]$ or $[-1]$. If $[g]$ is close to the identity matrix we use (standard) a two term taylor approximation in the angular parameters, with a third term as the remainder, for the matrix elements. This is often called the "infinitesimal rotation representation" for elements close to the identity. From the above estimates it follows that (by direct elementary calculations)

$$(*) \qquad \|[\tilde{g}]-\mathrm{I}\|_{\mathrm{M}} \leqq (r_2-r_1)\,\beta_1(r) \qquad \text{with} \quad \beta_1 \downarrow 0$$

as $\tau\downarrow 0$, $0<r_1\leqq r\leqq r_2<\tau<\mathrm{R}_0$.

Now since

$$\begin{aligned} &\|[\tilde{g}(r_1)]-[\tilde{g}(r_2)]\|_{\text{trace inner product norm}} \\ &\qquad -\|[\tilde{g}]^{-1}[\tilde{g}(r_2)][\tilde{g}]-[g(r_2)]\|_{\text{trace inner product norm}} \\ &\qquad \leqq \|[g(r_1)]-[\tilde{g}]^{-1}[g(r_2)][\tilde{g}]\|_{\text{trace inner product norm}} \\ &\qquad\qquad \leqq \tilde{\mathrm{C}}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})} \end{aligned}$$

$$\begin{aligned} &\|[g(r_1)]-[g(r_2)]\|_{\text{trace inner product norm}} \\ &\qquad \leqq \mathrm{C}\,\|[\tilde{g}]^{-1}[\tilde{g}(r_2)][\tilde{g}]-[g(r_2)]\|_{\mathrm{M}}+\mathrm{C}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})}. \end{aligned}$$

Now using estimate $(*)$ and the "infinitesimal" form of $[\tilde{g}]$ as above we see that:

$$\|[\tilde{g}]^{-1}[g(r_2)][\tilde{g}]-[g(r_2)]\|_{\mathrm{M}} \leqq O(r_2-r_1)\,.\,\tilde{\beta}(r)$$

where $\tilde{\beta}(r)\geqq 0$ and $\lim_{r\to 0}\tilde{\beta}(r)=0$. Thus:

$$\|[g(r_1)]-[g(r_2)]\|_{\text{trace inner product norm}} \leqq \tilde{\mathrm{C}}\,\|\mathrm{F}\|_{\mathrm{L}^1(\mathrm{B}_{r_2}-\mathrm{B}_{r_1})}+O(r_2-r_1)\,.\,\tilde{\beta}(r).$$

where $\tilde{\beta}\downarrow 0$ as $\tau\downarrow 0$, $0<r_1\leqq r\leqq r_2<\tau<\mathrm{R}_0$.

Using Hölder's inequality on the first right hand term we get:

$$\begin{aligned} &\|[g(r_1)]-[g(r_2)]\|_{\text{trace inner product norm}} \\ &\qquad \leqq k\,[(r_2-r_1)(r_2+r_1)]^{1+\varepsilon}\,\|\mathrm{F}\|_{\mathrm{L}^p} \\ &\qquad\quad +O(r_2-r_1)\,.\,\tilde{\beta}(r), \qquad \exists\,\varepsilon>0, \quad \exists\,p \text{ with } 2<p<\infty. \end{aligned}$$

Now, letting $0<r_1\leqq r\leqq r_2<\tau<\mathrm{R}_0$ and letting r_1-r_2 tend to zero we obtain

$$\left\| \lim_{r_1\to r_2}\left(\frac{[g(r_1)]-[g(r_2)]}{r_1-r_2}\right)\right\|_{\text{trace inner product norm}} \leqq \gamma(r),$$

with $\gamma(r)\geqq 0$ and $\gamma\downarrow 0$ as $\tau\downarrow 0$, $0<r_1\leqq r\leqq r_2<\tau<\mathrm{R}_0$ so that

$$\lim_{r\to 0}\left\|\frac{d}{dr}[\exp(h(r))]\right\|_{\text{trace inner product}} = 0$$

thus

$$\lim_{r \to 0} |h'(r)| \, \| [g(r)] \|_{\text{trace inner product}} = 0$$

But $\| [g(r)] \|_{\text{trace inner product}} = \sqrt{n}$, since $[g(r)] \in G \subset O(n)$.

Thus:

$$\lim_{r \to 0} h'(r) = 0$$

(we have used equivalence of finite dimensional norms).

Now note that; $\lim_{r \to 0} \left\{ \frac{d}{dr} (A_\theta(r, \theta)) \right\}_j^i = \lim_{r \to 0} (h'(r) \varphi'(\theta) \delta_{ij}) = 0;$

Since this follows from the formula for $(A_\theta(r, \varphi))^i_j$. [line 8 of the Proof (1-2)] in the proof of Theorem 1.1.

Q.E.D.

Remark. — Although Remark 1.2 shows that condition H is not implied by F in any L^P; $1 \leqq P \leqq \infty$ we show that if $F \in L_{1+\varepsilon}$ then holonomy limits (not necessarily the identity element) exist as $r \downarrow 0$.

COROLLARY 1.1. — *Let* $F \in L_{1+\varepsilon}(B_{R_0})$ *for some* $\varepsilon > 0$, *then* $\lim_{r \to 0} g(r)$ *exists.*

Proof. — We estimate the quantity $|g(r_1) - g(r_2)|_{\text{trace inner product norm}}$. First we assume the bundle is trivial over $B_{R_0} - \{ 0 \}$; we will remove this assumption at the end of the proof. Since this norm is $O(n)$-invariant we can, at no loss of generality, choose to evaluate it in a gauge where $\lim_{r \to 0} A(r, 0) = 0$. This gauge is constructed exactly as in the proof of Lemma 1.1, lines 1-11. Now the proof of Lemma 1.1 lines 23-140 followed exactly gives that

$$\| g(r_1) - g(r_2) \|_{\text{trace inner product norm}} \leqq C \| F \|_{L^1(B_{r_2} - B_{r_1})} + O(r_2 - r_1) . \tilde{\beta}(r)$$

where $\tilde{\beta}(r) \downarrow 0$ as $r \to 0, 0 < r_1 < r < r_2 < \tau < R_0$.

Since $F \in L^{1+\varepsilon}$ we have by Hölder that

$$\| g(r_1) - g(r_2) \|_{\text{trace inner product norm}} \leqq \tilde{C} Q(r_1 - r_2)$$

where $Q(r_1 - r_2) \downarrow 0$ uniformly as $r_1 - r_2 \downarrow 0$.

