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# Further remarks on the lower semicontinuity of polyconvex integrals

by

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ABSTRACT. – We study some lower semicontinuity properties of polyconvex integrals of the form $\int_\Omega f(M(\nabla u))\,dx$, where $\Omega \subset \mathbb{R}^n$, $u\colon\Omega \to \mathbb{R}^m$, and $M(\nabla u)$ denotes the family of the determinants of all minors of the gradient matrix $\nabla u$. In particular, we study the lower semicontinuity along sequences converging strongly in $L^1(\Omega, \mathbb{R}^m)$ when the integrand depends only on the minors of $\nabla u$ up to a given order, and the lower semicontinuity along sequences converging strongly in $L^1(\Omega, \mathbb{R}^n)$ and bounded in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ in the special case $m = n$.

*Key words and phrases:* Polyconvex functionals - Lower semicontinuity.

RÉSUMÉ. – Nous étudions la semicontinuité inférieure d'intégrales polyconvexes de la forme $\int_\Omega f(M(\nabla u))\,dx$, où $\Omega \subset \mathbb{R}^n$, $u\colon\Omega \to \mathbb{R}^m$, et $M(\nabla u)$ désigne le vecteur des déterminants de tous les mineurs de la matrice gradient $\nabla u$. En particulier, nous étudions la semicontinuité inférieure sur les suites convergentes fortement en $L^1(\Omega, \mathbb{R}^m)$ lorsque l'integrande dépend seulement des mineurs de $\nabla u$ jusqu'à un certain ordre et la semicontinuité inférieure sur les suites convergentes fortement en $L^1(\Omega, \mathbb{R}^n)$ et bornées en $W^{1,n-1}(\Omega, \mathbb{R}^n)$ dans le cas spécial $m = n$.

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## INTRODUCTION

In this paper we study some lower semicontinuity properties of the polyconvex functional

$$F(u) = \int_\Omega f\big(M_1^\ell(\nabla u)\big)\, dx\,,$$

where $\Omega$ is a bounded open set in $\mathbb{R}^n$, $u\colon\Omega \to \mathbb{R}^m$ varies in a suitable Sobolev space, $\nabla u$ is the gradient matrix of $u$, $M_1^\ell(\nabla u)$ is the vector composed of the determinants of all minors of $\nabla u$ of order less than or equal to $\ell$, and $f$ is non-negative, convex, and lower semicontinuous, with $f(0) < \infty$.

We prove (Theorem 2.2) that, if $f(\xi) \geq c_1|\xi|$ for a suitable constant $c_1 > 0$, then $F$ is lower semicontinuous on $W^{1,\ell}(\Omega,\mathbb{R}^m)$ along sequences converging strongly in $L^1(\Omega,\mathbb{R}^m)$. In the case $\ell = m \wedge n$ our theorem provides a new proof of a result of [1], while in the case $1 < \ell < m \wedge n$ the result is new. We prove also (Corollary 2.3) that, even if the condition $f(\xi) \geq c_1|\xi|$ is not satisfied, the functional $F$ is lower semicontinuous on $W^{1,\ell}(\Omega,\mathbb{R}^m)$ along sequences $(u_k)_{k\geq 1}$ converging strongly in $L^1(\Omega,\mathbb{R}^m)$ and such that $\big(|M_1^\ell(\nabla u_k)|\big)_{k\geq 1}$ is bounded in $L^1(\Omega)$. The latter condition is satisfied, in particular, when $(u_k)_{k\geq 1}$ is bounded in $W^{1,\ell}(\Omega,\mathbb{R}^m)$.

In the special case $m = n = \ell$ we can replace the boundedness in $W^{1,n}(\Omega,\mathbb{R}^n)$ with the boundedness in $W^{1,n-1}(\Omega,\mathbb{R}^n)$. We prove (Theorem 4.1) that, in this case, the functional $F$ is lower semicontinuous on $W^{1,n}(\Omega,\mathbb{R}^m)$ along sequences converging strongly in $L^1(\Omega,\mathbb{R}^m)$ and bounded in $W^{1,p}(\Omega,\mathbb{R}^m)$, $p \geq n-1$. This result was proved in [5] when $p > n-1$. The borderline case $p = n-1$ was proved in [6], if $f(M_1^n(\nabla u)) \geq c_1|\det \nabla u|$ for some constant $c_1 > 0$, and is new in the other cases, which require a completely different proof. A counterexample in [12] shows that the result is not true if $p < n-1$.

To simplify the exposition, we consider first (Theorem 3.1) the case where $f$ depends only on the determinant of $\nabla u$, and then (Theorem 4.1) we study the general case.

We remark that in all the previous results the space $W^{1,n}(\Omega,\mathbb{R}^n)$ can not be replaced by $W^{1,p}(\Omega,\mathbb{R}^n)$ for $n-1 \leq p < n$, as shown by a counterexample in [2]. For the same reason, the space $W^{1,\ell}(\Omega,\mathbb{R}^m)$ in Theorem 2.2 can not be replaced by $W^{1,p}(\Omega,\mathbb{R}^m)$ for $p < \ell$.

Our results are based on the following property (Lemma 1.2, Corollary 1.3, and Remark 1.4), obtained by M. Giaquinta, L. Modica, J. Souček in [8] by using methods of geometric measure theory: if $(u_k)_{k\geq 1}$

is a sequence in $\mathcal{C}^1(\Omega, \mathbb{R}^m)$, bounded in $L^\infty(\Omega, \mathbb{R}^m)$, converging to $u$ in $L^1(\Omega, \mathbb{R}^m)$, and such that $\left(|M_1^\ell(\nabla u_k)|\right)_{k\geq 1}$ is bounded in $L^1(\Omega)$, then there exist a vector-valued Radon measure $\mu$ on $\Omega$ and a subsequence, still denoted by $(u_k)_{k\geq 1}$, such that $M_1^\ell(\nabla u_k) \to \mu$ weakly in the sense of Radon measures on $\Omega$ and such that $M_1^\ell(\nabla u)$ is the (density of the) absolutely continuous part of the measure $\mu$.

In the case of Theorem 2.2, the lower semicontinuity along sequences in $\mathcal{C}^1(\Omega, \mathbb{R}^m)$, bounded in $L^\infty(\Omega, \mathbb{R}^m)$ and converging in $L^1(\Omega, \mathbb{R}^m)$, follows easily from this property and from a classical lower semicontinuity result in the space of Radon measures, for which we refer to [3], [11], [13]. The hypothesis of boundedness in $L^\infty(\Omega, \mathbb{R}^m)$ is dropped by adapting a sophisticated truncation argument introduced by E. De Giorgi in the theory of minimal surfaces. The assumption $u_k \in \mathcal{C}^1(\Omega, \mathbb{R}^m)$ is replaced with $u_k \in W^{1,\ell}(\Omega, \mathbb{R}^m)$ by a standard approximation argument.

The proofs of Theorems 3.1 and 4.1 follow essentially the same lines, with a new difficulty in the first step: since there is no coerciveness assumption, the hypotheses guarantee only that $\left(|M_1^{n-1}(\nabla u_k)|\right)_{k\geq 1}$ is bounded in $L^1(\Omega)$ for the sequences $(u_k)_{k\geq 1}$ considered in these theorems. Using some ideas from convex analysis, we obtain also the boundedness of $(\det \nabla u_k)_{k\geq 1}$, and hence we prove that $(|M_1^n(\nabla u_k)|)_{k\geq 1}$ is bounded in $L^1(\Omega)$. Then we can continue along the lines of the proof of Theorem 2.2, adapting the truncation lemma and the approximation argument to the new cases.

## 1. DEFINITIONS AND PRELIMINARY RESULTS

The aim of this section is to introduce the notation and to recall some basic definitions and results which will be used in the sequel.

We begin with some algebraic notation.

Given two integer numbers $m$, $n$, with $m, n \geq 2$, let $\mathbb{M}^{m\times n}$ be the linear space of all $m\times n$ matrices with real entries. For $A \in \mathbb{M}^{m\times n}$ we write $A = (a_j^i)$, $1 \leq i \leq m$, $1 \leq j \leq n$, where upper and lower indices correspond to rows and columns respectively. The euclidean norm of any $m\times n$ matrix $A$ (defined as the square root of the trace of $AA^* \in \mathbb{M}^{m\times m}$) will be denoted by $|A|$. Moreover, given $0 \leq \ell \leq m \wedge n = \min\{m, n\}$, let

$$\sigma(m,n,\ell) = \begin{cases} 0 & \text{if } \ell = 0 \\ \sum_{1\leq k\leq \ell} \binom{m}{k}\binom{n}{k} & \text{if } 1 \leq \ell \leq m \wedge n \end{cases}$$

be the number of all minors up to the order $\ell$ of any $m\times n$ matrix, and for $1 \leq h \leq \ell \leq m \wedge n$ and $A \in \mathbb{M}^{m\times n}$ let $M_h^\ell(A)$ be the vector in $\mathbb{R}^\tau$, $\tau = \sigma(m,n,\ell) - \sigma(m,n,h-1)$, whose components are given by the determinants of all minors of $A$ whose order $k$ satisfies $h \leq k \leq \ell$, taken with the appropriate sign, for which we refer to [1] and [9]. We point out, however, that the choice of this sign is irrelevant for most of our proofs.

For future purposes we notice also that the norm of the vector of the minors of the product of two matrices can be easily estimated. Indeed, let $A \in \mathbb{M}^{m\times n}$ and $B \in \mathbb{M}^{m\times m}$. Then, $|M_h^h(BA)| \leq |B|^h |M_h^h(A)|$ for $1 \leq h \leq m \wedge n$ and hence

$$|M_1^\ell(BA)| \leq \Big(\max_{1\leq h\leq \ell} |B|^h\Big)|M_1^\ell(A)| \qquad 1 \leq \ell \leq m \wedge n. \tag{1.1}$$

Moreover, (1.1) reduces to

$$|M_1^\ell(BA)| \leq |B|^\ell |M_1^\ell(A)| \qquad 1 \leq \ell \leq m \wedge n \tag{1.2}$$

as soon as $|B| \geq 1$.

Next, recall that for a square matrix $A \in \mathbb{M}^{n\times n}$, its *adjugate* matrix $\operatorname{adj} A$ is defined as the transpose of the cofactors of $A$ (see [4]). Hence, $\operatorname{adj} A$ satisfies $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)\mathbb{I}_n$, where $\mathbb{I}_n$ denotes the identity matrix of $\mathbb{M}^{n\times n}$.

We survey now some elementary properties of convex functions for which we refer to [14].

Let $f\colon \mathbb{R}^n \to (-\infty, \infty]$ be any proper, convex function and let $f^*\colon \mathbb{R}^n \to (-\infty, \infty]$ be the Young-Fenchel conjugate (or polar) of $f$ defined by

$$f^*(z) = \sup\{(\xi, z) - f(\xi) :\ \xi \in \mathbb{R}^n\} \qquad z \in \mathbb{R}^n,$$

where $(\cdot\,,\cdot)$ stands for the inner product of $\mathbb{R}^n$. It is well known that $f^*$ is proper, convex, and lower semicontinuous. Moreover, the conjugate of $f^*$ (called bipolar of $f$ and denoted by $f^{**}$) coincides with $f$ as soon as $f$ itself is lower semicontinuous. Then, recall that the recession function $f^\infty\colon \mathbb{R}^n \to (-\infty, \infty]$ of a proper, convex, and lower semicontinuous function $f$ is defined by

$$\begin{aligned} f^\infty(\xi) &= \sup_{\zeta\in \operatorname{dom}(f)} (f(\xi+\zeta) - f(\zeta)) \\ &= \lim_{t\to\infty} \frac{f(\xi_0 + t\xi)}{t} \qquad \xi \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

where $\operatorname{dom}(f) = \{\zeta \in \mathbb{R}^n :\ f(\zeta) < \infty\}$ is the effective domain of $f$ and $\xi_0 \in \mathbb{R}^n$ is any point of it. Then, $f^\infty$ turns out to be a proper, convex, lower semicontinuous, and positively homogeneous function of degree 1 (see [14], Theorem 8.5). The relationship between $f^*$ and $f^\infty$ is given by

$$f^\infty = \left(\chi_{\operatorname{dom}(f^*)}\right)^* \tag{1.4}$$

(see again [14], Theorem 13.3) where $\chi_A$ denotes the indicator function of the set $A$ defined by

$$\chi_A(\xi) = \begin{cases} 0 & \text{if } \xi \in A \\ \infty & \text{otherwise.} \end{cases}$$

We now turn to the function spaces and to the measures considered in this paper.

Let $\Omega$ be any bounded open subset of $\mathbb{R}^n$ and let $\mathcal{L}^n$ be the Lebesgue measure on $\Omega$. Denote by $\mathcal{B}(\Omega)$ the $\sigma$-algebra of all Borel subsets of $\Omega$ and by $\mathcal{B}_c(\Omega)$ the $\delta$-ring of all relatively compact Borel subsets of $\Omega$ whose closure is contained in $\Omega$. For $1 \leq p \leq \infty$, we denote by $W^{1,p}(\Omega, \mathbb{R}^m)$ the Sobolev space of all functions $(u^1, \ldots, u^m)$ in $L^p(\Omega, \mathbb{R}^m)$ whose distributional gradient can be identified with a function $\nabla u$ in $L^p(\Omega, \mathbb{M}^{m\times n})$.

