Semidifferentials, quadratic forms and fully nonlinear elliptic equations of second order

by

Michael G. CRANDALL

Department of Mathematics,
University of California, Santa Barbara
Santa Barbara, CA 93106

ABSTRACT. — We study the function $u(x) - v(y) - \dfrac{\lambda}{2} \| x - y \|^2$ to second order, when u is u. s. c. and v is l. s. c., near a point $(\hat{x}, \hat{y})$ where the maximum is attained. We obtain a sharpening of a result of P.-L. Lions and H. Ishii which implies comparison results for fully nonlinear elliptic equations of second order.

Key words : Viscosity solutions, nonlinear elliptic equations, semidifferentials.

RÉSUMÉ. — On étudie au second ordre la fonction $u(x) - v(y) - \dfrac{\lambda}{2} \| x - y \|^2$, quand u est s. c. s., v est s. c. i., au voisinage d'un point $(\hat{x}, \hat{y})$ où elle atteint son maximum. On en déduit notamment des résultats de comparaison pour des équations elliptiques non linéaires du second ordre.

INTRODUCTION AND STATEMENTS OF RESULTS

In this note we sharpen a bit a result of Ishii and Lions [3] concerning second order semidifferentials of functions which lies at the heart of the study of viscosity solutions of second order fully nonlinear elliptic equations. Moreover, around this kernel we weave an exposition of some basic facts concerning the comparison problem for viscosity solutions which we hope will make the recent important developments in this area more accessible than heretofore.

Let us recall the notions and attempt to motivate the statement. Let Ω be a bounded open subset of $\mathbb{R}^n$, $S^{n \times n}$ be the set of real symmetric $n \times n$ matrices and $u, v : \bar{\Omega} \to \mathbb{R}$ where $\bar{\Omega}$ is the closure of Ω. One of the principal methods in the viscosity theory of fully nonlinear elliptic equations was born in the study of first order equations and involves the consideration of the function $u(x) - v(y) - \dfrac{\lambda}{2} \|x - y\|^2$ near a maximum point $(\hat{x}, \hat{y}) \in \Omega \times \Omega$, so we assume that

$$(1) \quad u(x) - v(y) - \frac{\lambda}{2}\|x-y\|^2 \leqq u(\hat{x}) - v(\hat{y}) - \frac{\lambda}{2}\|\hat{x}-\hat{y}\|^2 \quad \text{for} \quad x, y \in \bar{\Omega}.$$

Above and later $\langle .,. \rangle$ and $\|.\|$ are used to denote the Euclidian inner-product and norm.

If we assume that u, v are twice differentiable at $(\hat{x}, \hat{y})$ in the sense of having second order expansions about $\hat{x}, \hat{y}$ of the form

$$(2) \quad \begin{aligned} u(x) &= u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x} \rangle + o(\|x-\hat{x}\|^2) \\ v(y) &= v(\hat{y}) + \langle q, y - \hat{y} \rangle + \frac{1}{2}\langle Y(y-\hat{y}), y-\hat{y} \rangle + o(\|y-\hat{y}\|^2) \end{aligned}$$

for some $X, Y \in S^{n \times n}$, $p, q \in \mathbb{R}^n$ and where the "little o" notation has its usual meaning, we would deduce from (1), (2) and the fact that first derivatives vanish and second derivatives are nonpositive at a maximum that $p = q = \lambda(\hat{x} - \hat{y})$ and

$$(3) \quad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqq \lambda \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

where the ordering is in the sense of quadratic forms. We ask to what extent we can make a similar statement if u, v are not smooth — indeed, we consider the situation when u is upper-semicontinuous and v is lower-semicontinuous.

