Ann. Inst. Henri Poincaré,

Vol. 11, n° 6, 1994, p. 613-632. *Analyse non linéaire*

Periodic solutions with prescribed energy for some Keplerian N-body problems [1]

by

Antonio AMBROSETTI and Kazunaga TANAKA

Scuola Normale Superiore,
Piazza dei Cavalieri, 56100 Pisa, Italy.

and

Enzo VITILLARO

Dipartimento di Matematica, Università di Perugia,
Via Vanvitelli 1, 06123 Perugia, Italy.

ABSTRACT. – We prove the existence of periodic solutions with prescribed energy for a class of N-body type problems with Keplerian like interaction.

Key words: Periodic solutions, N-body problems.

RÉSUMÉ. – Nous prouvons l'existence de solutions périodiques d'énergie prescrite pour une classe de problèmes à N corps avec interaction de type képlérien.

1. INTRODUCTION

In this paper we seek for periodic solutions of

$$\left.\begin{aligned} \ddot{x} + V'(x) = 0, \\ \frac{1}{2}|\dot{x}|^2 + V(x) = h, \end{aligned}\right\} \tag{1.1}$$

Classification A.M.S.: 58 E 05, 58 F 05.

[1] Work carried out while the second author was visiting the Scuola Normale Superiore of Pisa. He wishes to thank the Scuola Normale for support and hospitality. The first and third authors are supported by M.U.R.S.T.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 11/94/06/$ 4.00/

where $x = (x_1, \cdots, x_N)$, $x_j \in \mathbf{R}^k$ and

$$V(x) = \frac{1}{2} \sum_{i \neq j} V_{ij}(x_i - x_j). \tag{1.2}$$

Roughly, we deal with potentials like

$$V_{ij}(\xi) \simeq -\frac{1}{|\xi|^\alpha}, \qquad 0 < \alpha < 2. \tag{1.3}$$

When $V_{ij}(\xi)$ is as in (1.3) and $\alpha = 1$, (1.2) is the interaction potential of N-bodies of masses $m_1 = \cdots = m_N = 1$, and (1.1) is nothing but the Kepler N-body problem.

Periodic solutions of (1.1) with $N = 2$ have been widely investigated. *See* [3] and references therein.

When $N > 2$, the problem is more difficult because the lack of compactness arises in a stronger form. The breakdown of the Palais-Smale condition has been bypassed either assuming $V_{ij}(\xi) = V_{ji}(\xi)$, (*cf.* [5]) or using critical point at infinity and Morse theory in [4, 9], or employing critical point theory with boundary condition in [7]. Using this latter tool, solutions with fixed energy have been found in [8] for a class of $V_{ij}(\xi) \simeq -|\xi|^{-\alpha}$, $\alpha > 2$ and $h > 0$. However this does not cover the Kepler N-body problem, where, among other things, the natural value of energy is negative. When $V_{ij}(\xi) = V_{ji}(\xi)$, $V_{ij}(\xi) \simeq -|\xi|^{-\alpha}$, $0 < \alpha < 2$ and $h < 0$, the existence of periodic solutions of (1.1) has been proved in [2], but no results dealing with the general case, are known. In the present paper we address this situation and prove the existence of (generalized) solutions of (1.1) for a class of Keplerian-like K-body problem.

The usual functional framework to study (1.1) is to look for critical points of the Maupertuis-like functional:

$$I(u) = \frac{1}{2} \|\dot{u}\|_{L^2}^2 \int_0^1 (h - V(u))\, dt$$

defined on

$$\Lambda = \{ u \in H^{1,2}(S^1, \mathbf{R}^{kN}); \quad u_i(t) \neq u_j(t) \quad \text{for all } t \in \mathbf{R} \text{ and } i \neq j \}.$$

Our approach consists of 4 steps:

1° In order to control the behavior of I on $\partial\Lambda$, we consider the perturbed potentials $V_\varepsilon = V - \frac{1}{2} \cdot \varepsilon \sum_{i \neq j} |u_i - u_j|^{-2}$ and the corresponding functional I_ε.

2° In constrast with [8], I_ε is not bounded from below on Λ, because $h < 0$. To bypass this difficulty we use a device like in [1]. Namely we consider the manifold

$$M = \{ u \in \Lambda;\ \|\dot{u}\|_{L^2}^2 = 1 \}$$

and a suitable, related functional $J_\varepsilon(u)$ (*see* section 3) which is bounded below on M and such that the critical points of $J_\varepsilon(u)$ constrained on M correspond to critical points of $I_\varepsilon(u)$.

3° We show that the arguments used in [7, 8] to overcome the lack of Palais-Smale condition can be adapted here to obtain approximate solutions $x^\varepsilon(t)$ of (1.1) with V_ε instead of V.

4° We show that $x^\varepsilon(t) \rightharpoonup x(t)$, a weak solution of (1.1) in the sense of [3]. *See* also Definition 3.1 below.

We point out that the regularity of weak solutions will not be studied here. For this kind of results, when $N = 2$ and $k \geqq 3$, *see* [10].

2. MAIN RESULT

We assume that the potential $V(x) = \dfrac{1}{2} \sum_{i \neq j} V_{ij}(x_i - x_j)$ satisfies the following conditions:

(V1) $V_{ij} \in C^2(\mathbf{R}^k \setminus \{0\}, \mathbf{R})$, $V_{ij}(\xi) < 0$ for all $\xi \neq 0$;

(V2) $3 V_{ij}'(\xi)\xi + V_{ij}''(\xi)\xi \cdot \xi \neq 0$ for all $\xi \neq 0$;

(V3) There exists an $\alpha \in (0, 2)$ such that

$$V_{ij}'(\xi)\xi \geqq -\alpha V_{ij}(\xi) \quad \text{for all } \xi \neq 0;$$

(V4) There exist $\beta \in (0, 2)$ and $r_1 > 0$ such that

$$V_{ij}'(\xi)\xi \leqq -\beta V_{ij}(\xi) \quad \text{for all } 0 < |\xi| \leqq r_1;$$

(V5) $V_{ij}(\xi) + \dfrac{1}{2} V_{ij}'(\xi)\xi \to 0$ as $|\xi| \to \infty$;

(V6) There exist $\theta \in [0, \pi/2)$ and $r_2 > 0$ such that

$$\operatorname{ang}(V_{ij}'(\xi), \xi) \leqq \theta \quad \text{for all } |\xi| \geqq r_2.$$

Here

$$\operatorname{ang}(\xi, \eta) = \begin{cases} \arccos \dfrac{\xi \cdot \eta}{|\xi|\,|\eta|}, & \text{if } |\xi||\eta| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.1. – Without loss of generality, we can assume that

$$V_{ij}(\xi) = V_{ji}(-\xi).$$

Otherwise, consider $\bar{V}_{ij}(\xi) = \frac{1}{2}(V_{ij}(\xi) + V_{ji}(-\xi))$ instead of $V_{ij}(\xi)$.