Then, let $\mathcal{D}(\Omega)$ be the space of all infinitely differentiable functions with compact support in $\Omega$ and write $\mathcal{D}'(\Omega)$ for the space of distributions on $\Omega$. Furthermore, let $\mathcal{C}_c(\Omega, \mathbb{R}^m)$ be the space consisting of all continuous, $\mathbb{R}^m$-valued functions with compact support in $\Omega$ endowed with its usual topology. The dual space of $\mathcal{C}_c(\Omega, \mathbb{R}^m)$ is denoted by $\mathcal{M}(\Omega, \mathbb{R}^m)$, and we simply write $\mathcal{C}_c(\Omega)$ and $\mathcal{M}(\Omega)$ when $m = 1$. The elements of $\mathcal{M}(\Omega, \mathbb{R}^m)$ are called $\mathbb{R}^m$-valued *Radon measures* on $\Omega$. Each $\mathbb{R}^m$-valued Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ will be identified with the corresponding countably additive, $\mathbb{R}^m$-valued set function defined on $\mathcal{B}_c(\Omega)$. Hence, the duality pairing between $\mathcal{C}_c(\Omega, \mathbb{R}^m)$ and $\mathcal{M}(\Omega, \mathbb{R}^m)$ is given by integration:

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi \, d\mu \qquad \varphi \in \mathcal{C}_c(\Omega, \mathbb{R}^m), \quad \mu \in \mathcal{M}(\Omega, \mathbb{R}^m).$$

Furthermore, we write $|\mu|$ for the total variation of $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$. Recall also that $|\mu|$ can be extended to a unique non-negative Borel measure on $\Omega$ and that $|\mu|$ is finite if and only if the range of $\mu$ as a set function is bounded. If this is the case, $\mu$ is called a *bounded Radon measure* and we write $\mathcal{M}_b(\Omega, \mathbb{R}^m)$ for the subspace of all bounded Radon measures on $\Omega$. Moreover, given $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$, we denote by $(\mu_a, \mu_s)$ the Lebesgue decomposition of $\mu$ with respect to $\mathcal{L}^n$, with $\mu_a$ absolutely continuous and $\mu_s$ singular with respect to $\mathcal{L}^n$. We agree also that every function

$u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ will be identified with the $\mathcal{L}^n$-absolutely continuous Radon measure with density $u$ and, accordingly, every $\mathcal{L}^n$-absolutely continuous Radon measure will be identified with its Radon-Nikodym derivative with respect to $\mathcal{L}^n$.

Throughout this paper, $\mathcal{M}(\Omega, \mathbb{R}^m)$ is endowed with its weak* topology. Therefore, all topological concepts concerning Radon measures (convergence *in primis*) are to be referred to the weak* topology of $\mathcal{M}(\Omega, \mathbb{R}^m)$ and in particular we agree that for a sequence of functions $(u_k)_{k\geq 1}$ in $L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ convergence in the sense of Radon measures means that the Radon measures defined by $\mu_k = u_k$ are convergent in $\mathcal{M}(\Omega, \mathbb{R}^m)$. Furthermore, Banach-Alaoglu's theorem provides a useful criterion of compactness in $\mathcal{M}(\Omega, \mathbb{R}^m)$. Indeed, let $\mathcal{K}$ be a bounded subset of $\mathcal{M}(\Omega, \mathbb{R}^m)$, that is, $\sup\{|\mu|(K) \colon \mu \in \mathcal{K}\} < \infty$ for all compact sets $K \subset \Omega$. Then, $\mathcal{K}$ is relatively compact and also sequentially relatively compact.

We now recall a well known lower semicontinuity theorem for functionals defined on $\mathcal{M}(\Omega, \mathbb{R}^m)$. To this purpose, let $f \colon \mathbb{R}^m \to [0, \infty]$ be any proper, convex, and lower semicontinuous function and let $F \colon \mathcal{M}(\Omega, \mathbb{R}^m) \to [0, \infty]$ be defined by

$$F(\mu) = \int_\Omega f(\mu_a(x))d\mathcal{L}^n(x) + \int_\Omega f^\infty\left(\frac{d\mu_s}{d|\mu_s|}(x)\right)d|\mu_s|(x) \qquad (1.5)$$

for all $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$, where $(\mu_a, \mu_s)$ is the Lebesgue decomposition of $\mu$ and $\dfrac{d\mu_s}{d|\mu_s|}$ is the Radon-Nikodym derivative of $\mu_s$ with respect to $|\mu_s|$. Then, we have the following theorem, whose proof can be found in [3], [11] and [13].

THEOREM 1.1. – *Let $f \colon \mathbb{R}^k \to [0, \infty]$ be a proper, convex, and lower semicontinuous function and let $F$ be the functional defined by (1.5). Then, $F$ is sequentially lower semicontinuous on $\mathcal{M}(\Omega, \mathbb{R}^m)$.*

Next, we end this quick survey of the function spaces considered in this paper with $BV(\Omega, \mathbb{R}^m)$ and $BV_{\text{loc}}(\Omega, \mathbb{R}^m)$, the spaces of all $\mathbb{R}^m$-valued functions of bounded and locally bounded variation on $\Omega$ respectively. The former consists of all functions $u \in L^1(\Omega, \mathbb{R}^m)$ whose distributional gradient $Du$ can be identified with an $\mathbb{M}^{m\times n}$-valued bounded Radon measure on $\Omega$, while the latter consists of all $\mathbb{R}^m$-valued, locally integrable functions on $\Omega$ whose distributional gradient is now a possibly unbounded $\mathbb{M}^{m\times n}$-valued Radon measure on $\Omega$. For each $u \in BV_{\text{loc}}(\Omega, \mathbb{R}^m)$, the Radon-Nikodym derivative of the absolutely continuous part of $Du$ with respect to $\mathcal{L}^n$ will be denoted by $\nabla u \in L^1_{\text{loc}}(\Omega, \mathbb{M}^{m\times n})$, while the $\mathcal{L}^n$-singular part of $Du$ will be denoted by $D_s u \in \mathcal{M}(\Omega, \mathbb{M}^{m\times n})$. In particular, $\nabla u \in L^1(\Omega, \mathbb{M}^{m\times n})$ and $D_s u \in \mathcal{M}_b(\Omega, \mathbb{M}^{m\times n})$ provided $u \in BV(\Omega, \mathbb{R}^m)$. Hence, given any $u \in BV_{\text{loc}}(\Omega, \mathbb{R}^m)$, we have

$$Du(B) = \int_B \nabla u \, d\mathcal{L}^n + D_s u(B)$$

for all sets $B$ in $\mathcal{B}_c(\Omega)$ and

$$|Du|(B) = \int_B |\nabla u| \, d\mathcal{L}^n + |D_s u|(B)$$

for all Borel subsets $B$ of $\Omega$. It is plain that the first formula holds true for all Borel subsets $B$ of $\Omega$ as well, provided $u \in BV(\Omega, \mathbb{R}^m)$.

We point out that throughout this paper we consider $BV(\Omega, \mathbb{R}^m)$ and $BV_{\text{loc}}(\Omega, \mathbb{R}^m)$ as subspaces of $L^1(\Omega, \mathbb{R}^m)$ and $L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ respectively endowed with the relative topologies. Hence, $BV(\Omega, \mathbb{R}^m)$ and $BV_{\text{loc}}(\Omega, \mathbb{R}^m)$ are not complete with respect to $L^1(\Omega, \mathbb{R}^m)$- and $L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$-convergence. However, it is easy to check that whenever $(u_k)_{k\geq 1} \subset BV_{\text{loc}}(\Omega, \mathbb{R}^m)$ converges in $L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ to a function $u$ and the gradients $Du_k$ are bounded in $\mathcal{M}(\Omega, \mathbb{R}^m)$, then $u \in BV_{\text{loc}}(\Omega, \mathbb{R}^m)$ (see [10], Theorem 1.9). Of course, the same result holds true for $BV(\Omega, \mathbb{R}^m)$ provided $(u_k)_{k\geq 1}$ converges to $u$ in $L^1(\Omega, \mathbb{R}^m)$ and the gradients $Du_k$ have uniformly bounded total variation.

Finally, we prove a lemma concerning the convergence as Radon measures of the minors of a sequence of smooth functions. Roughly speaking, it states that, whenever a bounded sequence of continuously differentiable functions converges to a function $u$ in $L^1(\Omega, \mathbb{R}^m)$ and the sequence of all their minors converges to a measure $\mu$ in the sense of Radon measures, then $u$ is of locally bounded variation and the absolutely continuous part of $\mu$ with respect to $\mathcal{L}^n$ coincides with the vector of all minors of $\nabla u$. This result is actually the cornerstone for the subsequent sections of this paper. Its proof relies on some techniques of geometric measure theory introduced by Giaquinta, Modica, and Souček in [8] for the study of vectorial problems in the calculus of variations. See, in particular, Theorem 3 in [8] or Theorem 1.5 in [1], where some results of Giaquinta, Modica, and Souček are summarized. We refer to [7] and [16] for the notation and the results of geometric measure theory needed in the proof.

LEMMA 1.2. – *Set $\ell = m \wedge n$ and let the functions $u, u_k \colon \Omega \to \mathbb{R}^m$, $k \geq 1$ satisfy*

*(a)* $u_k \in C^1(\Omega, \mathbb{R}^m)$, $k \geq 1$;

*(b)* $(u_k)_{k\geq 1}$ *is bounded in* $L^\infty(\Omega,\mathbb{R}^m)$*;*

*(c)* $u_k \to u$ *in* $L^1(\Omega,\mathbb{R}^m)$*;*

*(d) there exists* $\mu \in \mathcal{M}(\Omega,\mathbb{R}^\sigma)$, $\sigma = \sigma(m,n,\ell)$, *such that* $M_1^\ell(\nabla u_k) \to \mu$ *in the sense of Radon measures on* $\Omega$.

*Then,* $u \in BV_{\rm loc}(\Omega,\mathbb{R}^m)$ *and* $\mu_a = M_1^\ell(\nabla u)$.

*Proof.* – As $\left(|M_1^\ell(\nabla u_k)|\right)_{k\geq 1}$ is bounded in $L^1_{\rm loc}(\Omega)$, it is clear that $u \in BV_{\rm loc}(\Omega,\mathbb{R}^m)$ (see [10], Theorem 1.9).

The remaining part of the proof can be carried out by a localization argument. Indeed, choose an open set $\Omega' \in \mathcal{B}_c(\Omega)$ and, for each $k \geq 1$, let $T_k$ be the $n$-rectifiable current of integer multiplicity on $U' = \Omega' \times \mathbb{R}^m$ defined as integration over the graph $\Gamma(u_k) = \{(x,u_k(x)) \in U' :\, x \in \Omega'\}$, which is an oriented, boundaryless $n$-manifold of class $C^1$ in $U'$. The $n$-current $T_k$ belongs to the space $\mathrm{Cart}\,(\Omega',\mathbb{R}^m)$ of cartesian currents on $\Omega'$ introduced by Giaquinta, Modica, and Souček (see again [8]). It is well known that for each $n$-form $\omega$ with smooth and compactly supported coefficients in $U'$ we have

$$T_k(\omega) = \int_\Omega \langle \tilde{M}_1^\ell(\nabla u_k(x)), \omega(x,u_k(x))\rangle \, d\mathcal{L}^n(x) \qquad k \geq 1,$$

where $\langle\cdot\,,\cdot\rangle$ denotes the duality pairing between $n$-vectors and $n$-covectors of $\mathbb{R}^{n+m}$ and where, for each matrix $A \in \mathbb{M}^{m\times n}$, $\tilde{M}_1^\ell(A)$ is defined as the $n$-vector of $\mathbb{R}^{n+m}$ whose components (with respect to the standard basis of simple $n$-vectors of $\mathbb{R}^{n+m}$) are the same as those of the vector $M_1^\ell(A)$. Similarly, let $\tilde{\mu}$ be the Radon measure on $\Omega$ with values in the space of $n$-vectors of $\mathbb{R}^{n+m}$ whose components are the same as those of $\mu$. Hence, $\tilde{M}_1^\ell(\nabla u_k) \to \tilde{\mu}$ in the sense of Radon measures on $\Omega'$ and the $n$-currents $T_k$ turn out to have uniformly bounded mass in $U'$. Therefore, (c) and the compactness properties of the space of cartesian currents yield a subsequence, still denoted by $(T_k)_{k\geq 1}$, which converges to a current $T \in \mathrm{Cart}\,(\Omega',\mathbb{R}^m)$ in the weak sense of cartesian $n$-currents on $\Omega'$ (see again [8]). Identifying each $T_k$ and $T$ with a bounded Radon measure on $U'$ with values in the space of the $n$-vectors of $\mathbb{R}^{n+m}$, it follows that $T_k \to T$ as Radon measures on $U'$. Now, let $p\colon U' \to \Omega'$ be the canonical projection of $U'$ onto $\Omega'$ and denote by $p_*T_k$ and $p_*T$ the image measures of $T_k$ and $T$. It is plain that

$$p_*T_k(\omega) = \int_\Omega \langle \tilde{M}_1^\ell(\nabla u_k(x)), \omega(x)\rangle \, d\mathcal{L}^n(x) \qquad k \geq 1$$

for all $n$-forms $\omega$ on $\Omega'$. Moreover, since, by (b), the graphs of the restrictions to $\Omega'$ of the functions $u_k$ lie in a bounded subset of $U'$, the same is true for the supports of the measures $T_k$. Hence, it follows that $p_*T_k \to p_*T$ in the sense of Radon measures on $\Omega'$. Thus, $p_*T = \tilde{\mu}$. Finally, an appeal either to Theorem 3 in [8] or Theorems 1.5 and 1.6 in [1] yields that $\tilde{\mu}_a = \tilde{M}_1^\ell(\nabla u)$ and hence $\mu_a = M_1^\ell(\nabla u)$ on $\Omega'$. The arbitrariness of the open set $\Omega' \in \mathcal{B}_c(\Omega)$ shows that the equation $\mu_a = M_1^\ell(\nabla u)$ holds in $\mathcal{M}(\Omega, \mathbb{R}^m)$. ■

In the previous lemma, it was explicitly assumed that all minors converge in the sense of measures. This is not actually needed; the convergence of all minors up to a given order $\ell \leq m \wedge n$ is enough. Indeed, we have the following result.

