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The existence of Ginzburg-Landau solutions on the plane by a direct variational method

by

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ABSTRACT. – This paper studies the existence and the minimization problem of the solutions of the Ginzburg-Landau equations in $\mathbf{R}^2$ coupled with an external magnetic field or a source current. The lack of a suitable Sobolev inequality makes it necessary to consider a variational problem over a special admissible space so that the space norms of the gauge vector fields of a minimization sequence can be controlled by the corresponding energy upper bound and a solution may be obtained as a minimizer of a modified energy of the problem. Asymptotic properties and flux quantization are established for finite-energy solutions. Besides, it is shown that the solutions obtained also minimize the original Ginzburg-Landau energy when the admissible space is properly chosen.

Key words: Minimization, Sobolev inequalities.

RÉSUMÉ. – Le but de ce travail est d'étudier l'existence et le problème de minimisation des solutions des équations de Ginzburg-Landau dans $\mathbf{R}^2$, couplées avec un champ magnétique externe ou une source de courant. L'absence d'une inégalité convenable de Sobolev amène à considérer un problème variationnel sur un espace admissible spécial. Sur cet espace, les normes des champs de vecteurs de la gauge d'une suite minimisante peuvent être contrôlées par la borne supérieure de l'énergie correspondante.

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Aussi, une solution peut être obtenue comme un minimiseur d'une énergie modifiée du problème. Des propriétés asymptotiques et une quantification de flux sont établies par des solutions d'énergie finie. Entre autres, il est prouvé que les solutions obtenues minimisent aussi l'énergie originale de Ginzburg-Landau lorsque l'espace admissible est convenablement choisi.

I. INTRODUCTION

Let $\phi : \mathbf{R}^n \to \mathbf{C}$ be a complex scalar field and $A : \mathbf{R}^n \to \mathbf{R}^n$ a vector field, where $n = 2, 3, 4$. The Ginzburg-Landau (GL) energy density in the presence of an external magnetic field F_{jk}^{ex} $(j, k = 1, 2, ..., n)$ takes the form

$$\mathcal{E}(\phi, A) = \frac{1}{4} F_{jk} F_{jk} + \frac{1}{2} (D_j^A \phi)^* (D_j^A \phi) + \frac{\lambda}{8} (|\phi|^2 - 1)^2 - \frac{1}{2} F_{jk} F_{jk}^{\mathrm{ex}}, \quad (1)$$

where $F_{jk} = \partial_j A_k - \partial_k A_j$ is the magnetic field induced from $A = (A_j)$-called the gauge vector field, $D_j^A \phi = \partial_j \phi - i A_j \phi$ is the gauge-covariant derivative of ϕ, * denotes the complex conjugate, and the summation convention is observed on repeated indices. There is a local gauge symmetry in (1):

$$\phi \mapsto \phi e^{i\omega}, \qquad A \mapsto A + \nabla \omega, \qquad \omega : \mathbf{R}^n \to \mathbf{R}. \quad (2)$$

When $n = 2, 3$, (1) defines the well-known GL model of superconductivity, while, when $n = 4$, it defines the Euclidean abelian Higgs model in classical field theory [DH]. Varying the energy $E(\phi, A) = \int \mathcal{E}(\phi, A)\, dx$ leads to the following GL equations

$$\left.\begin{aligned} & D_j^A D_j^A \phi + \frac{\lambda}{2} (1 - |\phi|^2) \phi = 0, \\ & \partial_k F_{kj} + \frac{i}{2} (\phi [D_j^A \phi]^* - \phi^* [D_j^A \phi]) \\ & = \frac{1}{2} \partial_k (F_{kj}^{\mathrm{ex}} - F_{jk}^{\mathrm{ex}}), \end{aligned}\right\} \qquad x \in \mathbf{R}^n. \quad (3)$$

A basic problem is: Prove the existence of a finite-energy solution for the GL system (3) assuming that the external field F_{jk}^{ex} carries a finite energy

$$\int_{\mathbf{R}^n} F_{jk}^{\mathrm{ex}} F_{jk}^{\mathrm{ex}}\, dx < \infty, \quad (4)$$

and, show that there are solutions which minimize the GL energy.

In the earlier work [Y1, Y2], existence and minimization theorems have been established for the dimensions $n = 3, 4$. Our solution of the problem there depends on a modification of the energy density (1) and the following Poincaré type inequality in $\mathbf{R}^n$ $(n > 2)$ (*see* [B, GT]):

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbf{R}^n)} \leqq C(n)\|\nabla u\|_{L^2(\mathbf{R}^n)}, \qquad u \in C_0^1(\mathbf{R}^n). \tag{5}$$

When $n = 2$, (5) fails and the techniques of [Y1, Y2] break down. However, the case $n = 2$ is actually a more interesting situation of the model because it gives rise to vortex-like or mixed-state solutions which is a fundamental phenomenon in superconductivity physics. The purpose of the present paper is to solve the $n = 2$ case. Our approach here is based on an inequality which can control the local L^2 norm by the Dirichlet type norm on the right-hand side of (5), of a function vanishing in a small ball. We are led to introducing a special but natural function space for the vector gauge vector A so that the minimizing sequence of the modified GL energy is weakly compact in the space $W^{1,2}(\Omega)$ for any bounded domain Ω in $\mathbf{R}^2$. Thus the major difficulty in the early attempts in solving the problem in $\mathbf{R}^2$ is overcome. A solution over the full plane can then be obtained by passing to the limit from bounded domains. The solutions found are all in the global Coulomb gauge. The flux quantization problem, typical in two dimensions, will also be briefly studied. It is shown that, when the external field has power-type decay estimate at infinity, so will the excited fields, which leads to a quantized magnetic flux as in [JT]. We will also show that the solutions obtained minimize the original GL energy among all field configurations in the Coulomb gauge when the admissible space of the gauge vector fields is properly chosen to allow a convenient disposal of the boundary terms arising from our energy comparison.

II. EXISTENCE BY THE CALCULUS OF VARIATIONS

We shall first find a solution of (3) in $\mathbf{R}^2$ by looking for a critical point of an energy stronger than E (*see* the discussion to follow). Analysis shows that the failure of (5) in the case $n = 2$ makes it difficult to control the local $W^{1,2}$ norms of a pair of field configurations in terms of its energy. However, it turns out that we can use the following inequality to tackle our problem here.