The variants of the notions of derivatives we need are described next. In view of the fact that we are dealing with semicontinuous functions, it

will be necessary to have a notation that records function values as well as first and second order information. A point $(u(z), p, S) \in \mathbb{R}^n \times S^{n \times n}$ is a (second order) superdifferential of u at $z \in \bar{\Omega}$ if

$$(4) \qquad u(x) \leqq u(z) + \langle p, x-z \rangle + \frac{1}{2} \langle S(x-z), x-z \rangle + o(\|x-z\|^2).$$

Observe that we allow the possibility that $z \in \partial \Omega$ (the boundary of Ω). We let $\mathscr{D}^{2,+} u(z)$ denote the set of second order superdifferentials of u at z, i.e.,

$$(5) \qquad \mathscr{D}^{2,+} u(z) = \{ (u(z), p, S) \in \mathbb{R}^n \times S^{n \times n} \text{ such that (4) holds} \}.$$

Likewise, one defines the subdifferentials $\mathscr{D}^{2,-} v(z)$ by replacing u by v and reversing the inequality in (4), which amounts to $\mathscr{D}^{2,-} v(z) = -\mathscr{D}^{2,+} (-v)(z)$). The functions $\mathscr{D}^{2,+}$, $\mathscr{D}^{2,-}$ map $\bar{\Omega}$ to subsets of $\mathbb{R} \times \mathbb{R}^n \times S^{n \times n}$. If M is any metric space and Γ is any map of $\bar{\Omega}$ to the subsets of M, we define its graph $G(\Gamma) = \{ (x, m) \in \bar{\Omega} \times M : m \in \Gamma(x) \}$ and its closure $\bar{\Gamma}$ by $G(\bar{\Gamma}) = \overline{G(\Gamma)}$ where the overbar always stands for closure. Thus the closure $\bar{\mathscr{D}}^{2,+}$, $\bar{\mathscr{D}}^{2,-}$ of $\mathscr{D}^{2,+}$, $\mathscr{D}^{2,-}$ are defined. We also use the notations $D\varphi(x)$ to denote the gradient of a (classically) differentiable function and $D^2 \varphi(x)$ to denote the second derivative matrix of a twice differentiable function φ. We are going to prove the following results:

THEOREM 1. — *Let* $u, v: \bar{\Omega} \to \mathbb{R}$, u *be bounded and upper-semicontinuous and* v *be bounded and lower-semicontinuous. Let* $\lambda > 0$ *and* $(\hat{x}, \hat{y}) \in \bar{\Omega} \times \bar{\Omega}$ *satisfy* (1). *Then there are* X, Y $\in S^{n \times n}$ *such that*

$$(6) \qquad -4\lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqq 2\lambda \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and

$$(7) \qquad (u(\hat{x}), \lambda(\hat{x}-\hat{y}), X) \in \bar{\mathscr{D}}^{2,+} u(\hat{x}), \quad (v(\hat{y}), \lambda(\hat{x}-\hat{y}), Y) \in \bar{\mathscr{D}}^{2,-} v(\hat{y}).$$

Moreover, there is a Z $\in S^{n \times n}$ *such that* (7) *holds with* X $=$ Y $=$ Z *and*

$$(8) \qquad -\lambda I \leqq Z \leqq \lambda I.$$

In order to indicate why this is an interesting result, let us quickly recall the comparison problem for viscosity solutions of fully nonlinear second order elliptic equations. Let $F: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^{n \times n} \to \mathbb{R}$ be continuous and satisfy

$$(9) \qquad F(x, r, p, X) \leqq F(x, s, p, Y) \quad \text{whenever } r \leqq s, \; Y \leqq X,$$

and the arguments are in the domain of F. We let u, v have the continuity and boundedness properties assumed in the theorem and, moreover, assume that $F(x, u(x), p, S) \leqq 0$ [respectively, $F(x, v(x), p, S) \geqq 0$] for all $x \in \Omega$ and $(u(x), p, S) \in \mathscr{D}^{2,+} u(x)$ [respectively, $x \in \Omega$ and $(v(x), p, S) \in \mathscr{D}^{2,-} v(x)$] — in other words, we assume that u (respectively, v)