COROLLARY 1.3. – *The result of Lemma 1.2 remains true if* $1 \leq \ell \leq m \wedge n$.

*Proof.* – If $\ell < m \wedge n$, choose any increasing $\ell$-tuple $(i_1, \ldots, i_\ell)$ out of $\{1, \ldots, m\}$ and set $v_k = (u_k^{i_1}, \ldots, u_k^{i_\ell})$, for $k \geq 1$, and $v = (u^{i_1}, \ldots, u^{i_\ell})$.

Each one of the minors of order less than or equal to $\ell$ of the functions $u_k$ converges, as $k \to \infty$, to a component of $\mu$. Let $\tilde{\mu}$ be the $\mathbb{R}^{\sigma(\ell,n,\ell)}$-valued Radon measure whose components are given by the components of $\mu$ which are limit of minors of the functions $u_k$ involving only the components $(u_k^{i_1}, \ldots, u_k^{i_\ell})$ of each $u_k$. By definition, $M_1^\ell(\nabla v_k) \to \tilde{\mu}$ in $\mathcal{M}(\Omega, \mathbb{R}^{\sigma(\ell,n,\ell)})$. Now, Lemma 1.2 with $m = \ell$ applies to $(v_k)_{k \geq 1}$ yielding $\tilde{\mu}_a = M_1^\ell(\nabla v)$. Since the $\ell$-tuple $(i_1, \ldots, i_\ell)$ is arbitrary, the conclusion follows. ■

*Remark 1.4.* – As a particular case of Corollary 1.3, by the compactness criterion in the space of Radon measures previously recalled, it is easy to check that, whenever $1 \leq \ell \leq m \wedge n$ and the functions $u$ and $(u_k)_{k \geq 1}$ satisfy the hypotheses (a), (b), (c) of Lemma 1.2 and

(e) $\left(|M_1^\ell(\nabla u_k)|\right)_{k \geq 1}$ is bounded in $L^1(\Omega)$,

then $u \in BV(\Omega, \mathbb{R}^m)$ and there exist a bounded Radon measure $\mu \in \mathcal{M}_b(\Omega, \mathbb{R}^\sigma)$, with $\sigma = \sigma(m, n, \ell)$, and a subsequence $(u_{k_h})_{h \geq 1}$ such that $M_1^\ell(\nabla u_{k_h}) \to \mu$ in the sense of Radon measures on $\Omega$ and such that $\mu_a = M_1^\ell(\nabla u)$.

## 2. POLYCONVEX FUNCTIONALS NOT DEPENDING ON ALL MINORS

Throughout this section, we consider polyconvex integral functionals which depend on all minors up to a given order $\ell$ and we investigate their lower semicontinuity properties with respect to convergence in $L^1(\Omega, \mathbb{R}^m)$. In the case $\ell = m \wedge n$, we give a new proof of the semicontinuity result proved in [1], Theorem 2.5, and in the case $\ell < m \wedge n$ we prove the same lower semicontinuity result under weaker hypotheses.

Let $m$, $n$, and $\ell$ be positive integers such that $m, n \geq 2$ and $1 \leq \ell \leq m \wedge n$. As $m$ and $n$ are kept fixed throughout this section, we shortly write $\sigma(\ell) = \sigma(m, n, \ell)$ for the number of the minors of order up to $\ell$ of any $m \times n$ matrix. Then, let $F: BV(\Omega, \mathbb{R}^m) \to [0, \infty]$ be the polyconvex integral functional defined by

$$F(u) = \int_\Omega f\big(M_1^\ell(\nabla u)\big)\, d\mathcal{L}^n \qquad u \in BV(\Omega, \mathbb{R}^m), \tag{2.1}$$

where $f: \mathbb{R}^{\sigma(\ell)} \to [0, \infty]$ is a function satisfying the following properties:

(i) $f$ is a convex function such that $f(0) < \infty$;

(ii) $f$ is lower semicontinuous on $\mathbb{R}^{\sigma(\ell)}$;

(iii) there exists $c_1 > 0$ such that $f(\xi) \geq c_1|\xi|$ for all $\xi \in \mathbb{R}^{\sigma(\ell)}$.

Occasionally, we will assume also that

(iv) there exists $c_2 > 0$ such that $f(\xi) \leq c_2(1 + |\xi|)$ for all $\xi \in \mathbb{R}^{\sigma(\ell)}$.

As far as lower semicontinuity is concerned, it is not restrictive to assume (iv), as pointed out by the following remark.

*Remark 2.1.* – Any function $f$ satisfying (i), (ii), (iii) can be approximated by an increasing sequence of functions $f_h: \mathbb{R}^{\sigma(\ell)} \to [0, \infty)$, $h \geq 1$, such that (i), (ii), (iii), (iv) hold for $h$ large enough. Indeed, it is enough to define

$$f_h(\xi) = \sup_{|z| \leq h} \big((\xi, z) - f^*(z)\big) \qquad \xi \in \mathbb{R}^{\sigma(\ell)}, \quad h \geq 1.$$

Then, $(f_h)_{h \geq 1}$ turns out to be an increasing sequence of proper, convex, and lower semicontinuous (actually continuous) functions such that $f_h \nearrow f$ pointwise on $\mathbb{R}^{\sigma(\ell)}$. Moreover, the definition of $f^*$ yields $f^*(z) \geq -f(0)$ for all $z \in \mathbb{R}^{\sigma(\ell)}$ and hence

$$f_h(\xi) \leq f(0) + h|\xi| \qquad \xi \in \mathbb{R}^{\sigma(\ell)}, \quad h \leq 1.$$

Thus, (iv) holds for $f_h$ with a suitable constant $c_2$ (depending on $h$). Finally, as $f(\xi) \geq c_1|\xi|$ for all $\xi \in \mathbb{R}^{\sigma(\ell)}$, its conjugate function satisfies

$$f^*(z) \leq \chi_{B(0,c_1)}(z) \qquad z \in \mathbb{R}^{\sigma(\ell)},$$

where $B(0, c_1)$ denotes the closed ball of $\mathbb{R}^{\sigma(\ell)}$ of radius $c_1$ centered at the origin. Therefore, for $h \geq c_1$ and $\xi \in \mathbb{R}^{\sigma(\ell)}$ we get

$$f_h(\xi) \geq \sup_{|z| \leq h} \big((\xi, z) - \chi_{B(0,c_1)}(z)\big) = \sup_z \big((\xi, z) - \chi_{B(0,c_1)}(z)\big) = c_1|\xi|,$$

so that $(f_h)_{h \geq 1}$ satisfies (iii) for $h \geq c_1$.

Now, the main result of this section reads as follows.

THEOREM 2.2. – *Let* $f: \mathbb{R}^{\sigma(\ell)} \to [0,\infty]$ *satisfy conditions (i), (ii), (iii) and let* $F$ *be the functional defined by (2.1). Assume that the functions* $u \in BV(\Omega, \mathbb{R}^m)$ *and* $(u_k)_{k\geq 1}$ *satisfy*

*(a)* $u_k \in W^{1,\ell}(\Omega, \mathbb{R}^m)$ , $k \geq 1$;

*(b)* $u_k \to u$ *in* $L^1(\Omega, \mathbb{R}^m)$.

*Then,* $F(u) \leq \liminf_{k\to\infty} F(u_k)$.

We notice that the lower semicontinuity of $F$ along sequences of functions in $W^{1,\ell}(\Omega, \mathbb{R}^m)$ converging in $L^1(\Omega, \mathbb{R}^m)$ can be established even if the integrand $f$ does not fulfill the growth condition (iii), provided we require the boundedness in $L^1(\Omega, \mathbb{R}^{\sigma(\ell)})$ of the sequence of all minors up to order $\ell$. This statement is easily seen to be completely equivalent to Theorem 2.2. Indeed, we have the following corollary.

COROLLARY 2.3. – *Let* $f: \mathbb{R}^{\sigma(\ell)} \to [0,\infty]$ *satisfy conditions (i), (ii). Assume that the functions* $u \in BV(\Omega, \mathbb{R}^m)$ *and* $(u_k)_{k\geq 1}$ *satisfy*

*(a)* $u_k \in W^{1,\ell}(\Omega, \mathbb{R}^m)$ , $k \geq 1$;

*(b)* $\left(|M_1^\ell(\nabla u_k)|\right)_{k\geq 1}$ *is bounded in* $L^1(\Omega)$;

*(c)* $u_k \to u$ *in* $L^1(\Omega, \mathbb{R}^m)$.

*Then,* $F(u) \leq \liminf_{k\to\infty} F(u_k)$.

*Proof.* – Let $\epsilon > 0$ be given. Set $f_\epsilon: \mathbb{R}^{\sigma(\ell)} \to [0,\infty]$, $f_\epsilon(\xi) = f(\xi) + \epsilon|\xi|$, so that $f_\epsilon$ satisfies (i), (ii), (iii), and define $F_\epsilon: BV(\Omega, \mathbb{R}^m) \to [0,\infty]$ by replacing $f$ with $f_\epsilon$ in (2.1). Then, Theorem 2.2 yields

$$F(u) \leq F_\epsilon(u) \leq \liminf_{k\to\infty} F_\epsilon(u_k) \leq \liminf_{k\to\infty} F(u_k) + \epsilon \sup_{k\geq 1} \int_\Omega |M_1^\ell(\nabla u_k)|\, d\mathcal{L}^n.$$

As $\epsilon > 0$ is arbitrary, the conclusion follows. ■

The proof of Theorem 2.2 will be accomplished through a chain of partial results. The main steps are the following. First, we prove Theorem 2.2 under the additional hypothesis that the functions $u_k$ are smooth and uniformly bounded. Then, we drop the boundedness requirement by a suitable truncation argument. Finally, we weaken the smoothness assumption on the sequence $(u_k)_{k\geq 1}$.

The first step is given by the following proposition.

PROPOSITION 2.4. – *Let* $f: \mathbb{R}^{\sigma(\ell)} \to [0,\infty]$ *satisfy conditions (i), (ii), (iii) and let* $w, w_k \in BV(\Omega, \mathbb{R}^m)$, $k \geq 1$. *Assume that*

*(a)* $w_k \in C^1(\Omega, \mathbb{R}^m)$, $k \geq 1$;

*(b)* $(w_k)_{k\geq 1}$ *is bounded in* $L^\infty(\Omega, \mathbb{R}^m)$*;*

*(c)* $w_k \to w$ *in* $L^1(\Omega, \mathbb{R}^m)$.

*Then,* $F(w) \leq \liminf_{k\to\infty} F(w_k)$.

*Proof.* –Let $\liminf_{k\to\infty} F(w_k) = c$. We may assume that $c < \infty$, otherwise nothing is left to prove. Then, choose a subsequence (still denoted by $(w_k)_{k\geq 1}$) such that $F(w_k) \to c$. Thus, $(F(w_k))_{k\geq 1}$ is a bounded sequence and this, together with (iii), yields that $\big(|M_1^\ell(\nabla w_k)|\big)_{k\geq 1}$ is bounded in $L^1(\Omega)$. Hence, by Remark 1.4, there exist a bounded Radon measure $\mu \in \mathcal{M}_b(\Omega, \mathbb{R}^{\sigma(\ell)})$ and a subsequence that we denote again by $(w_k)_{k\geq 1}$ such that $\big(M_1^\ell(\nabla w_k)\big)_{k\geq 1}$ converges to $\mu$ in the sense of Radon measures on $\Omega$. Moreover, $\mu_a = M_1^\ell(\nabla w)$. Therefore, recalling that the recession function $f^\infty$ of $f$ is non-negative and vanishes at zero, we get by Theorem 1.1

$$\begin{aligned} F(w) &= \int_\Omega f\big(M_1^\ell(\nabla w)\big)\, d\mathcal{L}^n \leq \int_\Omega f(\mu_a)\, d\mathcal{L}^n + \int_\Omega f^\infty\left(\frac{d\mu_s}{d|\mu_s|}\right) d|\mu_s| \\ &\leq \liminf_{k\to\infty} \int_\Omega f\big(M_1^\ell(\nabla w_k)\big)\, d\mathcal{L}^n = \lim_{k\to\infty} F(w_k) = c. \end{aligned}$$

This completes the proof. ■

Before going on with the second step in order to drop the boundedness requirement, we describe in the subsequent remark the truncating functions needed for the proof.

