Ann. Inst. Henri Poincaré,

Vol. 7, n° 6, 1990, p. 589-601. *Analyse non linéaire*

On the non-existence of energy stable minimal cones

by

Ulrich DIERKES

Fachbereich Mathematik, Universität des Saarlandes, D-6600 Saarbrücken
and
Sonderforschungsbereich 256, Institut für Angewandte Mathematik,
Universität Bonn, Wegelerstr. 6, D-5300 Bonn 1.

ABSTRACT. — We show that there are no non-trivial (potential) energy stable minimal cones in $\mathbb{R}^n \times \mathbb{R}^+$ with singularity at 0, if $2 \leqq n \leqq 5$. The sharpness of this result is demonstrated by proving that a certain six dimensional cone in $\mathbb{R}^7$ is stable. Moreover, we extend all results to the more general α-energy functional.

Key words : Stable cones.

RÉSUMÉ. — L'on démontre que, si $2 \leqq n \leqq 5$, il n'existe pas dans $\mathbb{R}^n \times \mathbb{R}^+$ de cônes minimaux stables en énergie. Ce résultat est optimal, car l'on exhibe dans $\mathbb{R}^7$ un cône de dimension 6 qui est stable. On étend également ces résultats à des fonctionnelles d'énergie plus générales.

A well known result due to J. Simons [SJ] states that there are no non-trivial n-dimensional stable minimal cones in $\mathbb{R}^{n+1}$ (with singularity at zero), provided $n \leqq 6$. One of the crucial ingrediences in his proof is an important identity for the Laplacian of the second fundamental form for minimal hypersurfaces. Using sharper estimates than had previously been

Classification A.M.S. : 49 F 22.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 7/90/06/589/13/$3.30/

realized, Schoen-Simon-Yau [SSY] gave a considerably simpler proof of Simons' result.

Simons also proved that the seven dimensional cone $x_1^2+\ldots+x_4^2=x_5^2+\ldots+x_8^2$ in $\mathbb{R}^8$ is stable and, in fact, it was proved by Bombieri-De Giorgi-Giusti [BDG] that it even minimizes area in $\mathbb{R}^8$. This result dashes the hope for general interior regularity of codimension one solutions to the problem of least area in $\mathbb{R}^8$.

In two papers [D 1] and [D 2] the author has investigated the cones

$$C_n^\alpha := \left\{ x=(x_1, \ldots, x_{n+1});\ 0 \leqq x_{n+1} \leqq \sqrt{\frac{\alpha}{n-1}}\,[x_1^2+\ldots+x_n^2]^{1/2} \right\} \subset \mathbb{R}^{n+1}$$

which have boundaries of least α-energy

$$\mathscr{E}_\alpha = \int x_{n+1}^\alpha \,|D\varphi_U| \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+,$$

provided one of the following conditions holds:

(i) $\alpha+p \geqq 6$, where $\alpha \geqq 2$ and $p:=n-1 \geqq 2$,

or

(ii) $\alpha+p \geqq 7$, for $\alpha \geqq 1$ and $p \geqq 1$.

Here $\mathbb{R}^+ = \{t \geqq 0\}$, $U \subset \mathbb{R}^n \times \mathbb{R}^+$, and $|D\varphi_U|$ is the n-dimensional Hausdorff measure restricted to the reduced boundary of U. Also a set $C \subset \mathbb{R}^n \times \mathbb{R}^+$ with characteristic function φ_C has a boundary of least α-energy in $\mathbb{R}^n \times \mathbb{R}^+$, if and only if for each $g \in L_{loc}^1(\mathbb{R}^n \times \mathbb{R}^+)$ with compact support $K \subset \mathbb{R}^n \times \mathbb{R}^+$ we have

$$\int_K x_{n+1}^\alpha |D\varphi_C| \leqq \int_K x_{n+1}^\alpha |D(\varphi_C+g)|.$$

Furthermore it could be shown in [D 2] that the six dimensional boundary of the cone C_6^1 does not minimize the energy $\mathscr{E}_1$ in $\mathbb{R}^6 \times \mathbb{R}^+$. Similarly, the cone C_2^5 does not minimize $\mathscr{E}_5$ in $\mathbb{R}^2 \times \mathbb{R}^+$.

We wish to emphasize the physical relevance of the problem. Namely if we regard the boundary $M=\partial U$ of U as a material surface of constant mass density, then $\mathscr{E}_1$ corresponds to the potential energy of M under gravitational forces. Here we have of course assumed that the gravitational force acts in the $-x_{n+1}$ direction. Therefore, we refer to $\mathscr{E}_\alpha$ as the α-energy, and, in particular, if $\alpha=1$ we shall simply omit the addition "α".