Remark 2.2. – From (V1)-(V4), it follows that

(i) $V'_{ij}(\xi)\,\xi > 0$ for all $\xi \neq 0$, (2.1)

(ii) For some $a > 0$,

$$V_{ij}(\xi) \leqq -\frac{a}{|\xi|^\alpha} \quad \text{for all } 0 < |\xi| \leqq r_1, \tag{2.2}$$

(iii) $V_{ij}(\xi) + \frac{1}{2}V'_{ij}(\xi)\,\xi \to -\infty$ as $|\xi| \to 0$. (2.3)

To state our result, we need to introduce the concept of weak periodic solutions of (1.1) as in Definition 10.1 of [3]. Roughly, it is a special class of generalized solutions which are found as limits of non-collision solutions of approximate problems. Since we need some notations to give a precise definition of weak solutions, we will give it in the next section.

Remark 2.3. – It is shown in [3] that every weak T-periodic solution $x(t)$ satisfies

(i) meas $\mathcal{C}(x) = 0$, where

$$\mathcal{C}(x) = \{t \in \mathbf{R};\ x_i(t) \neq x_j(t) \text{ for some } i \neq j\},$$

(ii) $x(t) \in C^2(\mathbf{R}\backslash\mathcal{C}(x), \mathbf{R}^{kN})$,

(iii) $x(t)$ satisfies (1.1) for all $t \in \mathbf{R}\backslash\mathcal{C}(x)$,

(iv) $\int_0^T \left[\frac{1}{2}|\dot{x}|^2 - V(x(t))\right] dt < \infty$.

In the sequel, a solution $x(t)$ will be called a *noncollision* solutions of (1.1) if $\mathcal{C}(x) = \emptyset$.

Now we can state our main result.

THEOREM 2.1. – *Suppose that* (V1)-(V6) *hold. Then for all* $h < 0$, *the problem* (1.1) *has a weak periodic solution.*

In the following sections, we will give a proof to Theorem 2.1.

3. VARIATIONAL FORMULATION

Throughout this paper, we use the following notation:

NOTATION:

$$H = H^{1,2}(S^1, \mathbf{R}^k),$$

$$[u] = \int_0^1 u(t)\,dt \qquad \text{for} \quad u \in H,$$

$$\|u\|_2^2 = \int_0^1 |u|^2\,dt \qquad \text{for} \quad u \in L^2(S^1, \mathbf{R}^{kN}),$$

$$\|u\|^2 = \int_0^1 |\dot{u}|^2\,dt + |[u]|^2 = \sum_{i=1}^N \left(\int_0^1 |\dot{u}_i|^2\,dt + |[u_i]|^2\right)$$

for all $u = (u_1, \cdots, u_N) \in H^N$,

$$E = \left\{ u = (u_1, \cdots, u_N) \in H^N;\ \sum_{i=1}^N [u_i] = 0 \right\},$$

$$\Lambda = \{ u \in E;\ u_i(t) \neq u_j(t) \text{ for all } t \text{ and all } i \neq j \},$$

$$(u|v) = \int_0^1 uv\,dt \quad \text{for } u,\ v \in L^2(S^1, \mathbf{R}^{kN}),$$

$\langle f|u \rangle$ = the duality product of $f \in E^*$ and $u \in E$.

For a sequence $(u^n)_{n=1}^\infty \subset E$, we write

$$u^n \rightharpoonup u^0$$

to indicate that u^n converges to u^0 weakly in E and uniformly on [0, 1].

We consider the following functional, $I : \Lambda \to \mathbf{R}$,

$$I(u) = \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 (h - V(u))\,dt.$$

It is well-known that critical point of $I(u)$ on Λ, such that $I(u) > 0$, would give a rise – after a suitable time scaling – to a non-collision periodic solution of (1.1). However unfortunately, it is difficult to deal with $I(u)$ directly and we need to introduce a modified functional $I_\varepsilon(u)$ for $\varepsilon \in (0, 1]$, by setting

$$V_\varepsilon(x) = V(x) - \frac{\varepsilon}{2}\sum_{i\neq j} \frac{1}{|x_i - x_j|^2} \qquad \text{for} \quad x \in \mathbf{R}^{kN},$$

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\|\dot{u}\|_2^2 \int_0^1 (h - V_\varepsilon(u))\,dt \\ &= I(u) + \frac{\varepsilon}{4}\|\dot{u}\|_2^2 \int_0^1 \sum_{i\neq j} \frac{1}{|u_i - u_j|^2}\,dt. \end{aligned}$$

The main different feature of V_ε is that: for every sequence $(u^n) \subset \Lambda$ such that $u^n \rightharpoonup u \in \partial\Lambda$, we have

$$\int_0^1 \sum_{i \neq j} \frac{1}{|u_i^n - u_j^n|^2} \, dy \to \infty, \tag{3.1}$$

that is, for $\varepsilon \in (0, 1]$

$$\int_0^1 -V_\varepsilon (u^n) \, dt \to \infty. \tag{3.2}$$

We remark here that if $v^\varepsilon \in \Lambda$ satisfies $I'_\varepsilon (v^\varepsilon) = 0$ and $I'_\varepsilon (v^\varepsilon) > 0$, then

$$x^\varepsilon (t) = v^\varepsilon(\omega_\varepsilon t), \qquad \omega_\varepsilon = \frac{2\sqrt{I_\varepsilon (v^\varepsilon)}}{\|\dot{v}^\varepsilon\|_2^2} \tag{3.3}$$

is a periodic solution of the perturbed problem:

$$\left.\begin{aligned} \ddot{x} + V'_\varepsilon (x) = 0, \\ \frac{1}{2}|\dot{x}|^2 + V_\varepsilon (x) = h, \end{aligned}\right\} \tag{3.4}$$

Now we can give a precise definition of a weak periodic solution $x(t)$ of (1.1).