*Remark 2.5.* – Let $\psi \in \mathcal{C}^1([0,\infty))$ be such that $\psi(t)$ is equal to $t$ for $0 \leq t \leq 1$ and vanishes for $t \geq 2$, and denote by $\mathrm{Lip}(\psi)$ its Lipschitz constant. Notice that $\mathrm{Lip}(\psi) > 1$. Then, the mapping $\Psi \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$ defined by

$$\Psi(y) = \begin{cases} \psi(|y|)\frac{y}{|y|} & \text{if } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

turns out to be a Lipschitz continuous function whose Lipschitz constant is just $\mathrm{Lip}(\psi)$. Notice also that $\Psi(y) = y$ for all $y \in \mathbb{R}^m$ with $|y| \leq 1$ and that $\Psi$ maps the complement of the open ball of radius 2 centered at 0 onto the singleton $\{0\}$.

For all $\rho > 0$, let $\Psi^\rho \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$ be defined by

$$\Psi^\rho(y) = \rho\Psi(\rho^{-1}y) \qquad y \in \mathbb{R}^m, \quad \rho > 0, \tag{2.2}$$

so that

$$L = \sup_y |\nabla\Psi^\rho(y)| = \sup_y |\nabla\Psi(y)| = \mathrm{Lip}(\psi) > 1 \qquad \rho > 0. \tag{2.3}$$

Notice again that $\Psi^\rho$ reduces to the identity map on the closed ball of $\mathbb{R}^m$ of radius $\rho$ centered at 0, and maps the complement of the open ball of radius $2\rho$ centered at 0 onto $\{0\}$.

Next, we prove the second step.

PROPOSITION 2.6. – *Let* $f: \mathbb{R}^{\sigma(\ell)} \to [0, \infty)$ *satisfy conditions (i), (ii), (iii), (iv) and let* $v, v_k \in BV(\Omega, \mathbb{R}^m)$, $k \geq 1$. *Assume that*

*(a)* $v_k \in C^1(\Omega, \mathbb{R}^m)$, $k \geq 1$;

*(b)* $v_k \to v$ *in* $L^1(\Omega, \mathbb{R}^m)$.

*Then,* $F(v) \leq \liminf_{k\to\infty} F(v_k)$.

*Proof.* – We may and do assume that $F(v_k) \to c < \infty$. It follows by (iii) that

$$\int_\Omega |M_1^\ell(\nabla v_k)|\, d\mathcal{L}^n \leq c_0 \qquad k \geq 1 \tag{2.4}$$

for some positive constant $c_0$. Now, let $\epsilon > 0$ be given and choose $s \in \mathbb{N}_+$ such that $c_0 c_2 L^\ell < s\epsilon$, where $L$ is the constant defined by (2.3). Then, for any $r_0 > 0$, set $r_i = 2^i r_0$, $0 \leq i \leq s$. By the choice of $s$ and by the inequalities

$$c_0 \geq \int_\Omega |M_1^\ell(\nabla v_k)|\, d\mathcal{L}^n \geq \sum_{1\leq i\leq s} \int_{\{r_{i-1}<|v_k|\leq r_i\}} |M_1^\ell(\nabla v_k)|\, d\mathcal{L}^n$$

we see that there exist $i_0 \in \{0, \ldots, s-1\}$ and a subsequence $(v_k)_{k\geq 1}$ such that

$$\int_{\{r_{i_0}<|v_k|\leq 2r_{i_0}\}} |M_1^\ell(\nabla v_k)|\, d\mathcal{L}^n \leq \frac{c_0}{s} < \epsilon c_2^{-1} L^{-\ell} \qquad k \geq 1. \tag{2.5}$$

Now, set $r = r_{i_0}$ and define

$$\begin{aligned} w^r &= \Psi^r \circ v, \\ w_k^r &= \Psi^r \circ v_k, \qquad k \geq 1, \end{aligned}$$

where $\Psi^r$ is the mapping defined by (2.2) for $\rho = r$. It is plain that $w^r$ is of bounded variation on $\Omega$, that $w_k^r \in C^1(\Omega, \mathbb{R}^m)$ for $k \geq 1$, and that $(w_k^r)_{k\geq 1}$ is bounded in $L^\infty(\Omega, \mathbb{R}^m)$. Moreover, as $\Psi^r$ is Lipschitz continuous, $(w_k^r)_{k\geq 1}$ converges to $w^r$ in $L^1(\Omega, \mathbb{R}^m)$. Now, recalling (iv) and the properties of $(w_k^r)_{k\geq 1}$, we get for each $k \geq 1$

$$F(w_k^r) \leq \int_{\{|v_k|\leq r\}} f\big(M_1^\ell(\nabla v_k)\big)\, d\mathcal{L}^n \\ + c_2 \int_{\{r<|v_k|\leq 2r\}} |M_1^\ell(\nabla \Psi^r(v_k)\nabla v_k)|\, d\mathcal{L}^n + c_2\mathcal{L}^n(\{|v_k| > r\}). \quad (2.6)$$

Then, taking (1.2) and (2.3) into account, we see that

$$|M_1^\ell(\nabla \Psi^r(v_k)\nabla v_k)| \leq L^\ell |M_1^\ell(\nabla v_k)| \quad (2.7)$$

pointwise on $\Omega$ for all $k \geq 1$. Hence, (2.6) and (2.7) together with (2.5) yield

$$F(w_k^r) \leq F(v_k) + \epsilon + c_2\mathcal{L}^n(\{|v_k| > r\}) \qquad k \geq 1. \quad (2.8)$$

Now, notice that for every $\eta > 0$ we have

$$\mathcal{L}^n(\{|v_k| > r\}) \leq \mathcal{L}^n(\{|v| > r_0 - \eta\}) + \mathcal{L}^n(\{|v - v_k| > \eta\}) \qquad k \geq 1,$$

and that $\mathcal{L}^n(\{|v - v_k| > \eta\}) \to 0$ as $k \to \infty$, since $(v_k)_{k\geq 1}$ converges to $v$ in $\mathcal{L}^n$-measure on $\Omega$. Thus, letting first $k \to \infty$ and then $\eta \to 0$, we get

$$\limsup_{k\to\infty} \mathcal{L}^n(\{|v_k| > r\}) \leq \mathcal{L}^n(\{|v| \geq r_0\}). \quad (2.9)$$

Therefore, applying Proposition 2.4 to the sequence $(w_k^r)_{k\geq 1}$ and taking (2.8) and (2.9) into account, we get

$$\int_{\{|v|<r_0\}} f\big(M_1^\ell(\nabla v)\big)\, d\mathcal{L}^n \\ \leq F(w^r) \leq \liminf_{k\to\infty} F(v_k) + \epsilon + c_2\mathcal{L}^n(\{|v| \geq r_0\}).$$

Finally, recalling that $v \in L^1(\Omega, \mathbb{R}^m)$ and noticing that the left hand side of the previous inequality converges monotonically to $F(v)$ as $r_0 \nearrow \infty$, we get $F(v) \leq \liminf_{k\to\infty} F(v_k) + \epsilon$. As $\epsilon > 0$ was arbitratily chosen, the proof is complete. ∎

Finally, we are left to prove the last step. This is done in the following proposition.

PROPOSITION 2.7. – *Let $f: \mathbb{R}^{\sigma(\ell)} \to [0, \infty)$ satisfy conditions (i), (ii), (iii), (iv). Assume that the functions $u \in BV(\Omega, \mathbb{R}^m)$ and $(u_k)_{k\geq 1}$ satisfy*

*(a) $u_k \in W^{1,\ell}(\Omega, \mathbb{R}^m)$, $k \geq 1$;*

*(b)* $u_k \to u$ *in* $L^1(\Omega, \mathbb{R}^m)$.

*Then,* $F(u) \leq \liminf_{k\to\infty} F(u_k)$.

*Proof.* – The mapping $w \in W^{1,\ell}(\Omega, \mathbb{R}^m) \to M_1^\ell(\nabla w) \in L^1(\Omega, \mathbb{R}^{\sigma(\ell)})$ is continuous and $F$ is finite on $W^{1,\ell}(\Omega, \mathbb{R}^m)$ by (iv). Hence, Carathéodory's continuity theorem shows that $F$ restricted to $W^{1,\ell}(\Omega, \mathbb{R}^m)$ is continuous with respect to the strong topology of $W^{1,\ell}(\Omega, \mathbb{R}^m)$. This, together with Meyers-Serrin's approximation theorem, yields a sequence of functions $(v_k)_{k\geq 1}$ in $\mathcal{C}^1(\Omega, \mathbb{R}^m) \cap W^{1,\ell}(\Omega, \mathbb{R}^m)$ such that

$$\|u_k - v_k\|_{W^{1,\ell}(\Omega, \mathbb{R}^m)} < 1/k, \qquad |F(u_k) - F(v_k)| < 1/k$$

for every $k \geq 1$. Thus, $v_k \to u$ in $L^1(\Omega, \mathbb{R}^m)$. Applying Proposition 2.6 to the sequence $(v_k)_{k\geq 1}$, we get

$$F(u) \leq \liminf_{k\to\infty} F(v_k) \leq \liminf_{k\to\infty} \left(F(u_k) + \frac{1}{k}\right) = \liminf_{k\to\infty} F(u_k). \quad \blacksquare$$

The proof of Theorem 2.2 reduces now to an easy consequence of the previous results.

*Proof of Theorem 2.2.* – Let $(f_h)_{h\geq 1}$ be the sequence of functions associated with $f$ by Remark 2.1 and let $F_h: BV(\Omega, \mathbb{R}^m) \to [0, \infty]$, $h \geq 1$, be the functionals defined by (2.1) with $f_h$ in place of $f$. On account of Proposition 2.7, we get

$$F_h(u) \leq \liminf_{k\to\infty} F_h(u_k) \qquad h \geq 1. \tag{2.10}$$

Since $f_h \leq f$ for all $h \geq 1$, the right hand side of (2.10) is not greater than $\liminf_{k\to\infty} F(u_k)$, while the left hand side converges monotonically to $F(u)$ as $h \to \infty$. Thus, $F$ is lower semicontinuous along the sequence $(u_k)_{k\geq 1}$. $\blacksquare$

## 3. CONVEX INTEGRAL FUNCTIONALS OF THE DETERMINANT

This section is devoted to the study of the lower semicontinuity properties with respect to $L^1(\Omega, \mathbb{R}^n)$-convergence of integral functionals depending only on the determinant. We aim at proving that, as soon as the integrand is proper, convex, and lower semicontinuous, the corresponding functional is lower semicontinuous on the space $W^{1,n}(\Omega, \mathbb{R}^n)$ along sequences converging in $L^1(\Omega, \mathbb{R}^n)$ and bounded in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq n-1$. This result cannot be improved by allowing $p < n-1$ as shown in [12].

In order to state our result, denote by $G: BV(\Omega, \mathbb{R}^n) \to [0, \infty]$ the polyconvex integral functional defined by

$$G(u) = \int_\Omega g(\det \nabla u)\, d\mathcal{L}^n \qquad u \in BV(\Omega, \mathbb{R}^n), \tag{3.1}$$

where $g: \mathbb{R} \to [0, \infty]$ is a function satisfying the following properties:

(i) $g$ is a convex function such that $g(0) < \infty$;

(ii) $g$ is lower semicontinuous on $\mathbb{R}$.

We shall prove the following lower semicontinuity result.

THEOREM 3.1. – *Let $g: \mathbb{R} \to [0, \infty]$ satisfy conditions (i) and (ii). Assume that the functions $u \in BV(\Omega, \mathbb{R}^n)$ and $(u_k)_{k \geq 1}$ satisfy*

*(a) $u_k \in W^{1,n}(\Omega, \mathbb{R}^n)$, $k \geq 1$;*

*(b) $(u_k)_{k \geq 1}$ is bounded in $W^{1,n-1}(\Omega, \mathbb{R}^n)$;*

*(c) $u_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$.*

*Then, $G(u) \leq \liminf_{k \to \infty} G(u_k)$.*

This theorem provides the lower semicontinuity result of [6] without any growth assumption on the integrand, at least in the case of an integrand independent of $x$ and $u$.

We begin by noticing that, if $g$ satisfies (i) and (ii), then either $g$ is constant, so that Theorem 3.1 becomes trivial, or there exist $c_1 > 0$ and $\alpha \geq 0$ such that $g$ satisfies at least one of the following growth conditions:

(iii+) $\quad g(t) \geq c_1 t^+ - \alpha \qquad t \in \mathbb{R}$,

(iii−) $\quad g(t) \geq c_1 t^- - \alpha \qquad t \in \mathbb{R}$,

where $t^+$ and $t^-$ denote the positive and the negative part of $t$ respectively. Notice also that, as soon as $g$ satisfies (i), (ii), and both (iii+) and (iii−), then Theorem 3.1 reduces to the case considered in [6].

