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Persistent homoclinic tangencies and the unfolding of cycles

by

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ABSTRACT. – We describe a new mechanism implying the persistence of homoclinic tangencies after the unfolding of a bifurcating cycle. The cycles we consider are heterodimensional: the index of the hyperbolic points involved in the cycle are different.

Key words : Bifurcation, cycle, heteroclinic point, homoclinic tangency, hyperbolic

RÉSUMÉ. – Nous décrivons un nouveau mécanisme impliquant la persistance de tangences homocliniques après le déploiement d'un cycle. Les cycles que l'on considère sont hétérodimensionnels : l'index des points hyperboliques impliqués dans le cycle sont différents.

1. INTRODUCTION

In this paper we are concerned with the problem of describing generic (in an open set) bifurcations of one-parameter families of diffeomorphisms leading to the phenomenon of persistence of homoclinic tangencies.

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Since Newhouse ([8]-[9], *see* [13] for a new proof) it is known that the generic unfolding of a homoclinic (or heteroclinic) tangency in arcs of surface diffeomorphisms implies persistence of homoclinic tangencies in intervals in the parameter line. This result was extended to higher dimensions first by Palis-Viana [14], who treated the codimension 1 case. The generalization to any codimension was more recently obtained by Romero [16].

In the unfolding of homoclinic tangencies on surfaces there are, essentially, three different possibilities according to the fractional dimension (Hausdorff dimension and thickness) of the hyperbolic set involved in the creation of the tangency:

1. if the Hausdorff dimension of the hyperbolic set is less than one, the family of intervals of persistent tangencies has density zero (in the Lebesgue sense) at the bifurcation value. More precisely, hyperbolicity corresponds to a set of density one at this parameter value, *see* Palis-Takens [12],
2. if the Hausdorff dimension of the hyperbolic set is bigger than one, the parameter values corresponding to hyperbolic diffeomorphisms is not of density one at the initial bifurcation value. Indeed there are "plenty" of parameter values corresponding to diffeomorphisms exhibiting homoclinic tangencies, *see* Palis-Yoccoz [15],
3. if the hyperbolic set is a thick set, *i.e.* the product of its stable and unstable thickness is bigger than one, the bifurcation value is in the boundary of an interval of persistence of homoclinic tangencies, *see* Newhouse ([8]-[9]).

For heteroclinic tangencies in surfaces the results are similar. *See* [13] for a comprehensive information on the subject above.

In the cases quoted above ([8]-[9], [14], [16]) the concept of thick horseshoe plays a key role to get persistence of tangencies: there are parameter values arbitrarely close to the bifurcating one having a thick horseshoe with a homoclinic tangency, from which the persistence of tangencies is obtained.

We also mention that in these cases the creation of a homoclinic/heteroclinic tangency implies the appearence of a cycle in which the index of the hyperbolic sets (*i.e.* dimension of the stable bundle) are equal. Such cycles are called *equidimensional.* Otherwise the cycle is *heterodimensional, i.e.* there is a pair P, Q of hyperbolic periodic points in the cycle with $\dim(W^s(P)) \neq \dim(W^s(Q))$. Notice that on surfaces every cycle is equidimensional.

Here we deal with heterodimensional cycles. We say that an arc of diffeomorphisms $(f_t)_{t\in I}$ defined on a n-dimensional manifold has a *heterodimensional cycle at* $t = b$ if there are hyperbolic saddle points P_t and Q_t with different index so that $W^u(P_b)$ meets $W^s(Q_b)$ quasitransversely along the orbit of a point and $W^s(P_b)$ intersects $W^s(Q_b)$ transversely.

If $W^s(P_b) \pitchfork W^u(Q_b)$ has a connected component being f_b^i-invariant we say that the cycle is *connected*. When $W^s(P_b) \pitchfork W^u(Q_b)$ contains a connected component γ_b so that $f_b^i(\gamma_b) \neq \gamma_b$ for every $i \neq 0$ the cycle is named *nonconnected*. Observe that there are cycles being simultaneously connected and nonconnected. The weak expanding eigenvalue of Q_b and the weak contracting eigenvalue of P_b are called the *connexion eigenvalues*.

In the present paper we describe a new mechanism leading to persistence of homoclinic tangencies after the unfolding of a cycle. We prove that for a large open class of arcs of diffeomorphisms $(f_t)_{t\in I}$ unfolding a connected heterodimensional cycle having a complex connexion eigenvalue, there is an open interval in the parameter line containing the bifurcation value b in its interior where the parameter values corresponding to homoclinic tangencies are dense. In such a case we say that the persistence of homoclinic tangencies is a *persistent* phenomenon.

We point out that in this paper we do not use concepts related to fractional dimensions (namely Hausdorff dimension and thickness). The main novelty of our proof of the persistence of tangencies is that it only involves the two hyperbolic periodic points in the cycle. Let us recall that in [8] to get persistence of tangencies it is considered a thick hyperbolic set with a homoclinic tangency.

Here to get persistence of homoclinic tangencies we analize the growth of the homoclinic points of P_t. In rough terms we prove the following: for t nearby b there is a subset $\tilde{\Lambda}_t$ of the tranversal homoclinic points of P_t being dense in a center-stable manifold of P_t, *see* Proposition. The set $\tilde{\Lambda}_t$ plays a similar role of the thick hyperbolic set in [8], *see* Section 3.2.

Heterodimensional cycles were introduced by Newhouse and Palis in the seventies, *see* [10]. In ([2]-[4]) the connected and nonconnected cases with real connexion eigenvalues in any dimension were studied. Connected heterodimensional cycles with complex connexion eigenvalues remain so far unexplored and here we give a contribution to the understanding of the dynamics in this case.

The unfolding of homoclinic/heteroclinic tangencies leads to the relevant dynamical phenomenon of abundance of Hènon-like attractors/repellors ([1], [7], [18]). On the other hand, the unique known geometric configurations leading to prevalence of parameter values with Hènon-like attractors are the dissipative critical saddle-node cycles studied in [11], *see* [6]. We think that in the sectionally dissipative cases considered in this paper our proof suggests that the heterodimensional cycles considered in this paper may be a good place to search for Hénon-like attractors with positive density at the bifurcation value.

2. STATEMENT OF RESULTS

Throughout this paper M denotes a compact n-dimensional ($n \geq 3$) boundaryless manifold and $\mathcal{P}^\infty(M)$ the space of arcs $(f_t)_{t\in I}$ of $\mathcal{C}^\infty$ diffeomorphisms equipped with the usual $\mathcal{C}^\infty$ topology.