Definition 3.1. – (*cf.* Definition 10.1 of [3]). $x(t)$ is said to be a *weak periodic solution* of (1.1) if there exist sequences $(v^n) \subset \Lambda$ and $\varepsilon_n \to 0$ such that

1° $v^n \in \Lambda$ is a critical point of I_{ε_n} such that $I_{\varepsilon_n} (v^n) > 0$, that is, if we set $x^n (t)$ as in (3.3), $x^n (t)$ is a periodic solution of (3.4).

2° There exists a constant $a > 0$ such that

$$0 < I_{\varepsilon_n} (v^n) \leqq a < \infty.$$

3° $\omega_n \to \omega \neq 0$, $v^n \rightharpoonup v \in E$ and $x(t) = v(\omega t)$.

4° There exists a $t_0 \in (0, 1/\omega]$ such that

$$x_i (t_0) \neq x_j (t_0) \quad \text{for all } i \neq j.$$

As anticipated before, it has been proved in Theorem 10.7 of [3] that any weak periodic solution satisfies the properties (i)-(iv) of Remark 2.3.

Next we define for $u \in \Lambda$ with $\dot{u} \not\equiv 0$ a positive number $\rho = \rho(u) > 0$ by

$$\frac{d}{d\rho} I_\varepsilon (\rho u) = 0. \tag{3.5}$$

Lemma 3.1. – *For any $u \in \Lambda$ with $\dot{u} \not\equiv 0$, the equation* (3.5) *has a unique solution $\rho = \rho(u) > 0$, which is independent of $\varepsilon \in (0, 1]$ and satisfies*

$$h = \int_0^1 \left[V(\rho u) + \frac{1}{2} V'(\rho u) \rho u \right] dt. \tag{3.6}$$

Proof. – For $u \in \Lambda$, a direct calculation gives us

$$\frac{d}{d\rho} I(\rho u) = \rho \|\dot{u}\|_2^2 \int_0^1 \left[h - V(\rho u) - \frac{1}{2} V'(\rho u) \rho u \right] dt.$$

Thus, for $u \in \Lambda$, with $\dot{u} \not\equiv 0$, (3.5) is equivalent to (3.6). We set for $u \in \Lambda$, and $\rho > 0$

$$\phi_u(\rho) = \int_0^1 \left[V(\rho u) + \frac{1}{2} V'(\rho u) \rho u \right] dt.$$

From (V2), (V5) and (2.3), it follows

$$\phi'_u(\rho) > 0 \quad \text{for all } \rho \in (0, \infty), \tag{3.7}$$

$$\phi_u(\rho) \to -\infty \quad \text{as } \rho \to 0,$$

$$\phi_u(\rho) \to 0 \quad \text{as } \rho \to \infty.$$

Thus there is a unique $\rho = \rho(u) > 0$ such that $\phi_u(\rho) = h$ for all $u \in \Lambda$ and $h < 0$. ■

Remark 3.1. – In what follows, we define $\rho(u) > 0$ for all $u \in \Lambda$ by (3.6). We state some properties of $\rho(u)$.

LEMMA 3.2. – (i) $\rho(u) \in C^1(\Lambda, \mathbf{R})$;

(ii) *If* $u^n \rightharpoonup u \in \Lambda$, *then* $\rho(u^n) \to \rho(u)$;

(iii) *For all* $u \in \Lambda$,

$$\begin{aligned} \int_0^1 [h - V(\rho(u) u)] \, dt &= \frac{1}{2} \int_0^1 V'(\rho(u) u) \rho(u) u \, dt \\ &\geqq \frac{1}{2} \left(\frac{1}{\alpha} - \frac{1}{2} \right)^{-1} (-h) \equiv C_0 > 0 \end{aligned} \tag{3.8}$$

Proof. – Properties (i) and (ii) easily follow from the implicit function theorem, using (3.6) and (3.7).

(iii) From (3.6) and (V3), we have

$$\begin{aligned} h &= \int_0^1 \left[V(\rho(u) u) + \frac{1}{2} V'(\rho(u) u) \rho(u) u \right] dt \\ &\leqq -\left(\frac{1}{\alpha} - \frac{1}{2} \right) \int_0^1 V'(\rho(u) u) \rho(u) u \, dt, \end{aligned}$$

and (3.8) follows. ■

We set

$$M = \{ u \in \Lambda;\ \|\dot{u}\|_2^2 = 1 \}$$

and

$$\begin{aligned} J_\varepsilon(u) &= I_\varepsilon(\rho(u)\, u) \\ &= \frac{1}{2}\rho(u)^2 \int_0^1 [h - V_\varepsilon(\rho(u)\, u)]\, dt \\ &= \frac{1}{2}\rho(u)^2 \int_0^1 [h - V(\rho(u)\, u)]\, dt + \frac{\varepsilon}{4}\int_0^1 \sum_{i\neq j} \frac{1}{|u_i - u_j|^2}\, dt \\ &= \frac{1}{4}\rho(u)^2 \int_0^1 V'(\rho(u)\, u)\, \rho(u)\, u\, dt \\ &\quad + \frac{\varepsilon}{4}\int_0^1 \sum_{i\neq j} \frac{1}{|u_i - u_j|^2}\, dt. \end{aligned} \tag{3.9}$$

We remark that $\rho(u) > 0$ and (3.8) imply

$$J_\varepsilon(u) \geqq \frac{C_0}{2}\rho(u)^2 > 0 \quad \text{on } M. \tag{3.10}$$

We also remark here that M is a submanifold of E of codimension 1 and

$$T_u M = \{ \varphi \in E;\ (\dot{u}|\dot{\varphi}) = 0 \},$$

$$(T_u M)^\perp = \operatorname{span}\{ u - [u] \}.$$

In particular, all constant functions belong to $T_u M$ for all $u \in M$.