Thus $g(r)$ is uniformly continuous on $(0, \tau)$ and has a unique limit and extension to $[0, \tau]$. Hence $\lim_{r \to 0} g(r)$ exists.

Now we drop the condition of triviality.

If the bundle is non-trivial we note that although the curvature on the bundle pulls down to several curvature forms over $B_{R_0} - \{ 0 \}$ and similarly for the connection form, the global definition of the curvature integral in

[Karcher] (*see* the Remark in our proof of Lemma 1.1 this paper), the gauge invariance of the L_1-norm of F, preserve the estimate $|v^1 - v^2|_{\text{fiber norm}} \leqq C \| F \|_{L^1(B_{r_2} - B_{r_1})}$.

The topology of bundles over $B_{R_0} - \{0\}$ is very simple since the base is homotopic to S^1. These bundles are classified by maps from the point $(R_0, 0)$ into G.

We can choose the local trivializations inducing pulled-down connection forms such that when $0 \leqq \theta < \theta_0 < \pi$ we have $A^1(r,\theta) = A_r^1(r,\theta)\,dr + A_\theta^2(r,\theta)\,d\theta$ and when $2\pi - \theta_1 < \theta \leqq 2\pi$ we have $A^2(r,\theta) = A_r^2(r,\theta)\,dr + A_\theta^2(r,\theta)\,d\theta$. The only change in our argument is that in the proof of Lemma 1.1 (*see* line 53) we have $v^2 = [\tilde{g}_2][g(r_2)][g].w$ where $[\tilde{g}_2]$ carries the parallel transport along the line segment σP when $\theta = 2\pi$. To estimate $\max_{[r_1, r_2]} \| [\tilde{g}_2(r)] \|_M$ and $\min_{[r_1, r_2]} \| [\tilde{g}_2(r)] \|_M$ we carry out the previous argument given in this remark for $A^2(r,\theta)$ exactly as we did before for $A(r,\theta)$ in the trivial bundle case. We obtain the same uniform estimate for $\| [g(r_1)] - g[(r_2)] \|_{\text{trace inner product norm}}$ hence $\lim_{r \to 0} [g(r)]$ exists.

Q.E.D.

DEFINITION 1.3. — We call the gauge defined by Lemma 1.1 the auxiliary gauge.

LEMMA 1.2. — *Let the conditions of Theorem* 1.1 *hold. Let the connection satisfy condition* H. *Let the curvature be in* $L^P(B_{R_0})$ $1 \leqq P < \infty$. *Then in the auxiliary gauge we have:* $\int_0^R |A_r(r,\theta)|^P r\,dr < \infty$, $0 < R < R_0$.

Proof. — In the auxiliary gauge we have:

$$\frac{\partial A_r}{\partial \theta} - \frac{\partial A_\theta}{\partial r} + \frac{1}{2}[A_r, A_\theta] = F_{r,\theta}$$

and

$$\int_0^{2\pi} \int_0^{R_0} \frac{|F_{r,\theta}|^P}{r^P} . r\,dr\,d\theta = \| F \|_{L^P(B_{R_0})}.$$

Fix S, $0<S\leqq R_0$ and integrate:

$$A_r(S,\theta)=A_r(S,0)+\int_0^\theta \frac{\partial A_\theta}{\partial r}(S,t)\,dt - \frac{1}{2}\int_0^\theta [A_r(S,t),A_\theta(S,t)]\,dt-\int_0^\theta F_{r,\theta}(S,t)\,dt$$

$0\leqq\theta<2\pi$.
Thus:

$$|A_r(S,\theta)|\leqq|A_r(S,0)|+\int_0^\theta\left|\frac{\partial A_\theta(S,t)}{\partial r}\right|dt + \int_0^\theta |F_{r,\theta}(S,t)|\,dt+2\int_0^\theta |A_r(S,t)||A_\theta(S,t)|\,dt \quad \text{for all S;}$$

$0<S<R_0$.

Thus, by Lemma 1.1 and since $F\in L_P$, $1\leqq P<\infty$. We obtain (by elementary computations):

$$\int_{R/2}^R |A_r(S,\theta)|^P S\,dS\leqq 0(R)+k\int_0^\theta\left[\int_{R/2}^R |A_r(S,t)|^P S\,dS\right].|A_\theta(S,t)|^P\,dt,$$

$0<R<R_0$, with K independent of R.

Now, apply Gronwall's inequality, p. 189 [AMR], to get:

$$(*)\quad \int_{R/2}^R |A_r(S,\theta)|^P S\,dS\leqq 0(R)\exp\left[K.\int_0^\theta |A_\theta(S,t)|^P\,dt\right] \leqq \tilde{K}.0(R),\qquad 0<R<R_0,$$

with K and $\tilde{K}$ independent of R, since $\lim_{S\to 0}|A_\theta(S,t)|=0$.

Now applying $(*)$ with R replaced by $R/2^m$, $m=1,2,\ldots$ and summing we obtain:

$$\int_0^R |A_r(S,\theta)|^P S\,dS\leqq 0(R).$$

Q.E.D.

LEMMA 1.3. – *Under the hypothesis of Lemma* 1.2 *in the auxilliary gauge we have* $\|A\|_{L^P(B_R-\{0\})}<0(R)$, $1\leqq P<\infty$.

Proof. – Apply Lemma 1.1 to estimate $\|A_\theta\|_{L^P(B_R-\{0\})}$ and Lemma 1.2 to estimate $\|A_r\|_{L^P(B_R-\{0\})}$. We note that for each $\varepsilon>0\,\exists_\zeta$ with $0<\varepsilon<\zeta<R$ such that:

$$A_\theta(r,\theta)=A_\theta(\varepsilon,\theta)+\left[\frac{d}{dr}[(A_\theta(r,\theta)]\right]\Bigg|_{r=\zeta}.(r-\varepsilon).$$

First letting $\varepsilon \downarrow 0$ and then dividing by r, we obtain:

$$\left|\frac{A_\theta(r,\theta)}{r}\right| \leqq \sup_{0<\zeta<r} \left|\left[\frac{d}{dr}[A_\theta(r,\theta]\right]\right| \Bigg|_{r=\zeta} \downarrow 0 \text{ as } r \downarrow 0$$

by Lemma 1.1.

Q.E.D.

2. SOME IMPROVEMENTS ON MORREY'S THEOREM

In this section we state some improved versions of Morrey's theorem in 2-dimensions that will be used later.