After these preliminaries, the proof of Theorem 3.1 can be carried out through the same steps described in Section 2. The only remarkable difference lies in the fact that the truncation argument of Section 2 is now to be performed by an orientation preserving mapping $\Psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $0 \leq \det \nabla\Psi \leq 1$ on $\mathbb{R}^n$.

We begin by proving a lemma concerning the relationship between convergence in the sense of distributions and in the sense of Radon measures. To this purpose, let $\mathcal{M}^+(\Omega)$ be the cone of non-negative elements of $\mathcal{M}(\Omega)$ and denote by $(\mu^+, \mu^-)$ the Jordan decomposition of $\mu \in \mathcal{M}(\Omega)$. Then, we have the following basic lemma.

LEMMA 3.2. – *Let* $(\mu_k)_{k\geq 1}$ *be a sequence in* $\mathcal{M}(\Omega)$. *Assume that*

*(a) there exists* $T \in \mathcal{D}'(\Omega)$ *such that* $\mu_k \to T$ *in the sense of distributions on* $\Omega$;

*(b) there exists* $\nu \in \mathcal{M}^+(\Omega)$ *such that* $\mu_k^+ \to \nu$ *in the sense of Radon measures on* $\Omega$.

*Then, there exists* $\mu \in \mathcal{M}(\Omega)$ *such that* $\mu = T$ *on* $\mathcal{D}(\Omega)$ *and* $\mu_k \to \mu$ *in the sense of Radon measures on* $\Omega$.

*Proof.* – Set $S = \nu - T$ so that $S \in \mathcal{D}'(\Omega)$ and $\mu_k^- = \mu_k^+ - \mu_k \to S$ in $\mathcal{D}'(\Omega)$. Thus, $S$ is a positive distribution on $\Omega$ and therefore it is actually the restriction to $\mathcal{D}(\Omega)$ of a positive Radon measure on $\Omega$ that we still denote by $S$ (see [15], Chapter 1, Theorem 5). Moreover, $\mu = \nu - S$ belongs to $\mathcal{M}(\Omega)$ and agrees with $T$ on $\mathcal{D}(\Omega)$. In order to see that $\mu_k \to \mu$ as Radon measures on $\Omega$, let $K$ be any compact subset of $\Omega$ and let $\vartheta \in \mathcal{D}(\Omega)$ be any function such that $0 \leq \vartheta \leq 1$ on $\Omega$ and $\vartheta = 1$ on $K$. Then, choose any subsequence $(\mu_{k_h})_{h\geq 1}$ and notice that

$$0 \leq \mu_{k_h}^-(K) \leq \int_\Omega \vartheta \, d\mu_{k_h}^- \qquad h \geq 1.$$

As $\int_\Omega \vartheta \, d\mu_{k_h}^- \to \langle S, \vartheta \rangle$ as $h \to \infty$, we see that $\left(\mu_{k_h}^-(K)\right)_{h\geq 1}$ is bounded for all compact sets $K \subset \Omega$. Thus, $\left(\mu_{k_h}^-\right)_{h\geq 1}$ has a convergent subsequence in $\mathcal{M}(\Omega)$ whose limit is $S$ itself. It follows that the whole sequence $\left(\mu_k^-\right)_{k\geq 1}$ converges to $S$ in $\mathcal{M}(\Omega)$. Therefore, $\mu_k \to \mu$ in the sense of Radon measure on $\Omega$. ■

As in Section 2, for $1 \leq h \leq n$, write $\sigma(h) = \sigma(n, n, h)$ for the number of all minors up to order $h$ of any $n \times n$ matrix. Then, as a consequence of Lemma 3.2 and Lemma 1.2, we get the following corollary.

COROLLARY 3.3. – *Let the functions* $w, w_k : \Omega \to \mathbb{R}^m$, $k \geq 1$, *satisfy*

*(a)* $w_k \in C^1(\Omega, \mathbb{R}^n)$, $k \geq 1$;

*(b)* $(w_k)_{k\geq 1}$ *is bounded in* $W^{1,n-1}(\Omega, \mathbb{R}^n)$;

*(c)* $(w_k)_{k\geq 1}$ *is bounded in* $L^\infty(\Omega, \mathbb{R}^n)$;

*(d)* $\sup_{k\geq 1} \int_\Omega (\det \nabla w_k)^+ d\mathcal{L}^n < \infty$.

*Then, there exists* $\mu \in \mathcal{M}(\Omega, \mathbb{R}^{\sigma(n)})$ *and a subsequence* $(w_{k_h})_{h\geq 1}$ *with the property that* $M_1^n(\nabla w_{k_h}) \to \mu$ *as* $h \to \infty$ *in the sense of Radon measures on* $\Omega$. *If, in addition,*

*(e)* $w_k \to w$ *in* $L^1(\Omega, \mathbb{R}^n)$,

*then* $w \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ *and* $\mu_a = M_1^n(\nabla w)$.

*Proof.* – First, notice that (b) implies that there exists a subsequence, still denoted by $(w_k)_{k\geq 1}$, such that $\left(M_1^{n-1}(\nabla w_k)\right)_{k\geq 1}$ converges in $\mathcal{M}(\Omega, \mathbb{R}^{\sigma(n-1)})$. Then, by (a), the equation

$$\det \nabla w_k = \sum_{1\leq j\leq n} D_j\Big(w_k^1(\operatorname{adj}\nabla w_k)_1^j\Big) \qquad k\geq 1$$

holds in the sense of distributions and (b) and (c) together yield that the sequence $\Big(w_k^1(\operatorname{adj}\nabla w_k)_1^j\Big)_{k\geq 1}$ is bounded in $L^1(\Omega)$ for $1\leq j\leq n$. Hence, by passing to a subsequence $(w_k)_{k\geq 1}$, we get that

$$w_k^1(\operatorname{adj}\nabla w_k)_1^j \to T^j \qquad \text{in } \mathcal{D}'(\Omega), \qquad 1\leq j\leq n$$

as $k\to\infty$, so that $\det\nabla w_k \to \sum_{1\leq j\leq n} D_j T^j$ in $\mathcal{D}'(\Omega)$. Moreover, (d) implies that, passing once more to a subsequence $(w_k)_{k\geq 1}$, the positive parts of the determinants of the gradients of the functions $w_k$ converge in $\mathcal{M}(\Omega)$. Thus, Lemma 3.2 yields that $(\det\nabla w_k)_{k\geq 1}$ is convergent in $\mathcal{M}(\Omega)$. Since we have already proved that $\left(M_1^{n-1}(\nabla w_k)\right)_{k\geq 1}$ converges in $\mathcal{M}(\Omega, \mathbb{R}^{\sigma(n-1)})$, it follows that there exist $\mu \in \mathcal{M}(\Omega, \mathbb{R}^{\sigma(n)})$ and a subsequence $(w_{k_h})_{h\geq 1}$ which satisfy the first part of the conclusions of the corollary. Finally, the second part follows immediately from (e) and Lemma 1.2. ■

Applying Corollary 3.3, we derive the lower semicontinuity of $G$ along bounded sequences of continuously differentiable functions on $\Omega$.

PROPOSITION 3.4. – *Let* $g\colon \mathbb{R} \to [0,\infty]$ *satisfy conditions (i), (ii), and either (iii+) or (iii−). Assume that the functions* $w \in BV(\Omega, \mathbb{R}^n)$ *and* $(w_k)_{k\geq 1}$ *satisfy*

*(a)* $w_k \in C^1(\Omega, \mathbb{R}^n)$, $k\geq 1$;

*(b)* $(w_k)_{k\geq 1}$ *is bounded in* $W^{1,n-1}(\Omega, \mathbb{R}^n)$;

*(c)* $(w_k)_{k\geq 1}$ *is bounded in* $L^\infty(\Omega, \mathbb{R}^n)$;

*(d)* $w_k \to w$ *in* $L^1(\Omega, \mathbb{R}^n)$.

*Then,* $G(w) \leq \liminf_{k\to\infty} G(w_k)$.

*Proof.* – We give the proof in the case (iii+). Assume that $G(w_k) \to c < \infty$ so that the positive parts of the determinants of the functions $w_k$ are bounded in $L^1(\Omega)$. Now, all hypotheses of Corollary 3.3 are fulfilled. Hence, there exist a subsequence, still denoted by $(w_k)_{k\geq 1}$, and a Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^{\sigma(n)})$, with $\mu_a = M_1^n(\nabla w)$, such that $M_1^n(\nabla w_k) \to \mu$

in the sense of Radon measures on $\Omega$. In particular, denoting by $\mu'$ the last component of $\mu$, we get that $\det \nabla w_k \rightarrow \mu'$ in $\mathcal{M}(\Omega)$ with $\mu'_a = \det \nabla w$. Therefore, applying Theorem 1.1 and recalling that $g^\infty$ is non-negative and vanishes at zero, we derive that $G(w) \leq \liminf_{k\rightarrow\infty} G(w_k)$ by the very same argument of Proposition 2.4. ■

Next, we remove the requirement of boundedness in $L^\infty(\Omega, \mathbb{R}^n)$ by a truncation argument. In order to do this, choose a non-negative, non-decreasing function $\psi \in C^1([0,\infty))$, with $0 \leq \psi' \leq 1$, such that $\psi(t)$ is equal to $t$ on the interval $[0,1]$ and is constant on the interval $[2,\infty)$. Then, let $\Psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ be defined by

$$\Psi(y) = \begin{cases} \psi(|y|)\frac{y}{|y|} & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The function $\Psi$ is Lipschitz continuous on $\mathbb{R}^n$ with $\mathrm{Lip}(\Psi) = \mathrm{Lip}(\psi) \leq 1$ and $\Psi$ is the identity map on the closed unit ball of $\mathbb{R}^n$. Moreover, $0 \leq \det \nabla\Psi \leq 1$ on $\mathbb{R}^n$ and $\det \nabla\Psi$ vanishes on the complement of the closed ball of radius 2 centered at zero. Finally, for all $\rho > 0$, set

$$\Psi^\rho(y) = \rho\Psi(\rho^{-1}y) \qquad y \in \mathbb{R}^n, \tag{3.2}$$

so that $\Psi^\rho \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\Psi^\rho(y) = y$ for all $|y| \leq \rho$ and $\nabla\Psi^\rho(y) = \nabla\Psi(\rho^{-1}y)$, $y \in \mathbb{R}^n$.

Then, we can prove the counterpart of Proposition 2.6 for the functional $G$.

PROPOSITION 3.5. – *Let $g: \mathbb{R} \rightarrow [0,\infty]$ satisfy conditions (i), (ii), and either (iii+) or (iii−). Assume that the functions $v \in BV(\Omega, \mathbb{R}^n)$ and $(v_k)_{k\geq 1}$ satisfy*

*(a) $v_k \in C^1(\Omega, \mathbb{R}^n)$, $k \geq 1$;*

*(b) $(v_k)_{k\geq 1}$ is bounded in $W^{1,n-1}(\Omega, \mathbb{R}^n)$;*

*(c) $v_k \rightarrow v$ in $L^1(\Omega, \mathbb{R}^n)$.*

*Then, $G(v) \leq \liminf_{k\rightarrow\infty} G(v_k)$.*

*Proof.* – Assume once more that (iii+) holds and that $G(v_k) \rightarrow c < \infty$. Then, choose $r > 1$ and set

$$w^r = \Psi^r \circ v,$$
$$w^r_k = \Psi^r \circ v_k, \qquad k \geq 1,$$

where $\Psi^r$ is defined by (3.2) with $\rho = r$. It is plain that $w^r$ and $(w^r_k)_{k\geq 1}$ satisfy all the hypotheses previously fulfilled by $v$ and $(v_k)_{k\geq 1}$, and in addition $(w_k^r)_{k\geq 1}$ is a bounded sequence in $L^\infty(\Omega, \mathbb{R}^n)$. Thus, $G(w^r) \leq \liminf_{k\to\infty} G(w_k^r)$ by Proposition 3.4. Then, recalling that $0 \leq \det \nabla\Psi^r \leq 1$ on $\mathbb{R}^n$, the convexity of $g$ yields

$$g(\det \nabla w_k^r) = g(\det \nabla\Psi^r(v_k) \det \nabla v_k) \leq g(\det \nabla v_k) + g(0)$$

pointwise on $\Omega$ for all $k \geq 1$ and hence

$$\begin{aligned} G(w_k^r) \leq & \int_{\{|v_k|\leq r\}} g(\det \nabla w_k)\, d\mathcal{L}^n \\ & + \int_{\{r<|v_k|\leq 2r\}} g(\det \nabla w_k^r)\, d\mathcal{L}^n + g(0)\mathcal{L}^n(\{|v_k| > 2r\}) \\ & \leq G(v_k) + g(0)\mathcal{L}^n(\{|v_k| > r\}) \end{aligned}$$

by the properties of $\Psi^r$. It follows that

$$\int_{\{|v|<r\}} g(\det \nabla v)\, d\mathcal{L}^n \leq G(w^r) \leq \liminf_{k\to\infty} G(v_k) + g(0)\mathcal{L}^n(\{|v| \geq r\}).$$

Finally, letting $r \to \infty$, the lower semicontinuity of $G$ along $(v_k)_{k\geq 1}$ follows. ■

We are now left to drop the smoothness assumption on the sequence. This is done in the following proposition.