In what follows, for a functional $F(u) \in C^1(\Lambda, \mathbf{R})$, we denote by $F'(u)$ its gradient in Λ, and for $u \in M$ we denote by $\nabla_M F(u)$ its gradient constrained on M, *i. e.*, $\nabla_M F(u) \in E^*$ is a vector satisfying

$$\langle \nabla_M F(u)|\varphi\rangle = \langle F'(u)|\varphi\rangle \quad \text{for all } \varphi \in T_u M,$$

$$\langle \nabla_M F(u)|\varphi\rangle = 0 \quad \text{for all } \varphi \in (T_u M)^\perp.$$

More precisely, it is given by

$$\langle \nabla_M F(u)|\varphi\rangle = \langle F'(u)|\varphi\rangle - \langle F'(u)|u - [u]\rangle (\dot{u}|\dot{\varphi}).$$

Here we state some properties of $I_\varepsilon(u)$ and $J_\varepsilon(u)$.

1° It follow from (3.5) that

$$\langle I'_\varepsilon(\rho(u)\, u)|u\rangle = 0 \quad \text{for all } u \in \Lambda. \tag{3.11}$$

2° For $u \in \Lambda$ and $\varphi \in E$,

$$\begin{aligned}\langle J_\varepsilon'(u)|\varphi\rangle &= \rho(u)\langle I_\varepsilon'(\rho(u)u)|\varphi\rangle + \langle I_\varepsilon'(\rho(u)u)|u\rangle\langle \rho'(u)|\varphi\rangle \\ &= \rho(u)\langle I_\varepsilon'(\rho(u)u)|\varphi\rangle,\end{aligned}$$

that is,

$$J_\varepsilon'(u) = \rho(u) I_\varepsilon'(\rho(u)u). \tag{3.12}$$

3° For $u \in E$, define $f_u \in E^*$ by

$$\langle f_u|\varphi\rangle = (\dot{u}|\dot{\varphi}).$$

Then $\nabla_M J_\varepsilon(u)$ can be written as

$$\begin{aligned}\nabla_M J_\varepsilon(u) &= J'(u) - \nu f_u \\ &= \rho(u) I_\varepsilon'(\rho(u)u) - \nu f_u \end{aligned} \tag{3.13}$$

with $\nu = \langle J_\varepsilon'(u)|u - [u]\rangle$.

4° Assume $\nabla_M J_\varepsilon(u) = 0$ for $u \in M$. By 1° and 3°, we have

$$\begin{aligned}0 &= \langle \nabla_M J_\varepsilon(u)|u\rangle \\ &= \rho(u)\langle I_\varepsilon'(\rho(u)u)|u\rangle - \nu = -\nu.\end{aligned}$$

Hence $I_\varepsilon'(\rho(u)u) = 0$.

Thus we have

LEMMA 3.3. – *Let $u^\varepsilon \in M$ be a critical point of J_ε on M, that is, $\nabla_M J_\varepsilon(u^\varepsilon) = 0$. Then*

(i) *$I_\varepsilon'(\rho_\varepsilon u^\varepsilon) = 0$, where $\rho_\varepsilon = \rho(u^\varepsilon)$.*

(ii) *Set $\omega_\varepsilon^2 = 4 I_\varepsilon(\rho_\varepsilon u^\varepsilon)/\rho_\varepsilon^4$ and $x^\varepsilon(t) = \rho_\varepsilon u^\varepsilon(\omega_\varepsilon t)$. Then $x^\varepsilon(t)$ is a noncollision solution of* (3.4). ■

Remark 3.2. – Since $\rho_\varepsilon > 0$ and (3.10) holds, ω_ε is well-defined and $\omega_\varepsilon > 0$.

In the following sections, we shall find a critical point u^ε of J_ε on M.

4. A CRITICAL POINT LEMMA

It is known that $J_\varepsilon(u)$ does not satisfy the Palais-Smale condition on M (*cf.* [4, 7, 8]). To overcome this difficulty, we follow the procedure of [7, 8] (*see* also [6]). We set

$$g(u) = \sum_{i=1}^{N} |[u_i - u_j]|^2 \qquad \text{for} \quad u \in E,$$
$$M^b = \{ u \in M;\ g(u) \leqq b \}.$$

The following is nothing but Lemma 2.1 of [8] in our setting.

LEMMA 4.1. – *Assume that there are constants c and $\tilde{c}$ with $c < \tilde{c}$ and $b \in \mathbf{R}$ such that*

(H1) *If $(u^n) \subset M$ satisfies $u^n \rightharpoonup u^0 \in \partial\Lambda$ and $g(u^n)$ is bounded, then*

$$J_\varepsilon(u^n) \to \infty.$$

(H2) $\nabla_M g(u) \neq 0$ *for all* $g(u) = b$, $J_\varepsilon(u) = c$.

(H3) *If $(u^n) \subset M$ satisfies $J_\varepsilon(u^n) \to c$, $\limsup g(u^n) \leqq b$ and $\nabla_M J_\varepsilon(u^n) \to 0$, then (u^n) possesses a convergent subsequence.*

(H4) *If $(u^n) \subset M$ satisfies $J_\varepsilon(u^n) \to c$, $g(u^n) \to b$ and*

$$\nabla_M J_\varepsilon(u^n) - \lambda_n \nabla_M(u^n) \to 0$$

for some $\lambda_n \geqq 0$, then (u^n) possesses a convergent subsequence.

(H5) $\nabla_M J_\varepsilon(u) \neq \lambda \nabla_M g(u)$ *for all $u \in M$ with $J_\varepsilon(u) = c$, $g(u) = b$ and for all $\lambda > 0$.*

(H6) *For any $\delta > 0$ with $c + \delta < \tilde{c}$, the set*

$$\{u \in M;\ J_\varepsilon(u) \leqq c+\delta\} \cup (\{u \in M;\ J_\varepsilon(u) \leqq \tilde{c}\} \cap \{u \in M;\ g(u) \geqq b\})$$

is not deformable in M into

$$\{u \in M;\ J_\varepsilon(u) \leqq c-\delta\} \cup (\{u \in M;\ J_\varepsilon(u) \leqq \tilde{c}\} \cap \{u \in M;\ g(u) \geqq b\}).$$

Then $J_\varepsilon(u)$ has a least one critical point $u \in M$ such that $J_\varepsilon(u) = c$ and $g(u) \leqq b$. ■

We are going to verify the conditions (H1)-(H6) for suitable c, $\tilde{c}$, $b > 0$. First of all, we remark by (3.1) and (3.9) that if $u^n \rightharpoonup u^0 \in \partial\Lambda$ then

$$J_\varepsilon(u^n) \geqq \frac{\varepsilon}{4} \int_0^1 \sum_{i \neq j} \frac{1}{|u_i - u_j|^2}\, dt \to \infty. \tag{4.1}$$

Therefore (H1) holds. Moreover, since

$$\langle \nabla_M g(u) | [u] \rangle = 2g(u) = 2b \neq 0 \qquad \text{and} \qquad [u] \in T_u M$$

for all $g(u) = b > 0$,

$$\nabla_M g(u) \neq 0 \quad \text{for all } g(u) = b > 0.$$

That is, (H2) holds for all $b > 0$ and $\tilde{c} \in \mathbf{R}$.