We say that $(f_t)_{t\in I}$ *exhibits a heterodimensional cycle at* $t = b$ if there are hyperbolic periodic points P_t and Q_t such that

(1) $W^s(P_t)) \pitchfork W^u(Q_t)) \neq \emptyset$,

(2) $W^u(P_b) \cap W^s(Q_b) = \{f_b^i(r_b)\}_{i\in\mathbb{Z}}$ and

$$\dim(T_{r_b}W^u(P_b) + T_{r_b}W^s(Q_b)) = n - 1.$$

Condition (2) means that x_b is a quasitranversal heteroclinic point and $\dim(W^s(P_t))$ $+\dim(W^u(Q_t)) = n+1$. We lose no generality by assuming that $r_b = f_b^{k_0}(W^u_{\text{loc}}(P_b)) \cap W^s_{\text{loc}}(Q_b)$ for some k_0 and that $b = 0$.

We say that $(f_t)_{t\in I}$ *unfolds generically the cycle* above if there are a $\mathcal{C}^1$ curve $(r_t)_{t\in I}$ and a $\mathcal{C}^1$ map $C: I \to \mathbb{R}^+$ with $r_t \in f_t^{k_0}(W^u_{\text{loc}}(P_t))$ and $C(0) \neq 0$ such that

(1) $\mathrm{d}(r_t, W^s_{\text{loc}}(Q_t)) = |t|C(t)$,

(2) $T_{r_0}(W^s(Q_0)) \oplus T_{r_0}(W^u(P_0)) \oplus V = T_{r_0}(M)$, where V denotes the space spanned by $(r_t)'_{|t=0}$.

We proceed to describe the set of arcs of diffeomorphisms we consider here. Let $\mathcal{H}(M)$ be the subset of $\mathcal{P}^\infty(M)$ consisting of arcs that unfolds generically a heterodimensional cycle. From now on, for simplicity, we assume that P_t and Q_t are fixed. We consider arcs $(f_t)_{t\in I}$ satisfying:

(CI) Linearizing coordinates: Let $\{\lambda_i(t)\}_{i=1,\dots,n}$ and $\{\beta_i(t)\}_{i=1,\dots,n}$ be the eigenvalues of $Df_t(P_t)$ and $Df_t(Q_t)$, respectively. Assume that $|\lambda_1(0)| \leq \dots \leq |\lambda_{n-2}(0)| < |\lambda_{n-1}(0)| < 1 < |\lambda_n(0)|$ and $|\beta_1(0)| \leq \dots \leq |\beta_{n-2}(0)| < 1 < |\beta_{n-1}(0)| = |\beta_n(0)|$ where $\beta_n(0) \in (\mathbb{C} \setminus \mathbb{R})$.

From now on write $\lambda_s = \lambda_{n-2}$, $\lambda_c = \lambda_{n-1}$ and $\lambda_u = \lambda_n$.

(1) f_t, $t \in I$, is C^1-linearizable at P_t and Q_t

and

(2a) $\dfrac{|\lambda_s(0)|}{|\lambda_c(0)|} > \dfrac{|\lambda_c(0)|}{|\lambda_u(0)|}$ and $|\lambda_s(0)| < |\lambda_c(0)|^2$,

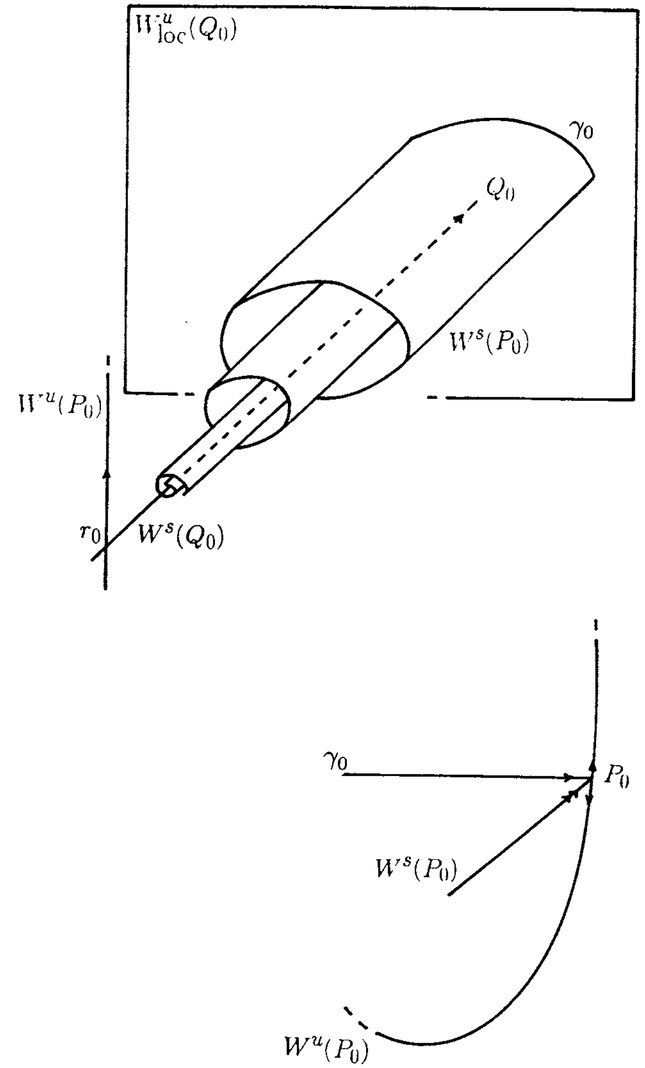

FIG. 1. – Heterodimensional cycle with a complex connexion eigenvalue.

or

(2b) $|\lambda_s(0)| < |\lambda_c(0)|^2$ and $|\lambda_c(0)||\lambda_u(0)| > 1$.

By (CI) there is strong stable foliation $\mathcal{F}_t^{ss}$ in $W^s(P_t)$ with $\dim(\mathcal{F}_t^{ss}) = 1$. From now on $F_t^{ss}(x)$ denotes the leave of $\mathcal{F}_t^{ss}$ containing x.

We say that a curve $\alpha_t \subset W^s(P_t)$ *does not have s-criticalities* if $\alpha_t \pitchfork_x F_t^{ss}(x)$ in $W^s(P_t)$ for every $x \in \alpha_t$.

(CII) There is an f_t-invariant connected component γ_t of $W^s(P_t) \pitchfork W^u(Q_t)$ depending continuously on t such that γ_t has not neither s-criticalities nor radial criticalities, *see* definition below.

Let $\mathcal{C}(M)$ be the subset of $\mathcal{H}(M)$ consisting of arcs $(f_t)_{t\in I}$ satisfying (CI-II).

THEOREM. – *There is an open and dense subset $\mathcal{T}(M)$ of $\mathcal{C}(M)$ so that for every $(f_t)_{t\in I} \in \mathcal{T}(M)$ there is $\varepsilon = \varepsilon((f_t)_{t\in I}) > 0$ and a dense subset T_ε in $[-\varepsilon, \varepsilon]$ such that f_s has a homoclinic tangency for every $s \in T_\varepsilon$.*

We point out that our arguments give that $(f_t)_{t\in I}$ unfolds generically a homoclinic tangency for every $s \in T_\varepsilon$.