In this paper we will employ the method of Schoen-Simon-Yau [SSY] to obtain a result on the non-existence of non-trivial α-stable minimal cones in $\mathbb{R}^n \times \mathbb{R}^+$ *i.e.*, cones which are stable with respect to the α-energy $\mathscr{E}_\alpha$. We in fact prove (Theorem 2) that such a result holds true provided that

$$\alpha+p<3+\sqrt{8}, \qquad p=n-1.$$

On the other hand we show in Theorem 1 that the cones $x_{n+1} = +\sqrt{\frac{\alpha}{p}}\left[x_1^2 + \ldots + x_n^2\right]^{1/2}$ are α-stable, if

$$\alpha + p \geqq 3 + \sqrt{8}.$$

Note that this in particular implies stability, if $\alpha = 1$, $n = 6$, or $\alpha = 5$, $n = 2$, but because of [D 2] the boundaries of the set C_6^1 or C_2^5 do not minimize the corresponding α-energy in $\mathbb{R}^n \times \mathring{\mathbb{R}}^+$. In fact, we might even obtain a field of α-stable and non-minimizing minimal cones, e. g. the two-dimensional cones $x_3 = \sqrt{\alpha}\,[x_1^2 + x_2^2]^{1/2}$ in $\mathbb{R}^3$ where $2 + \sqrt{8} \leqq \alpha \leqq 5$.

Acknowledgement. I would like to thank Leon Simon for directing my attention to the problem which is treated in this paper.

1. NOTATIONS AND RESULTS

In this section we set up our terminology and, in particular, we give simple expressions for the first and second variation of the α-energy. Finally, we formulate our main results.

Let M be an n-dimensional submanifold of class C^2 contained in the open half-space $\mathbb{R}^n + \mathring{\mathbb{R}}^+ \subset \mathbb{R}^{n+1}$, $\mathring{\mathbb{R}}^+ = \{t > 0\}$, and let $U \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ be open with $U \cap M \neq \varnothing$, $(\text{clos}\, M - M) \cap U = \varnothing$, $\mathscr{H}_n(M \cap K) < \infty$ for each compact set $K \subset U$; here $\mathscr{H}_t$, $t \geqq 0$, denotes t-dimensional Hausdorff measure. We consider one parameter families $\{\Phi_t\}$, $-1 \leqq t \leqq 1$, of diffeomorphisms from U into U, with the following properties:

$$\Phi(t, x) =: \Phi_t(x) \in C^2((-1, 1) \times U, U), \tag{1}$$

$$\Phi_0(x) = x \quad \text{for all } x \in U, \tag{2}$$

$$\Phi_t(x) = x \quad \text{for all } t \in (-1, 1) \text{ and all } x \in U - K \text{ for some compact set } K \subset U. \tag{3}$$

Put

$$X(x) := \frac{\partial \Phi}{\partial t}(t, x)\big|_{t=0},$$

and

$$Z(x) := \frac{\partial^2 \Phi}{\partial t^2}(t, x)\big|_{t=0},$$

to denote the initial velocity and acceleration vectors of Φ_t respectively. Then, because of (3), X and Z have compact support $K \subset U$, and furthermore

$$\Phi_t(x) = x + t\,X(x) + \frac{t^2}{2} Z(x) + o(t^2).$$

Let $M_t := \Phi_t(M)$ denote the image of $M = M_0$ under Φ_t; then we are interested in the first and second variation of the α-energy functional

$$\mathscr{E}_\alpha(M) = \int_M x_{n+1}^\alpha \, d\mathscr{H}_n,$$

where $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+$, $\alpha > 0$, *i.e.* we want to compute

$$\delta\mathscr{E}_\alpha(M, X) = \frac{d}{dt} \int_{M_t} x_{n+1}^\alpha \, d\mathscr{H}_n \big|_{t=0},$$

and

$$\delta^2 \mathscr{E}_\alpha(M, X, Z) = \frac{d^2}{dt^2} \int_{M_t} x_{n+1}^\alpha \, d\mathscr{H}_n \big|_{t=0}.$$

Choose a local field of orthonormal frames $\tau^1, \ldots, \tau^n, \nu$ such that $\tau^1, \ldots, \tau^n \in T_x M$ are tangent to M. For a given vectorfield Y on M (not necessarily tangential) we denote by $D_{\tau^i} Y$ the directional derivative of Y in the direction τ^i. Also

$$\operatorname{div} Y = \sum_{i=1}^n (D_{\tau^i} Y) \tau^i$$

stands for the divergence on M, and

$$\nabla f = \sum_{i=1}^n (D_{\tau^i} f) \tau^i$$

denotes the gradient of the function $f \in C^1(M, \mathbb{R})$ respectively. We shall also employ the symbol Δ to denote the Laplacian on M, *i.e.* $\Delta = \nabla_i \nabla_i$ where $\nabla_i = D_{\tau^i}$, and $Y^\perp = Y - \sum_{j=1}^n (Y \cdot \tau^j) \tau^j$ stands for the normal part of Y.