In Section 5, we verify (H3) and (H4) which are local versions of Palais-Smale condition, and in Section 6 we will get (H5) and (H6).

5. PALAIS-SMALE CONDITION

To verify (H3), (H4), some lemmas are in order. First we need some properties of $\rho(u)$.

LEMMA 5.1. – *For $c > 0$, there are $k_1 = k_1(c) > 0$ and $k_2 = k_2(c) > 0$ independent of $\varepsilon \in (0, 1]$ such that for $u \in M$*

(i) $J_\varepsilon(u) \leqq c$ *implies* $\rho(u) \leqq k_1(c)$,

(ii) $J_\varepsilon(u) \geqq c$ *implies* $\rho(u) \geqq k_2(c)$.

Proof. – (i) follows from (3.10) easily. We prove (ii) here. We argue indirectly and assume there are sequences $(u^n) \subset M$ and $(\varepsilon_n) \subset (0, 1]$ such that

$$J_{\varepsilon_n}(u^n) \geqq c \qquad \text{and} \qquad \rho_n \equiv \rho(u^n) \to 0.$$

We set $w^n = \rho_n u^n$. Since $\rho^n \to 0$ and

$$c \leqq J_{\varepsilon_n}(u^n) = \frac{1}{2}\rho_n^2 \int_0^1 \left[h - V(w^n) + \frac{\varepsilon_n}{4} \sum_{i \neq j} \frac{1}{|w_i^n - w_j^n|^2} \right] dt,$$

it follows

$$\int_0^1 \left[h - V(w^n) + \frac{\varepsilon_n}{4} \sum_{i \neq j} \frac{1}{|w_i^n - w_j^n|^2} \right] dt \to \infty.$$

In particular, for some $i \neq j$, one has

$$\min_{t \in [0, 1]} |w_i^n(t) - w_j^n(t)| \to 0.$$

On the other hand, $\|\dot{w}^n\|_2 = \rho_n \to 0$ and hence

$$\max_{t \in [0, 1]} |w_i^n(t) - w_j^n(t)| \to 0.$$

Then $w_i^n - w_j^n \to 0$ uniformly in [0, 1] and therefore, by (3.6) and (2.3)

$$h = \int_0^1 \left[V(w^n) + \frac{1}{2} V'(w^n) w^n \right] dt \to -\infty.$$

This is a contradiction. ■

Recall that $\sum_{i=1}^{N} [u_i] = 0$ for $u \in E$. Therefore M^b is a bounded set of E for all b. Thus using also (4.1), we infer

LEMMA 5.2. – *For* $\varepsilon \in (0, 1]$, *suppose* $(u^n) \subset M$ *satisfies for* $b > 0$, $c > 0$

(i) $u^n \in M^b$, i.e., $g(u^n) \leqq b$,

(ii) $J_{\varepsilon_n}(u^n) \leqq c$.

Then (u^n) *has a subsequence – still denoted by* u^n *– such that*

$$u^n \rightharpoonup u^0 \in \Lambda. \quad ■$$

Next we prove (H3) and (H4).

LEMMA 5.3. – *Suppose* $\varepsilon \in (0, 1]$. *Then*

(i) (H3) *holds for all* $b > 0$ *and* $c > 0$,

(ii) (H4) *holds for all* $b > 0$ *and* $c > 0$.

Proof. – (i) We assume

$$u^n \in M^b,$$

$$J_\varepsilon(u^n) \to c > 0,$$

$$\nabla_M J_\varepsilon(u^n) \to 0 \quad \text{strongly in } E^*.$$

Our goal is to prove there is a strongly convergent subsequence of (u^n) such that $u^n \to u^0 \in M$. By Lemma 5.2, (u^n) possesses a weakly convergent subsequence $u^n \rightharpoonup u^0 \in \Lambda$. Thus it suffices to show the convergence is strong, that is, $\|\dot{u}^n\|_2 \to \|\dot{u}^0\|_2$, *i.e.*, $\|\dot{u}^0\|_2 = 1$.

First, by Lemma 5.1, we remark that

$$\rho_n \equiv \rho(u^n) \in [k_1, k_2],$$

where $k_1, k_2 > 0$ are independent of n and hence $\rho_n \to \rho_0 = \rho(u^0) \neq 0$.

Since $\nabla_M J_\varepsilon(u^n) \to 0$, by (3.13) there exists a sequence $(\nu_n) \subset \mathbf{R}$ such that

$$\rho_n I'_\varepsilon(\rho_n u^n) - \nu_n f_{u^n} \to 0 \quad \text{strongly in } E^*.$$

Taking a scalar product with u^n, we infer $\nu_n \to 0$ by (3.11). Thus,

$$I'_\varepsilon(\rho_n u^n) \to 0 \quad \text{strongly in } E^*.$$

In particular, we have $\langle I'_\varepsilon(\rho_n u^n)|u^0\rangle \to 0$, *i.e.*,

$$\rho_n (\dot{u}^n|\dot{u}^0) \int_0^1 [h - V_\varepsilon(\rho_n u^n)]\, dt - \frac{1}{2}\rho_n^2 \int_0^1 V'_\varepsilon(\rho_n u^n)\, u^0\, dt \to 0.$$

Then

$$\rho_0 \|\dot{u}^0\|_2^2 \int_0^1 [h - V_\varepsilon(\rho_0 u^0)]\, dt - \frac{1}{2}\rho_0^2 \int_0^1 V'_\varepsilon(\rho_0 u^0)\, u^0\, dt = 0. \quad (5.1)$$

On the other hand, by (3.11), $\langle I'_\varepsilon(\rho_n u^n)|u^n\rangle = 0$, *i.e.*,

$$\rho_n \int_0^1 [h - V_\varepsilon(\rho_n u^n)]\, dt - \frac{1}{2}\rho_n^2 \int_0^1 V'_\varepsilon(\rho_n u^n)\, u^0\, dt = 0.$$

Taking a limit as $n \to \infty$, we have

$$\rho_0 \int_0^1 [h - V_\varepsilon(\rho_0 u^0)]\, dt - \frac{1}{2}\rho_0^2 \int_0^1 V'_\varepsilon(\rho_0 u^0)\, u^0\, dt = 0. \quad (5.2)$$

Comparing (5.1) and (5.2) and recalling (3.8), we have $\|\dot{u}^0\|_2 = 1$, *i.e.*, $u^n \to u^0 \in \Lambda$ strongly in E.