is a viscosity subsolution (respectively, supersolution) of the equation

(10) $F(x, w, Dw, D^2 w) = 0$

in Ω. Hereafter we usually drop the term "viscosity" and simply speak of sub- and supersolutions. For example, if $F(x, r, p, S) = r - (\text{trace}(S))^3 - f(x)$, then (10) is the equation $w - (\Delta w)^3 = f(x)$. Notice that if F is lower-semicontinuous (respectively, upper-semicontinuous), then $\mathscr{D}^{2,+} \mathscr{D}^{2,-}$ may be replaced by $\bar{\mathscr{D}}^{2,+}$ (respectively, $\bar{\mathscr{D}}^{2,-}$) in the inequalities defining subsolutions (respectively, supersolutions), so we may do both since F is assumed to be continuous. The comparison problem for the Dirichlet problem (for example) for (10) is to show that if u and v are a subsolution and a supersolution of (10) and $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω. The strategy for doing this is to assume to the contrary that $u(x) > v(x)$ for some $x \in \Omega$. It is then easy to see (using the assumption at the boundary) that for large $\lambda > 0$ if $(\hat{x}, \hat{y})$ has the property (1) then $\hat{x}, \hat{y} \in \Omega$. In view of the Theorem 1 and the assumptions, we then have

(11) $F(\hat{x}, u(\hat{x}), \lambda(\hat{x} - \hat{y}), X) \le 0 \le F(\hat{y}, v(\hat{y}), \lambda(\hat{x} - \hat{y}), Y)$

for some X, Y satisfying (6), and then one proceeds by deducing a contradiction as $\lambda \to \infty$ (which, of course requires some structure conditions on F, etc.). We give examples of this in Section 2. We remark that in practice it seems that the simpler structure of the assertions concerning the existence of a Z as at the end of the theorem suffices to treat many possibly degenerate equations, while the more detailed information in (6) is needed in strictly elliptic equations. The constants are sharp in (8) and perhaps not in (6) — one worsens the bound above from (3) in order to have a lower bound, and we do not know if the constant in the lower bound is sharp for a given upper bound, but this does not seem important. The reader will see from the proof that if the constant 2 in upper bound in (6) is replaced by $\dfrac{1}{1-2\varepsilon}$ and the 4 in the lower bound is replaced by $\dfrac{1}{\varepsilon}$ where $0 < 2\varepsilon < 1$, then the result remains true $\left((6) \text{ results from } \varepsilon = \dfrac{1}{4}\right)$. The application of Theorem 1 indicated above does not use the possibility that $(\hat{x}, \hat{y}) \in \partial(\Omega \times \Omega)$ allowed in the Theorem. However, this possibility is important for discussing more general boundary conditions as was pointed out to us by H. Ishii, and we are grateful to him for recommending the more general formulation (see also Remark 3 below for a still more general variant suggested by Ishii).

The first part of Theorem 1 is a sharpened version of Lemma IV. 1 of Ishii and Lions [3] which in turn was deduced from an argument of Ishii [2] which in turn relied on results of Jensen [4]. The statement of Theorem 1 is less mysterious than that of Lemma IV. 1 of [3] and has the virtues that

the lower bound does not contain a large parameter $\dfrac{1}{\varepsilon}$ depending on u, v in an uncontrolled way and the statement concerns every $(\hat{x}, \hat{y})$ for which (1) holds. While the range of application to equations like (10) is not really increased by these ameliorations, the tasks of writing proofs and discovering structure are less onerous when working with the cleaner statement and this is pleasing. Indeed, Theorem 1 (including such variants as will appear as the need arises) appears to us to be the right way to summarize the information needed for the uniqueness theory of viscosity solutions. As regards this theory, there have been recent significant advances following the path breaking (and now obsolete) results of Jensen [4]. Ishii and Lions [3] present a comprehensive overview from a point of view consistent (but necessarily a bit more complex in the absence of Theorem 1) with the remarks above, while Jensen [5] offers a competing presentation from another point of view as does Trudinger [8] in a special case. The reader is invited to consult these works and their references to obtain a balanced image of the area (the bibliography of [3] is large).