First we state Morrey's theorem in 2-dimensions.

THEOREM 2.1 (Morrey's theorem in 2-dimensions) [MO]. — *Let* $u \in H_2^1(\Omega)$ *with* $u \geqq 0$ *and suppose that:* Ω *is a locally Lipshitz domain in* R^2, *and* $\int_\Omega \nabla u \nabla \xi + f.u\, dx \leqq 0$ *for all non-negative* $\xi \in C_0^\infty(\Omega)$. *Let f satisfy the Morrey Condition:*

$$\int_{B_R \subset \Omega} |f|^{1+\varepsilon} dx \leqq c R^\beta$$

for all $B_R \subset \Omega$ *and some* ε, $\beta > 0$ *then*

$$\sup_{B(x_0, \rho)} |u(x)|^2 \leqq \frac{c}{a^2} \int_{B(x_0, \rho+a)} |u(y)|^2 dy$$

for all $B(x_0, \rho) \subset B(x_0, \rho+a) \subset \Omega$.

Proof. — Identical to the proof of Theorem 5.3.1 of [MO], p. 137, except that we need our somewhat stronger Morrey condition because the inequality

$$\int g |w|^2 \leqq c_n \left[\int |\nabla w|^2 dx + \int |g|^{n/2} dx\right]$$

fails in 2-dimensions due to critical Sobolov exponents.

We would now like to note that if $u \in C^\infty(\Omega)$ we can state an improvement of Morrey's estimate involving $\frac{K}{a^2} \int_{B(x_0, \rho+a)} |u(y)| dy$. This improvement follows from an iteration argument of E. Bombieri. *See* [BO], p. 66.

THEOREM 2.2 (Bombieri). — *Let Ω be compact. Let $u \in C^\infty$ in Ω and let $u \geqq 0$. Let u satisfy:*

$$\sup_{B_\rho} (u(x))^2 \leqq \frac{c}{(R-\rho)^2} \int_{B_R} u^2 \, dx$$

for all concentric B_R, $B_\rho \subset \Omega$, $0 < \rho < R$. Then $\sup_{B_\rho} u(x) \leqq \frac{c}{(R-\rho)^2} \int_{B_R} u \, dx$ where B_R and B_ρ are as above.

Proof. — Use the iteration at the top of p. 66 of [BO].

Q.E.D.

3. A REGULARITY THEOREM FOR THE HIGGS FIELD

In this section we assume that the Higgs field is a C^∞ solution of the field equation:

$$\text{(YMH 2)} \; D^* D\varphi = \frac{\lambda}{2}(|\varphi|^2 - m^2)\varphi$$

in the punctured unit ball $B^2 - \{0\}$. As in [Sb 2] the assumptions on φ near the origin depend on the sign of λ.

Because of the criticality of the Sobolov exponent $\frac{2n}{n-2}$ for L_2 functions in 2-dimensions, we require several technical changes from the argument in [SB2]. This is where we use the estimates of section 2.

The main result of this section is:

THEOREM 3.1. — *Let φ be a C^∞ solution of* (YMH2) *in* $B^2 - \{0\}$ *in* R^2. *We assume:*

(*a*) $\varphi \in H_2^1(B^2)$ *if* $\lambda > 0$

(*b*) $\varphi \in H_2^1(B^2)$ *if* $\lambda = 0$

(*c*) $\varphi \in L^{2+\varepsilon}(B^2)$ *for some* $\varepsilon > 0$ *and*

$$\overline{\lim_{t \to 0}} \int_{B_1/B_t} \frac{|\varphi|^2}{|x|^2 \log^2(1/t)} = 0, \qquad \text{if} \;\; \lambda < 0.$$

Then $\varphi \in L^\infty(B^2 - \{0\})$.

Remark 3.1. — That condition (*c*) is natural follows by considering the case when the structure group is commutative (*i. e.*, the real numbers) and looking at the scalar inequality

$$\Delta u + u^3 \geqq 0.$$

then, $u=-\ln r+r$ is an unbounded function satisfying the above inequality and $-\ln r+r$ is in all L^p $1\leqq p<\infty$. Also note that, if r is small enough, the function $-\sqrt{-\ln r+r}=u$ also satisfies the above inequality and is in all L^p $1\leqq p<\infty$.

Also note that our condition (c) is weaker than $\varphi=o(|\log|x||^{1/2})$ and that $\varphi\in O(|\log|x||^{1/2})$ is weaker than (c).

Similarly, we see that conditions (b) and (a) are natural by considering $\Delta u=0$ in $B_2^2-\{0\}$. Then $u=\ln\left(\dfrac{|x|}{2}\right)$ is an unbounded solution of $\Delta u+u^3\leqq 0$ with $u\notin H_2^1(B^2)$.

To prove 7.1 we make strong use of the fact that $u=|\varphi|$ is a weak solution in $B^2-\{0\}$ of: $(\Delta|\varphi|)\geqq\frac{\lambda}{2}(|\varphi|^2-m^2)|\varphi|$, where Δ is the ordinary Laplacian on functions. This follows from Weitzenblock – like identities and details may be found in [Sb2] (formula 2 and Lemma 1.2).

At no loss of generality we assume $u\geqq 1$.

For example, in case (b) the function $|\varphi|$ is subharmonic. We dispose of case (b).

Proof [case (b)]. – First we show that u is a weak solution of $\int_{B^2}\nabla u.\nabla\eta\,dx\leqq 0$ for all $\eta\in C_0^\infty(B^2)$, $\eta\geqq 0$. Let $\varepsilon>0$. Let ψ_ε be in $C_0^\infty(B^2)$ with $\psi_\varepsilon=\psi_\varepsilon(|x|)$, $\psi_\varepsilon=1$ on B_ε, $\psi_\varepsilon=0$ on $B_{2\varepsilon}$, ψ_ε monotone decreasing in $|x|$, $|\nabla\psi_\varepsilon|\leqq\dfrac{K}{\varepsilon}$. We multiply $\Delta u\geqq 0$ by $\eta\psi_\varepsilon$ to obtain:

$$\int_{B^2}(\nabla u.\nabla\eta)\,\psi_\varepsilon\,dx\leqq\sqrt{\int_{\operatorname{supp}\psi_\varepsilon}|\nabla u|^2\,dx}\,\sqrt{\int_{\operatorname{supp}\psi_\varepsilon}|\nabla\psi_\varepsilon|^2\,dx}.$$