LEMMA 1. – *Let $u(x)$ $(x \in \mathbf{R}^2)$ be a differentiable function with support contained in $B^c = \{x \in \mathbf{R}^2 | 1 < |x| < \infty\}$. Then*

$$\int_{|x|>1} \frac{u^2(x)}{|x|^2 \ln^2 |x|} \, dx \leqq 4 \int_{|x|>1} |\nabla u(x)|^2 \, dx, \qquad x \in \mathbf{R}^2. \tag{6}$$

Proof. – The above inequality can be found in [L] where the function u is assumed to be compactly supported in B^c. However, this result is also valid for functions in our generality which will be used in the study of (3) in two dimensions. We proceed as follows.

First, it is easily checked that there holds the identity

$$\nabla \cdot \left(\frac{u^2}{|x|^2 \ln |x|} x \right) = -\frac{u^2}{|x|^2 \ln^2 |x|} + \frac{\nabla (u^2) \cdot x}{|x|^2 \ln |x|}, \qquad x \in B^c. \tag{7}$$

Integrating (7) on $B_R^c = \{w \in \mathbf{R}^2 | 1 < |x| < R\}$, we get

$$\begin{aligned} \int_{B_R^c} \frac{u^2}{|x|^2 \ln^2 |x|} \, dx &= 2 \int_{B_R^c} \frac{u \nabla u \cdot x}{|x|^2 \ln |x|} \, dx - \int_{|x|=R} \frac{u^2}{|x|^2 \ln |x|} \, x \cdot n \, ds \\ &\leqq 2 \left(\int_{B_R^c} \frac{u^2}{|x|^2 \ln^2 |x|} \, dx \right)^{\frac{1}{2}} \left(\int_{B_R^c} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus (6) follows from taking $R \to \infty$ in the above inequality. □

Let $\mathcal{B}$ be the set of vector fields on $\mathbf{R}^2$ with differentiable components and finite $\| \quad \|_{\mathcal{H}}$-norm. Here, for $A \in \mathcal{B}$,

$$\|A\|_{\mathcal{H}}^2 = \int_{|x|<3} \{(\partial_j A_k)(\partial_j A_k) + |A|^2\} \, dx + \int_{|x|>1} (\partial_j [\rho A_k])(\partial_j [\rho A_k]) \, dx,$$

where ρ is a smooth function over $\mathbf{R}^2$ so that

$$0 \leqq \rho \leqq 1, \quad \rho(x) = 1 \quad \text{for } |x| > 2, \quad \operatorname{supp}(\rho) \subset \{x \in \mathbf{R}^2 | |x| > 1\}.$$

The completion of $\mathcal{B}$ under the norm $\| \quad \|_{\mathcal{H}}$ is denoted by $\mathcal{H}$. It is clear that $\mathcal{H}$ is a Hilbert space.

LEMMA 2. – *For any $A = (A_j) \in \mathcal{H}$, there holds*

$$\int_{|x|>1} \frac{\rho^2 |A|^2}{|x|^2 \ln^2 |x|} \, dx \leqq 4 \int_{|x|>1} [\partial_j (\rho A_k)] [\partial_j (\rho A_k)] \, dx. \tag{8}$$

Proof. – For $A \in \mathcal{H}$, there is a sequence $\{A^n\} \subset \mathcal{B}$ so that $\|A - A^n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. By Lemma 1, we know that (8) is valid for any element of $\mathcal{B}$. Thus

$$\begin{aligned}\int_{|x|>1} \frac{\rho^2 |A^m - A^n|^2}{|x|^2 \ln^2 |x|} dx &\leqq 4 \int_{|x|>1} (\partial_j [\rho (A_k^m - A_k^n)]) \\ &\qquad \times (\partial_j [\rho (A_k^m - A_k^n)]) \, dx \\ &\leqq 4 \|A^m - A^n\|_{\mathcal{H}}^2.\end{aligned}$$

Letting $m \to \infty$, we obtain

$$\int_{|x|>1} \frac{\rho^2 |A - A^n|^2}{|x|^2 \ln^2 |x|} dx \leqq 4 \|A - A^n\|_{\mathcal{H}}^2. \tag{9}$$

As a consequence of the Minkowski inequality and (9), we have

$$\begin{aligned}\left(\int_{|x|>1} \frac{\rho^2 |A|^2}{|x|^2 \ln^2 |x|} dx\right)^{\frac{1}{2}} &\leqq 2\|A - A^n\|_{\mathcal{H}} \\ &\quad + 2\left(\int_{|x|>1} (\partial_j [\rho A_k^n]) (\partial_j [\rho A_k^n]) \, dx\right)^{\frac{1}{2}}.\end{aligned}$$

Taking $n \to \infty$, we arrive at (8).

We are now ready to solve (3). For simplicity, we assume that the external field is smooth. We rewrite the GL energy density (1) in the form

$$\mathcal{E}(\phi, A) = \mathcal{E}_0(\phi, A) - \frac{1}{2} F_{jk} F_{jk}^{\mathrm{ex}}, \qquad j, k = 1, 2. \tag{10}$$

The main result of this section is

THEOREM 3. – *In $\mathbf{R}^2$, the GL system (3) has a finite-energy smooth solution (ϕ, A) in the Coulomb gauge $\partial_j A_j = 0$ and $E_0(\phi, A) = \int \mathcal{E}_0(\phi, A) \, dx < \infty$.*

Proof. – The original energy density (10) is not good enough to work on. As in [Y1], we start from the modified density functions

$$\left.\begin{aligned}\mathcal{I}_0(\phi, A) &= \frac{1}{2} (\partial_j A_k)(\partial_j A_k) + \frac{1}{2} (D_j^A \phi)^* (D_j^A \phi) \\ &\quad + \frac{\lambda}{8} (|\phi|^2 - 1)^2, \\ \mathcal{I}(\phi, A) &= \mathcal{I}_0(\phi, A) - \frac{1}{2} F_{jk} F_{jk}^{\mathrm{ex}},\end{aligned}\right\} \tag{11}$$

which are obviously stronger than $\mathcal{E}$ and $\mathcal{E}_0$.

Set $I = \int \mathcal{I}\, dx$ and $I_0 = \int \mathcal{I}_0\, dx$ over $\mathbf{R}^2$. Consider the variational problem

$$I_{\min} \equiv \min \{I(\phi, A) | (\phi, A) \in \mathcal{J} \times \mathcal{H}\}, \tag{12}$$

where $\mathcal{J}$ is the set of complex-valued functions on $\mathbf{R}^2$ with components lying in $W^{1,2}_{\text{loc}}(\mathbf{R}^2)$. Since $F^{\text{ex}}_{jk} \in L^2(\mathbf{R}^2)$ [*see* (4)], it is easily checked that $I_{\min}$ in (12) is finite.