To the expert in this subject, the principal contributions of this note consist, perhaps, of the formulation of Theorem 1 and the following lemma concerning quadratic forms which corresponds to the fact that the theorem is true in the quadratic case:

LEMMA 2. — *Let* X, $Y \in S^{n \times n}$ *satisfy*

$$(12) \qquad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqq \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Then for $\varepsilon < \dfrac{1}{2}$, $(I - \varepsilon X)$ *and* $(I + \varepsilon Y)$ *are invertible and if*

$$(13) \qquad X^\varepsilon = X (I - \varepsilon X)^{-1}, \qquad Y_\varepsilon = Y (I + \varepsilon Y)^{-1},$$

then

$$(14) \qquad X \leqq X^\varepsilon \leqq Y_\varepsilon \leqq Y$$

and

$$(15) \qquad -\frac{1}{\varepsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqq \begin{pmatrix} X^\varepsilon & 0 \\ 0 & -Y_\varepsilon \end{pmatrix} \leqq \frac{1}{(1 - 2\varepsilon)} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Moreover, there is a $Z \in S^{n \times n}$ *such that*

$$(16) \qquad X \leqq Z \leqq Y \qquad and \qquad -I \leqq Z \leqq I.$$

We will prove the lemma in Section 2, while Theorem 1 is proved in Section 1 assuming the validity of the lemma. Addressing the experts, we point out that a principal feature of the lemma is the following: while the set of matrices satisfying (12) is unbounded, the regularization processes

(13) produce matrices which are bounded as indicated in (15) and favorably related (as in (14)) to the original ones in the ordering from the point of view of semidifferentials and this provides ample compactness for various arguments. Finally we mention that there is another path to the proof of the first part of Theorem 1 which parallels the route taken by Lions and Ishii rather closely and this path is also outlined in Section 2.

Having dispensed with the experts, we offer the novice something more. We will outline a full proof of Theorem 1 (although we will call on two results — one from Aleksandrov[1] and another from Jensen[4]) in a way we like (certainly no real news here, as this proof is largely a cleaning up of arguments from the references given above), thereby conveniently collecting the essential points needed to understand the uniqueness theory of viscosity solutions. Indeed, for example, with this proof, Theorem 1 and the sample use of this theorem in this context given in Section 2 in hand, one may peruse [3] quite comfortably and simplify other of the arguments there as well. In the very brief Section 3 we formulate a version of Theorem 1 appropriate to the discussion of fully nonlinear parabolic equations.

We have kept this note as brief as possible consistent with a certain completeness and remark that we intend to consider appropriate variants arising from changes of variables, unbounded domains, infinite dimensional considerations, etc., elsewhere.

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1. THE PROOF OF THEOREM 1

We begin the proof of Theorem 1 by noting that we may replace u, v by u/λ, v/λ and reduce to the case $\lambda = 1$ and we do so herafter. We will sketch the proof in a sequence of steps.

Step 0. — We may assume that $(\hat{x}, \hat{y}) \in \Omega$. Indeed, if we choose a sufficiently large constant M and extend u, v to functions u_e, v_e on $\mathbb{R}^n$ by setting $u_e(x) = -M$, $v_e(x) = M$ on $\mathbb{R}^n/\bar{\Omega}$, then a check of the definitions shows that (1) holds for u_e, v_e in place of u, v and a full neighborhood N of $\bar{\Omega}$ in place of $\bar{\Omega}$ and, moreover, for $\hat{x} \in \bar{\Omega}$, $(u_e(\hat{x}), p, X) \in \mathscr{D}^{2,+} u_e(\hat{x})$ (closed as a graph on N) if and only if $(u(\hat{x}), p, X) \in \bar{\mathscr{D}}^{2,+} u(\hat{x})$, etc.

Remark 3. — In fact, $\mathscr{D}^{2,+}$, $\mathscr{D}^{2,-}$ are local objects by their definitions and the above reduction does not require "$\bar{\Omega}$" to be the closure of an open set. Thus one can treat the situation in which u, v are defined on an arbitrary locally compact subset of $\mathbb{R}^n$.