PROPOSITION 3.6. – *Let $g: \mathbb{R} \to [0, \infty)$ satisfy conditions (i), (ii), either (iii+) or (iii−), and in addition:*

*(iv) there exists $c_2$ such that $g(t) \leq c_2(1 + |t|)$ for all $t \in \mathbb{R}$.*

*Assume that the functions $u \in BV(\Omega, \mathbb{R}^n)$ and $(u_k)_{k\geq 1}$ satisfy*

*(a) $u_k \in W^{1,n}(\Omega, \mathbb{R}^n)$, $k \geq 1$;*

*(b) $(u_k)_{k\geq 1}$ is bounded in $W^{1,n-1}(\Omega, \mathbb{R}^n)$;*

*(c) $u_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$.*

*Then, $G(u) \leq \liminf_{k\to\infty} G(u_k)$.*

*Proof.* – Just repeat the proof of Proposition 2.7. ■

Finally, we give the proof of Theorem 3.1.

*Proof of Theorem 3.1.* – Assume that $g$ is not constant. By the argument described in Remark 2.1, an increasing sequence of functions $g_h: \mathbb{R} \to [0, \infty)$, $h \geq 1$, can be found with the properties that $(g_h)_{h\geq 1}$ converges to $g$ pointwise on $\mathbb{R}$ and each $g_h$ satisfies (i), (ii), either (iii+) or (iii−), and (iv). The conclusion follows now from Proposition 3.6 and from the approximation argument used in the proof of Theorem 2.2. ■

## 4. POLYCONVEX FUNCTIONALS: THE CASE $m = n$

The arguments developed in Sections 2 and 3 team up yielding a lower semicontinuity result for polyconvex integral functionals on $BV(\Omega, \mathbb{R}^n)$ with respect to $L^1(\Omega, \mathbb{R}^n)$-convergence. We point out that in this case the polyconvex functionals may depend either on all or only on some of the minors. Indeed, we prove that every polyconvex integral functional with convex, proper, and lower semicontinuous integrand is lower semicontinuous on the space $W^{1,n}(\Omega, \mathbb{R}^n)$ along sequences converging in $L^1(\Omega, \mathbb{R}^n)$ and bounded in $W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq n-1$. Notice that the result presented in [5] is contained in the previous statement and, as previously remarked in Section 3, the counterexample in [12] shows that this result is sharp.

To begin with, as $n$ is kept fixed throughout this section, shortly write $\sigma(h) = \sigma(n, n, h)$ for $1 \leq h \leq n$. Then, denote by $F: BV(\Omega, \mathbb{R}^n) \to [0, \infty]$ the polyconvex integral functional defined by

$$F(u) = \int_\Omega f(M_1^n(\nabla u))\, d\mathcal{L}^n \qquad u \in BV(\Omega, \mathbb{R}^n), \tag{4.1}$$

where $f: \mathbb{R}^{\sigma(n)} \to [0, \infty]$ satisfies the following properties:

(i) $f$ is a convex function such that $f(0) < \infty$;

(ii) $f$ is lower semicontinuous on $\mathbb{R}^{\sigma(n)}$.

In the sequel, we identify $\mathbb{R}^{\sigma(n)}$ with $\mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$, so that we regard $f$ either as a function of $\xi \in \mathbb{R}^{\sigma(n)}$ or as a function of $(\zeta, \eta) \in \mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$. In particular, we freely write either $f(M_1^n(A))$ or $f\big(M_1^{n-1}(A), \det A\big)$ for any matrix $A \in \mathbb{M}^{n \times n}$.

We shall sometimes assume also that

(iii) there exists $c_1 > 0$ such that $f(\zeta, \eta) \geq c_1 |\zeta|$ for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$.

We shall prove the following lower semicontinuity result.

THEOREM 4.1. – *Let $f: \mathbb{R}^{\sigma(n)} \to [0, \infty]$ satisfy conditions (i) and (ii). Assume that the functions $u \in BV(\Omega, \mathbb{R}^n)$ and $(u_k)_{k \geq 1}$ satisfy*

*(a) $u_k \in W^{1,n}(\Omega, \mathbb{R}^n)$, $k \geq 1$;*

*(b) $(u_k)_{k \geq 1}$ is bounded in $W^{1,n-1}(\Omega, \mathbb{R}^n)$;*

*(c) $u_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$.*

*Then, $F(u) \leq \liminf_{k \to \infty} F(u_k)$.*

*Remark 4.2.* – Theorem 4.1 still remains true if we replace hypothesis (b) with the weaker assumption of boundedness in $L^1(\Omega)$ of the sequence $\big(|M_1^{n-1}(\nabla u_k)|\big)_{k \geq 1}$.

Moreover, as soon as $f$ satisfies the growth condition (iii), the lower semicontinuity of $F$ can be established without any boundedness assumption on the sequence $(u_k)_{k\geq 1}$.

COROLLARY 4.3. – *Let $f:\mathbb{R}^{\sigma(n)} \to [0,\infty]$ satisfy conditions (i), (ii), (iii). Assume that the functions $u \in BV(\Omega,\mathbb{R}^n)$ and $(u_k)_{k\geq 1}$ satisfy*

*(a) $u_k \in W^{1,n}(\Omega,\mathbb{R}^n)$, $k \geq 1$;*

*(b) $u_k \to u$ in $L^1(\Omega,\mathbb{R}^n)$.*

*Then, $F(u) \leq \liminf_{k\to\infty} F(u_k)$.*

*Proof.* – Assume as usual that $F(u_k) \to c < \infty$ and notice that (iii) implies that the sequence $\left(|M_1^{n-1}(\nabla u_k)|\right)_{k\geq 1}$ is bounded in $L^1(\Omega)$. Therefore, Theorem 4.1 and Remark 4.2 can be applied. ■

Notice that Corollary 4.3 generalizes Theorem 2.5 of [1], at least in the case of an integrand independent of $x$ and $u$.

The proof of Theorem 4.1 is based on the following lemma concerning convex functions.

LEMMA 4.4. – *Let $f:\mathbb{R}^k\times\mathbb{R} \to [0,\infty]$ be a proper, convex, and lower semicontinuous function. Then, the following statements hold true:*

*(a) if $f^\infty(0,1) = f^\infty(0,-1) = 0$, then there exists a convex, proper, and lower semicontinuous function $f_0:\mathbb{R}^k \to [0,\infty]$ such that $f(\zeta,\eta) = f_0(\zeta)$ for all $(\zeta,\eta) \in \mathbb{R}^k\times\mathbb{R}$;*

*(b) if $f^\infty(0,1) > 0$, then for all $\alpha < f^\infty(0,1)$ there exist $a, b > 0$ such that*

$$f(\zeta,\eta) + a|\zeta| + b \geq \alpha\eta^+ \qquad (\zeta,\eta) \in \mathbb{R}^k\times\mathbb{R};$$

*(c) if $f^\infty(0,-1) > 0$, then for all $\alpha < f^\infty(0,-1)$ there exist $a, b > 0$ such that*

$$f(\zeta,\eta) + a|\zeta| + b \geq \alpha\eta^- \qquad (\zeta,\eta) \in \mathbb{R}^k\times\mathbb{R};$$

*(d) if $f^\infty(0,1) > 0$ and $f^\infty(0,-1) > 0$, then for all $\alpha < f^\infty(0,1) \wedge f^\infty(0,-1)$ there exist $a, b > 0$ such that*

$$f(\zeta,\eta) + a|\zeta| + b \geq \alpha|\eta| \qquad (\zeta,\eta) \in \mathbb{R}^k\times\mathbb{R}.$$

*Proof.* – ($a$) Suppose that $f^\infty(0,1) = f^\infty(0,-1) = 0$. Thus, $f^\infty(0,\eta) = 0$ for all $\eta \in \mathbb{R}$ by the positive 1-homogeneity of $f^\infty$. Then, recalling that

$$0 = f^\infty(0,\eta) = \sup\{f(\zeta',\eta'+\eta) - f(\zeta',\eta') : (\zeta',\eta') \in \operatorname{dom}(f)\},$$

we see that $f(\zeta', \eta' + \eta) \leq f(\zeta', \eta')$ for all $\eta \in \mathbb{R}$ and $(\zeta', \eta') \in \operatorname{dom}(f)$. This shows that $\operatorname{dom}(f)$ contains the straight line $(\zeta', \eta' + \eta)$, $\eta \in \mathbb{R}$, as soon as $(\zeta', \eta') \in \operatorname{dom}(f)$. Moreover, letting $\eta = -\eta'$, we get $f(\zeta', 0) \leq f(\zeta', \eta')$ for all $(\zeta', \eta') \in \operatorname{dom}(f)$. Replacing $\eta'$ with 0 and $\eta$ with $\eta'$ we get the reverse inequality. Thus, $f$ is actually independent of its last variable.

(*b*) Choose $\alpha < f^{\infty}(0, 1)$. It is enough to consider the case $\alpha > 0$, the other cases being trivial. Recalling (1.4) and the definition of Young-Fenchel conjugate function, we see that there exists $(z_0, t_0) \in \operatorname{dom}(f^*)$ such that $t_0 > \alpha$. Choose $b > f^*(z_0, t_0)$ and let $a > \alpha t_0^{-1}|z_0|$. Now, set $g(\zeta, \eta) = a|\zeta| + b$, $h(\zeta, \eta) = \alpha\eta^+$ for all $(\zeta, \eta) \in \mathbb{R}^k \times \mathbb{R}$, and notice that we are left to prove that $(f + g)^* \leq h^*$. In order to do this, recall that the conjugate functions of $g$ and $h$ are given by

$$g^* = \chi_{B(0,a)\times\{0\}} - b, \qquad h^* = \chi_{[(0,0)\,,\,(0,\alpha)]},$$

where $B(0, a)$ stands for the closed ball of $\mathbb{R}^k$ with radius $a$ centered at zero and where $[(\zeta_1, \eta_1)\,,\,(\zeta_2, \eta_2)]$ denotes the closed segment in $\mathbb{R}^k \times \mathbb{R}$ whose extreme points are $(\zeta_1, \eta_1)$ and $(\zeta_2, \eta_2)$. Moreover, as $f$ and $g$ are convex, proper, and lower semicontinuous, and as $\operatorname{dom}(g) = \mathbb{R}^k \times \mathbb{R}$, it follows that

$$(f + g)^*(z, t) = \inf_{(z', t')} (f^*(z - z'\,,\, t - t') + g^*(z'\,,\, t'))$$

(see, for instance, [14], Theorem 16.4).

Now, the definition of $a$ and $b$ yields for all $0 \leq t \leq \alpha$:

$$\begin{aligned}\inf_{|z|\leq a} f^*(z, t) &\leq f^*((t/t_0)z_0, t)\\ &\leq (1 - (t/t_0))f^*(0, 0) + (t/t_0)f^*(z_0, t_0) \leq f^*(z_0, t_0) < b,\end{aligned}$$

and hence

$$\begin{aligned}(f + g)^*(0, t) &= \inf_{(z', t')} \big(f^*(-z', t - t') + \chi_{B(0,a)\times\{0\}}(z', t') - b\big)\\ &= \inf_{|z'|\leq a} (f^*(z', t) - b) \leq 0\end{aligned}$$

for all $0 \leq t \leq \alpha$. Therefore, $(f + g)^* \leq h^*$ on $\mathbb{R}^k \times \mathbb{R}$ and the proof of (b) is complete.

(*c*) It is enough to repeat the proof of (b) with obvious modifications.

(*d*) Property (d) follows immediately from (b) and (c). ■

Now, the proof of Theorem 4.1 can be carried out through the same steps described in Section 2. Therefore, we begin by proving the lower semicontinuity of $F$ along bounded sequences of continuously differentiable functions.

PROPOSITION 4.5. – *Let* $f:\mathbb{R}^{\sigma(n)} \to [0,\infty]$ *satisfy conditions (i) and (ii). Assume that the functions* $w \in BV(\Omega,\mathbb{R}^n)$ *and* $(w_k)_{k\geq 1}$ *satisfy*

*(a)* $w_k \in C^1(\Omega,\mathbb{R}^n)$, $k \geq 1$;

*(b)* $(w_k)_{k\geq 1}$ *is bounded in* $W^{1,n-1}(\Omega,\mathbb{R}^n)$;

*(c)* $(w_k)_{k\geq 1}$ *is bounded in* $L^\infty(\Omega,\mathbb{R}^n)$;

*(d)* $w_k \to w$ *in* $L^1(\Omega,\mathbb{R}^n)$.

*Then,* $F(w) \leq \liminf_{k\to\infty} F(w_k)$.