LEMMA 1. – *Let* M, $\Phi_t : U \to U$ *and*

$$X(x) = (X_1(x), \ldots, X_{n+1}(x)),$$
$$Z(x) = (Z_1(x), \ldots, Z_{n+1}(x))$$

be defined as above. Then

$$\delta\mathscr{E}_\alpha(M, X) = \int_M \{ x_{n+1}^\alpha \operatorname{div} X + \alpha x_{n+1}^{\alpha-1} X_{n+1} \} \, d\mathscr{H}_n \tag{4}$$

and

$$\delta^2 \mathscr{E}_\alpha(M, X, Z) = \int_M \{ \alpha(\alpha-1) x_{n+1}^{\alpha-2} X_{n+1}^2 + \alpha x_{n+1}^{\alpha-1} Z_{n+1} + 2\alpha x_{n+1}^{\alpha-1} X_{n+1} \operatorname{div} X + x_{n+1}^\alpha [\operatorname{div} Z + (\operatorname{div} X)^2 + \sum_{i=1}^n |(D_{\tau^i} X)^\perp|^2 - \sum_{i,j=1}^n (\tau^i D_{\tau^j} X)(\tau^j D_{\tau^i} X)] \} d\mathscr{H}_n. \quad (5)$$

Proof. — From the general area formula (*see* e. g. [SL], § 8, or [FH] 3.2.20 Cor.), we infer that

$$\mathscr{E}_\alpha(\Phi_t(M \cap K)) = \int_{M \cap K} (\psi_t)_{n+1}^\alpha \, J\psi_t \, d\mathscr{H}_n,$$

where $\psi_t = \Phi_t|_{M \cap U}$, $J\psi_t$ denotes the Jacobian of ψ_t and $(\psi_t)_{n+1}^\alpha$ is the $(n+1)$-th component of ψ_t to the power α. The Jacobian $J\psi_t$ may be computed as in [SL], p. 50,

$$J\psi_t = 1 + t.\operatorname{div} X + \frac{t^2}{2}\left\{ \operatorname{div} Z + (\operatorname{div} X)^2 + \sum_{i=1}^n |(D_{\tau^i} X)^\perp|^2 - \sum_{i,j=1}^n (\tau^i D_{\tau^j} X)(\tau^j D_{\tau^i} X) \right\} + o(t^2).$$

Similarly we find

$$(\psi_t)_{n+1}^\alpha = x_{n+1}^\alpha + t\alpha x_{n+1}^{\alpha-1} X_{n+1} + \frac{t^2}{2}[\alpha(\alpha-1) x_{n+1}^{\alpha-2} X_{n+1}^2 + \alpha x_{n+1}^{\alpha-1} Z_{n+1}] + o(t^2).$$

Now the result follows immediately by computing the coefficients of t and $\frac{t^2}{2}$ in the product $(\psi_t)_{n+1}^\alpha . J\psi_t$. □

Lemma 1 motivates the following definition.

DEFINITION 1. — A C^1-submanifold $M \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ is called α-stationary in $U \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$, if $\mathscr{E}_\alpha(M \cap K) < \infty$ for all compact sets $K \subset U$, and

$$\int_M \{ x_{n+1}^\alpha \operatorname{div} X + \alpha x_{n+1}^{\alpha-1} X_{n+1} \} d\mathscr{H}_n = 0, \quad (6)$$

for all vector fields $X \in C_c^1(U, \mathbb{R}^{n+1})$.

LEMMA 2. — *Suppose* M *is α-stationary in* U *and of class* C^2. *Then the mean curvature* H *of* M *with respect to the unit normal* $\nu = (\nu_1, \ldots, \nu_{n+1})$ *is given by*

$$H(x) = \alpha x_{n+1}^{-1} \nu_{n+1} \quad \textit{for all } x \in M \cap U.$$

Proof. — Take some arbitrary function $\xi \in C_c^1(M, \mathbb{R})$ with compact support in U and put $X = \xi . \nu$. Then we infer from (6)

$$0 = \int_M \{ x_{n+1}^\alpha \operatorname{div}(\xi . \nu) + \alpha x_{n+1}^{\alpha-1} \xi . \nu_{n+1} \} d\mathcal{H}_n$$
$$= \int_M \{ \operatorname{div}(x_{n+1}^\alpha \xi . \nu) + \alpha x_{n+1}^{\alpha-1} \xi . \nu_{n+1} \} d\mathcal{H}_n,$$
$$= - \int_M \{ x_{n+1}^\alpha \xi . \nu . \underline{\underline{H}} - \alpha x_{n+1}^{\alpha-1} \xi . \nu_{n+1} \} d\mathcal{H}_n,$$

where $\underline{\underline{H}} = \nu H$ is the mean curvature vector of M. The lemma follows by applying the fundamental lemma in the calculus of variations. □