Step 1. — We show that the result holds if u and v are twice differentiable at $(\hat{x}, \hat{y})$ in the sense that (2) holds. In this case we know that

$p = q = \hat{x} - \hat{y}$ and (3) holds. Since

$$(u(\hat{x}), \hat{x} - \hat{y}, X) \in \mathscr{D}^{2,\,+} u(\hat{x}), \ (v(\hat{y}), \hat{x} - \hat{y}, Y) \in \mathscr{D}^{2,\,-} v(\hat{y})$$

and we have (14), we will also have (directly from the definitions) that

$$(u(\hat{x}), \hat{x} - \hat{y}, X^{\varepsilon}) \in \mathscr{D}^{2,\,+} u(\hat{x}), \qquad (v(\hat{y}), \hat{x} - \hat{y}, Y_{\varepsilon}) \in \mathscr{D}^{2,\,-} v(\hat{y}).$$

Putting $\varepsilon = \dfrac{1}{4}$ and using (15) we find that the desired relations (6) hold with X_{ε}, Y^{ε} in place of X, Y. In order to produce Z with the properties of the final assertions of the Theorem, we simply use the final assertion of the lemma. Notice that we don't need to use the closures of $\mathscr{D}^{2,\,+}$, $\mathscr{D}^{2,\,-}$ here.

To obtain the general case, we make several approximations and take limits.

Step 2. — We show that if the result holds when the inequality (1) is strict for $(x, y) \neq (\hat{x}, \hat{y})$, then it holds in general. We replace $u(x)$, $v(y)$ by

$$u_{\delta}(x) = u(x) - \delta \|x - \hat{x}\|^{2}, \qquad v_{\delta}(y) = v(y) + \delta \|y - \hat{y}\|^{2}$$

where $\delta > 0$ so that $(\hat{x}, \hat{y})$ becomes a strict maximum after this replacement. Assuming that the result is true in this event, we have the existence of $X(\delta)$, $Y(\delta)$ such that

(17) $\quad (u_{\delta}(\hat{x}), \hat{x} - \hat{y}, X(\delta)) \in \bar{\mathscr{D}}^{2,\,+} u_{\delta}(\hat{x}), \qquad (v_{\delta}(\hat{y}), \hat{x} - \hat{y}, Y(\delta)) \in \bar{\mathscr{D}}^{2,\,-} v_{\delta}(\hat{y})$

and

(18) $$-4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqq \begin{pmatrix} X(\delta) & 0 \\ 0 & -Y(\delta) \end{pmatrix} \leqq 2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

However, recalling the definitions of u_{δ}, v_{δ}, (17) amounts to

$$(u(\hat{x}), \hat{x} - \hat{y}, X(\delta) + 2\,\delta I) \in \bar{\mathscr{D}}^{2,\,+} u(\hat{x}), \ (v(\hat{y}), \hat{x} - \hat{y}, Y(\delta) - 2\,\delta I) \in \bar{\mathscr{D}}^{2,\,-} v(\hat{y})$$

and since the set of symmetric matrices in any fixed interval in the ordering is compact, we may pass to a subsequential limit as $\delta \downarrow 0$ to obtain the existence of X, Y with the desired properties. The existence of Z follows similarly.

Step 3. — In this "step" we recall some standard facts. While not all of these facts are needed here, they tell a complete story. The utility of the approximation about to be introduced in this arena was first pointed out in Lions and Souganidis [7] and Jensen, Lions and Souganidis [6]. Let $\mathcal{O} \subset \mathbb{R}^{m}$ be open and bounded. Let $W : \bar{\mathcal{O}} \to \mathbb{R}$ be upper-semicontinuous and bounded

(19) $$|W| \leqq C.$$

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For $\varepsilon > 0$ we define $W^\varepsilon : \bar{\mathcal{O}} \to \mathbb{R}$ by

$$(20) \qquad W^\varepsilon(x) = \sup_{y \in \bar{\mathcal{O}}} \left(W(y) - \frac{1}{\varepsilon} \| x - y \|^2 \right).$$

Then the following things are true (and the first three are obvious).

(i) $W \leq W^\varepsilon \leq$ where C is from (19).

(ii) The map $x \mapsto W^\varepsilon(x) + \frac{1}{\varepsilon} \| x \|^2$ is the supremum of linear functions and is therefore convex.