Let $\varepsilon\downarrow 0$ and note that the right hand side tends to zero. By Lebesques dominated convergence theorem we have:

$$\int_{B^2}(\nabla u.\nabla\eta)\,dx\leqq 0.$$

Now, we apply the argument of [Giaq] p. 119 and the "reverse" Sobolov estimate [Giaq], p. 122, or equivalently apply Theorem 2.1 p. 136 of [Giaq] to get $\nabla u\in L_{2+\varepsilon}(B_{1/2}^2)$. Since $u\in H_2^1(B^2)\subset R^2$ Sobolov's embedding theorem gives $u\in L_{2+\varepsilon}(B_{1/2}^2)$. Now extend u to $\hat{u}$ where $\hat{u}$ is defined in B_1^2 with compact support and with

$$\|\hat{u}\|_{H_{2+\varepsilon}^1}(B_1^2)\leqq K\|u\|_{H_{2+\varepsilon}^1}(B_{1/2}).$$

Thus, by the Sobolov embedding theorem and the inequality of [GT] p. 155 we have:

$$\sup_{B^2_{1/2}} |u| \leqq \sup_{B^2_1} |\hat{u}| \leqq C \| \hat{u} \|_{H^1_{2+\varepsilon}} (B_1) \leqq K \| u \|_{H^1_{2+\varepsilon}} (B_{1/2}).$$

Since u is bounded in $B^2_2 - B^2_{1/2}$ we see that u is bounded in B^2_2.

Q.E.D.

We now dispose of case (*a*).

Proof. [case (*a*)]. – In case (*a*) we have that that $u = |\varphi|$ solves $\Delta u \geqq \frac{\lambda}{2}(u^2 - m^2) u$ with $\lambda > 0$. Thus: $\Delta u \geqq \frac{\lambda}{2}(u^2 - m^2) u$. Now consider the two sets.

$$A = \{ x \in B^2 - \{ 0 \} \text{ such that } u \leq m \},$$
$$B = \{ x \in B^2 - \{ 0 \} \text{ such that } u > m \}.$$

These sets are pairwise disjoint. Now, because $u \in C^\infty$ on $B^2 - \{ 0 \}$, the set B is open.

Cover B by a countable collection of small balls, each contained in B. Then on any such small ball in B we have $\Delta u \geqq 0$ and by the estimate above used in the proof of case (*b*) we obtain:

$$\sup_B u \leqq K \| u \|_{H^1_2 (B^2 - \{ 0 \})}.$$

Now on A, u is bounded above by m. Hence u is bounded on $B^2 - \{ 0 \}$.

Q.E.D.

We now prove case (*c*). This requires some work because the proof of Proposition 2.3 of [Sb2] fails in 2-dimensions. The main problem is that when $n = 2$ inequality (1.14), p. 7 of [Sb2], fails since $\frac{2n}{n-2} = \infty$ and $c_n = \infty$ when $n = 2$. Nevertheless we establish the same estimate as in the conclusion of Proposition 2.3 of [Sb2] using a modified technique.

First we prove the following proposition.

PROPOSITION 3.1 (*cf.* Prop. 2.3 of [Sb2]). – If condition (*c*) is satisfied, either we have:

$$\int_{B^2} \eta^2 |\nabla u|^2 \, dx \leqq K \int_{B^2} |\nabla \eta|^2 u^2 \, dx$$

for all test functions η in $C^\infty_0 (B^2)$ or u is bounded.

Proof. – We use a sequence η_K of test functions that vanish for $|x| \leqq \varepsilon_K$, tend to 1 as ε_K tends to zero and such that $\int |\nabla \eta_K|^2 \, dx \to 0$ as $K \to \infty$.

These are defined *cf.* [G], p. 547 bottom, by:

$$\bar{\eta}_K = \bar{\eta}^{\varepsilon_K}(|x|) = \left\{ \begin{array}{ll} 0 & \text{for } |x| \leqq \varepsilon_K \\ 1 & \text{for } |x| \geqq 1 \\ \dfrac{1}{\log(1/\varepsilon_K)} . \log\left[\dfrac{1 \times 1}{\varepsilon_K}\right] & \text{for } \varepsilon_K < |x| < 1 \end{array} \right\}.$$

Remark 3.2. – Note that our growth condition in case (*c*) is chosen exactly to insure that $\int_{B_2} |u|^2 |\nabla \bar{\eta}_K| \to 0$ as $K \to \infty$.

Now let η be C_0^∞ and let $\bar{\eta}$ be a C^∞ function vanishing in a neighborhood of the origin. Use the test function $\tau = (\eta\bar{\eta})^2 (u)$ as ξ in: $\int \nabla u . \nabla \xi \, dx \leqq \int hu \xi \, dx$ for all non-negative $\xi \in C_0^\infty (B^2 - \{0\})$, where $h = -\dfrac{\lambda}{2}(|\varphi|^2 - m^2)$ and $u = |\varphi|$. We get:

$$J_1(\bar{\eta}) = K \int (\eta\bar{\eta})^2 |\nabla u|^2 \, dx$$

$$\leqq \int |2\eta\bar{\eta} \nabla u| |\nabla(\eta\bar{\eta}) u| \, dx + \int (\eta\bar{\eta}) hu^2 \, dx = I_1 + I_2.$$

Now, $I_1 \leqq \mu \int (\eta\bar{\eta})^2 |\nabla u|^2 \, dx + C(\mu) \int |\nabla(\eta\bar{\eta})|^2 |u|^2 \, dx$ and the first term on the right may be absorbed into the left hand side. Also,

$$\int |\nabla(\eta\bar{\eta})|^2 u^2 \, dx \leqq K \left[\int |\nabla \eta|^2 u^2 \, dx + \int |\nabla \bar{\eta}|^2 u^2 \, dx \right].$$

Note that $\int |\nabla \bar{\eta}|^2 \, dx \to 0$ if we set $\bar{\eta} = \eta_K$ and let $K \to \infty$. Do this. Thus in the limit as $K \to \infty$, $I_1 \leqq \int |\nabla \eta|^2 |u|^2 \, dx$. Now,

$$I_2 = \int (\eta\bar{\eta})^2 hu^2 \, dx \leqq \int_{\operatorname{sup} \eta \cap \operatorname{sup} \bar{\eta}} (\eta\bar{\eta})^2 hu^2 \, dx.$$

Since $\lambda \leqq 0$ we have $h = \dfrac{-\lambda}{2}(|\varphi|^2 - m^2) \leqq \dfrac{-\lambda}{2}(|\varphi|^2)$.

$$I_2 \leqq K \int_{\operatorname{supp} \eta \cap \operatorname{supp} \bar{\eta}} (\eta\bar{\eta})^2 |\varphi|^2 |u|^2 \, dx = J_2.$$

We now estimate J_2:

Remark. – The estimate of I_2 in the proof of Proposition 2.3, p. 11 of [SB2], is based on the inequality: $\int g w^2\,dx \leqq C_n \|g\|_{n/2} \int |\nabla w|^2\,dx$ which is proved using Sobolev's inequality. This inequality estimates I_2 from above by a sum of terms, the first of which is proportional to $\|\varphi\|_{L^2}$. Then use is made of conformal scaling to make $\|\varphi\|_{L^2}$ small.