We take again the special variation $X = \xi . \nu$, $\xi \in C_c^1(M, \mathbb{R})$ and find successively,

$$\operatorname{div} X = - X . \underline{\underline{H}} = - \alpha \nu_{n+1} x_{n+1}^{-1} \xi$$
$$\sum_{i=1}^n |(D_{\tau^i} X)^\perp|^2 = \sum_{i=1}^n |\nu D_{\tau^i} \xi|^2 = |\nabla \xi|^2,$$

and

$$\sum_{i,\, j=1}^n (\tau^i D_{\tau^j} X)(\tau^j D_{\tau^i} X) = \xi^2 |A|^2,$$

where $|A|$ denotes the length of the second fundamental form $A = h_{ij} \tau^i \otimes \tau^j$, *i.e.*

$$|A|^2 = \sum_{i,\, j=1}^n h_{ij}^2.$$

Thus we have proved

LEMMA 3. — *Suppose* $M \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ *is a submanifold of class* C^2 *which is* α*-stationary in* $U \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$, $(\text{clos } M - M) \cap U = \varnothing$. *If* $X = \xi . \nu$ *for some function* $\xi \in C_c^1(M, \mathbb{R})$ *with compact support in* U, *then the second variation is given by*

$$\delta^2 \mathcal{E}(M, \xi) = \int_M \{ x_{n+1}^\alpha |\nabla \xi|^2 - \alpha x_{n+1}^{\alpha-2} \nu_{n+1}^2 \xi^2 - x_{n+1}^\alpha |A|^2 \xi^2 \} d\mathcal{H}_n. \quad \square$$

Hence it is reasonable to define stability as follows.

DEFINITION 2. — Suppose $M \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ is a n-dimensional submanifold of class C^2 which is α-stationary in $U \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$, $(\text{clos } M - M) \cap U = \varnothing$. Then M is called α-stable in U, if

$$\int_M \{ x_{n+1}^\alpha |\nabla \xi|^2 - \alpha x_{n+1}^{\alpha-2} \nu_{n+1}^2 \xi^2 - x_{n+1}^\alpha |A|^2 \xi^2 \} d\mathcal{H}_n \geqq 0 \qquad (7)$$

for each $\xi \in C_c^1(M, \mathbb{R})$ with compact support in U. In particular, if $\mathscr{C} = \operatorname{clos} M$ is a cone in $\mathbb{R}^n \times \mathbb{R}^+$ with singularity at $\{0\}$, and if $M = \mathscr{C} - \{0\} \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ is α-stationary in $\mathbb{R}^n \times \mathring{\mathbb{R}}^+$, then $\mathscr{C}$ is called α-stable if (7) holds for all $\xi \in C_c^1(M, \mathbb{R})$.

Put $c_n^\alpha(y) = \sqrt{\frac{\alpha}{p}}[y_1^2 + \ldots + y_n^2]^{1/2}$, $y \in \mathbb{R}^n$, $\alpha > 0$, $p = n-1$ and define the cones

$$\mathscr{C}_n^\alpha = \{(y, c_n^\alpha(y)) : y \in \mathbb{R}^n\},$$

then we have

THEOREM 1. — *The cones $\mathscr{C}_n^\alpha$ are α-stable, if $\alpha + p \geqq 3 + \sqrt{8}$.*

Observe that the critical number $3 + \sqrt{8}$ also enters the discussion of the ordinary differential system [11] in [D 1]. Here, it appears as a necessary, though not sufficient condition for the construction of a minimal foliation about the cone $\mathscr{C}_n^\alpha$.

THEOREM 2. — *Suppose $\mathscr{C} \subset \mathbb{R}^n \times \mathbb{R}^+$ is an α-stable n-dimensional cone with singularity at $\{0\}$. If $\alpha + p < 3 + \sqrt{8}$ then $\mathscr{C}$ is a hyperplane $\mathscr{P}$. Furthermore, $\mathscr{P}$ must be perpendicular to the plane $\{x_{n+1} = 0\}$.*

COROLLARY. — *In particular, if $2 \leqq n \leqq 5$ there are no non-trivial (potential-) energy stable cones in $\mathbb{R}^n \times \mathbb{R}^+$ with singularity at $\{0\}$.*