3. PERSISTENCE OF TANGENCIES: PROOF OF THE THEOREM

From now on we assume that $(f_t)_{t\in I} \in \mathcal{C}(M)$.

We begin by remarking that up to a finite number of nonresonance conditions on the eigenvalues of f_0 at P_0 and Q_0 the linearizations of f_t at P_t and Q_t, say φ_{P_t} and φ_{Q_t}, can be taken, and we do, to depend differentially on the parameter, (*see* [17]), and defined on neighbourhoods $\mathcal{U}_P$ and $\mathcal{U}_Q$ of P_t and Q_t independent of t.

We take a metric in M so that $\mathrm{d}(x,y) = \mathrm{e}(\varphi_{R_t}(x), \varphi_{R_t}(y))$ for every x, $y \in \mathcal{U}_R$, ($R = P$, Q), where e denotes the euclidean metric. From now on given a curve I, $|I|$ means its lenght.

Given a set A and $x \in A$, $C(x, A)$ denotes the connected component of A containing x. Let $W^i_{\text{loc}}(R_t) = C(R_t, W^i(R_t) \cap \mathcal{U}_R)$, $i = s, u$, $R = P, Q$. We suppose that $\varphi_{P_0}(W^s_{\text{loc}}(P_t)) = \{(x_s, x_c, 0),\ x_s \in \mathbb{R}^{n-2},\ x_c \in \mathbb{R}\}$, $\varphi_{P_0}(W^u_{\text{loc}}(P_t)) = \{(0, \ldots, 0, x_u),\ x_u \in \mathbb{R}\}$, $\varphi_{Q_t}(W^u_{\text{loc}}(Q_t)) = \{(0, \ldots, 0, x_c, x_u),\ x_i \in \mathbb{R}\}$ and $\varphi_{Q_t}(W^s_{\text{loc}}(Q_t)) = \{(x_s, 0, 0),\ x_s \in \mathbb{R}^{n-2}\}$. By (CI-II) we can choose φ_{P_t} so that $\varphi_{P_t}(\gamma_t \cap \mathcal{U}_P) = \{(0, \ldots, 0, x_c, 0),\ x_c \in \mathbb{R}^-\}$.

Let $H(P_t)$ denote the set of transversal homoclinic points related with P_t. We say that S_1 and S_2 are *ξ-transverse at x*, denoted by $S_1 \pitchfork^\xi_x S_2$, if $S_1 \pitchfork S_2$ and $\angle_x(S_1, S_2) \geq \xi$, where $\angle$ denotes the angle. Define $H_\xi(P_t)$ as the subset of $H(P_t)$ consisting of points x so that $W^s(P_t) \pitchfork^\xi_x W^u(P_t)$.

We say that γ_t *has no radial criticalities* (*see* CII) if $\varphi_{Q_t}(\gamma_t)$ is transverse to the pencil of straighlines through $(0, \ldots, 0)$. This definition does not depend on φ_{Q_t}.

The Theorem follows from the arguments in the proof of the next Proposition.

PROPOSITION. – *Suppose that* $\beta_u(0) = \beta_u e^{i\theta}$ *where* $\theta \in (\mathbb{R} \setminus \mathbb{Q})$, $\beta_u \in \mathbb{R}$ *and* $\frac{d}{dt}\lambda_c(0) \neq 0$.

Fixed $\tau > 0$ *there is* $t_0 > 0$ *such that for every* $t \in (-t_0, t_0)$ $\pi_t^{ss}(H_\tau(P_t))$ *is dense in* γ_t, *where* π_t^{ss} *denotes the projection from* $W^s(P_t)$ *to* γ_t *along the leaves of the strong stable foliation of* $W^s(P_t)$.

3.1. Proof of the Proposition

The main tool to proof the proposition is the following result that extends a previous lemma in [3].

LEMMA 1. – *Let* $m \in \mathbb{N}$ *and* $x^0, \ldots, x^m \in \mathbb{R}$ *so that* $x^m < x^{m-1} < \ldots x^1 < x^0$. *For any* $I = (i_1, \ldots, i_n)$, $i_k \in \{0, 1, \ldots, m\}$, $n \geq 0$, *(if* $n = 0$, $I = \emptyset$*), and* $J = (j_1, \ldots, j_n)$, $j_k \geq 1$, *consider the sequences* $\{x_{J,j}^{s,I,r}\}_{j\geq 1}$, r, $s \in \{0, 1, \ldots, m\}$, *so that*

(1) $x_{J,j}^{s,I,r} \to x_J^{s,I}$ *as* $j \to \infty$ *for every* $r \in \{0, 1, \ldots, m\}$,

(2) $x_{J,1}^{m,I,s} < x_J^{I,s+1}$ *for every* $s \in \{0, 1, \ldots, m-1\}$,

(3) $\operatorname{diam}(\{x_{J,j}^{s,I,r}\}_{j\geq 1}) \to 0$ *as* $|J| \to \infty$, *where* $|J| = n$ *if* $J = (j_1, \ldots, j_n)$.

Let $\Lambda = \{x_{j_1,\ldots,j_n}^{i_1,\ldots,i_{n+1}}, n \in \mathbb{N}, i_k \in \{0, \ldots, m\}, j_k \geq 1\}$. *Then* $[x_1^{mm}, x^0] \subset \overline{\Lambda}$.

Proof. – Take $x \in [x_1^{mm}, x^0]$. If $x \in \Lambda$ there is nothing to prove. Otherwise either $x^{i+1} < x < x^i$ for some $i \in \{1, \ldots, m-1\}$ or $x_1^{mm} < x < x^m$. Let us assume that the first case occurs, the other one follows similarly. By (1) and (2) there j_1 so that $x_{j_1+1}^{m,i} < x < x_{j_1}^{m,i}$. Inductively we get a sequence $\{j_n\}$ with $x_{j_1,\ldots,j_n,j_{n+1}}^{m,\ldots,m,i} < x < x_{j_1,\ldots,j_n}^{m,\ldots,m,i}$. Now (3) implies that $z_n = x_{j_1,\ldots,j_n}^{m,\ldots,m,i} \to x$ as $n \to \infty$. This ends the proof of the lemma. □

Now our target is, roughly speaking, the following:

Denote by $\prec$ the natural ordering in γ_0 so that $Q_0 \prec P_0$. Fixed $\tau > 0$ we construct a family of sequences of τ-tranverse points (x_J^I) related with P_0 so that $\pi_0^{ss}(x_J^I)$ satisfies the hypotheses in Lemma 1.

To prove the proposition we need some technical definitions and constructions. Without loss of generality we can assume that $\lambda_c(0) > 0$, otherwise it is enough to replace f_t by f_t^2, and $C(0) > 0$. From now on we write λ_i instead of $\lambda_i(0)$.