In two dimensions however, the Sobolev estimate has a critical exponent and constant c_n corresponding to this exponent is *infinite*. Thus we need a new argument. This new estimate is contained in the proof of the following sublemma.

SUBLEMA 3.1. – *Let* $B^2 - \{0\} \supset \Omega \supset \operatorname{supp} \eta \cap \operatorname{supp} \bar{\eta}$. *Then:*

$$J_2 \leqq C\left[\int_\Omega |\varphi|^2\,dx\right] \cdot \left[\int_\Omega (\eta\bar{\eta}\,u)^2\,dx + \int_\Omega |\nabla(\eta\bar{\eta}\,u)^2\,dx\right].$$

Remark. – The idea of the proof is that $V = |\varphi|^2$ is a weak subsolution (in fact a C^∞ solution) of an elliptic equation on $\operatorname{supp}\eta \cap \operatorname{supp}\bar{\eta} = \Omega_0$. Thus by a Morrey-like estimate (Bombieri's lemma) we can estimate $\sup_{B(R)\subset\Omega_0} |\varphi| \leqq \frac{C}{R}\left[\int_{B(2R)\subset\Omega_0} |\varphi|^2\,dx\right]^{1/2}$. The sublemma then follows from a covering theorem. We do it now.

Let $V = |\varphi|^2$, let all balls $B(r)$ be contained in Ω. Choose the balls B_R so that $B_R \subset B_{2R} \subset B_{3R} \subset \Omega_0$. Then Ω_0 is covered by a finite number of such balls. Since u is C^∞ in Ω_0 we can at no loss of generality assume that $u \geqq 1$ on Ω_0. (If no such Ω_0 exists then u is bounded.) Recall that $u = |\varphi|$ is a subsolution of $\Delta u \geqq \frac{\lambda}{2}(|u|^2 - m^2)|u| \geqq \frac{\lambda}{2}(|u|^2)|u|$ in Ω_0 since $\lambda < 0$. Thus $\Delta u - \frac{\lambda}{2}|u|^3 \geqq 0$ in Ω_0. Now since $u \geqq 1$, $u \in C^\infty$ on Ω_0, we have: $\Delta(|u|^2\} = 2u\,\Delta u + 2|\nabla u|^2 \geqq \Delta u$. Thus $V = |u|^2$ is a C^∞ subsolution in Ω_0 of $\Delta V + \left(\frac{-\lambda}{2}|u|\right)V \geqq 0$. Note that $\left(\frac{-\lambda}{2}|u|\right)$ is in $L_{1+\varepsilon,\,\exists_\varepsilon,\,\varepsilon>0}$ [by our growth assumption $\varphi \in L_{2+\varepsilon}(B^2)$]. Now we apply Theorem 6.1 (Morrey's

Theorem in 2-dimensions) and Theorem 6.2 (Bombieri's lemma) to get

$$\sup_{B(R)\subset\Omega_0} V \leqq \frac{C}{R^2}\int_{B(2R)\subset\Omega_0} |V|^1\,dx,$$

$\forall_{B(R),\,B(2R)}$ concentric in Ω_0.

Thus

$$\sup_{B(R)\subset\Omega_0} |\varphi| \leqq \frac{C}{R}\left[\int_{B(2R)\subset\Omega_0} |\varphi|^2\,dx\right]^{1/2}.$$

We now use the above inequality and Holder's inequality to achieve our estimate of J_2. Using Holder's inequality with $p=1+\frac{\varepsilon}{2}$, $q=\frac{2+\varepsilon}{\varepsilon}$ we get:

$$J_2 \leqq |\sup_{B(R)\subset\Omega_0} |\varphi||^{2\varepsilon/(2+\varepsilon)}\left[\int_{B(R)\subset\Omega_0} (\eta\bar{\eta}\,u)^{2.((2+\varepsilon)/\varepsilon)}\right]^{(\varepsilon/(2+\varepsilon))} \times\left[\int_{B(R)\subset\Omega_0} |\varphi|^2\right]^{(2+\varepsilon)/\varepsilon} = J_3.$$

Now extend $\eta\bar{\eta}\,u$ to B_{4R} with the extension $E(\eta\bar{\eta}\,u)$ equal to zero on B_{4R}/B_{3R} and $\|E\eta\bar{\eta}\,u\|_{H_2^1(B_{4R})} \leqq \hat{K}\,\|\eta\bar{\eta}\,u\|_{H_2^1(B_{2R})}$. We can do this by Theorem 3.4.3 p. 74 [MO]. Thus:

$$J_3 \leqq \left[\int_{B_R} \sup|\varphi|\right]^{2\varepsilon/(2+\varepsilon)}\left[\int_{B_R} |\varphi|^2\right]^{2/(2+\varepsilon)} \times\left[\int_{B_{4R}} E(\eta\bar{\eta}\,u)^{2\,((2+\varepsilon)/\varepsilon)}\right]^{2\,(\varepsilon/(2+\varepsilon))\,(1/2).2}.$$

Now use Sobolev's inequality in the form:

$$\left[\int_{B_{4R}} u^t\,dx\right]^{1/t} \leqq CR^{2/t}\left[\int_{B_{4R}} |\nabla u|^2\,dx\right]^{1/2} \qquad \text{where} \quad t \geqq 2$$

for $u\in H_2^1(B_{4R})$ with $u=0$ on B_{4R}/B_{3R}. We let $u=E(\eta\bar{\eta}\,u)$ and $t=(2(2+\varepsilon))/\varepsilon$ to get:

$$\left[\int_{B_{4R}} |E(\eta\bar{\eta}\,u)|^{2\,(2+\varepsilon)/\varepsilon}\right]^{\varepsilon/(2+\varepsilon)} \leqq c\,R^{2\varepsilon/(2+\varepsilon)}\left[\int_{B_{4R}} |\nabla E(\eta\bar{\eta}\,u)|^2\,dx\right]$$

and thus

$$\int_{B_R\subset\Omega_0} |\varphi|^2\eta^2\bar{\eta}^2u^2\,dx \leqq (\sup_{B_R}|\varphi|)^{2\varepsilon/(2+\varepsilon)} \times\left[\int_{B_{4R}} |\nabla E(\eta\bar{\eta}\,u)|^2\,dx\right].\left[\int_{B_R} |\varphi|^2\,dx\right]^{2/(2+\varepsilon)}[CR^{2\varepsilon/(2+\varepsilon)}].$$