2. PROOFS

Let $\xi \in C_c^1(\mathscr{C}_n^\alpha - \{0\}, \mathbb{R})$ be arbitrary and put $X(x) = x . |x|^{-2} \xi^2$ for $x \in \mathbb{R}^n \times \mathring{\mathbb{R}}^+$ where $|x|^2 = (x_1^2 + \ldots + x_{n+1}^2)$. A standard calculation yields (*see* [SL], § 17)

$$\operatorname{div} X = \sum_{i=1}^n (D_{\tau^i} X)\,\tau^i = 2\,|x|^{-2}\,(x \nabla \xi)\,\xi + (n-2)\,\xi^2\,|x|^{-2} + 2\,|x|^{-2}\,\xi^2\,|(D\,|x|)^\perp|^2.$$

Since $\mathscr{C}_n^\alpha - \{0\}$ is α-stationary in $\mathbb{R}^n \times \mathring{\mathbb{R}}^+$, we conclude from (6) that

$$\int_{\mathscr{C}_n^\alpha - \{0\}} x_{n+1}^\alpha \,\{2\,|x|^{-2}\,(x \nabla \xi)\,\xi + (n-2+\alpha)\,|x|^{-2}\,\xi^2\}\, d\mathscr{H}_n \leqq 0.$$

We apply Schwarz inequality and obtain

$$\left(\frac{n-2+\alpha}{2}\right)^2 \int_{\mathscr{C}_n^\alpha - \{0\}} x_{n+1}^\alpha\,|x|^{-2}\,\xi^2\, d\mathscr{H}_n \leqq \int_{\mathscr{C}_n^\alpha - \{0\}} x_{n+1}^\alpha\,|\nabla \xi|^2\, d\mathscr{H}_n.$$

Therefore $\mathscr{C}_n^\alpha$ is α-stable, if

$$\left(\frac{n-2+\alpha}{2}\right)^2 \geqq |x|^2\,|A|^2+\alpha\,x_{n+1}^{-2}\,|x|^2\,\nu_{n+1}^2. \tag{8}$$

An elementary calculation shows that for the cone $\mathscr{C}_n^\alpha$ the length of the second fundamental form is given by

$$|A|^2=\frac{\alpha p}{\alpha+p}\,r^{-2}=\alpha\,|x|^{-2}\quad \text{for all } x\in\mathscr{C}_n^\alpha-\{0\},$$

where we have put $r^2=(x_1^2+\ldots+x_n^2)$. Then along $\mathscr{C}_n^\alpha$, $x_{n+1}=\sqrt{\frac{\alpha}{p}}\,r$ and we infer from (8) that $\mathscr{C}_n^\alpha$ is stable, if

$$\left(\frac{n-2+\alpha}{2}\right)^2 \geqq \alpha+\alpha\,x_{n+1}^{-2}\,|x|^2\,\nu_{n+1}^2$$
$$=\alpha+\frac{\alpha p}{\alpha+p}\left[1+\frac{r^2}{x_{n+1}^2}\right]=\alpha+p.$$

This is true in case that $\alpha+p\geqq 3+\sqrt{8}$. Theorem 1 follows.

Proof of Theorem 2. – In the following we shall always assume that $M=\mathscr{C}-\{0\}$ is an α-stable cone in $\mathbb{R}^n\times\mathring{\mathbb{R}}^+$, so that in particular (7) holds true. Replacing ξ by $|A|\,\xi$ in (7) we get

$$\int_M \{x_{n+1}^\alpha\,|A|^4\,\xi^2+\alpha\,x_{n+1}^{\alpha-2}\,\nu_{n+1}^2\,|A|^2\,\xi^2\}\,d\mathscr{H}_n$$
$$\leqq \int_M x_{n+1}^\alpha\,\{|A|^2\,|\nabla\xi|^2+|\nabla|A||^2\,\xi^2+2\,\xi\,|A|\,(\nabla\xi\,\nabla|A|)\}\,d\mathscr{H}_n. \tag{9}$$

Now

$$2\int_M x_{n+1}^\alpha\,|A|\,\xi\,(\nabla\xi\,\nabla|A|)\,d\mathscr{H}_n=\int_M x_{n+1}^\alpha\,(\nabla\xi^2)\,\nabla\left(\frac{1}{2}|A|^2\right)d\mathscr{H}_n$$
$$=-\int_M x_{n+1}^\alpha\,\xi^2\,\Delta\left(\frac{1}{2}|A|^2\right)d\mathscr{H}_n-\int_M \xi^2\,(\nabla x_{n+1}^\alpha)\left(\nabla\frac{1}{2}|A|^2\right)d\mathscr{H}_n. \tag{10}$$

In order to conclude further we need a sharp estimate for the Laplacian of $|A|^2$. This will be provided by the following

LEMMA 4 ([SSY], [SL, appendix B]). – *If* $M=\mathscr{C}-\{0\}$ *is a cone, then*

$$-\frac{1}{2}\Delta\,|A|^2\leqq |A|^4-2\,|x|^{-2}\,|A|^2-|\nabla|A||^2-h_{ij}H_{,ij}-H\,h_{mi}h_{mj}h_{ij}\quad (^1)$$

(1) The summation convention is used freely here!