Next, it is clear that the functions in (11) are no longer invariant under the general gauge symmetry (2). However, they are invariant if ω in (2) is a linear function of x_1, x_2. Thus, for a pair $(\phi, A) \in \mathcal{J} \times \mathcal{H}$ with $I_0(\phi, A) < \infty$, define (ϕ', A') so that

$$\phi' = \phi\, e^{i\omega}, \qquad A' = A + \nabla \omega,$$
$$\omega = -a_1 x_1 - a_2 x_2 \quad \text{where } (a_1, a_2) = \frac{1}{9\pi} \int_{|x|<3} A(x)\, dx.$$

Then the average of A' on $\{x \in \mathbf{R}^2 | \, |x| < 3\}$ vanishes, *i. e.*,

$$\underline{A}' \equiv \frac{1}{9\pi} \int_{|x|<3} A'(x)\, dx = 0.$$

Moreover, the linearity of ω in the variables x_1, x_2 implies that $A' \in \mathcal{H}$ because A' is a translation of A by a constant vector.

Let $\{(\phi^n, A^n)\}$ be a minimizing sequence of the problem (12). The Schwarz inequality implies that

$$C I_0(\phi^n, A^n) \leqq \sup_n I(\phi^n, A^n) + \|F^{\text{ex}}_{jk}\|^2_{L^2(\mathbf{R}^2)}. \tag{13}$$

Here, and in the sequel, $C > 0$ is an irrelevant constant. From (13) and the definition of I_0, we see the finiteness of the quantity

$$M = \sup_n \int_{\mathbf{R}^2} (\partial_j A^n_k)(\partial_j A^n_k)\, dx. \tag{14}$$

By the above discussion, we may also assume that $\underline{A}^n = 0$, $n = 1, 2, \ldots$ Thus the Poincaré inequality applied on $\{x \in \mathbf{R}^2 | \, |x| < 3\}$ implies

$$\sup_n \int_{|x|<3} \{(\partial_j A^n_k)(\partial_j A^n_k) + |A^n|^2\}\, dx \leqq CM. \tag{15}$$

Recall that, in the definition of the norm $\| \quad \|_{\mathcal{H}}$, the truncation function satisfies $\rho(x) = 1$ for $|x| > 2$. As a consequence, the combination of (14) and (15) gives us the boundedness of $\{A^n\}$ in $\mathcal{H}$. Using Lemma 2 and the definition of $\| \quad \|_{\mathcal{H}}$, we conclude that the components of $\{A^n\}$ are bounded sequences in $W^{1,2}(\Omega)$ for any bounded domain $\Omega \subset \mathbf{R}^2$. Thus $\{A^n\}$ is also bounded in $L^p(\Omega)$ for any $p \geqq 1$. On the other hand, the definition of the gauge-covariant derivative gives

$$|D_j^{A^n} \phi^n|^2 \geqq \frac{1}{2}|\partial_j \phi^n|^2 - C(|A^n|^4 + |\phi^n|^4 + 1).$$

Thus $\{\phi^n\}$ is a bounded sequence in $W^{1,2}(\Omega)$ as well for any bounded domain Ω in $\mathbf{R}^2$. Using the compact embedding $W^{1,2}(\Omega) \to L^p(\Omega)\,(p \geqq 1)$ and by passing to a subsequence if necessary (a diagonal subsequence argument), we may assume the existence of a pair $(\phi, A) \in W^{1,2}_{\text{loc}}(\mathbf{R}^2)$ so that

$$\begin{aligned} (\phi^n, A^n) &\to (\phi, A) \quad \text{weakly in } W^{1,2}(\Omega), \\ (\phi^n, A^n) &\to (\phi, A) \quad \text{strongly in } L^p(\Omega) \end{aligned} \tag{16}$$

for every bounded domain $\Omega \in \mathbf{R}^2$ and any $p \geqq 1$. Of course A is also the weak limit of $\{A^n\}$ in the space $\mathcal{H}$. Thus $(\phi, A) \in \mathcal{J} \times \mathcal{H}$. We are left to show that (ϕ, A) solves the variational problem (12).

Let $\mathcal{I}_0, \mathcal{I}$ be as defined in (11) and $B_R = \{x \in \mathbf{R}^2 \,|\, |x| < R\}\,(R > 0)$. The functionals $I_0(\phi, A; R) = \int \mathcal{I}_0(\phi, A)\,dx$ and $I(\phi, A; R) = \int \mathcal{I}(\phi, A)\,dx$ are integrals over B_R. The condition (4) says that for given $\varepsilon > 0$ there is $R_0 > 0$ to ensure

$$\int_{\mathbf{R}^2 - B_R} F_{jk}^{\text{ex}} F_{jk}^{\text{ex}}\,dx < \varepsilon^2 \quad \text{whenever } R > R_0.$$

Thus for $R > R_0$,

$$\begin{aligned} I(\phi^n, A^n; R) &\leqq I(\phi^n, A^n) + \frac{1}{2}\int_{\mathbf{R}^2 - B_R} F_{jk}^n F_{jk}^{\text{ex}}\,dx \\ &\leqq I(\phi^n, A^n) + \varepsilon\sqrt{M}, \qquad n = 1, 2, \ldots, \end{aligned}$$

where $M \geqq 0$ is as given in (14). Using (16), the weak lower semicontinuity of the terms in $I(.,.; R)$, and $I(\phi^n, A^n) \to I_{\min}$ (as $n \to \infty$), we arrive at

$$I(\phi, A; R) \leqq I_{\min} + \varepsilon\sqrt{M}, \qquad R > R_0.$$

Namely,

$$\begin{aligned} I_0(\phi, A; R) &\leqq I_{\min} + \varepsilon\sqrt{M} + \frac{1}{2}\int_{\mathbf{R}^2} F_{jk} F_{jk}^{\text{ex}}\,dx - \frac{1}{2}\int_{\mathbf{R}^2 - B_R} F_{jk} F_{jk}^{\text{ex}}\,dx \\ &\leqq I_{\min} + 2\varepsilon\sqrt{M} + \frac{1}{2}\int_{\mathbf{R}^2} F_{jk} F_{jk}^{\text{ex}}\,dx, \qquad R > R_0. \end{aligned}$$

Setting $R \to \infty$ in the above, we get $I(\phi, A) \leqq I_{\min}$ by the arbitrariness of $\varepsilon > 0$. Hence $I(\phi, A) = I_{\min}$ and (12) is solved.