Recall that $\sup_{B_R}|\varphi| \leqq (K/R)\left[\int_{B_{2R}} |\varphi|^2\, dx\right]^{1/2}$. Thus combining all our estimates we get:

$$\int_{B_R \subset \Omega_0} |\varphi|^2 \eta^2 \bar{\eta}^2 u^2 \leqq \left[\frac{C}{R}\left(\int_{B_{4R}} |\varphi|^2\, dx\right)^{1/2}\right]^{2\varepsilon/(2+\varepsilon)} \cdot [CR^{2\varepsilon/(2+\varepsilon)}]$$

$$\times \left[\int_{B_R} |\varphi|^2\, dx\right]^{2/(2+\varepsilon)} \cdot \left[\int_{B_{4R}} |\nabla(E\eta\bar{\eta} u)|^2\, dx\right]$$

$$\leqq \tilde{K}\left[\int_{B_{2R}} |\varphi|^2\, dx\right]\left[\int_{B_{4R}} (E(\eta\bar{\eta} u))^2\, dx + \int_{B_{4R}} |\nabla(E\eta\bar{\eta} u)|^2\, dx\right]$$

$$\leqq K\left[\int_{B_R} |\varphi|^2\, dx\right]\left[\int_{B_{2R}} (\eta\bar{\eta} u)^2\, dx + \int_{B_{2R}} |\nabla(\eta\bar{\eta} u)|^2\, dx\right].$$

Now using Besocovitch's covering lemma and changing constants appropriately we have

$$\int_\Omega |\varphi|^2 (\eta\bar{\eta})^2 \leqq C\left[\int_\Omega |\varphi|^2\right]\left[\int_\Omega (\eta\bar{\eta} u)^2\, dx + \int_\Omega |\nabla(\eta\bar{\eta} u)|^2\, dx.\right.$$

This completes the proof of the Sublemma.

Q.E,D. Sublemma

Now we return to the main proof and use the Sublemma. We have, using the Sublemma and recalling that conformal invariance implies that we may choose $\left[\int |\varphi|^2\, dx\right]^{1/2} < \gamma$ (where γ may chosen small) that (II):

$$J_1(\bar{\eta}_k) = K\int_\Omega (\eta\bar{\eta}_k)^2 |\nabla u|^2\, dx \leqq \int_\Omega |\nabla\eta|^2 u^2\, dx + g(k)$$

$$+ c(\gamma)\left[\left(\int_\Omega |\nabla(\eta\bar{\eta}_k u)|^2\, dx\right) + C\left(\int_\Omega |(\eta\bar{\eta}_k u)|^2\, dx\right)\right]$$

with $c(\gamma)\downarrow 0$ if $\gamma \to 0$ and $\lim_{k\to\infty} g(k) = 0$.

Note that:

$$\text{(III)} \quad \int_\Omega |\nabla(\eta\bar{\eta}_k u)|^2\, dx \leqq 2\int_\Omega \eta^2 \bar{\eta}_k^2 |\nabla u|^2\, dx + 2\int_\Omega |\nabla(\eta\bar{\eta}_k)|^2 u^2\, dx$$

so from (II) and (III) we obtain

$$\text{(IV)}\quad K\int_\Omega (\eta\bar{\eta}_k)^2|\nabla u|^2\,dx \leqq \int_\Omega |\nabla\eta|^2 u^2\,dx + g(k) + 2C(\gamma)\left[\int_\Omega \eta^2\bar{\eta}_k^2|\nabla u|^2\,dx + 2\int_\Omega |\nabla\eta\bar{\eta}_k|^2 u^2\,dx\right] + C(\gamma)\left[\tilde{K}\int_\Omega (\eta\bar{\eta}_k u)^2\,dx\right].$$

Now choosing γ small enough we absorb the term $2C(\gamma)\int_\Omega |\eta\bar{\eta}_k|^2|\nabla u|^2\,dx$ in the left hand side of (IV) and we get:

$$\text{(V)}\quad K\int_\Omega |\eta\bar{\eta}_k|^2|\nabla u|^2\,dx \leqq g(k) + \int |\nabla\eta|^2 u^2\,dx + C(\gamma)\left[2\int_\Omega |\nabla\eta\bar{\eta}_k|^2 u^2 + K\int_\Omega (\eta\bar{\eta}_k u)^2\,dx.\right.$$

Now using growth condition c, we have

$$\int_\Omega |\nabla(\eta\bar{\eta}_k)|^2 u^2\,dx \leqq 2\int |\nabla\eta|^2 u^2\,dx + 2\int \eta^2(|\nabla\bar{\eta}_k|)^2 u^2\,dx \leqq 2\int_\Omega |\nabla\eta|^2 u^2\,dx + 2\int |u^2|\nabla\bar{\eta}_k|^2 . \sup_\Omega \eta \leqq 2\int |\nabla\eta|^2 u^2\,dx + h(k)$$

where $h(k)\downarrow 0$ as $k\to\infty$. Using this in (V) we obtain

$$\text{(VI)}\quad K\int_\Omega |\eta\bar{\eta}_k|^2|\nabla u|^2\,dx \leqq h(k) + g(k) + \tilde{K}\int_\Omega |\nabla\eta|^2 u^2\,dx + KC(\gamma)\int_\Omega (\eta\bar{\eta}_k u)^2\,dx$$

with $C(\gamma)\downarrow 0$ if $\gamma\to 0$. But

$$\int_\Omega (\eta\bar{\eta}_k u)^2\,dx = \int_{B^2} (\eta\bar{\eta}_k)^2 u^2\,dx \leqq 2\int_{B^2} |\nabla(\eta\bar{\eta}_k)|^2 u^2\,dx + 2\int_{B^2} \eta^2\bar{\eta}_k^2|\nabla u|^2\,dx.$$

Thus

$$\text{(VII)}\quad K\int_{B^2}|\eta\bar{\eta}_k|^2|\nabla u|^2\,dx \leqq h(k)+g(k)+\tilde{K}\int_{B^2}|\nabla n|^2|u|^2\,dx + 2K(\gamma)\int_{B^2}|\nabla(\eta\bar{\eta}_k)|^2u^2\,dx + 2K(\gamma)\int_{B^2}\eta^2\bar{\eta}_k^2|\nabla u|^2\,dx$$

[again with $K(\gamma)\downarrow 0$ as $\gamma\downarrow 0$.]