Let $B = f_0^{k_0}(\mathcal{U}_Q)$. Consider the extension of $\mathcal{F}_0^{uu}$ to the whole B, $\mathcal{F}_0^u$, defined by $\mathcal{F}_0^u = f_0^{k_0}(\varphi_{P_0}^{-1}(\{(a_1, \ldots, a_{n-1}, x), x \in \mathbb{R}\}))$. Write $F(r_0) = C(r_0, W^u(P_0) \cap \mathcal{U}_Q)$.

We extend $\mathcal{F}_t^{ss}$ to the whole $W^s(P_t)$. By (CI-II),

$$\lim_{n\to\infty} C(f_t^{-n}(x), F_t^{ss}(f_t^{-n}(x)\cap\mathcal{U}_Q)) \to W^s_{\mathrm{loc}}(Q_t)$$

for every $x\in\gamma_t$, where the convergence turns out to be C^1 on compacta. Now consider a C^1-extension of F_0^{ss} to $\mathcal{U}_Q$, say $\mathcal{F}_0^s$. Let π_0^s be the projection along the leaves of $\mathcal{F}_0^s$ from $\mathcal{U}_Q$ to $W^u_{\mathrm{loc}}(Q_0)$.

Define $F^i = \pi_0^s(F(r_0))\cap(\varphi_{Q_0}^{-1}(\{(x_1,\dots,x_{n-1},x_n),\ x_n\in\mathbb{R}^i\}))$, $i=+,-$. We can choose φ_{P_0} in such a way that $\varphi_{P_0}(F^+)$ is tangent to $(0,\dots,0,x_u)$ at $(0,\dots,0)$. For simplicity let us suppose that $\varphi_{P_0}(F^+)=(0,\dots,0,x_u)$.

Let us assume that γ_0 spirals anti-clockwise. Let $F^+\cap\gamma_0=\bigcup_{i\geq 0}\hat{x}_i$, where $\mathrm{d}(\hat{x}_{i+1},Q_0)<\mathrm{d}(\hat{x}_i,Q_0)$. Otherwise the proof follows similarly by considering the intersections between F^- and γ_0.

Pick the fundamental domain D of γ_0 in $\mathcal{U}_P$ so that $\varphi_{P_0}(D)=\{(0,\dots,0,x,0),\ x\in[-1,-\lambda_c]\}$.

Given $x\in\gamma_t$ and a curve $\alpha\subset\gamma_t$, let

$$n(x)=\min\{i\in\mathbb{N}\text{ such that } f_0^i(x)\in D\},\quad n(\alpha)=\min\{n(x),\ x\in\alpha\}.$$

Let E be the fundamental domain of γ_0 bounded by $\hat{x}_0$ and $f_0^{-1}(\hat{x}_0)$. Without loss of generality we can assume, and we do, that $f_0^{n(\hat{x}_0)}(E)=D$. Write $n(\hat{x}_0)=k_1$.

Define $\kappa(i)$ by the natural number so that $f_0^{\kappa(i)}(\hat{x}_i)\in(f_0^{-1}(\hat{x}_0),\hat{x}_0]\subset\gamma_0$. Remark that κ is strictly increasing.

Let γ^i be the arc in γ_0 bounded by $\hat{x}_i$ and $\hat{x}_{i+1}$. There are Δ and δ so that

$$(2.0)\qquad \delta\beta_u^{\kappa(i)}<|D_x(f_0^{n(\alpha)}/\gamma_0)|<\Delta\beta_u^{\kappa(i)},\qquad \forall x\in\alpha\subset\gamma^i.$$

Define $\Sigma(x,\delta)\subset\gamma_0$ by the curve of length 2δ centered at x and $\Sigma_i(\delta)=\Sigma(\hat{x}_i,\beta_u^{-\kappa(i)}\delta)$. Remark that there is δ_0 such that $\Sigma_i(\delta)$ is $\frac{\mathcal{T}}{2}$-transversal to $\mathcal{F}^v$ for every $0<\delta\leq\delta_0$, here $\mathcal{F}^v$ is the foliation given by vertical lines in the linerizing coordinates.

Take a small neighbourhood $\mathcal{V}$ of γ_0 containing r_0. Fixed $A\subset(\gamma_0\cap\mathcal{V})$ let $A^s=\bigcup_{x\in A}C(x,F_0^{ss}(x)\cap\mathcal{V})$.

Let $\overline{x}^i=\Sigma_i^s(\delta)\pitchfork f_0^{k_0}(W^u_{\mathrm{loc}}(P_0))$, $W^u=W^u(P_0)\cup W^u(Q_0)$ and $\Gamma^i(\delta)=C(\overline{x}^i,\Sigma_i^s(\delta)\cap W^u)$. For each $i\geq 0$ define projections along the leaves of $\mathcal{F}_0^{ss}$,

$$\pi_i^s\colon\quad \Gamma^i(\delta)\to\Sigma_i(\delta),\quad x\mapsto F_0^{ss}\pitchfork\Sigma_i(\delta).$$

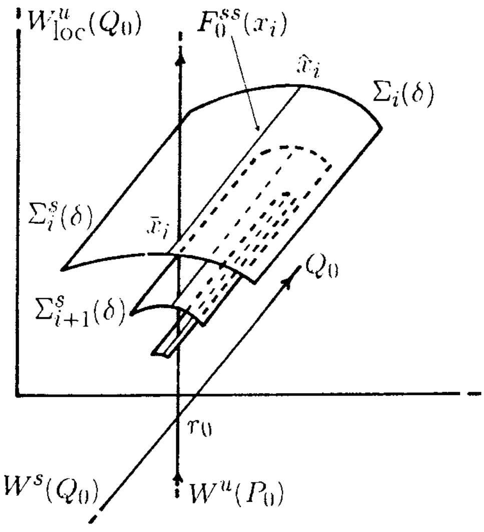

FIG. 2.

Notice that $\hat{x}_i = \pi_i^s(\overline{x}_i)$ (recall the choice of F^+) and that there are constants K_1^s and K_2^s with

$$(2.1)\qquad K_1^s|\alpha| < |\pi_i^s(\alpha)| < K_2^s|\alpha|, \quad \forall \alpha \subset \Gamma^i(\delta), \quad i \geq 0.$$

Let $\Pi_i(\delta) = [z_i(\delta), P_0] \subset \gamma_0 \cap \mathcal{U}_P$ be the maximal segment in γ_0 where the function π_i bellow is well defined, *see* Figure 3,

$$\pi_i\colon \quad \Pi_i(\delta) \to \Gamma_i(\delta), \quad x \mapsto C(x, F_0^{uu}(x) \cap B) \pitchfork \Gamma_i(\delta).$$

Remark that $\overline{x}^i = \pi_i(P_0)$ and that there are constants K_1^u and K_2^u with

$$(2.2)\qquad K_1^u|\alpha| < |\pi_i(\alpha)| < K_2^u|\alpha|, \quad \forall \alpha \subset \Pi_i(\delta), \quad i \geq 0.$$