Finally, varying the modified energy I in $\mathcal{J} \times \mathcal{H}$ by compactly supported test functions around the obtained minimizer (ϕ, A) of (12) and using the standard elliptic regularity theory, we see that (ϕ, A) is a smooth solution of the system

$$\left.\begin{aligned} &D_j^A D_j^A \phi + \frac{\lambda}{2}(1 - |\phi|^2)\phi = 0, \\ &\nabla^2 A_j + \frac{i}{2}(\phi [D_j^A \phi]^* - \phi^* [D_j^A \phi]) \\ &= \frac{1}{2}\partial_k (F_{kj}^{\text{ex}} - F_{jk}^{\text{ex}}), \end{aligned}\right\} \quad x \in \mathbf{R}^2. \tag{17}$$

We can use the two equations in (17) to show that $\partial_j A_j$ is harmonic in $\mathbf{R}^2$. However, since $\partial_j A_j \in L^2(\mathbf{R}^2)$, the Liouville theorem says that $\partial_j A_j \equiv 0$. As a consequence, it follows that $\nabla^2 A_j = \partial_k \partial_k A_j = \partial_k (\partial_k A_j - \partial_j A_k) = \partial_k F_{kj}$, $j = 1, 2$. In other words, (17) recovers the GL equations (3) in $\mathbf{R}^2$. Thus (ϕ, A) solves (3) as well and the theorem is proved. □

III. THE SOURCE CURRENT CASE

In our discussion of the GL theory in Section 2, the external field is a magnetic field, F_{jk}^{ex}. A more general situation is that the external field may be coupled into the model through the form of a smooth source current density, $J^{\text{ex}} = (J_j^{\text{ex}})$ (say). In this case the energy density takes the following form

$$\mathcal{E}(\phi, A) = \mathcal{E}_0(\phi, A) + A_j J_j^{\text{ex}}, \tag{18}$$

and the corresponding GL equations are

$$\left.\begin{aligned} D_j^A D_j^A \phi + \frac{\lambda}{2}(1 - |\phi|^2)\phi = 0, \\ \partial_k F_{kj} + \frac{i}{2}(\phi [D_j^A \phi]^* - \phi^* [D_j^A \phi]) = J_j^{\text{ex}}. \end{aligned}\right\} \tag{19}$$

In order to observe consistency in (19), we must impose the current conservation law on J^{ex} :

$$\partial_j J_j^{\text{ex}} = 0. \tag{20}$$

For $n = 3, 4$, the existence of solutions of (19) as minimizers of the GL energy has been established in [Y1, Y2] under the condition (20) and

$$J_j^{\text{ex}} \in L^{\frac{2n}{n+2}}(\mathbf{R}^n), \qquad j = 1, \ldots, n; \qquad n = 3, 4.$$

An obvious reason for imposing the above condition is due to the Poincaré type inequality (5) so that the GL energy is ensured to be bounded from below. For the case $n = 2$, since (5) is no longer valid, we need some new conditions on J^{ex} to solve (19). In this section, we assume in addition to (20) that

$$\int_{\mathbf{R}^2} |x|^2 \ln^2 |x| \, |J^{\text{ex}}|^2 \, dx < \infty, \tag{21}$$

$$R \, x_j \, J_j^{\text{ex}}(x) \to 0 \quad \text{as } |x| = R \to \infty, \qquad j = 1, 2. \tag{22}$$

THEOREM 4. – *Assume that (20)-(22) hold. The GL equations (19) over* $\mathbf{R}^2$ *subject to the general source current* $J^{\text{ex}} = (J_j^{\text{ex}})$ *have a finite-energy smooth solution* (ϕ, A) *in the Coulomb gauge,* $\partial_j A_j \equiv 0$.

Proof. – Again the energy density (18) is not good enough to work on and we have to use the modified energy $I_0 = \int \mathcal{I}_0 \, dx$ where $\mathcal{I}_0$ is as defined in (11). Set $I = I_0 + \int A_j \, J_j^{\text{ex}} \, dx$. Consider the problem

$$I_{\min} = \min \{ I(\phi, A) | (\phi, A) \in \mathcal{J} \times \mathcal{H}, \, I_0(\phi, A) < \infty \}. \tag{23}$$

Since for $A \in \mathcal{H}$,

$$\int_{|x| \geqq 2} \frac{|A|^2}{|x|^2 \ln^2 |x|} \, dx \leqq C \, \|A\|_{\mathcal{H}}^2,$$

therefore the Schwarz inequality and (21) imply that $A_j \, J_j^{\text{ex}} \in L(\mathbf{R}^2)$.

We first show that I is invariant under the gauge transformation (2) when $\omega(x) = \alpha_1 x_1 + \alpha_2 x_2$ is a linear function. In fact, we have already observed the invariance of I_0. On the other hand, from (20) and the divergence theorem, we have for $A' = A + \nabla \omega$,

$$\begin{aligned} \int_{B_R} A_j \, J_j^{\text{ex}} \, dx &= \int_{B_R} (A'_j - \partial_j \, \omega) \, J_j^{\text{ex}} \, dx \\ &= \int_{B_R} A'_j \, J_j^{\text{ex}} \, dx - \frac{1}{R} \int_{|x|=R} \omega \, x_j \, J_j^{\text{ex}} \, ds. \end{aligned}$$

Thus, using (22) and letting $R \to \infty$, we see that

$$\int_{\mathbf{R}^2} A_j \, J_j^{\text{ex}} \, dx = \int_{\mathbf{R}^2} A'_j \, J_j^{\text{ex}} \, dx. \tag{24}$$

Namely, the invariance of I holds as well.

We next show that $I_{\min}$ in (23) is finite. For given (ϕ, A) in the admissible set of (23), we obtain as in Section II a modified pair (ϕ', A') so that $\phi' = \phi \, e^{i\omega}$, $A' = A + \nabla \omega$, ω is a linear function, and the average of A' on $B_3 = \{x \in \mathbf{R}^2 \mid |x| < 3\}$ is zero. Using the Poincaré inequality on B_3 and Lemma 2, we have

$$\int_{|x|<3} |A'|^2 \, dx + \int_{|x|>2} \frac{|A'|^2}{|x|^2 \ln^2 |x|} \, dx \leqq C_1 + C_2 \int_{\mathbf{R}^2} \partial_j A'_k \, \partial_j A'_k \, dx. \tag{25}$$

On the other hand, there holds for any $\varepsilon > 0$,

$$\int_{\mathbf{R}^2} A'_j \, J_j^{\mathrm{ex}} \, dx > -\varepsilon \left(\int_{|x|<3} |A'|^2 \, dx + \int_{|x|>2} \frac{|A'|^2}{|x|^2 \ln^2 |x|} \, dx \right) - \varepsilon^{-1} \left(\int_{|x|<3} |J^{\mathrm{ex}}|^2 \, dx + \int_{|x|>2} |x|^2 \ln^2 |x| \, |J^{\mathrm{ex}}|^2 \, dx \right). \tag{26}$$

Inserting (25) and (26) into (24) yields

$$I(\phi, A) \geqq C_1 \, I_0(\phi, A) - C_2, \tag{27}$$

where $C_1, C_2 > 0$ are constants so that C_2 depends only on J^{ex}. In particular, the finiteness of $I_{\min}$ follows.