*Proof.* – As usual, we may and do assume that $F(w_k) \to c < \infty$. Then, notice that either $f^\infty(0,1) = f^\infty(0,-1) = 0$ or at least one of the values $f^\infty(0,1)$ and $f^\infty(0,-1)$ is positive. In the former case, $f$ is actually independent of its last variable by Lemma 4.4 and the statement reduces to a particular case of Corollary 2.3. In the latter case, choose $0 < \alpha < f^\infty(0,1) \vee f^\infty(0,-1)$ and let $a, b > 0$ be so chosen as to satisfy either (b) or (c) of Lemma 4.4. Therefore, as $(F(w_k))_{k\geq 1}$ is bounded and $(w_k)_{k\geq 1}$ is bounded in $W^{1,n-1}(\Omega,\mathbb{R}^m)$, it follows that either the positive or the negative parts of the determinants of the gradients of the functions $w_k$ are bounded in $L^1(\Omega)$. Hence, Corollary 3.3 yields $\mu \in \mathcal{M}(\Omega,\mathbb{R}^{\sigma(n)})$ such that $\mu_a = M_1^n(\nabla w)$ and a subsequence, still denoted by $(w_k)_{k\geq 1}$, such that $M_1^n(\nabla w_k) \to \mu$ in the sense of Radon measures on $\Omega$. Now, the conclusion follows from Theorem 1.1 and from the argument of Proposition 2.4. ■

Next, we remove the requirement of boundedness in $L^\infty(\Omega,\mathbb{R}^n)$ by the combined action of an approximation and a truncation argument. Indeed, unless $f$ is independent of its last variable, $f$ itself satisfies either (b), (c), or possibly (d) of Lemma 4.4, for some positive constants $a$, $b$, and $\alpha$. Then, $f$ can be approximated by an increasing sequence of convex functions $(f_h)_{h\geq 1}$ satisfying one of the following growth conditions:

$$0 \leq f_h(\zeta,\eta) \leq c_h(1+|\zeta|+\eta^+) \qquad (\zeta,\eta)\in\mathbb{R}^{\sigma(n-1)}\times\mathbb{R}, \quad h\geq 1, \tag{4.2}$$

$$0 \leq f_h(\zeta,\eta) \leq c_h(1+|\zeta|+\eta^-) \qquad (\zeta,\eta)\in\mathbb{R}^{\sigma(n-1)}\times\mathbb{R}, \quad h\geq 1, \tag{4.3}$$

$$0 \leq f_h(\zeta,\eta) \leq c_h(1+|\zeta|+|\eta|) \qquad (\zeta,\eta)\in\mathbb{R}^{\sigma(n-1)}\times\mathbb{R}, \quad h\geq 1, \tag{4.4}$$

for some positive constant $c_h$, according to the validity of either (b), (c), or (d) of Lemma 4.4. Hence, it will be enough to prove the lower semicontinuity of the functionals $F_h$ defined by (4.1) with $f$ replaced by $f_h$. This will be accomplished by truncating the functions of the sequence $(v_k)_{k\geq 1}$, along which the lower semicontinuity of $F_h$ is investigated, by means of orientation preserving mappings (as in Section 3), which, in addition, map a suitably chosen unbounded set onto a finite subset of $\mathbb{R}^n$ (as in Section 2).

The approximation argument and the truncation mappings mentioned above are described in detail in the following Lemma 4.6 and Remark 4.7.

LEMMA 4.6. – *Let* $f: \mathbb{R}^{\sigma(n-1)} \times \mathbb{R} \to [0, \infty]$ *satisfy conditions (i), (ii), and let either* $f^\infty(0,1)$ *or* $f^\infty(0,-1)$ *be positive. Then, there exists an increasing sequence of convex functions* $f_h: \mathbb{R}^{\sigma(n-1)} \times \mathbb{R} \to [0, \infty)$, $h \geq 1$, *converging to* $f$ *pointwise on* $\mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$ *as* $h \to \infty$ *and satisfying the following properties:*

*(a) if* $f^\infty(0,1) > 0$ *and* $f^\infty(0,-1) = 0$, *then* $f_h^\infty(0,1) > 0$ *and* $f_h^\infty(0,-1) = 0$ *for* $h$ *large enough, and there exist* $c_h > 0$ *such that (4.2) holds for all* $h \geq 1$;

*(b) if* $f^\infty(0,1) = 0$ *and* $f^\infty(0,-1) > 0$, *then* $f_h^\infty(0,1) = 0$ *and* $f_h^\infty(0,-1) > 0$ *for* $h$ *large enough, and there exist* $c_h > 0$ *such that (4.3) holds for all* $h \geq 1$;

*(c) if* $f^\infty(0,1) > 0$ *and* $f^\infty(0,-1) > 0$, *then* $f_h^\infty(0,1) > 0$ *and* $f_h^\infty(0,-1) > 0$ *for* $h$ *large enough, and there exist* $c_h > 0$ *such that (4.4) holds for all* $h \geq 1$.

*Proof.* – Assume that $f^\infty(0,1) > 0$, $f^\infty(0,-1) = 0$ and set

$$f_h(\zeta, \eta) = \sup\{(\zeta, z) + \eta t - f^*(z,t) : |z| \leq h, \ 0 \leq t \leq h\}$$

for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$ and all $h \geq 1$. Then, $(f_h)_{h \geq 1}$ turns out to be an increasing sequence of convex and lower semicontinuous functions such that $f_h \leq f$ for all $h \geq 1$. Each function $f_h$ is non-negative, since $f^*(0,0) \leq 0$ yields $f_h(\zeta, \eta) \geq -f^*(0,0) \geq 0$ for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$ and $h \geq 1$. Moreover, as $f_h(0,0) \geq -f^*(z,t)$ for all $|z| \leq h$, $0 \leq t \leq h$, it follows that

$$(\zeta, z) + \eta t - f^*(z,t) \leq f_h(0,0) + h|\zeta| + h\eta^+ \qquad |z| \leq h, \quad 0 \leq t \leq h.$$

Hence, $f_h$ satisfies (4.2) for some positive constant $c_h$. Finally, in order to see that $f_h \nearrow f$ pointwise on $\mathbb{R}^{\sigma(n)}$, notice that (1.4) implies that

$$0 = f^\infty(0,-1) = \sup_{(z,t)} \big(-t - \chi_{\operatorname{dom}(f^*)}(z,t)\big),$$

which means that $\operatorname{dom}(f^*) \subset \{(z,t) : t \geq 0\}$. Therefore, the value of each $f_h$ at $(\zeta, \eta) \in \mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$ can be also computed by

$$f_h(\zeta, \eta) = \sup\{(\zeta, z) + \eta t - f^*(z,t) : |z| \leq h, \ |t| \leq h\} \qquad h \geq 1$$

and this yields $f_h \nearrow f$ pointwise on $\mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$ as $h \to \infty$. On account of this result and the definition of recession function (1.3), it is easy to check that also $f_h^\infty \nearrow f^\infty$ pointwise on $\mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$ as $h \to \infty$. Thus, $f_h^\infty(0,-1) = 0$ for all $h \geq 1$ and $f_h^\infty(0,1) > 0$ for $h$ large enough.

Finally, the case $f^\infty(0,1) = 0$, $f^\infty(0,-1) > 0$ can be treated similarly, while in the case $f^\infty(0,1) > 0$ and $f^\infty(0,-1) > 0$ we can argue as in Remark 2.1. ■

*Remark 4.7.* – Let us define for any positive $r > 0$ the following subsets of $\mathbb{R}^n$

$$
\begin{aligned}
Q_r &= \{y \in \mathbb{R}^n \colon |y^i| \leq r \quad i = 1, \ldots, n\}, \\
C_r &= \{y \in \mathbb{R}^n \colon y^1 < r\} \setminus Q_{2r}, \\
D_r &= \{y \in \mathbb{R}^n \colon y^1 \geq 2r\}.
\end{aligned}
$$

We aim at constructing a bounded, Lipschitz continuous mapping $\Psi \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ with the following properties:

$$\Psi(y) = y \qquad \text{if } y \in Q_1; \tag{4.5}$$

$$\det \nabla\Psi(y) \geq 0 \qquad \text{for all } y \in \mathbb{R}^n; \tag{4.6}$$

$$\nabla\Psi(y) = 0 \qquad \text{if } y \in C_1 \cup D_1. \tag{4.7}$$

We are going to define $\Psi$ as the composition of three mappings $\Psi_i \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $i = 1, 2, 3$.

(a) *Construction of* $\Psi_1$. Choose a non-decreasing function $\psi_1 \in \mathcal{C}^1(\mathbb{R})$ such that $\psi_1(t) = t$ if $|t| \leq 1$, $\psi_1(t) = 2$ if $t \geq 2$ and $\psi_1(t) = -2$ if $t \leq -2$. Then, let $\Psi_1 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ be the bounded, Lipschitz continuous mapping defined by

$$\Psi_1(y) = \big(\psi_1(y^1), \ldots, \psi_1(y^n)\big) \qquad y \in \mathbb{R}^n.$$

It is plain that $\Psi_1$ satisfies (4.5), (4.6), and that $\Psi_1$ maps $\mathbb{R}^n$ onto $Q_2$. Moreover, the sets $C_1$ and $D_1$ are mapped by $\Psi_1$ onto $\{y \in \partial Q_2 \colon y^1 < 1\}$ and $\{y \in \partial Q_2 \colon y^1 = 2\}$ respectively.

(b) *Construction of* $\Psi_2$. Choose a Lipschitz continuous function $\psi_2 \in \mathcal{C}^1(\mathbb{R}^n)$ such that $\psi_2(y) = y^1$ if $y \in Q_1$, $\psi_2(y) = -2$ if $y \in \partial Q_2$ with $y^1 \leq 1$, $\psi_2(y) = 2$ if $y^1 = 2$, and such that the partial derivative of $\psi_2$ with respect to $y^1$ is non-negative on $\mathbb{R}^n$. Then, set

$$\Psi_2(y) = \big(\psi_2(y), y^2, \ldots, y^n\big) \qquad y \in \mathbb{R}^n,$$

so that $\Psi_2 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ is Lipschitz continuous and satisfies (4.5) and (4.6). Moreover, the sets $Q_2$ and $\{y \in \partial Q_2 \colon y^1 = 2\}$ are invariant with respect to $\Psi_2$ while $\{y \in \partial Q_2 \colon y^1 \leq 1\}$ is mapped by $\Psi_2$ onto $\{y \in \partial Q_2 \colon y^1 = -2\}$.

(c) *Construction of* $\Psi_3$. Choose a function $\psi_3 \in \mathcal{C}^1(\mathbb{R})$ with $0 \leq \psi_3 \leq 1$, which is equal to 1 when $|t| \leq 1$ and vanishes when $|t| \geq 2$. Then, let $\Psi_3 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ be defined by

$$\Psi_3(y) = \left(y^1, \psi_3(y^1)y^2, \ldots, \psi_3(y^1)y^n\right) \qquad y \in \mathbb{R}^n.$$

It is plain that $\Psi_3$ satisfies (4.5) and a routine calculation shows that (4.6) too holds true. Finally, we notice that $\Psi_3$ fails to be Lipschitz continuous on $\mathbb{R}^n$. Nevertheless, we have

$$\sup_{y \in Q_2} |\nabla \Psi_3(y)| < \infty. \tag{4.8}$$

Now, let $\Psi \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ be defined by $\Psi = \Psi_3 \circ \Psi_2 \circ \Psi_1$. On account of $\Psi_2 \circ \Psi_1(\mathbb{R}^n) = Q_2$ and (4.8), it follows that $\Psi$ is a bounded, Lipschitz continuous mapping satisfying (4.5) and (4.6). As far as (4.7) is concerned, recall that $C_1$ and $D_1$ are mapped by $\Psi_2 \circ \Psi_1$ onto $\{y \in \partial Q_2 \colon y^1 = -2\}$ and $\{y \in \partial Q_2\colon y^1 = 2\}$ respectively and that these sets in turn are mapped by $\Psi_3$ onto the constant vectors $-2e_1$ and $2e_1$, where $(e_1, \ldots, e_n)$ denotes the standard basis of $\mathbb{R}^n$. Hence, $\Psi$ satisfies (4.7).

Finally, for all positive $\rho$, let $\Psi^\rho \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ be the bounded, Lipschitz continuous mapping defined by

$$\Psi^\rho(y) = \rho\Psi(\rho^{-1}y) \qquad y \in \mathbb{R}^n. \tag{4.9}$$

The following properties of each $\Psi^\rho$ follows immediately from the corresponding properties of $\Psi$:

$$\Psi^\rho(y) = y \qquad \text{if } y \in Q_\rho; \tag{4.10}$$

$$\det \nabla \Psi^\rho(y) \geq 0 \qquad \text{if } y \in \mathbb{R}^n; \tag{4.11}$$

$$\nabla \Psi^\rho(y) = 0 \qquad \text{if } y \in C_\rho \cup D_\rho. \tag{4.12}$$

Moreover, let $L \geq 1$ be such that

$$\sup_{y \in \mathbb{R}^n} |\nabla \Psi^\rho(y)| = \sup_{y \in \mathbb{R}^n} |\nabla \Psi(y)| = L < \infty \qquad \rho > 0. \tag{4.13}$$

Now, we prove the following proposition.