(ii) Next we assume $(u^n) \subset M$ satisfies

$$g(u^n) \to b > 0, \quad (5.3)$$

$$J_\varepsilon(u^n) \to c > 0, \quad (5.4)$$

$$\nabla_M J_\varepsilon(u^n) - \mu_n \nabla_M g(u^n) \to 0, \quad (5.5)$$

with $\mu_n \geqq 0$.

By Lemma 5.2, we may assume $u^n \to u^0 \in \Lambda$ and again it suffices to show $\|\dot{u}^0\|_2 = 1$. Again we note that

$$\rho_n \to \rho_0 \neq 0.$$

By the definition of ∇_M and (5.5), there exists $(\nu_n) \subset \mathbf{R}$ such that

$$\rho_n I'_\varepsilon(\rho_n u^n) - \mu_n g'(u^n) + \nu_n f_{u^n} \to 0 \quad \text{strongly in } E^*. \quad (5.6)$$

Taking a product of (5.6) and u^n, we get from (3.11)

$$-\mu_n \langle g'(u^n)|u^n\rangle + \nu_n \langle f_{u^n}|u^n\rangle \to 0,$$

i.e.,

$$-2\mu_n g(u^n) + \nu_n \to 0. \quad (5.7)$$

Taking a product of (5.6) and $[u^n]$, we also get

$$\rho_n \langle I'_\varepsilon (\rho_n u^n) | [u^n] \rangle - 2 \mu_n g(u^n) \to 0. \tag{5.8}$$

Since $\rho_n u^n \rightharpoonup \rho_0 u^0 \in \Lambda$ uniformly in [0, 1] and weakly in E, we can see $\langle I'_\varepsilon (\rho_n u^n) | [u^n] \rangle$ stays bounded as $n \to \infty$. Thus by (5.3), (5.7), (5.8), μ_n and ν_n stay bounded as $n \to \infty$. Therefore we may assume $\mu = \lim_{n \to \infty} \mu_n$ and $\nu = \lim_{n \to \infty} \nu_n$ exist. Here we remark

$$\nu = \lim_{n \to \infty} \nu_n = 2b \lim_{n \to \infty} \mu_n \geqq 0. \tag{5.9}$$

As in the proof of (i), we take scalar products of (5.6) and u^n (resp. u^0) and take limits as $n \to \infty$. Then we have

$$\rho_0^2 \int_0^1 [h - V_\varepsilon (\rho_0 u^0)] \, dt - \rho_0^3 \int_0^1 V'_\varepsilon (\rho_0 u^0) u^0 \, dt - 2b\mu + \nu = 0$$

and

$$\rho_0^2 \|\dot{u}^0\|_2^2 \int_0^1 [h - V_\varepsilon (\rho_0 u^0)] \, dt - \rho_0^3 \times \int_0^1 V'_\varepsilon (\rho_0 u^0) u^0 \, dt - 2b\mu + \nu \|\dot{u}^0\|_2^2 = 0.$$

Recalling (3.8) and (5.9), we get $\|\dot{u}^0\|_2 = 1$. ■

6. SOLUTIONS OF (3.4)

Next we deal with (H5) and (H6). The arguments of the proofs are similar to those of [8].

LEMMA 6.1. – *For any* $0 < c_1 < c_2$ *there exists* $B_0 = B_0(c_1, c_2) > 0$ *independent of* $\varepsilon \in (0, 1]$ *such that*

$$\nabla_M J_\varepsilon(u^n) \neq \lambda \nabla_M g(u)$$

for all $\lambda > 0$ *and* $u \in M$ *with* $J_\varepsilon(u) \in [c_1, c_2]$, $g(u) \geqq B_0$.

Proof. – Arguing indirectly, we assume that there exist $(\varepsilon_n) \subset (0, 1]$, $(u^n) \subset M$ and $(\lambda_n) \subset (0, \infty)$ such that

$$\nabla_M J_{\varepsilon_n}(u^n) = \lambda_n \nabla_M g(u^n), \tag{6.1}$$

$$J_{\varepsilon_n}(u^n) \in [c_1, c_2], \tag{6.2}$$

$$g(u^n) \to \infty. \tag{6.3}$$

We set $z^n = g(u^n)^{-1/2}[u^n]$. Clearly (z^n) is a bounded sequence and we may assume $z^0 = \lim_{n\to\infty} z^n$ exists. We remark $g(z^0) = 1$ and $z^0 \in T_{u^n} M$ for all n. We will show

$$\langle \nabla_M J_{\varepsilon_n}(u^n) | z^0 \rangle \leqq 0, \tag{6.4}$$

$$\langle \nabla_M g(u^n) | z^0 \rangle > 0, \tag{6.5}$$

for large n. Clearly they are incompatible with (6.1) and $\lambda_n > 0$. By (6.2) and Lemma 5.1, we have

$$\rho_n \equiv \rho(u^n) \in [k_1(c_1),\ k_2(c_2)].$$

Note that if $z_i^0 \neq z_j^0$, then $|[u_i^n] - [u_j^n]| \to \infty$. For such $i \neq j$, we have from (6.3)

$$\begin{aligned} |\rho_n(u_i^n(t) - u_j^n(t))| &\geqq \rho_n|[u_i^n] - [u_j^n]| - 2\rho_n\|\dot{u}^n\|_2 \\ &\geqq k_1(c_1)|[u_i^n] - [u_j^n]| - k_2(c_2) \\ &\to \infty \quad \text{as } n \to \infty, \end{aligned} \tag{6.6}$$

$$\begin{aligned} &|g(u^n)^{-1/2}\rho_n(u_i^n(t) - u_j^n(t)) - \rho_n(z_i^0 - z_j^0)| \\ &\quad \leqq k_2(c_2)\, g(u^n)^{-1/2}\|\dot{u}^n\|_2 + 2k_2(c_2)|z^n - z^0| \to 0 \end{aligned} \tag{6.7}$$

as $n \to \infty$ uniformly in t.