Let $\{(\phi^n, A^n)\}$ be a minimizing sequence of (23). Our earlier discussion on the invariance of the terms in the energy functional allows us to assume, after a suitable gauge transformation, that A^n has zero average on B_3. Hence A^n verifies (25). Consequently (27) implies that $\{A^n\}$ is a bounded sequence in $\mathcal{H}$. So it follows by passing to a subsequence if necessary that we may assume (16) for $\{(\phi^n, A^n)\}$.

Given $\varepsilon > 0$, let $R_0 > 2$ be such that

$$\int_{\mathbf{R}^2 - B_R} |x|^2 \ln^2 |x| \, |J^{\mathrm{ex}}|^2 \, dx < \varepsilon^2, \qquad R > R_0.$$

Then the integral $I(\phi^n, A^n; R) = \int (\mathcal{I}_0(\phi^n, A^n) + A_j^n \, J_j^{\mathrm{ex}}) \, dx$ over B_R satisfies

$$I(\phi^n, A^n; R) \leqq I(\phi^n, A^n) + \varepsilon \sqrt{M}, \qquad n = 1, 2, \ldots,$$

where

$$M = \sup_n \int_{|x|>2} \frac{|A^n|^2}{|x|^2 \ln^2 |x|} \, dx,$$

which is finite according to (25) and (27). We can then duplicate the steps in proving Theorem 3 to show that (ϕ, A) is a minimizer of (23) which also solves the GL equations (19) because of (20). □

IV. ASYMPTOTIC BEHAVIOR AND FLUX QUANTIZATION

In this section, we assume that (ϕ, A) is a solution of (3) or (19) with finite modified energy $I_0(\phi, A) < \infty$ and in the Coulomb gauge. Using the methods of [Y1, Y2], it is not hard to establish the following L^2-estimates.

THEOREM 5. – *Suppose in* (3) ((19)) *that the external field* F^{ex}_{jk} (J^{ex}_j) *lies in the space* $W^{1,2}(\mathbf{R}^2)$ $(L^2(\mathbf{R}^2))$ $(j, k = 1, 2)$. *Then* $D^A_j D^A_k \phi \in L^2(\mathbf{R}^2)$, $|D^A_j \phi| \in W^{1,2}(\mathbf{R}^2)$ $(j, k = 1, 2)$ *and* $1 - |\phi|^2 \in W^{1,2}(\mathbf{R}^2)$ *for any* $p > 1$. *Besides,* $|\phi|^2 \to 1$ *as* $|x| \to \infty$ *and* $|\phi| < 1$ *in* $\mathbf{R}^2$ *or otherwise* $|\phi| \equiv 1$. *If, furthermore,* $F^{\text{ex}}_{jk} \in W^{2,2}(\mathbf{R}^2)$ $(J^{\text{ex}}_j \in W^{1,2}(\mathbf{R}^2))$ $(j, k = 1, 2)$, *then* $D^A_j \phi, \partial_j A_k \in W^{2,2}(\mathbf{R}^2)$ $(j, k = 1, 2)$.

We skip the proof here.

From the well-known Sobolev embedding $W^{1,2}(\mathbf{R}^2) \to L^p(\mathbf{R}^2)$ $(\forall p > 1)$ and the fact that functions in $W^{1,p}(\mathbf{R}^2)$ $(p > 2)$ vanish at infinity, we obtain

COROLLARY 6. – *For* $F^{\text{ex}}_{jk} \in W^{2,2}(\mathbf{R}^2)$ $(J^{\text{ex}}_j \in W^{1,2}(\mathbf{R}^2))$ $(j, k = 1, 2)$, *there hold*

$$D^A_j \phi \to 0, \qquad \partial_j A_k \to 0 \quad \text{as } |x| \to \infty, \qquad j, k = 1, 2.$$

We have shown in Section II (III) that (ϕ, A) can be obtained as a minimizer of the problem (12) ((23)) which is a finite-energy smooth solution of the GL equations (3) ((19)). In order to calculate the magnetic flux carried by the solution, some minimal decay assumption for the external field has to be made. The main reason for doing this is that decay estimates of the quantities $1 - |\phi|^2$, $D^A_j \phi$, and F_{jk} can thus be established so that undesired boundary terms will be eliminated in the integral of our flux evaluation.

For this purpose, we impose the following power type decay properties for the external field:

$$|\partial_k (F^{\text{ex}}_{kj} - F^{\text{ex}}_{jk})| (\text{or } |J^{\text{ex}}_j|) \leqq C\,|x|^{-\alpha}, \tag{28}$$

$$|\partial_j \partial_k (F_{jk}^{\text{ex}} - F_{jk}^{\text{ex}})|(\text{or } |\partial_j J_k^{\text{ex}} - \partial_k J_j^{\text{ex}}|) \leqq C |x|^{-\beta}, \tag{29}$$

where $\alpha, \beta > 0$, $j, k = 1, 2$, and $|x|$ is large.

THEOREM 7. – *Suppose* $F_{jk}^{\text{ex}} \in W^{2,2}(\mathbf{R}^2)$ $(J_j^{\text{ex}} \in W^{1,2}(\mathbf{R}^2))$ $(j = 1, 2)$. *If* F_{jk}^{ex} (J_j^{ex}) *decays according to (28), then asymptotically*

$$|D_j^A \phi| \leqq C |x|^{-\alpha}, \qquad 0 \leqq 1 - |\phi|^2 \leqq C |x|^{-2\alpha}, \qquad j = 1, 2.$$

If, in addition, F_{jk}^{ex} (J_j^{ex}) *satisfies (29), then*

$$|F_{jk}| \leqq C|x|^{-\gamma}, \quad \gamma = \min\{2\alpha, \beta\}, \quad j, k = 1, 2.$$

Proof. – We shall only demonstrate the case when the external field is the source current, J^{ex}.

Set $g_j = D_j^A \phi$ and $|g|^2 = g_j g_j^*$. It is crucial to derive the decay estimate for $|g|$ first.