Here $H_{,ij}$ denote the second covariant derivatives of the mean curvature H with respect to τ^i and τ^j, and, as above, h_{ij} are the coefficients of A.

Proof of Lemma 4. – B.8 Lemma and B.9 Lemma in [SL] yield the relations

$$\Delta\left(\frac{1}{2}|A|^2\right)=\sum_{i,j,k} h_{ij,k}^2-|A|^4+h_{ij}H_{,ij}+H\,h_{mi}h_{mj}h_{ij},$$

here $H=h_{kk}=\text{trace } A$ and $h_{ij,k}$ denotes the covariant derivative of A with respect to τ^k; also

$$\sum_{i,j,k} h_{ij,k}^2-|\nabla|A||^2\geqq 2\,|x|^{-2}\,|A|^2 \quad \text{for all } x\in M.$$

Both relations imply Lemma 4. □

From (9), (10) and Lemma 4 we conclude that

$$\int_M \xi^2\left\{2x_{n+1}^\alpha\,|x|^{-2}|A|^2+\alpha x_{n+1}^{\alpha-2}\,|A|^2\,\nu_{n+1}^2\right.$$
$$\left.+\nabla(x_{n+1}^\alpha)\,\nabla\left(\frac{1}{2}|A|^2\right)+x_{n+1}^\alpha\,h_{ij}H_{,ij}+x_{n+1}^\alpha\,H\,h_{mi}h_{mj}h_{ij}\right\}d\mathcal{H}_n$$
$$\leqq\int_M x_{n+1}^\alpha\,|A|^2|\nabla\xi|^2\,d\mathcal{H}_n. \quad (11)$$

Relation (11) will be of crucial importance in what follows.

To begin, select an orthonormal frame $\tau^1,\ldots,\tau^n\in T_xM$ so that $\tau^n=\frac{x}{|x|}$ and $\tau^1,\ldots,\tau^n$ are constant along the ray through x. Also we can assume that $\tau_{n+1}^1=\tau_{n+1}^2=\ldots=\tau_{n+1}^{n-1}=0$. Then $h_{in}=h_{ni}=0$ for $i\in\{1,\ldots,n\}$ and, since $h_{ij}(\lambda x)=\lambda^{-1}h_{ij}(x)$, $\lambda>0$, we have $h_{ij,n}=-|x|^{-1}h_{ij}$.

We first compute the expression

$$(\nabla x_{n+1}^\alpha)\left(\nabla\frac{1}{2}|A|^2\right)=\alpha x_{n+1}^{\alpha-1}\,(D_{\tau^k}x_{n+1})\left(D_{\tau^k}\left(\frac{1}{2}|A|^2\right)\right)$$
$$=\alpha x_{n+1}^{\alpha-1}h_{ij}h_{ij,k}\,\tau_{n+1}^k=-\alpha x_{n+1}^\alpha\,|x|^{-2}\,|A|^2, \quad (12)$$

and then

$$
\begin{aligned}
\frac{1}{\alpha} \mathrm{H}_{,ij} &= \frac{1}{\alpha} \nabla_i \nabla_j \mathrm{H} = \nabla_i \nabla_j \left(\frac{\nu_{n+1}}{x_{n+1}} \right) \\
&= \nabla_i \{ -x_{n+1}^{-2} (\nabla_j x_{n+1}) \nu_{n+1} + x_{n+1}^{-1} \nabla_j \nu_{n+1} \} \\
&= 2\, x_{n+1}^{-3} \nabla_i x_{n+1} \nabla_j x_{n+1} \nu_{n+1} - x_{n+1}^{-2} (\nabla_i \nabla_j x_{n+1}) \nu_{n+1} \\
&\quad - x_{n+1}^{-2} \nabla_j x_{n+1} \nabla_i \nu_{n+1} \\
&\quad - x_{n+1}^{-2} \nabla_i x_{n+1} \nabla_j \nu_{n+1} + x_{n+1}^{-1} \nabla_i \nabla_j \nu_{n+1} \\
&= 2\, x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j \nu_{n+1} \\
&\quad - x_{n+1}^{-2} \nabla_i \tau_{n+1}^j \nu_{n+1} - x_{n+1}^{-2} \tau_{n+1}^j \nabla_i \nu_{n+1} \\
&\quad - x_{n+1}^{-2} \tau_{n+1}^i \nabla_j \nu_{n+1} + x_{n+1}^{-1} \nabla_i \nabla_j \nu_{n+1}.
\end{aligned}
$$