PROPOSITION 4.8. – *Let* $f \colon \mathbb{R}^{\sigma(n)} \to [0, \infty]$ *satisfy conditions (i) and (ii). Assume that the functions* $v \in BV(\Omega, \mathbb{R}^n)$ *and* $(v_k)_{k \geq 1}$ *satisfy* *(a)* $v_k \in C^1(\Omega, \mathbb{R}^n)$, $k \geq 1$;

*(b)* $(v_k)_{k\geq 1}$ *is bounded in* $W^{1,n-1}(\Omega, \mathbb{R}^n)$;

*(c)* $v_k \to v$ *in* $L^1(\Omega, \mathbb{R}^n)$.

*Then,* $F(v) \leq \liminf_{k\to\infty} F(v_k)$.

*Proof.* – To begin with, notice that we may and do assume that at least one of the values $f^\infty(0,1)$ and $f^\infty(0,-1)$ is positive. Otherwise, $f$ is independent of its last variable by Lemma 4.4 and the conclusion follows by Corollary 2.3. Then, let $\Phi: \mathbb{R} \to [0, \infty)$ be the function defined according to the values of $f^\infty$ at the points $(0,1)$ and $(0,-1)$ by

$$\Phi(t) = \begin{cases} t^+ & \text{if } f^\infty(0,1) > 0 \text{ and } f^\infty(0,-1) = 0 \\ t^- & \text{if } f^\infty(0,1) = 0 \text{ and } f^\infty(0,-1) > 0 \\ |t| & \text{if } f^\infty(0,1) > 0 \text{ and } f^\infty(0,-1) > 0 \end{cases}$$

and notice that we may assume also that $f$ satisfies the following growth assumption:

(iv) there exists $c_2 < \infty$ such that $0 \leq f(\zeta, \eta) \leq c_2(1 + |\zeta| + \Phi(\eta))$ for all $(\zeta, \eta) \in \mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$.

If not, let $(f_h)_{h\geq 1}$ be the increasing sequence of convex functions associated with $f$ by Lemma 4.6, and let $F_h$ be the functional defined on $BV(\Omega, \mathbb{R}^n)$ by (4.1) with $f$ replaced by $f_h$. Each $f_h$ satisfies (iv) for some positive constant $c_h$. Once the lower semicontinuity of each functional $F_h$ along $(v_k)_{k\geq 1}$ has been proved, we get

$$F_h(v) \leq \liminf_{k\to\infty} F_h(v_k) \leq \liminf_{k\to\infty} F(v_k) \qquad h \geq 1.$$

Letting $h \to \infty$, the lower semicontinuity of $F$ along $(v_k)_{k\geq 1}$ follows by the monotone convergence theorem.

Therefore, assume from now on that (iv) holds true and that $F(v_k) \to c < \infty$. Then, choose positive numbers $a$, $b$ and $\alpha$ according to Lemma 4.4 in such a way that $f(\zeta, \eta) + a|\zeta| + b \geq \alpha\Phi(\eta)$ holds on $\mathbb{R}^{\sigma(n-1)} \times \mathbb{R}$. It follows from (b) and the boundedness of $(F(v_k))_{k\geq 1}$ that $(\Phi \circ \det \nabla v_k)_{k\geq 1}$ is bounded in $L^1(\Omega)$. Hence, let $c_0 < \infty$ be such that

$$\int_\Omega \{|M_1^{n-1}(\nabla v_k)| + \Phi(\det \nabla v_k)\}\, d\mathcal{L}^n \leq c_0 \qquad k \geq 1.$$

Next, let $\epsilon > 0$ be given and choose $s \in \mathbb{N}_+$ such that $2c_0c_2L^n < \epsilon s$, where $L$ is the constant associated with $\Psi$ by (4.13). Then, pick $r_0 > 0$ and set $r_i = 2^i r_0$, $1 \leq i \leq s$. Let $(E_{r_i})_{1\leq i\leq s}$ be the sets defined by

$$E_{r_i} = \mathbb{R}^n \setminus (C_{r_i} \cup D_{r_i} \cup Q_{r_i}) \qquad 1 \leq i \leq s,$$

where the sets $C_{r_i}$, $D_{r_i}$, and $Q_{r_i}$ are those defined in Remark 4.7 with $r = r_i$. Write each $E_{r_i}$ as a disjoint union of

$$A_{r_i} = \{y \in E_{r_i}\colon\ r_i \le y < y2r_i\},$$
$$B_{r_i} = E_{r_i} \setminus A_{r_i},$$

and notice that both families $(A_{r_i})_{1\le i\le s}$ and $(B_{r_i})_{1\le i\le s}$ consist of pairwise disjoint sets. Hence, each point of $\mathbb{R}^n$ is contained in at most two sets $E_{r_i}$, $1 \le i \le s$, and this yields for each $k \ge 1$

$$\begin{aligned} 2c_0 &\ge 2\int_\Omega \{|M_1^{n-1}(\nabla v_k)| + \Phi(\det \nabla v_k)\}\, d\mathcal{L}^n \\ &\ge \sum_{1\le i\le s} \int_{\{v_k\in E_{r_i}\}} \{|M_1^{n-1}(\nabla v_k)| + \Phi(\det \nabla v_k)\}\, d\mathcal{L}^n. \end{aligned}$$

Now, the argument described in Proposition 2.6 yields an index $i_0 \in \{1, \dots, s\}$ and a subsequence $(v_k)_{k\ge1}$ such that

$$\int_{\{v_k\in E_{r_{i_0}}\}} \{|M_1^{n-1}(\nabla v_k)| + \Phi(\det \nabla v_k)\}\, d\mathcal{L}^n \le \frac{2c_0}{s} \qquad k \ge 1. \tag{4.14}$$

Set $r = r_{i_0}$ and define

$$w^r = \Psi^r \circ v,$$
$$w_k^r = \Psi^r \circ v_k, \qquad k \ge 1,$$

where $\Psi^r$ is the mapping defined by (4.9) with $\rho = r$. Now, recalling the properties of smoothness and Lipschitz continuity of $\Psi^r$, it is easy to check that $w_r \in BV(\Omega, \mathbb{R}^n)$, that $w_k^r \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$, $k \ge 1$, and that $(w_k^r)_{k\ge1}$ is a bounded sequence in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ converging to $w_r$ in $L^1(\Omega, \mathbb{R}^n)$. Moreover, $(w_k^r)_{k\ge1}$ is now bounded in $L^\infty(\Omega, \mathbb{R}^n)$. Indeed, the range of each $w_k^r$ is contained in $Q_{2r}$.

Next, we evaluate $F$ on each $w_k^r$. Recalling (iv), the properties of $\Psi^r$, and the positive homogeneity of $\Phi$, we get for all $k \ge 1$

$$\begin{aligned} F(w_k^r) &\le \int_{\{v_k\in Q_r\}} f\big(M_1^{n-1}(\nabla v_k), \det \nabla v_k\big)\, d\mathcal{L}^n \\ &+ c_2 \int_{\{v_k\in E_r\}} \{|M_1^{n-1}(\nabla\Psi^r(v_k)\nabla v_k)| + \det \nabla\Psi^r(v_k)\Phi(\det \nabla v_k)\}\, d\mathcal{L}^n \\ &+ c_2\mathcal{L}^n(\{v_k \notin Q_r\}). \end{aligned}$$

Now, recalling (1.2), we get

$$|M_1^{n-1}(\nabla\Psi^r(v_k)\nabla v_k)| \le L^{n-1}M_1^{n-1}(\nabla v_k),$$
$$0 \le \det \nabla\Psi^r(v_k) \le L^n$$

pointwise on $\Omega$ for all $k \ge 1$, and hence

$$F(w_k^r) \le F(v_k) + c_2 L^n \int_{\{v_k\in E_r\}} \left\{|M_1^{n-1}(\nabla v_k)| + \Phi(\det\nabla v_k)\right\} d\mathcal{L}^n$$
$$+ c_2\mathcal{L}^n(\{v_k \notin Q_r\})$$

for all $k \ge 1$. Therefore, the previous estimate together with (4.14) and the choice of $s \in \mathbb{N}_+$ yields

$$F(w_k^r) \le F(v_k) + \epsilon + c_2\mathcal{L}^n(\{v_k \notin Q_r\}) \qquad k \ge 1. \tag{4.15}$$

Now, Proposition 4.5 can be applied to the sequence $(w_k^r)_{k\ge 1}$. It follows that $F(w^r) \le \liminf_{k\to\infty} F(w_k^r)$. Letting $k \to \infty$ in (4.15), we get

$$F(w^r) \le \liminf_{k\to\infty} F(w_k^r) \le \liminf_{k\to\infty} F(v_k) + \epsilon + \mathcal{L}^n(\{v \notin Q_r\})$$

as $v_k \to v$ in $\mathcal{L}^n$-measure. We have thus proved that

$$\int_{\{v\in Q_{r_0}\}} f\big(M_1^{n-1}(\nabla v), \det\nabla v\big)\, d\mathcal{L}^n$$
$$\le \liminf_{k\to\infty} F(v_k) + \epsilon + \mathcal{L}^n(\{v \notin Q_r\})$$

with $r > 0$ arbitrarily chosen. As $v \in L^1(\Omega, \mathbb{R}^n)$, the monotone convergence theorem yields $F(v) \le \liminf_{k\to\infty} F(v_k) + \epsilon$ as $r \to \infty$ monotonically. As $\epsilon > 0$ was arbitrary, the proof is complete. ■

Finally, we drop the smoothness assumption on the sequence in the case $f$ satisfies (iv) too.

PROPOSITION 4.9. – *Let $f: \mathbb{R}^{\sigma(n)} \to [0, \infty]$ satisfy conditions (i), (ii), and (iv). Assume that the functions $u \in BV(\Omega, \mathbb{R}^n)$ and $(u_k)_{k\ge 1}$ satisfy*

*(a) $u_k \in W^{1,n}(\Omega, \mathbb{R}^n)$, $k \ge 1$;*

*(b) $(u_k)_{k\ge 1}$ is bounded in $W^{1,n-1}(\Omega, \mathbb{R}^n)$;*

*(c) $u_k \to u$ in $L^1(\Omega, \mathbb{R}^n)$.*

*Then, $F(u) \le \liminf_{k\to\infty} F(u_k)$.*

*Proof.* – Notice that (iv) in particular implies that $0 \leq f(\xi) \leq c_2'(1+|\xi|)$, $\xi \in \mathbb{R}^{\sigma(n)}$ for some positive constant $c_2'$. Therefore, the argument of Proposition 2.7 can be applied once more. ■

Now, the proof of our main result is straightforward.

*Proof of Theorem 4.1.* – Approximate $f$ by the increasing sequence $(f_h)_{h\geq 1}$ associated with $f$ by Lemma 4.6. Each functional $F_h$ defined on $BV(\Omega, \mathbb{R}^m)$ by (4.1), with $f$ replaced by $f_h$, is lower semicontinuous along $(u_k)_{k\geq 1}$ by Proposition 4.9. The conclusion follows as in the proof of Theorem 2.2. ■

## REFERENCES


[1] E. Acerbi and G. Dal Maso, New lower semicontinuity results for polyconvex integrals, *Calc. Var.* (to appear).

[2] J. M. Ball and F. Murat, $W^{1,p}$-quasiconvexity and variational problems for multiple integrals, *J. Funct. Anal.*, Vol. **58**, 1984, pp. 225-253; Vol. **66**, 1986, pp. 439.

[3] G. Buttazzo, Semicontinuity, relaxation and integral representation problems in the calculus of variations, *Pitman Res. Notes in Math.*, Vol. **207**, Longman, Harlow, 1989.

[4] B. Dacorogna, *Direct methods in the calculus of variation*, Springer-Verlag, Berlin, 1989.

[5] B. Dacorogna and P. Marcellini, Semicontinuité pour des intégrands polyconvexes sans continuité des déterminants, *C. R. Acad. Sci. Paris*, Sér. I, Vol. **311**, 1990, pp. 393-396.

[6] G. Dal Maso and C. Sbordone, Weak lower semicontinuity of policonvex integrals: a borderline case, *Math. Z.* (to appear).

[7] H. Federer, Geometric measure theory, Springer-Verlag, Berlin, 1969.

[8] M. Giaquinta, G. Modica and J. Souček, Cartesian currents, weak diffeomorphisms and nonlinear elasticity, *Arch. Rational Mech. Anal.*, Vol. **106**, 1989, pp. 97-159; Vol. **109**, 1990, pp. 385-392.

[9] M. Giaquinta, G. Modica and J. Souček, The Dirichlet integral for mappings between manifolds: cartesian currents and homology, *Math. Ann*, Vol. **294**, 1992, pp. 325-386.

[10] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston, 1984.

[11] C. Goffman and J. Serrin, Sublinear functions of measures and variational integrals, *Duke Math. J.*, Vol. **31**, 1964, pp. 159-178.

[12] J. Maly, Weak lower semicontinuity of polyconvex integrals, *Proc. Roy. Soc.*, Edinburgh, Vol. **123A**, 1993, pp. 681-691.

[13] Yu. G. Reshetnyak, Weak convergence of completely additive vector functions on a set, *Siberian Math. J.*, Vol. **101**, 1961, pp. 139-167.

[14] T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

[15] L. Schwartz, *Théorie des distributions*, Hermann, Paris,1966.

[16] L. M. Simon, Lectures on geometric measure theory, *Proc. of the Centre for Mathematical Analysis*, Australian National University, Vol. **3**, Canberra, 1983.


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