(iii) W^ε is twice differentiable a. e. on $\mathcal{O}$. This follows from (ii) and the result of Aleksandrov[1] asserting that convex functions are twice differentiable a. e.

(iv) Since W is upper-semicontinuous, for each $x \in \bar{\mathcal{O}}$ there is a point $y_{\varepsilon, x} \in \bar{\mathcal{O}}$ such that

$$(21) \qquad W^\varepsilon(x) = W(y_{\varepsilon, x}) - \frac{1}{\varepsilon} \| y_{\varepsilon, x} - x \|^2$$

and any such point satisfies [by (i) and (21)]

$$(22) \qquad \| y_{\varepsilon, x} - x \|^2 \leq \varepsilon\, 2\, C.$$

(v) For $x \in \bar{\mathcal{O}}$, $W^\varepsilon(x) \downarrow W(x)$ and $\frac{1}{\varepsilon} \| y_{\varepsilon, x} - x \|^2 \to 0$ as $\varepsilon \downarrow 0$. This follows from (21), (22), (i) and the upper-semicontinuity of W.

(vi) If $\varphi \in C^2(\mathbb{R}^m)$, $\hat{x} \in \mathcal{O}$ and

$$(23) \qquad W^\varepsilon(x) - \varphi(x) \leq W^\varepsilon(\hat{x}) - \varphi(\hat{x}) \qquad \text{for} \quad x \in \mathcal{O},$$

then

$$(24) \quad W(y) - \varphi(y + \hat{x} - y_{\varepsilon, \hat{x}}) \leq W(y_{\varepsilon, \hat{x}}) - \varphi(\hat{x}) \quad \text{for} \quad y \in (\mathcal{O} + y_{\varepsilon, \hat{x}} - \hat{x}) \cap \mathcal{O}.$$

In particular, if $y_{\varepsilon, \hat{x}} \in \mathcal{O}$ so that y may range over a neighborhood of $y_{\varepsilon, \hat{x}}$, $y \to W(y) - \varphi(y + \hat{x} - y_{\varepsilon, \hat{x}})$ has a maximum at $y_{\varepsilon, \hat{x}}$

$$(25) \qquad (W(y_{\varepsilon, \hat{x}}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \in \mathscr{D}^{2, +} W(y_{\varepsilon, \hat{x}}),$$

and, as a consequence, if $x, y_{\varepsilon, x} \in \mathcal{O}$, then

$$(26) \quad (W^\varepsilon(x), p, X) \in \mathscr{D}^{2, +} W^\varepsilon(x) \text{ implies } (W(y_{\varepsilon, x}), p, X) \in \mathscr{D}^{2, +} W(y_{\varepsilon, x}).$$

The reader will deduce (24) from (23) quickly upon using the definition of W^ε, while (25) follows from the immediate fact that if $\psi \in C^2$ and $W - \psi$ has a maximum at z, then $(D\psi(z), D^2\psi(z)) \in \mathscr{D}^{2, +} W(z)$ and (26) then from the fact that all the values of $\mathscr{D}^{2, +} W$ can be so obtained.

(vii) Let $B_r(y)$ be the ball of radius r centered at y and $B_r = B_r(0)$. If $\varphi \in C^2(\mathbb{R}^m)$, $\hat{x} \in \mathcal{O}$ and (23) with strict inequality for $x \neq \hat{x}$ holds, then for

every $\delta > 0$ the set

$\{ \tilde{x} \in \mathcal{O} \cap B_\delta (\hat{x}) : \exists \, p \in B_\delta \ni x$

$\qquad\qquad \mapsto W^\varepsilon (x) - \varphi (x) + \langle p, \, x \rangle$ has a maximum at $\tilde{x} \}$

has positive measure. This follows from (ii) and Lemma 3.10 of Jensen [4] (which is itself a variant of earlier results by Aleksandrov and others). Recall, in applying this lemma, which may be read quite independently of the main text of [4], that convex functions are locally Lipschitz continuous.