Let ψ be an arbitrary complex scalar function. By the definition of the gauge-covariant derivatives, we easily verify the useful curvature identity

$$D_j^A D_k^A \psi - D_k^A D_j^A \psi = -\mathrm{i}\, F_{jk} \psi.$$

From this and the equations (19), we have after a lengthy calculation

$$\begin{aligned} D_k^A D_k^A g_j = &-\frac{\lambda}{2}(1 - |\phi|^2) g_j + \frac{\lambda + 1}{2} |\phi|^2 g_j \\ &+ \frac{\lambda - 1}{2} \phi^2 g_j^* - 2\mathrm{i} F_{kj} g_k - \mathrm{i} J_j^{\text{ex}} \phi. \end{aligned} \tag{30}$$

Applying (30) in

$$\nabla^2 |g|^2 = 2 D_k^A g_j (D_k^A g_j)^* + 2 \operatorname{Re}\{g_j^* D_k^A D_k^A g_j\},$$

we find that

$$\nabla^2 |g|^2 \geqq ([\lambda + 1 - |\lambda - 1|]|\phi|^2 - [\lambda(1 - |\phi|^2) + 4|F_{12}|]) |g|^2 - 2|J^{\text{ex}}| |g|.$$

Since $|\phi| \to 1$, $F_{12} \to 0$ as $|x| \to \infty$ (*see* Corollary 6) and $\lambda > 0$, the above inequality has the reduction

$$\nabla^2 |g|^2 \geqq C_1 |g|^2 - C_2 |J^{\text{ex}}|^2, \qquad |x| > R, \tag{31}$$

where $C_1, C_2 > 0$ are constants and $R > 0$ is sufficiently large.

Introduce the comparison function

$$\sigma(x) = C_3 |x|^{-2\alpha}, \qquad |x| \geqq R. \tag{32}$$

Then $\nabla^2 \sigma = 4\alpha^2 |x|^{-2} \sigma \, (|x| > R)$. Inserting this equation and (28) (for J^{ex}) into (31) yields

$$\begin{aligned} \nabla^2 (\sigma - |g|^2) &\leqq 4\alpha^2 |x|^{-2} \sigma - C_1 |g|^2 + C_2 |x|^{-2\alpha} \\ &= \left(4\alpha^2 |x|^{-2} + \frac{C_2}{C_3}\right) \sigma - C_1 |g|^2 \\ &\leqq \left(4\alpha^2 |x|^{-2} + \frac{C_2}{C_3}\right) (\sigma - |g|^2), \qquad |x| > R, \end{aligned} \tag{33}$$

where we have assumed that R and C_3 are such that $4\alpha^2 R^{-2} + C_2/C_3 \leqq C_1$. Obviously, for fixed R, we may choose $C_3 > 0$ in (32) so large that $(\sigma - |g|^2)|_{|x|=R} > 0$. From this boundary condition, the property $\sigma - |g|^2 \to 0$ as $|x| \to \infty$, (33), and the maximum principle, we see that $|g|^2 < \sigma$ for $|x| > R$. Namely $|g| = O(|x|^{-\alpha})$.

Similarly, the asymptotic estimate of $1 - |\phi|^2$ follows from using the above argument in the equation

$$\nabla^2 (1 - |\phi|^2) = \lambda |\phi|^2 (1 - |\phi|^2) - 2|g|^2$$

while treating $-2|g|^2$ as a decaying source term.

Now suppose (29) holds in addition. Differentiating (19) gives

$$\nabla^2 (\partial_k A_j) = \frac{\mathrm{i}}{2} (g_j g_k^* - g_j^* g_k) + \frac{\mathrm{i}}{2} (\phi^* [D_k^A g_j] - \phi [D_k^A g_j]^*) + \partial_k J_j^{ex}. \tag{34}$$

Thus, using (34) and the aforementioned curvature identity, it is seen that $F_{12} = \partial_1 A_2 - \partial_2 A_1$ satisfies the equation

$$\nabla^2 F_{12} = |\phi|^2 F_{12} + \mathrm{i}(g_1^* g_2 - g_1 g_2^*) + (\partial_1 J_2^{ex} - \partial_2 J_1^{ex}).$$

Thus

$$\begin{aligned} \nabla^2 F_{12}^2 &\geqq 2|\phi|^2 F_{12}^2 - 4|g|^2 |F_{12}| - 2|\partial_1 J_2^{ex} - \partial_2 J_1^{ex}| \, |F_{12}| \\ &\geqq C_1 F_{12}^2 - C_2 |x|^{-2\gamma}, \qquad |x| > R, \end{aligned}$$

where $C_1, C_2 > 0$ are constants, R is a large number, and $\gamma = \min\{2\alpha, \beta\}$ with β being given in (29). Thus we can prove as before that $F_{12}^2 \leqq C|x|^{-2\gamma}$.

The proof of the theorem is complete. □

Concerning the total magnetic flux of the solution (ϕ, A), we have as in [JT] the following statement.

THEOREM 8. – *Assume the condition* (28) *where* $\alpha > 1$ *and let* $H = F_{12}$ *denote the induced magnetic field. Then the total flux* Φ *is quantized according to the expression*

$$\Phi \equiv \lim_{R\to\infty} \int_{|x|<R} H\,dx = 2\pi N, \tag{35}$$

where N is an integer which is recognized as the winding number of the order parameter ϕ *on the circle at infinity of the plane.*

Proof. – Let $R_0 > 0$ be such that $|\phi| > \frac{1}{2}$ for $|x| > R_0$ (Theorem 5 or 7). Then the winding number N of ϕ at infinity obeys

$$2\pi N = \int_{|x|=R} d\arg\phi = -\mathrm{i}\int_{|x|=R} d\ln\phi \quad \text{for any} \quad R > R_0. \tag{36}$$

On the other hand, the divergence theorem and Theorem 7 imply that

$$\left|\int_{|x|<R} H\,dx + \mathrm{i}\int_{|x|=R} d\ln\phi\right| \leqq \left|\int_{|x|=R} \phi^{-1} D_j^A \phi\,dx_j\right| \leqq \mathrm{C}\,R^{-(\alpha-1)}, \qquad R > R_0. \tag{37}$$

Inserting (36) into (37) and letting $R \to \infty$, we find $\Phi = 2\pi N$ as expected. □

Thus, although the flux of the external field may take any prescribed value, the excited flux can only assume quantized values which is a typical phenomenon in superconductivity theory.