In the sequel we take $\varepsilon \leq \frac{\delta}{4}$. From (2.2) and (2.0) there are C_1 and C_2

$$(2.3)\qquad |z_i(\varepsilon)| \in (C_1\beta_u^{-\kappa(i)}\varepsilon, C_2\beta_u^{-\kappa(i)}\varepsilon).$$

Let $m_i(\varepsilon) = n_i(\varepsilon) + 2$, where $n_i(\varepsilon)$ is defined by

$$(2.4)\qquad \lambda_c^{n_i(\varepsilon)-1} > C_1\beta_u^{-\kappa(i)}\varepsilon \geq \lambda_c^{n_i(\varepsilon)}.$$

Let $D_{-1} = f_0^{-1}(D)$. Define $D_i(\varepsilon)$ by the convex hull of

$$(f_0^{m_i(\varepsilon)}(D_{-1}) \cup P_0), i.e.$$

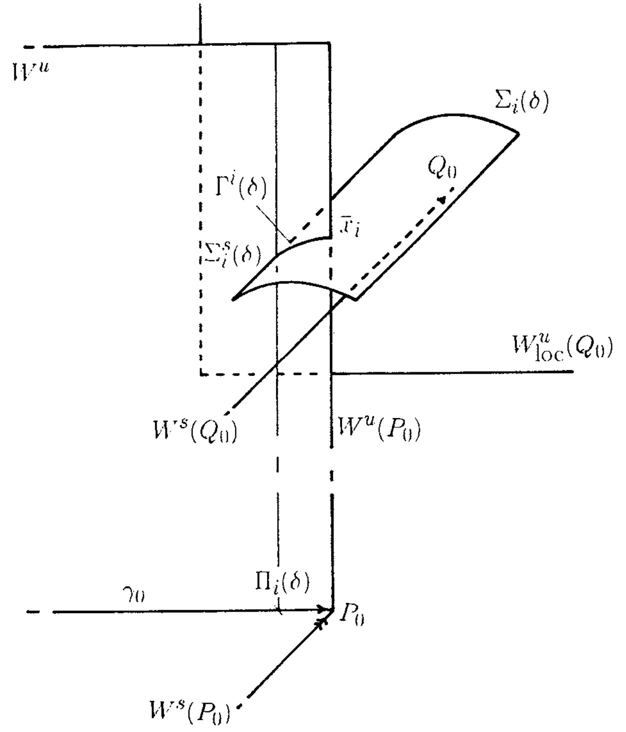

FIG. 3.

$$\varphi_{P_0}(D_i(\varepsilon)) = \{(0, \ldots, x, 0),\, x \in [-\lambda_c^{m_i(\varepsilon)-1}, 0]\},$$

and

$$\hat{D}_i(\varepsilon) = f_0^{m_i(\varepsilon)}(D_{-1} \cup D).$$

From the definition of $m_i(\varepsilon)$, $(D_i(\varepsilon) \cup \hat{D}_i(\varepsilon)) \subset \Pi_i(\varepsilon)$. Hence we can define $\pi_i(f_0^{m_i(\varepsilon)+j}(y))$ for every $y \in D_{-1} \cup D$ and $j \geq 0$.

By the definition of $D_i(\varepsilon)$, $|D_i(\varepsilon)| = \lambda_c^{n_i(\varepsilon)+1}$. On the other hand, from (2.3) $C_1 \beta_u^{-\kappa(i)} \varepsilon \leq |\Pi(\varepsilon)| C_2 \beta_u^{-\kappa(i)} \varepsilon$. Now, from (2.4)

$$(2.5) \qquad \begin{cases} \lambda_c^2 \dfrac{C_1}{C_2} \leq \dfrac{|D_i(\varepsilon)|}{|\Pi_i(\varepsilon)|} \leq \lambda_c, \\ \lambda_c^3 C_1 \beta_u^{-\kappa(i)} \varepsilon \leq \mathrm{d}(y, P_0) \leq C_1 \beta_u^{-\kappa(i)} \varepsilon, \quad \forall y \in \hat{D}_i(\varepsilon). \end{cases}$$

From (2.2-5)

$$(2.6) \begin{cases} K_1^u \lambda_c^2 \dfrac{C_1^2}{C_2} \beta_u^{-\kappa(i)} \varepsilon \leq |\pi_i(D_i(\varepsilon))| \leq K_2^u \lambda_c C_2 \beta_u^{-\kappa(i)} \varepsilon, \\ K_1^u \lambda_c^3 C_1 \beta_u^{-\kappa(i)} \varepsilon \leq \mathrm{d}(\pi_i(y)), \bar{x}^i) \leq K_2^u C_1 \beta_u^{-\kappa(i)} \varepsilon, \quad \forall y \in \hat{D}_i(\varepsilon). \end{cases}$$

Let

$$L = \max\{|D_x f_{|\gamma_0}^{n(x_0)}|, x \in E\}, \quad \ell = \min\{|D_x f_{|\gamma_0}^{n(x_0)}|, x \in E\}.$$

From (2.6), (2.0) and the definitions of Δ and δ

$$(2.7) \quad \ell\delta K_1^u K_1^s \lambda_c^2 \frac{C_1}{C_2}\varepsilon < |f_0^{\kappa(i)+k_1}(\pi_i^s(\pi_i(D_i(\varepsilon))))| < L\Delta K_2^u K_2^s \lambda_c C_2 \varepsilon.$$

Take

$$\varepsilon < \frac{\lambda_c^{-1} - 1}{4\Delta L \lambda_c C_2 K_2^u K_2^s}.$$

The choice of ε and (2.7) imply that

$$\mathrm{d}(f_0^{\kappa(i)+k_1}(\pi_i^s(\pi_i(f_0^{m_i(\varepsilon)+j}(y)))), f_0^{\kappa(i)+k_1}(\hat{x}_i)) < \lambda_c^{-1} - 1$$

for every $y \in D_{-1} \cup D$ and $i \in I(\varepsilon)$. Since by the definition of $\kappa(i)$ and k_1, $f_0^{\kappa(i)+k_1}(\hat{x}_i)) \in D^s$ (recall the definition of A^s) the last inequality gives $f_0^{\kappa(i)+k_1}(\pi_i^s(\pi_i(f_0^{m_i(\varepsilon)+j}(y)))) \in D_{-1}^s \cup D^s$ for every $y \in D_{-1} \cup D$ and $j \geq 0$.

We have proved the following

Remark. – Given $y \in D_{-1} \cup D_1$ and $j \geq 1$ let $y_j^i = f_0^{\kappa(i)+k_1}(\pi_i^s(\pi_i(f_0^{m_i(\varepsilon)+j}(y))))$. Then

(1) y_j^i is well defined for every $i \geq 0$ and $j \geq 1$,

(2) $y_j^i \in D_{-1}^s \cup D^s$.