(viii) If $\varphi \in C^2 (\mathbb{R}^m)$ and $W^\varepsilon - \varphi$ has a maximum at $\hat{x} \in \mathcal{O}$, then for every $\delta > 0$ there exists $p \in B_\delta$ such that $x \mapsto W^\varepsilon (x) - \varphi (x) + \langle p, \, x \rangle$ has a maximum at a point $x_p \in B_\delta (\hat{x})$ and W^ε is twice differentiable at x_p. This follows at once from (vii) and (iv).

(ix) If

$$W (x) - \varphi (x) \leqq W (\hat{x}) - \varphi (\hat{x}) \quad \text{for} \ \ x \in \bar{\mathcal{O}}$$

holds with strict inequality for $x \neq \hat{x}$ and x^ε is a maximum point of $W^\varepsilon - \varphi$, then $x^\varepsilon \to \hat{x}$, $W^\varepsilon (x^\varepsilon)$, $W (y_{\varepsilon, \, x^\varepsilon})$, $\to W (\hat{x})$ as $\varepsilon \downarrow 0$. To see this, observe that

$$W (x) - \varphi (x) \leqq W^\varepsilon (x) - \varphi (x) \leqq W^\varepsilon (x^\varepsilon) - \varphi (x^\varepsilon) \leqq W (y_{\varepsilon, \, x^\varepsilon}) - \varphi (x^\varepsilon)$$

so that if $\hat{x}$ is any limit of x^ε (and hence $y_{\varepsilon, \, x^\varepsilon}$) as $\varepsilon \downarrow 0$, then

$$W (x) - \varphi (x) \leqq \liminf_{\varepsilon \downarrow 0} W^\varepsilon (x^\varepsilon) - \varphi (x^\varepsilon)$$

$$\leqq \liminf_{\varepsilon \downarrow 0} W (y_{\varepsilon, \, x^\varepsilon}) - \varphi (x^\varepsilon) \leqq W (\tilde{x}) - \varphi (\tilde{x})$$

and we conclude that $\tilde{x} = \hat{x}$. *Moreover, choosing* $x = \hat{x} = \tilde{x}$, we learn that $W (y_{\varepsilon, \, x^\varepsilon})$, $W^\varepsilon (x^\varepsilon) \to W (\hat{x})$.

(x) The reader may skip this point at this time as it is not used except in an alternate proof of Theorem 1. If $\mathcal{O}$ is convex and λ, ε, $\delta > 0$, then

$$(27) \qquad\qquad (\lambda \, W^\varepsilon)^\delta (x) = \lambda \, W^{\varepsilon + \delta \lambda} (x) \qquad \text{for} \ \ x \in \bar{\mathcal{O}}.$$

The relation (28) follows from appropriately composing the identities

$$(\lambda \, W)^\varepsilon = \lambda W^{\lambda \varepsilon}$$

(which is trivially true for all ε, $\lambda > 0$) and if $\gamma > 0$, then

$$(W^\varepsilon)^\gamma (x) = W^{\varepsilon + \gamma} (x),$$

which is seen to be true upon changing the order in the iterated supremum process on the left and using the definitions provided that $\mathcal{O}$ is convex (or that a point y for which one has

$$(28) \qquad\qquad (W^\varepsilon)^\gamma (x) = W^\varepsilon (y) - \frac{1}{\gamma} \| y - x \|^2$$

lies in $\mathcal{O}$).

Step 4. — We complete the proof. Assume that $u(x) - v(y) - \dfrac{1}{2} \| x - y \|^2$

has a strict maximum at $(\hat{x}, \hat{y})$ in $\Omega \times \Omega$. Set $\mathcal{O} = \Omega \times \Omega$ and $W(x, y) = u(x) - v(y)$ on $\overline{\mathcal{O}}$ (so the pair (x, y) plays the role of x in the discussion of Step 3). Then (by (ix) above), $W^\varepsilon(x, y) - \dfrac{1}{2} \| x - y \|^2$ has a maximum at a point $(x_\varepsilon, y_\varepsilon)$ and $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{y})$ as $\varepsilon \downarrow 0$. Moreover, by (viii) and (ix), there are points $p_\varepsilon, q_\varepsilon \in \mathbb{R}^n$ of norm at most ε such that