V. MINIMIZATION OF ENERGY

In Sects. II and III, we proved the existence of finite-energy smooth solutions of the GL system (3) or (19) in $\mathbf{R}^2$ by obtaining minimizers of the problem (12) or (23) where the energy functional I takes a modified form and is not physical. For example, the invariance under the general gauge symmetry (2) is no longer valid. Since the GL energy E is actually given in (10) or (18), it will be important to find solutions of (3) or (19) so that they also minimize the GL energy. Note that, in general, the GL equations may have multiple solutions and the physical states are represented by energy minimizers. The purpose of this section is to get solutions of the GL equations that minimize the energy in a suitable admissible class.

Our starting point now is to find a comparison of the GL energy E and the modified energy I defined in (12) or (23). It appears that the space $\mathcal{H}$ used in Sects. II and III is not proper for such a comparison and it is necessary to consider a subspace of $\mathcal{H}$, which is restrictive enough so that we can compare E and I and, at the same time, large enough so that the GL equations can be fulfilled by energy minimizers of E. The following study will follow this line.

Let $\mathcal{C} \subset \mathcal{B}$ (*see* Sect. II) be the set of vector fields in $\mathbf{R}^2$ satisfying the condition that, for each $A \in \mathcal{C}$, there is a constant vector $A^0 \in \mathbf{R}^2$ such that $A - A^0$ is of compact support. The closure of $\mathcal{C}$ in $\mathcal{H}$ is denoted by $\mathcal{H}_1$. A useful property of $\mathcal{H}_1$ is that there holds the identity

$$\left.\begin{array}{c}\displaystyle\int_{\mathbf{R}^2} (\partial_j A_k)(\partial_j A_k)\, dx = \int_{\mathbf{R}^2} \{(\partial_1 A_2 - \partial_2 A_1)^2 + (\partial_j A_j)^2\}\, dx,\\ A = (A_j) \in \mathcal{H}_1.\end{array}\right\} \tag{38}$$

In fact, it is straightforward to verify (38) in $\mathcal{C}$. However, since both sides of (38) are continuous with respect to the norm of $\mathcal{H}$, we see that (38) is true in $\mathcal{H}_1$ in general.

We are now ready to obtain a solution pair of the equations (3) or (19) as a GL energy minimizer in the following sense.

THEOREM 9. – *Suppose that* (4) *or* (20)-(22) *hold. Then the GL equations* (3) *or* (19) *over* $\mathbf{R}^2$ *have a smooth solution* $(\tilde{\phi}, \tilde{A})$ *which minimizes the GL energy* $E = \int \mathcal{E}\, dx$ *among all field configurations in the admissible set* $\mathcal{J} \times \mathcal{H}_1$ *and in the global Coulomb gauge. Here* $\mathcal{E}$ *is as defined in (10) or (18).*

Proof. – We can proceed to show as in Sect. II or III that the optimization problem (12) or (23), with $\mathcal{H}$ replaced by $\mathcal{H}_1$, has a solution $(\tilde{\phi}, \tilde{A})$, that this solution satisfies the GL equations (3) or (19), and that $\partial_j \tilde{A}_j \equiv 0$ in $\mathbf{R}^2$. We claim that

$$E(\tilde{\phi}, \tilde{A}) = \min\{E(\phi, A) | (\phi, A) \in \mathcal{J} \times \mathcal{H}_1,\ \partial_j A_j = 0\}. \tag{39}$$

In fact, the identity (38) says that $I(\phi, A) = E(\phi, A)$ for $(\phi, A) \in \mathcal{J} \times \mathcal{H}_1$ with $\partial_j A_j = 0$. Thus $E(\tilde{\phi}, \tilde{A}) = I(\tilde{\phi}, \tilde{A}) \leq I(\phi, A) = E(\phi, A)$ for $(\phi, A) \in \mathcal{J} \times \mathcal{H}_1$ and $\partial_j A_j = 0$ as expected. □

Note. Another technical reason for the choice of $\mathcal{H}_1$ (or $\mathcal{C}$) is that the space is invariant under translations by constant vectors. This property is crucial because it allows us to make gauge transformations (2) in the modified energy I in which ω is a linear function of the variables. The discussion in Sects. II and III showed the importance of such a feature in extracting a locally weakly convergent minimizing sequence.

VI. THE GENERAL SCALAR POTENTIAL CASE

We have assumed in our Lagrangian density (1) that the potential function of the order parameter ϕ takes the form $V(|\phi|) = \frac{\lambda}{8}(|\phi|^2 - 1)^2$. In fact our method in the existence proofs applies to the more general situation that the non-negative function $V(s)\,(s \geqq 0)$ satisfies

$$s^p - C_1 \leqq V(s) \leqq s^q + C_2, \qquad p,\, q > 1,$$

where $C_1, C_2 \geqq 0$ are constants. Part of the interest in such an extension is to accommodate in the family of finite-energy solutions the normal state $\phi = 0$, $F_{jk} = F_{jk}^{\mathrm{ex}}$. For example, we may choose $V(|\phi|) = \frac{\lambda}{8}|\phi|^2(|\phi|^2 - 1)^2$ as in the self-dual Chern-Simons-Higgs theory [HKP, JW, SY]. In this case the generalized GL equations in $\mathbf{R}^2$ are

$$\left.\begin{aligned} D_k^A D_k^A \phi + \frac{\lambda}{2}|\phi|^2(1 - |\phi|^2)\phi - \frac{\lambda}{4}(1 - |\phi|^2)^2\phi = 0, \\ \partial_k F_{kj} + \frac{\mathrm{i}}{2}(\phi\,[D_j^A \phi]^* - \phi^*\,[D_j^A \phi]) = J_j^{\mathrm{ex}}. \end{aligned}\right\} \tag{40}$$

Here we consider only the source current case. We can prove as in Sect. V, for example, the existence of a smooth solution of the equations which is also a GL energy minimizer among all field configurations in the Coulomb gauge. In this section, we restrict our attention on the study of the asymptotic behavior of a finite-energy solution. In view of (40), a lengthy calculation shows that $g_j = D_j^A \phi$ satisfies

$$\begin{aligned} D_k^A D_k^A g_j = &-\mathrm{i}\, J_j^{\mathrm{ex}}\phi - 2\,\mathrm{i}\, F_{kj}\, g_k + \left(\frac{\lambda}{4} + \left[\frac{1}{2} - 2\lambda\right]|\phi|^2 + \frac{9}{4}\lambda\,|\phi|^4\right) g_j \\ &+ \frac{1}{2}(3\lambda\,|\phi|^2 - [2\lambda + 1])\,\phi^2 g_j^*. \end{aligned} \tag{41}$$