Take

$$\nu = \nu(\varepsilon) = \frac{\ell\delta\lambda_c^3 C_1 \varepsilon}{4}.$$

Since $\theta \in (\mathbb{R} \setminus \mathbb{Q})$ there is a finite subset $I(\varepsilon) = \{i_0, i_1, \ldots, i_{e(\varepsilon)}\}$, $i_0 = 0$, $i_k < i_{k+1}$, of $\mathbb{Z}$ such that

(1) $\{y_j\}_{j \in I(\varepsilon)}$ is $\frac{\nu}{2}$-dense in D, where $y_k = f_0^{n(\hat{x}_k)}(\hat{x}_k)$,

(2) $y_{i_{k+1}} \prec y_{i_k}$, where $\prec$ means the natural ordering in γ_0 satisfying $Q_0 \prec P_0$.

Now we are ready to construct the sequences $\{y_{J,j}^{s,I,r}\}_{j \geq 1}$ in the Lemma 1.

Let $\{y_{J,j}^{s,I,r}\}_{j \geq 1} = \{\pi_0^s(x_{J,j}^{s,I,r})\}_{j \geq 1}$, where $\{x_{J,j}^{s,I,r}\}_{j \geq 1}$ is defined as follows. For $i \in I(\varepsilon)$ take $x^i = f_0^{\kappa(i)+k_1}(\overline{x}^i)$. For every $k \in I(\varepsilon)$ and $j \geq 1$ let

$$\overline{x}_j^{k,i} = C(f_0^{m_i(\varepsilon)+j}(x^k), W^u(P_0) \cap B) \pitchfork \Sigma_i^s(\varepsilon),$$
$$x_j^{k,i} = f_0^{\kappa(i)+k_1}(\overline{x}_j^{k,i}).$$

By construction $x_j^{k,i} \to x^i$ as $j \to \infty$ for every k, $i \in I(\varepsilon)$.

Given $I = (i_1, i_2, \dots, i_m)$ write $|I| = m$. Suppose already defined $\{x_J^I\}_{j\geq 1}$ for every $|I| \leq n+1$, $|J| \leq n$, $|I| = |J|+1$ and that $x_{J,j}^{k,I} \to x_J^I$ as $j \to \infty$ for every $k \in I(\varepsilon)$. Let

$$\begin{aligned} \overline{x}_{J,j}^{r,I,i} &= C(f_0^{m_i(\varepsilon)+j}(x_J^{r,I}), W^u(P_0) \cap B) \pitchfork \Sigma_i^s(\varepsilon), \\ x_{J,j}^{r,I,i} &= f_0^{\kappa(i)+k_1}(\overline{x}_{J,j}^{r,I,i}). \end{aligned}$$

Again by construction $x_{J,j}^{r,I,i} \to x_J^{I,i}$ as $j \to \infty$ for every $r \in I(\varepsilon)$.

Now we claim

Claim: *The sequences $\{z_{J,j}^{r,s,I}\}_{j\geq 1}$, ($z = x, y$) are well defined and $\{y_{J,j}^{r,s,I}\}_{j\geq 1}$ satisfies the hypotheses in Lemma 1.*

Proof of the Claim. – Given $x, y \in W^s(P_0)$ write $x \prec y$ meaning $\pi^s(x) \prec \pi^s(y)$. Notice that π_i preserves the ordering $\prec$.

We say that $(\alpha, \mathcal{H})$ is a *pair* if α is a curve in $W^s(P_0)$ and $\mathcal{H} = \{H(x)\}_{x\in\alpha}$ is a family of 1-disks depending differentially on x with $H(x) \pitchfork_x W^s(P_0)$. A pair $(\alpha', \mathcal{H}')$ is a *subpair* of $(\alpha, \mathcal{H})$ if $\alpha' \subset \alpha$ and $H'(x) = H(x)$ for every $x \in \alpha'$.

Remark that there is $\mu > 0$ such that for every pair $(\alpha, \mathcal{H})$ $\beta_u^{-\kappa(i)}\varepsilon\mu$-$\mathcal{C}^1$-close to a subpair of $(\Pi_i(\varepsilon), \{F_0^{uu}(x)\}_{x\in\Pi_i(\varepsilon)})$ (where the $\mathcal{C}^1$-proximity is defined in the obvious way) we can define the projection $\rho_{(\alpha,\mathcal{H})}$ as follows

$$\rho_{(\alpha,\mathcal{H})}\colon \quad \alpha \to \Sigma_i^s(\varepsilon), \quad x \mapsto H(x) \pitchfork \Sigma_i^s(\varepsilon).$$

By shrinking $\varepsilon < \delta$ we can assume that $\rho_{(\alpha,\mathcal{H})}$ preserves the ordering $\prec$ and by (2.1-2)

$$\text{(2.8)} \qquad \begin{cases} K_1^s < \dfrac{|\tilde{\pi}_i^s(\rho_{(\alpha,\mathcal{H})}(\omega))|}{|\rho_{(\alpha,\mathcal{H})}(\omega)|} < K_2^s, \\ K_1^u < \dfrac{|\rho_{(\alpha,\mathcal{H})}(\omega)|}{|\omega|} < K_2^u, \end{cases} \qquad \forall \omega \subset \alpha,$$

where $\tilde{\pi}_i^s$ denotes the projection along the leaves of $\mathcal{F}_0^{ss}$ from $\Sigma_i^s(\delta)$ to $\Sigma_i(\delta)$.

A pair $(\alpha, \mathcal{H})$ is called *ξ-pair* if

(1) $\alpha \subset D_{-1}^s \cup D^s$,

(2) $\alpha \pitchfork^{\xi} \mathcal{F}_0^{ss}$,

(3) $H(x)$ is ξ-$\mathcal{C}^1$-close to $F_0^u(x)$ for every $x \in \alpha$.

For $i \geq 0$ define $(\alpha^i, \mathcal{H}^i)$ by $\alpha^i = f_0^i(\alpha)$ and $\mathcal{H}^i = \{\mathcal{H}^i(f_0^i(x))\}$, where $H^i(f_0^i(x)) = C(f_0^i(x), f_0^i(H(x)) \cap B)$.

Now let us suppose that (CI(2a)) occurs, the case (CI(2b)) follows analogously, so we omit the details.