By virtue of

$$\nabla_i \nu = -h_{il} \tau^l \qquad \text{and} \qquad \nabla_i \tau^j = h_{ij} \nu$$

we obtain

$$
\begin{aligned}
\frac{1}{\alpha} \mathrm{H}_{,ij} = 2\, x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j \nu_{n+1} & \\
- x_{n+1}^{-2} h_{ij} \nu_{n+1}^2 + x_{n+1}^{-2} \tau_{n+1}^j h_{il} \tau_{n+1}^l & \\
+ x_{n+1}^{-2} \tau_{n+1}^i h_{jl} \tau_{n+1}^l - x_{n+1}^{-1} \nabla_i [h_{jl} \tau_{n+1}^l]. &
\end{aligned}
$$

Using the Codazzi equations we conclude

$$
\begin{aligned}
\nabla_i [h_{jl} \tau_{n+1}^l] &= h_{jl,\, i} \tau_{n+1}^l + h_{jl} \nabla_i \tau_{n+1}^l \\
&= h_{ij,\, l} \tau_{n+1}^l + h_{jl} h_{il} \nu_{n+1},
\end{aligned}
$$

whence

$$
\begin{aligned}
\frac{1}{\alpha} h_{ij} \mathrm{H}_{,ij} = 2\, x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j h_{ij} \nu_{n+1} - x_{n+1}^{-2} |\mathrm{A}|^2 \nu_{n+1}^2 & \\
+ x_{n+1}^{-2} h_{ij} h_{il} \tau_{n+1}^j \tau_{n+1}^l + x_{n+1}^{-2} h_{ij} h_{jl} \tau_{n+1}^i \tau_{n+1}^l & \\
- x_{n+1}^{-1} h_{ij} h_{ij,\, l} \tau_{n+1}^l - x_{n+1}^{-1} h_{ij} h_{jl} h_{il} \nu_{n+1}. &
\end{aligned}
$$

Thus

$$\frac{1}{\alpha} h_{ij} \mathrm{H}_{,ij} = -x_{n+1}^{-2} |\mathrm{A}|^2 \nu_{n+1}^2 + |x|^{-2} |\mathrm{A}|^2 - x_{n+1}^{-1} h_{ij} h_{jl} h_{il} \nu_{n+1},$$

and finally

$$h_{ij} \mathrm{H}_{,ij} = -\alpha\, x_{n+1}^{-2} |\mathrm{A}|^2 \nu_{n+1}^2 + \alpha\, |x|^{-2} |\mathrm{A}|^2 - \mathrm{H}\, h_{ij} h_{jl} h_{il}. \tag{13}$$

(12), (13), and (11) yield the relation

$$2\int_{M} x_{n+1}^{\alpha}\,|x|^{-2}\,|A|^2\,\xi^2\,d\mathscr{H}_n \leqq \int_{M} x_{n+1}^{\alpha}\,|A|^2\,|\nabla\xi|^2\,d\mathscr{H}_n \qquad (14)$$

for all $\xi \in C_c^1(M, \mathbb{R})$.

If ξ does not have compact support in $M = \mathscr{C} - \{0\}$ then (14) continues to hold, if only

$$\int_{M} x_{n+1}^{\alpha}\,|x|^{-2}\,|A|^2\,\xi^2\,d\mathscr{H}_n < \infty. \qquad (15)$$

In fact, replace ξ by $\xi.\gamma_\varepsilon$ where γ_ε is a suitable cut off function with

$$\gamma_\varepsilon = \begin{cases} 1 & \text{for } |x| \in (\varepsilon, \varepsilon^{-1}) \\ 0 & \text{for } |x| < \dfrac{\varepsilon}{2} \text{ or } |x| > 2\varepsilon^{-1} \end{cases}$$

and $0 \leqq \gamma_\varepsilon \leqq 1$, $|\nabla\gamma_\varepsilon(x)| \leqq 3\,|x|^{-1}$ in all of $\mathbb{R}^n \times \mathbb{R}^+$. Then $\xi.\gamma_\varepsilon$ is admissible in (14) and the assertion follows by letting $\varepsilon \to 0^+$ and using (15).