Now choose γ small enough and absorb the last right hand term on the left hand side.

$$\text{(VIII)}\quad K\int_{B^2}|\eta\bar{\eta}_k|^2|\nabla u|^2\,dx \leqq h(k)+g(k)+\tilde{K}\int_{B^2}|\nabla\eta|^2u^2\,dx + 2K(\gamma)\int_{B^2}|\nabla(\eta\bar{\eta}_k)|^2u^2\,dx.$$

But, $\int_{B^2}|\nabla(\eta\bar{\eta}_k)|^2u^2\,dx=\int_\Omega|\nabla(\eta\bar{\eta}_k)^2u^2\,dx=A$ (we have already shown $A\leqq 2\int_\Omega|\nabla\eta|^2u^2\,dx+h(k)$, with $h(k)\downarrow 0$ as $k\to\infty$. Thus combining terms we obtain

$$\text{(IX)}\qquad K\int_{B^2}|\eta\bar{\eta}_k|^2|\nabla u|^2\,dx \leqq m(k)+\tilde{K}\int_{B^2}|\nabla\eta|^2u^2\,dx$$

with $m(k)\downarrow 0$ if $k\to\infty$.

Now, let $k\to\infty$ and we get:

$$\text{(X)}\qquad \int_{B^2}|\eta|^2|\nabla u|^2\,dx\leqq K\int_{B^2}|\nabla\eta|^2u^2\,dx$$

with K independent of u.

Q.E.D.

Now we prove Theorem 3.1.

Proof. — Theorem 3.1 now follows from De-Georgi iteration, p. 76 [LU] which uses the estimate of Proposition 3.1 as its basic inequality.

Q.E.D.

We now conclude this section with a final corollary.

COROLLARY 3.1. — *Under the hypothesis of Theorem* 3.1, $D\varphi$ *is in* $L^2(B^2)$.

Proof. – This is the same as the proof of Corollary 2.4 of [Sb2].

Q.E.D.

4. A GROWTH ESTIMATION FOR F

We show F is actually in L_P for all $1 \leqq P < \infty$, in any smooth gauge over $B_R - \{0\}$, $0 < R < R_0$.

THEOREM 4.1. – *Under the conditions of theorem* M, F *is in any* L^P, $1 \leqq P < \infty$, *in any smooth gauge over* $B_R - \{0\}$, $0 < R < R_0$.

Proof. – Since *F is a smooth function on the punctured ball it follows from inequality 6.7, p. 269 of [JT] and from YMH1 that:

$$\| d |{}^* F| \|_{L^2} \leqq 2\,\mathrm{MAX} |\varphi| \, \| D\varphi \|_{L^2} < \infty$$

by our estimates on φ. Now, since $F \in L^1$ is follows that *F is in $W_1^1(B_R - \{0\})$. Thus by Sobolev's theorem F is in $L^2(B_R - \{0\})$. Now, we have $F \in H_2^1(B_R - \{0\})$ and Sobolev's embedding theorem gives that $F \in L^P(B_R - \{0\})$ for all P, $1 \leqq P < \infty$.

Q.E.D.

5. PROOF OF THE MAIN THEOREM

COROLLARY 5.1. – (F, φ) *is a weak solution of the field equations in the full ball* B_R, $0 < R < R_0$.

Proof. – Same as Corollary 5.3 of [Sb2].

LEMMA 5.1. – $\displaystyle\int_{B_R} |D\varphi| \, dr \leqq k R$, $0 < R < R_0$.

Q.E.D.

Proof. – $D\varphi \in L^2(B_R)$; apply Holder's inequality.

THEOREM 5.1. – *Under the conditions of theorem* M, *there exists a smooth gauge over* $B_{R_0} - \{0\}$ *in which the induced covariant derivative is* $d + A$ *and* $A \in H_q^1(B_{R_0})$ *with* $q > 2$.

Proof. – By lemma 1.3 we have an auxillary gauge in which the induced covariant derivative is

$$d + A_{aux} \quad \text{and} \quad \| A_{aux} \|_{L^P(B_R - \{0\})} \leqq 0(R); \qquad 0 < R \leqq R_0, \quad 1 \leqq p < \infty.$$

As in the proof of Corollary 4.3 of [U3] we solve (by now this is standard): $d^*(g^{-1} dg + g^{-1} A_{aux} g) = 0$ for g in the space

$L_P^1(B^2-\{0\}, G) \subset C^0(B_{R_0}-\{0\}, G)$; if $P>2$. Let $P>4$. Note, in this gauge, the connection form (again denoted by A), is in $L_P(B_{R_0}-\{0\})=L_P(B_{R_0})$. Now, in this gauge, as in Proposition 5.4 [Sb2], we have $dA=F-\frac{1}{2}[A, A] \Rightarrow dA \in L_{P/2}(B_{R_0})$. Since $\delta A=0$ in $B_{R_0}\{0\}$ we have $\nabla A \in L^{P/2}(B_{R_0})$. Thus $A \in H_q^1(B_{R_0}) \subset C^0(B_{R_0})$; with $q>2$.

Q.E.D.

Remark. – Note that for clarity and consistency we have followed the function space notation in [Sb2]. In more precise notation $A \in L^P(B_{R_0}-\{0\})$ would be $A \in L^P(B_{R_0}-\{0\}, \mathfrak{G} \otimes \Lambda^{1*} R^2)$, $A \in H_q^1(B_{R_0})$ would be $A \in H_q^1(B_{R_0}, \mathfrak{G} \otimes \Lambda^{1*} R^2)$, etc.

THEOREM 5.2. – *Under the conditions of Theorem* M, *there exists a smooth gauge over* B_{R_0} *in which the induced curvature form* F *and the induced connection from* A *satisfy:*

1. $\delta A=0$.
2. $A \in H_q^1(B_{R_0})$ *with* $q>2$.
3. $\|A\|_{H_2^1(B_{R_0})} \leq C \|F\|_{L^2(B_{R_0})}$.