Let $\psi = \dfrac{|\lambda_s|}{|\lambda_c|} > 1$ and $\eta = \dfrac{\log(|\lambda_s|)}{\log(|\lambda_c|)} - 1 > 1$. Using the linearizing coordinates one gets that $(\alpha^j, \mathcal{H}^j)$ is $C_3\xi\psi^j$-$\mathcal{C}^1$-close to a subpair of $([-\lambda_c^{j-1}, P_0], \{F_0^{uu}(x)\}_{x\in[\lambda_c^{j-1}, P_0]})$, here C_3 does not depend on j. In particular, $(\alpha^{m_i(\varepsilon)}, \mathcal{H}^{m_i(\varepsilon)})$ is $C_3\xi\psi^{m_i(\varepsilon)}$-close to a subpair of $(\Pi_i(\varepsilon), \{F_0^{uu}(x)\}_{x\in\Pi_i(\varepsilon)})$. From (2.4) and $|\lambda_s| < |\lambda_c|^2$ [*see* CI(2a)], $\psi^{m_i(\varepsilon)} < C_4(\beta_u^{-\kappa(i)}\varepsilon)^\eta$. By shrinking ε, then increasing $m_i(\varepsilon)$, $\psi^{m_i(\varepsilon)} < \beta_u^{-\kappa(i)}\varepsilon\mu$, where μ is defined as above. The choice of μ allow us to define, for every $i \in I(\varepsilon)$ and $j \geq 0$, the projection

$$(\rho_{(\alpha,\mathcal{H})})_j^i\colon \quad \alpha^{m_i(\varepsilon)+j} \to \Sigma_i^s(\varepsilon), \quad (\rho_{(\alpha,\mathcal{H})})_j^i = \rho_{(\alpha^{m_i(\varepsilon)+j}, \mathcal{H}^{m_i(\varepsilon)+j})}$$

where $(\rho_{(\alpha,\mathcal{H})})_j^i$ satisfies (2.8).

We define the i-j-sucessor of $(\alpha, \mathcal{H})$, denoted by $(\alpha_j^i, \mathcal{H}_j^i)$, as follows

$$\alpha_j^i = f_0^{\kappa(i)+k_1}((\rho_{(\alpha,\mathcal{H})})_j^i(f_0^{m_i(\varepsilon)+j}(\alpha)),$$

$$\begin{aligned} &H_j^i(f_0^{\kappa(i)+k_1}((\rho_{(\alpha,\mathcal{H})})_j^i(f_0^{m_i(\varepsilon)+j}(x)))) \\ &\quad = f_0^{\kappa(i)+k_1}(C((\rho_{(\alpha,\mathcal{H})})_j^i(f_0^{m_i(\varepsilon)+j}(x)), f_0^{m_i(\varepsilon)+j}(H(x)) \cap B))). \end{aligned}$$

The choice of ε implies straighforwardly that $\alpha_j^i \subset D_{-1}^s \cup D^s$.

From (2.8), (2.4) and the definitions of Δ and L

$$\begin{aligned} |\pi_0^s(\alpha_j^i)| &= |f_0^{\kappa(i)+k_1}(\pi_i^s(\rho_{(\alpha,\mathcal{H})})_j^i(f_0^{m_i(\varepsilon)+j}(\alpha)))| \\ &\leq \Delta L \beta_u^{\kappa(i)} K_2^u K_2^s \lambda_c^{m_i(\varepsilon)}|\alpha| \leq \Delta L K_2^u K_2^s C_5 \varepsilon |\alpha|. \end{aligned}$$

By shrinking ε one gets

$$\max\{|\pi_0^s(\alpha_j^i)|, |\alpha_j^i|\} < \frac{|\alpha|}{2}, \qquad \forall i \in I(\varepsilon), \quad j \geq 1. \tag{2.9}$$

By hypothesis, $f_0^{k_0}(W_{\text{loc}}^u(P_0)) \pitchfork_{x^i}^\tau \Sigma_i^s(\varepsilon)$, for every $i \in I(\varepsilon)$. Hence there is C_6 so that $f_0^{\kappa(i)+k_1}(W_{\text{loc}}^u(P_0)) \pitchfork_{x^i}^{C_6\tau} (D_1 \cup D)^s$.

Consider any $C_6\tau$-pair $(\alpha, \mathcal{H})$. From the arguments above, by shrinking ε, we can assume that the i-j-sucessor of $(\alpha, \mathcal{H})$ is also a $C_6\tau$-pair. Arguing inductively, suppose that for every $I = (i_1, i_2, \ldots, i_n)$, $i_k \in I(\varepsilon)$ and $J = (j_1, j_2, \ldots, j_n)$, $j_k \geq 1$, $(\alpha_J^I, \mathcal{H}_J^I)$ is a $C_6\tau$-pair.

We define $(\alpha^{I,i_{n+1}}_{J,j_{n+1}}, \mathcal{H}^{I,i_{n+1}}_{J,j_{n+1}})$ as the i_{n+1}-j_{n+1}-sucessor of $(\alpha^I_J, \mathcal{H}^I_J)$. From the construction above, $(\alpha^{I,i_{n+1}}_{J,j_{n+1}}, \mathcal{H}^{I,i_{n+1}}_{J,j_{n+1}})$ is a $C_6\tau$-pair and $(\rho_{(\alpha^I_J,\mathcal{H}^I_J)})^{i_{n+1}}_{j_{n+1}}$ satisfies (2.8) and $\max\{|\pi^s_0(\alpha^I_J)|, |\alpha^I_J|\} < \left(\frac{1}{2}\right)^{|I|} |\alpha|$.

Now we are ready to finish the proof of the Claim. For simplicity let us assume that $I(\varepsilon) = \{0, 1, 2, \ldots, e\}$. Take the $C_6\tau$-pair $(\alpha, \mathcal{H})$, where α is a curve joining $x^{e,i}_1$ and x^{i+1}, $H(x^{e,i}_1) = C(x^{e,i}_1, W^u(P_0) \cap \mathcal{V})$ and $H(x^{i+1}) = C(x^{i+1}, W^u(P_0) \cap \mathcal{V})$. From the arguments above

$$x^{e,i,r}_{1,j} = f_0^{m_r(\varepsilon)+k_1}((\rho_{(\alpha,\mathcal{H})})^r_j)(x^{e,i}_1),$$
$$x^{i+1,r}_j = f_0^{m_r(\varepsilon)+k_1}((\rho_{(\alpha,\mathcal{H})})^i_j)(x^{i+1}_j).$$