From (41) we can prove that $D_j^A D_k^A \phi \in L^2(\mathbf{R}^2)$, $|D_j^A \phi| \in W^{1,2}(\mathbf{R}^2)$ if (ϕ, A) is of finite energy and $J_j^{\mathrm{ex}} \in L^2(\mathbf{R}^2)$. Moreover, since $\partial_k A_j$

satisfies (34), we obtain $\partial_k A_j,\ D_j^A \phi \in W^{2,2}(\mathbf{R}^2)$ if $J_j^{\mathrm{ex}} \in W^{1,2}(\mathbf{R}^2)$. Therefore the simple inequality

$$|\partial_j (|\phi| [|\phi|^2 - 1])| \leqq |D_j^A \phi| \, | |\phi|^2 - 1| + 2 |\phi|^2 |D_j^A \phi|$$

and the bound $|\phi| \leqq 1$ imply that $|\phi| (|\phi|^2 - 1) \in W^{1,p}(\mathbf{R}^2)$ for any $p > 1$. As a consequence (*see* Sect. IV), we can state the following asymptotic behavior of ϕ, $D_j^A \phi$, F_{jk}.

THEOREM 10. – *Suppose* $J_j^{\mathrm{ex}} \in W^{1,2}(\mathbf{R}^2)$ *for* $j = 1, 2$ *and* (ϕ, A) *is a finite-energy solution of (40) in the Coulomb gauge. Then*

$$0 \leqq |\phi| (1 - |\phi|^2) \to 0, \quad D_j^A \phi \to 0, \quad \partial_k A_j \to 0 \quad as\ |x| \to \infty.$$

Furthermore, if J_j^{ex} *fulfills* (28)-(29), *then the solutions satisfying* $|\phi| = 1$ *at infinity have the decay properties described in Theorem 7. On the other hand, the solutions satisfying* $|\phi| = 0$ *at infinity obey the exponential decay property*

$$|\phi(x)|^2 = O(e^{-\sqrt{\frac{\lambda}{2}}|x|}) \quad for\ large\ |x|. \tag{42}$$

In this situation $|D_j^A \phi|$ *decays as in the* $|\phi| \to 1$ *case.*

Proof. – It suffices to elaborate on the second part of the statement. As in Sect. IV, (41) gives us the inequality

$$\nabla^2 |g|^2 \geqq \left(\frac{\lambda}{2} + [1 - 4\lambda]|\phi|^2 + \frac{9}{2}\lambda |\phi|^4\right) |g|^2$$
$$-|3\lambda|\phi|^2 - (2\lambda + 1)||\phi|^2 |g|^2 - 4|F_{12}||g| - 2|J^{\mathrm{ex}}||g|. \tag{43}$$

Since either $|\phi| \to 0$ or $|\phi| \to 1$ as $|x| \to \infty$, the inequality (43) always leads to (31). Thus $|g| = O(|x|^{-\alpha})$.

Assume first that $|\phi| \to 1$ as $|x| \to \infty$. Then

$$\nabla^2 (1 - |\phi|^2) = -2\,\mathrm{Re}\{\phi^* D_k^A D_k^A \phi\} - 2|g|^2$$
$$= \lambda \left(|\phi|^4 - \frac{1}{2}|\phi|^2 [1 - |\phi|^2]\right) (1 - |\phi|^2) - 2|g|^2.$$

Because the coefficient of $1 - |\phi|^2$ on the right-hand side of above equation becomes positive when $|x|$ is large, we see that $1 - |\phi|^2$ decays like $|g|^2$ as in Sect. IV.

Assume next that $|\phi| \to 0$ as $|x| \to \infty$. Then

$$\begin{aligned}\nabla^2 |\phi|^2 &= 2\,\mathrm{Re}\,\{\phi^* D_k^A D_k^A \phi\} + 2|g|^2 \\ &\geqq \frac{\lambda}{2}(1-|\phi|^2)(1-3|\phi|^2)|\phi|^2.\end{aligned}$$

As $|x| \to \infty$, the coefficient of $|\phi|^2$ on the right-hand side above goes to $\frac{\lambda}{2}$. Hence it is standard that $|\phi(x)|^2 = O(e^{-\sqrt{\frac{\lambda}{2}}|x|})$ for large $|x|$. □

Note. The accurate statement of (42) is that for any $0 < \varepsilon < 1$, there is $C(\varepsilon) > 0$, so that

$$|\phi(x)|^2 \leqq C(\varepsilon)\, e^{-\sqrt{\frac{\lambda}{2}}(1-\varepsilon)|x|}, \qquad x \in \mathbf{R}^2.$$

It is interesting to see that the decay rate of $|\phi|^2$ in this case is independent of the property of the source term at infinity. This reveals a difference of solutions which are asymptotically the symmetric vacuum, characterized by $|\phi| = 0$, from the solutions which are asymptotically asymmetric vacua, characterized by $|\phi| = 1$. The former are called in the Chern-Simons model case non-topological solutions [JLW] in contrast to the latter, topological solutions, for which the integer N given in the flux formula (35) in Sect. IV is a topological invariant.

VII. A CONSTRAINED MINIMIZATION PROBLEM

We now go back to the classical case that the bare energy density $\mathcal{E}_0$ is as defined in (10) so that a finite-energy solution goes to the asymmetric vacua at infinity. From the discussion in Sect. IV, we see that such a solution carries a quantized magnetic flux given in (35). The integer N is unknown to us. In particular, it is not clear whether every integer N can be realized by the flux of a finite-energy solution according to the expression (35). Thus we are led to the following question.

Given an integer N, can we find a solution of the problem

$$\left.\begin{aligned} &\text{Minimize}\, E(\phi, A) = \int_{\mathbf{R}^2} \left\{ \mathcal{E}_0(\phi, A) - \frac{1}{2} F_{jk} F_{jk}^{\mathrm{ex}} \right. \\ &\qquad\qquad\qquad \left. (\text{or } A_j J_j^{\mathrm{ex}}) \right\} dx \\ &\text{subject to} \int_{\mathbf{R}^2} F_{12}\, dx = 2\pi N? \end{aligned}\right\} \tag{44}$$

There are only available results when the external field F_{jk}^{ex} or J_j^{ex} is absent. The problem is best understood in the self-dual case [Bo] when $\lambda = 1$ due to the work in [T1, T2, JT]: For any N, (44) has a $2N$-parameter family of solutions with the parameters characterizing the locations of the zeros of the scalar field ϕ. These solutions represent N non-interacting vortices. When $\lambda \neq 1$, it is shown in [P, BC] that, for each N, the energy E has a radially symmetric critical point satisfying the constraint in (44). However, it is not clear as to whether these radial solutions are absolute energy minimizers in the constraint class.

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