Proof. – Using the gauge given by Theorem 5.1, apply Lemma 1.3 of [U1]. Note that $\|F\|_{q, B_{R_0}} < k(n)$ as required in Lemma 1.3, if R_0 is small enough, since $F \in L_P(B_{R_0})$ for all $1 \leq P < \infty$ (apply Holder's inequality).

Q.E.D.

At this point, the proof of Thorem M follows exactly the proof on the last two pages of [p. 15-16] of [Sb2].

Q.E.D.

We are finished.

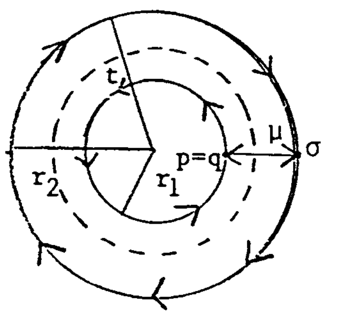

FIG. 1

$$c_1(S)= \begin{cases} (O, r_1); & 0 \leq S \leq \bar{\varepsilon} \\ \left(\left(\frac{2\pi}{2\pi-2\bar{\varepsilon}}\right)S+\frac{2\pi\bar{\varepsilon}}{2\bar{\varepsilon}-2\pi}, r_1\right); & \bar{\varepsilon} \leq S \leq 2\pi-\bar{\varepsilon} \\ (O, r_1); & 2\pi-\bar{\varepsilon} \leq S \leq 2\pi \end{cases}$$

$$c_2(S)=\begin{cases}\left(O,\dfrac{(r_2-r_1)}{\bar{\varepsilon}}S+r_1\right); & 0\leqq S\leqq\bar{\varepsilon}\\ \left(\left(\dfrac{2\pi}{2\pi-2\bar{\varepsilon}}\right)S+\dfrac{2\pi\bar{\varepsilon}}{2\bar{\varepsilon}-2\pi},r_1\right); & \bar{\varepsilon}\leqq S\leqq 2\pi-\bar{\varepsilon}\\ \left(O,\dfrac{(r_1-r_2)}{\bar{\varepsilon}}S+\dfrac{2\pi(R_2-R_1)}{\bar{\varepsilon}}+r_1\right); & 2\pi-\bar{\varepsilon}\leqq S\leqq 2\pi;\end{cases}$$

where $(-,-)$ is the polar co-ordinate of the image in R^2. $\bar{\varepsilon}$ is arbitrary such that $0<\bar{\varepsilon}<2\pi$.

Geometrically $c_1(S)$ holds constant equal to p for $0\leqq S\leqq\bar{\varepsilon}$, $c_1(S)$ goes around the inner circle clockwise r linearly in S for $\bar{\varepsilon}\leqq S\leqq 2\pi-\varepsilon$, then $c_1(S)$ holds constant equal to $q=p$ for $2\pi-\bar{\varepsilon}\leqq S\leqq 2\pi$.

Geometrically $c_2(S)$ goes linearly in S from p to σ along $p\sigma$ when $0\leqq S\leqq\bar{\varepsilon}$, then goes clockwise linearly in S around the outer circle when $\bar{\varepsilon}\leqq S\leqq 2\pi-\varepsilon$, then r goes linearly in S from σ to p along σp when $2\pi-\bar{\varepsilon}\leqq S\leqq 2\pi$.

REFERENCES

[AMR] R. Abraham, J. Marsden and T. Rativ, *Manifolds, Tensor Analysis and Applications*, Addision Wesley, 1983.

[Bell] E. Beckenbuch and R. Bellman, *Inequalities Ergebnisseder Math.*, Vol. **30**, 1971, Springer-Verlag.

[BO] E. Bombieri, An introduction to Minimal Currents and Parametric Variational Problems, *Proc. Beijing Conference on PDE and Geomtry*, 1980.

[Giaq] M. Giaquintia, *Multiple Integrals in the Calculus of Variations*, Princeton Press, 1985.

[GS] B. Gidas and J. Spruck, Global and Local Behavior, *Comm. Pure Appl. Math.*, Vol. **4**, 1981, pp. 525-598.

[GT] D. Gilbarg and N. Trudinger, *Quasilinear Elliptic P.D.E.*, Springer-Verlag, Second edition, 1983.

[H] P. Hartman, *Ordinary Differential Equations*, Birkhauser, 1982.

[JT] A. J. Jaffe and C. Taubes, Vortices and Monopoles, *Prog. Phys.*, Vol. **2**, 1980, Birkhauser, Boston.

[Karch] P. Buser and H. Karcher, Gromov's Almost Flat Manifolds, *Astérique*, Vol. **81**, Soc. Math. France.

[KN1] S. Kobayashi and K. Nomizo, *Found. Diff. Geo.*, Wiley, 1963.

[LU] N. Ladyzenskya and N. Uralaltizva, *Quasi Linear Elliptic Equations*, Academic Press, 1968.

[MO] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer, New York, 1966.

[P] T. Parker, Gauge Theories on Foar Dimensional Manifolds, *Comm. Math. Phys.*, Vol. **85**, 1982.

[Sb1] L. M. Sibner, Removable Singularities of Yang-Mills Fields in R^3, *Composito Math.*, Vol. **53**, 1984, pp. 91-104.

[Sb2] L. M. Sibner and R. J. Sibner, Removable Singularities of Coupled Yang-Mills Fields in R^3, *Comm. Math. Phys.*, Vol. **93**, 1984, pp. 1-17.

[Sb3] L. M. SIBNER, The Isolated Point Singularity Problem for the Yang-Mills Equations in Higher Dimensions, *Math. Ann.*, 1985 (to appaear).

[SM] P. D. SMITH, *Removable Singularities for the Yang-Mills Higgs equation in* 2 *dimensions*, Max Planck Institute, Preprint, MPI/87-35.

[U1] K. UHLENBECK, Removable Singularities in Yang-Mills Fields, *Comm. Math. Phys.*, Vol. **83**, 1982, pp. 11-29.

[U2] K. UHLENBECK, Connections with L^p Bounds on Curvature, *Comm. Math. Phys.*, **83**, 1982, pp. 31-42.

[U3] K. UHLENBECK, The Chern Classes of Sobolev Connections, *Comm. Math. Phys.*, Vol. **101**, 1985, pp. 449-457.

[UF] K. UHLENBECK and D. FREED, *Instantons and Four Manifolds*, MSRI Publications, Springer, 1984.

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