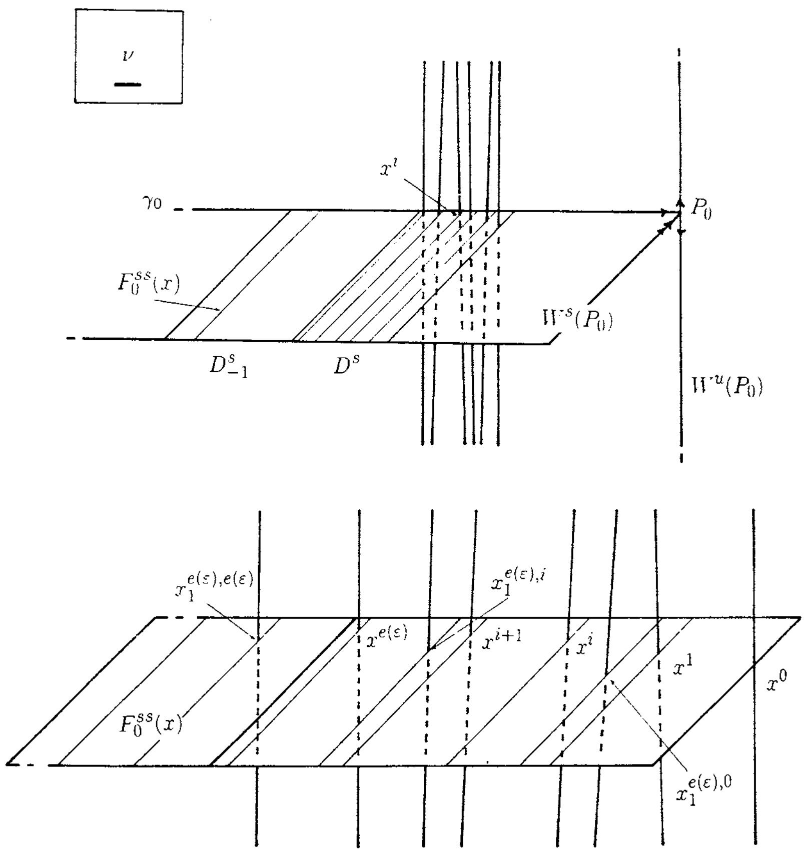

FIG. 4.

where $x_{1,j}^{e,i,r} \prec x_j^{i+1,r}$. Now the inductive pattern to get (2) in Lemma 1 is obvious, so we omit the details.

Finally, $\mathrm{diam}(\{y_J^I\}) \to 0$ as $|I| \to \infty$ follows from the arguments in the proof of (2.9) by reduzing ε. Now the proof of our claim is complete. □

Now we prove the Proposition. Our construction allow us to define $z_J^I(t)$, $z = x, y$, for every $I = (i_1, \dots, i_n, i_{n+1})$, $i_k \in \{0, \dots, e\}$ and $n \geq 0$, and $J = (j_1, \dots, j_n)$, $j_k \geq 1$, $n \geq 0$ in the natural way for every $t \in [-t_0, t_0]$, t_0 is small. Moreover these sequences satisfy Lemma 1.

Fix I and J, now the definition of $z_J^I(t)$, $z = x, y$, only involves compact parts of the invariant manifolds of P_t, hence $z_J^I : [-t_0, t_0] \to M$, $t \mapsto z_J^I(t)$ depends C^1 on t. Moreover, by construction, $(y_1^{ee}(t), y^0(t))$ contains a fundamental domain of γ_t for every $t \in [-t_0, t_0]$.

From the λ-lemma estimates with eigenvalue depending differentially on t one gets that if ε is taken small enough

$$\frac{d}{dt} x_J^I(s) < C_7, \qquad \forall s \in [-t_0, t_0].$$

Let $W_-^u(t) = \bigcup_{x \in \gamma_t \cap \mathcal{U}_P} C(x, F_t^{uu}(x) \cap \mathcal{U}_P)$. Observe that $W_-^u(0) \cap W^s(P_0)$ contains a family of curves $\zeta_i(0)$ as in Figure 5. Fixed i there is $t(i) \in (0, t_0]$

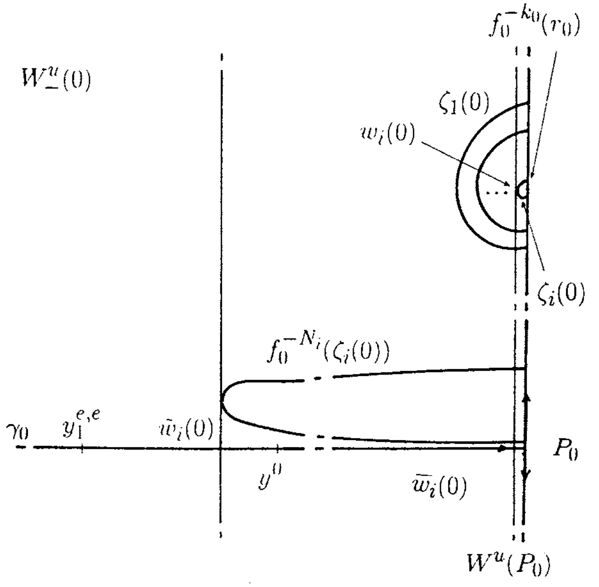

FIG. 5.

such that we can define the continuation $\zeta_i(t)$ of $\zeta_i(0)$ depending differentially on t for every $t \in (-t(i), t(i))$. Let $w_i(t)$ be the point of quadratic contact between $\zeta_i(t)$ and the $\mathcal{F}_t^{uu}$. Write $\overline{w}_i(t) = F_t^{uu}(w_i(t)) \cap \gamma_t$, *see* Figure 5.

Since $(y_1^{ee}(0), y^0(0))$ contains a fundamental domain of γ_0 there is N_i so that $f_0^{N_i}(\overline{w}_i(0)) \in (y_1^{ee}(0), y^0(0))$. Observe that $N_i \to \infty$ as $i \to \infty$. By shrinking $t(i)$, $f_t^{-N_i}(\overline{w}_i(t)) \in (y_1^{ee}(t), y^0(t))$ for every $t \in (-t(i), t(i))$. Let $\tilde{w}_i(t) = f_t^{-N_i}(\overline{w}_i(t))$. From $\frac{d}{dt}\lambda_c(0) \neq 0$ follows

$$\frac{d}{dt}\tilde{w}_i(s) > C_8 \lambda_c(0)^{-N_i}, \qquad \forall s \in [-t(i), t(i)],$$

where C_8 does not depend on t and i.

Take i so that $C_8 \lambda_c(0)^{-N_i} >> C_7$ for every $t \in (-t(i), t(i))$. Given $\nabla > 0$ we get I and J as above and $s \in (t - \nabla, t + \nabla)$ such that $\tilde{w}_i(s) = y_J^I(s)$. Now it is not hard to see that fixed $t \in (-t(i), t(i))$ and $\nabla > 0$ there is $s \in (t - \nabla, t + \nabla)$ such that $W^u(P_s)$ is tangent to $C(x_J^I(s), W^u(P_s) \cap \mathcal{U}_P)$. Now the proof of the Proposition is complete. □

3.2 Proof of the Theorem

To get the Theorem just observe that in the proof of the Proposition the hypothesis $\theta \in (\mathbb{R} \setminus \mathbb{Q})$ is only used to obtain the $\frac{\nu}{2}$-dense subset $\{\tilde{y}_i\}_{i \in I(\varepsilon)}$ in D for suitable ν. However, if $\tilde{\theta} \in \mathbb{R}$ is close enough to θ we can define continuations $\{\tilde{y}_{i_0}, \dots, \tilde{y}_{i_e}\}$ being a $\frac{\nu}{2}$-dense subset of D. Now the Theorem follows from the arguments in the proof of the Proposition. □

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