Thus from (6.6)-(6.7), we can see

$$\text{ang}\,(\rho_n(u_i^n(t) - u_j^n(t)),\ z_i^0 - z_j^0) \to 0,$$

$$\min_{t\in[0,1]} |\rho_n(u_i^n(t) - u_j^n(t))| \to \infty$$

as $n \to \infty$ uniformly in t. Thus by (V6)

$$\begin{aligned} \langle \nabla_M J_{\varepsilon_n}(u^n) | z \rangle &= \rho_n \langle I'_\varepsilon(\rho_n u^n) | z \rangle \\ &= -\frac{1}{2}\rho_n^2 \int_0^1 \left[\sum_{i\neq j} V_{ij}(\rho_n(u_i^n - u_j^n))(z_i^0 - z_j^0) \right. \\ &\quad \left. + \frac{\varepsilon_n}{4} \sum_{i\neq j} \frac{(u_i^n - u_j^n)(z_i^0 - z_j^0)}{|u_i^n - u_j^n|^4} \right] dt \\ &< 0 \end{aligned}$$

for large n. Thus we get (6.4). On the other hand,

$$\langle \nabla_M g(u^n) | z^0 \rangle = g(u^n)^{1/2} \langle g'(z^n) | z^0 \rangle \to \infty.$$

Therefore we get (6.5). Thus (6.1) cannot take place. ■

As in [7, 8], we define admissible sets. Let $\mathcal{H}$ be the set of deformations of Λ in E into the space of constant functions $\mathbf{R}^{kN}$;

$$\mathcal{H} = \{ \eta \in C([0, 1] \times \Lambda, E);\ \eta(0, \cdot) = \mathrm{id},\ \eta(1, \Lambda) \subset \mathbf{R}^{kN} \}.$$

We also use the notation, for $i \neq j$,

$$\Gamma_{ij} = \{ u \in E;\ u_i(t) = u_j(t) \text{ for some } t \}.$$

DEFINITION 6.1. – Let A be a closed subset of Λ. We say A is *admissible* if for any $\eta \in \mathcal{H}$ there exists $u \in A$ such that for any $i \neq j$ there exists a sequence $i_1, \cdots, i_m \in \{ 1, \cdots, N \}$ satisfying

1° $i_1 = i,\ i_m = j$;

2° $i_k \neq i_{k+1}$ for all $k = 1, \cdots, m-1$;

3° $\eta([0, 1] \times \{ u \}) \cap \Gamma_{i_k i_{k+1}} \neq \emptyset$ for all $k = 1, \cdots, m-1$.

We denote by $\mathcal{A}$ the class of admissible sets.

It is shown in [7, 8] that there is a compact admissible set, which does not contain constant functions, and

(A1) If $A \subset B$ and $A \in \mathcal{A}$, then $B \in \mathcal{A}$,

(A2) If B is a deformation of $A \in \mathcal{A}$ in Λ, then $B \in \mathcal{A}$.

We set

$$\mathcal{A}_M = \{ A \subset M;\ A \in \mathcal{A} \}.$$

Plainly $\mathcal{A}_M \neq \emptyset$, indeed it contains any radial projection on M of $A \in \mathcal{A}$ with $A \cap \{ u \in \Lambda;\ \dot{u} \equiv 0 \} = \emptyset$.

The following property is important for our argument.

LEMMA 6.2. – *There exists $B_1 > 0$ such that*

$$\{ u \in M;\ g(u) \geqq B_1 \} \notin \mathcal{A}_M.$$

Proof. – It suffices to show for any $A \in \mathcal{A}_M$ there exists $u \in A$ such that $g(u) \leqq 2N(N-1)^3$.

Let $\eta_0 \in \mathcal{H}$ be a deformation such that

$$\eta_0(s, u) = [u] + (1-s)(u - [u]).$$

By the definition of admissible sets, there is a $u \in A$ such that for any $i \neq j$ there exists a sequence $i_1, \cdots, i_m$ satisfying the properties 1°-3° of Definition 6.1. We remark that we may assume $m \leqq N$.

By 3° of Definition 6.1, $\eta_0([0, 1] \times \{u\}) \cap \Gamma_{i_k i_{k+1}} \neq \emptyset$ for all $k = 1, \cdots, m-1$. Thus for some $s_k \in [0, 1]$ and t_k, we have

$$[u_{i_k} - u_{i_{k+1}}] + (1 - s_k)(u_{i_k}(t_k) - u_{i_{k+1}}(t_k) - [u_{i_k} - u_{i_{k+1}}]) = 0.$$

Thus

$$|[u_{i_k} - u_{i_{k+1}}]| \leqq (1 - s_k)\|\dot{u}_{i_k} - \dot{u}_{i_{k+1}}\|_2 \leqq \sqrt{2}.$$

Therefore

$$|[u_i - u_j]| \leqq \sum_{k=1}^{m-1} |[u_{i_k} - u_{i_{k+1}}]| \leqq \sqrt{2}(N-1).$$

Since the pair (i, j) with $i \neq j$ is arbitrary, we have

$$g(u) = \sum_{i \neq j} |[u_i - u_j]|^2 \leqq 2N(N-1)^3.$$

We also have

LEMMA 6.3. – *For any given $b' > 0$ there is a $\gamma = \gamma(b') > 0$ independent of $\varepsilon \in (0, 1]$ such that*

$$\{u \in M;\ J_\varepsilon(u) \leqq \gamma\} \subset \{u \in M;\ g(u) \geqq b'\}.$$