Note that (15) is satisfied, if

$$\int_{M} |x|^{\alpha-2}\,|A|^2\,\xi^2\,d\mathscr{H}_n < \infty. \qquad (16)$$

From the coarea formula we infer that

$$\int_{M} \varphi(x)\,d\mathscr{H}_n(x) = \int_0^\infty r^{n-1} \int_{\Sigma} \varphi(r\omega)\,d\mathscr{H}_{n-1}\,dr \qquad (17)$$

for all non-negative $\varphi \in C^0(M)$, where $\Sigma = M \cap S^n$, and $S^n \subset \mathbb{R}^{n+1}$ denotes the unit n-phere. Also, since M is a cone, we find

$$|A(x)|^2 = |x|^{-2}\,|A(x/|x|)|^2 \quad \text{for all } x \in M.$$

Hence, we readily infer from (17) and (16) that

$$\xi = |x|^{1+\varepsilon-\alpha}.\,|x|_1^{1+\alpha-(n/2)-2\varepsilon},$$

where

$$|x|_1 = \max(1, |x|),$$

is admissible in (14), if $\varepsilon > \dfrac{\alpha}{2}$ (where we have of course assumed that $n \geqq 2$).

Furthermore we find

$$|\nabla\xi|^2 \leqq \begin{cases} (1+\varepsilon-\alpha)^2\,|x|^{2\varepsilon-2\alpha} & \text{in } M \cap B_1(0), \quad B_1(0) = \{|x| < 1\}, \\ \left(2 - \dfrac{n}{2} - \varepsilon\right)^2 |x|^{2-n-2\varepsilon} & \text{in } (\mathbb{R}^{n+1} - B_1(0)) \cap M \end{cases}$$

and (14) implies

$$
\begin{aligned}
2 \int_{M \cap B_1} & x_{n+1}^{\alpha} |A|^2 |x|^{2\varepsilon - 2\alpha} d\mathcal{H}_n \\
& + 2 \int_{M \cap (\mathbb{R}^{n+1} - B_1)} x_{n+1}^{\alpha} |A|^2 |x|^{2-n-2\varepsilon} d\mathcal{H}_n \\
& \leqq (1+\varepsilon-\alpha)^2 \int_{M \cap B_1} x_{n+1}^{\alpha} |A|^2 |x|^{2\varepsilon-2\alpha} d\mathcal{H}_n \\
& + \left(2 - \frac{n}{2} - \varepsilon\right)^2 \int_{M \cap (\mathbb{R}^{n+1} - B_1)} x_{n+1}^{\alpha} |A|^2 |x|^{2-n-2\varepsilon} d\mathcal{H}_n.
\end{aligned}
$$

We would like to choose n, ε, α so that

$$
\varepsilon > \frac{\alpha}{2}, \qquad (1+\varepsilon-\alpha)^2 < 2 \qquad \text{and} \qquad \left(\frac{n}{2} + \varepsilon - 2\right)^2 < 2. \tag{18}
$$

(18) is equivalent to

$$
-1 - \sqrt{2} + \alpha < \varepsilon < \sqrt{2} + \alpha - 1 \qquad \text{and} \qquad \frac{\alpha}{2} < \varepsilon < 2 + \sqrt{2} - \frac{n}{2}. \tag{19}
$$

If $\alpha + n < 4 + 2\sqrt{2}$ then a suitable choice of ε is

$$
\varepsilon = \frac{\alpha}{2} + \delta,
$$

where

$$
\delta = N^{-1} \left[2 + \sqrt{2} - \frac{n}{2} - \frac{\alpha}{2} \right] > 0
$$

with $N \in \mathbb{N}$ large. Thus we conclude that $|A|^2 \equiv 0$ *i.e.* M is a hyperplane $\mathcal{P}$. Because of $0 = H = \alpha \dfrac{\nu_{n+1}}{x_{n+1}}$ we must have $\nu_{n+1} = 0$ as required.

REFERENCES

[BDG] E., Bombieri, E. De Giorgi, and E. Giusti, Minimal Cones and the Bernstein Problem, *Invent. Math.*, Vol. 7, 1969, pp. 243-268.

[D1] U. Dierkes, Minimal Hypercones and $C^{0,1/2}$-Minimizers for a Singular Variational Problem, *Indiana Univ. Math. J.*, Vol. **37**, No. 4, 1988.

[D2] U. Dierkes, A Classification of Minimal Cones in $\mathbb{R}^n \times \mathbb{R}^+$ and a Counter-Example to Interior Regularity of Energy Minimizing Functions, *Manuscripta Math.*, Vol. **63**, 1989, pp. 173-192.

[FH] H. Federer, *Geometric Measure Theory*, Springer Berlin-Heidelberg-New York, 1969, Grundlehren, 153.

[SSY] R. Schoen, L. Simon and S. T. Yau, Curvature Estimates for Minimal Hypersurfaces, *Acta Math.*, Vol. **134**, 1975, pp. 276-288.

[SL] L. Simon, Lectures on Geometric Measure Theory, *Proc. of the Centre for Math. Analysis*, Australian Nat. Univ., Vol. **3**, 1983.

[SJ] J. Simons, Minimal Varieties in Riemannian Manifolds, *Ann. Math.*, Vol. **88**, 1968, pp. 62-105.

(*Manuscript received September* 4, 1989.)