Proof. – We argue indirectly and assume there are sequences $(u^n) \subset M$ and $(\varepsilon_n) \subset (0, 1]$ such that

$$J_{\varepsilon_n}(u^n) \to 0,$$

$$g(u^n) \leqq b'.$$

By (3.10), we have

$$\rho_n \equiv \rho(u^n) \to 0.$$

Since $M^{b'}$ is bounded in E, we can see

$$\rho_n u^n \to 0 \quad \text{strongly in } E.$$

But this is incompatible with (3.6) and (2.3). ■

COROLLARY 6.4. – *Let $B_1 > 0$ be a number given in Lemma 6.2. Then*

$$\{u \in M; J_\varepsilon(u) \leqq \gamma(B_1)\} \cup \{u \in M; g(u) \geqq B_1\} = \{u \in M;\ g(u) \geqq B_1\}.$$

Now we define minimax values c_ε^* by

$$c_\varepsilon^* = \inf_{A \in \mathcal{A}_M} \sup_{u \in A} J_\varepsilon(u).$$

By the definition of c_ε^*, it is clear from (A1)-(A2) that for any $\delta > 0$

$$\{ u \in M;\ J_\varepsilon(u) \leqq C_\varepsilon^* + \delta \} \in \mathcal{A}_M. \tag{6.8}$$

Thus we see from Corollary 6.4 that

$$\gamma(B_1) \leqq c_\varepsilon^* + \delta.$$

Fix $C^* > c_1^*$ and let $B_2 = B_0(\gamma(B_1),\ C^*)$ [let $B_0(\cdot,\ \cdot)$ be given in Lemma 6.1] and set

$$d = \max\{ B_1,\ B_2\}.$$

LEMMA 6.5. – *$J_\varepsilon|_M$ has a critical point u^ε such that*
(i) $J_\varepsilon(u^\varepsilon) \in [\gamma(B_1),\ C^*]$,
(ii) $u^\varepsilon \in M^d$.

Proof. – Set

$$c_\varepsilon = \inf\{ c \in \mathbf{R};\ \{ u \in M;\ J_\varepsilon(u) \leqq c \} \cup \{ u \in M;\ g(u) \geqq d \} \in \mathcal{A}_M \}.$$

Then clearly

$$c_\varepsilon \leqq c_\varepsilon^* < C^* \quad \text{for all } \varepsilon \in (0,\ 1].$$

From Corollary 6.4 it follows

$$\begin{aligned} &\{ u \in M;\ J_\varepsilon(u) \leqq \gamma(B_1) \} \cup \{u \in M;\ g(u) \geqq d\} \\ &\subset \{u \in M;\ g(u) \leqq B_1 \} \cup \{u \in M;\ g(u) \geqq d\} \\ &= \{u \in M;\ g(u) \geqq d\}. \end{aligned}$$

Since $\{u \in M;\ g(u) \geqq d\} \notin \mathcal{A}_M$, (A1) yields

$$\{ u \in M;\ J_\varepsilon(u) \leqq \gamma(B_1) \} \cup \{u \in M;\ g(u) \geqq d\} \notin \mathcal{A}_M.$$

Thus we have

$$\gamma(B_1) < c_\varepsilon \quad \text{for all } \varepsilon \in (0,\ 1].$$

Now it is easy to see all assumptions (H1)-(H6) of Lemma 4.1 are satisfied with $c = c_\varepsilon$, $\tilde{c} = C^*$ and $b = d$. ■

7. LIMITING PROCESS

In previous sections, we have shown that for any $\varepsilon \in (0, 1]$ there exists a critical point u^ε such that

$$\nabla_M J_\varepsilon(u^\varepsilon) = 0,$$

$$J_\varepsilon(u^\varepsilon) = c_\varepsilon,$$

$$u^\varepsilon \in M^d.$$

We set

$$\rho_\varepsilon \equiv \rho(u^\varepsilon)$$
$$v^\varepsilon(t) = \rho_\varepsilon u^\varepsilon(t),$$
$$\omega_\varepsilon^2 = \frac{\int_0^1 [h - V_\varepsilon(v^\varepsilon)]\,dt}{\frac{1}{2}\rho_\varepsilon^2} = \frac{J_\varepsilon(u^\varepsilon)}{\frac{1}{4}\rho_\varepsilon^4}$$

From the arguments of Sections 5 and 6 one deduces:

$$c_\varepsilon \in [\gamma(B_1),\, C^*] \tag{7.1}$$

$$\rho(u^\varepsilon) \in [k_1(\gamma(B_1)),\, k_2(C^*)], \tag{7.2}$$

$$\omega_\varepsilon^2 \in \left[\frac{4\gamma(B_1)}{k_2(C^*)^4},\, \frac{C^*}{k_1(\gamma(B_1))^4}\right]. \tag{7.3}$$

Since $u^\varepsilon \in M^d$ and M^d is a bounded subset of E, then, up to a subsequence,

$$u^\varepsilon \rightharpoonup u. \tag{7.4}$$

Moreover, by (7.2), it follows that

$$\rho_\varepsilon \to \rho \neq 0, \tag{7.5}$$

while, by (7.3), $\omega_\varepsilon \to \omega$. We set

$$x(t) = \rho\, u(\omega t).$$

LEMMA 7.1. – *There exists a* $t_0 \in (0, 1]$ *such that*

$$x_i(t_0) \neq x_j(t_0) \quad \textit{for all } i \neq j. \tag{7.6}$$

Proof. – By (7.4) and (7.5), $v^\varepsilon \rightharpoonup v := \rho\, u$. Since

$$J_\varepsilon(u^\varepsilon) = \frac{1}{2}\rho_\varepsilon^2 \int_0^1 [h - V_\varepsilon(z^\varepsilon)]\,dt,$$

we have by (7.1) and (7.2)

$$\int_0^1 [h - V_\varepsilon(v^\varepsilon)]\, dt = \frac{2\, J_\varepsilon(u^\varepsilon)}{\rho_\varepsilon^2} \in \left[\frac{2\,\gamma(B_1)}{k_2\,(C^*)^2}, \frac{2\,C^*}{k_1\,(\gamma(B_1))^2}\right]$$

for all $\varepsilon \in (0, 1]$. It is easy to see that, *via* the Fatou's Lemma, this implies the existence of $t_0 \in (0, 1]$ satisfying (7.6). ■

Proof of Theorem 2.1 completed. – It suffices to note that $v_\varepsilon \rightharpoonup v$, $v = \rho\, u$ and properties 1°-3° of Definition 3.1 are satisfied, while, property 4° is nothing but Lemma 7.1. ■

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(Manuscript received November 14, 1993.)