$W^\varepsilon(x, y) - \dfrac{1}{2} \| x - y \|^2 + \langle p_\varepsilon, x \rangle + \langle q_\varepsilon, y \rangle$ has a maximum and two derivatives at a point $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)$ which is within ε of $(x_\varepsilon, y_\varepsilon)$. Since $W^\varepsilon(x, y) = u^\varepsilon(x) - v_\varepsilon(y)$ (where u^ε is defined via (20) and $v_\varepsilon = -(-v^\varepsilon)$), we conclude from Step 1 that as soon as ε is sufficiently small (as it also must be several places below) that there are $X(\varepsilon)$, $Y(\varepsilon)$, such that

$$(29) \qquad \begin{cases} (u^\varepsilon(\tilde{x}_\varepsilon), \tilde{x}_\varepsilon - \tilde{y}_\varepsilon - p_\varepsilon, X(\varepsilon)) \in \mathscr{D}^{2,+} u^\varepsilon(\tilde{x}_\varepsilon) \\ (v(\tilde{y}_\varepsilon), \tilde{x}_\varepsilon - \tilde{y}_\varepsilon + q_\varepsilon, Y(\varepsilon)) \in \mathscr{D}^{2,-} v_\varepsilon(\tilde{y}_\varepsilon) \end{cases}$$

$$(30) \qquad -4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqq \begin{pmatrix} X(\varepsilon) & 0 \\ 0 & -Y(\varepsilon) \end{pmatrix} \leqq 2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

while if

$$W^\varepsilon(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) = W(x_{\varepsilon, \tilde{x}_\varepsilon}, y_{\varepsilon, \tilde{y}_\varepsilon}) - \dfrac{1}{2} \| \tilde{x}_\varepsilon - x_{\varepsilon, \tilde{x}_\varepsilon} \|^2 - \dfrac{1}{2} \| \tilde{y}_\varepsilon - y_{\varepsilon, \tilde{y}_\varepsilon} \|^2$$

(26) then yields

$$(31) \qquad \begin{cases} (u(x_{\varepsilon, \tilde{x}_\varepsilon}), \tilde{x}_\varepsilon - \tilde{y}_\varepsilon - p_\varepsilon, X(\varepsilon)) \in \mathscr{D}^{2,+} u(x_{\varepsilon, \tilde{x}_\varepsilon}) \\ (v(y_{\varepsilon, \tilde{y}_\varepsilon}), \tilde{x}_\varepsilon - \tilde{y}_\varepsilon + q_\varepsilon, Y(\varepsilon)) \in \mathscr{D}^{2,-} v(y_{\varepsilon, \tilde{y}_\varepsilon}). \end{cases}$$

Since $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{y})$ as $\varepsilon \downarrow 0$ we also have $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) \to (\hat{x}, \hat{y})$ and then $(x_{\varepsilon, \tilde{x}_\varepsilon}, y_{\varepsilon, \tilde{y}_\varepsilon}) \to (\hat{x}, \hat{y})$ [by (iv)]. One shows that $u(x_{\varepsilon, \tilde{x}_\varepsilon}) \to u(\hat{x})$, $v(x_{\varepsilon, \tilde{x}_\varepsilon}) \to v(\hat{y})$ just as in Step 3 (ix) and then concludes that any limit point (X, Y) of $(X(\varepsilon), Y(\varepsilon))$ as $\varepsilon \downarrow 0$ satisfies (6), (7) (recall that we put $\lambda = 1$). The existence of Z is demonstrated similarly, and it's over.

2. THE PROOF OF LEMMA 2 AND AN APPLICATION OF THEOREM 1

To prove Lemma 2, we observe several things. First, if (12) holds, then

(32)
$$\begin{cases} \text{(i)} \ X < I, \\ \text{(ii)} \ -I < Y, \end{cases}$$

and

(33)
$$\begin{cases} \text{(i)} \ X \leqq X(1-X)^{-1} \leqq Y, \\ \text{(ii)} \ X \leqq Y(I+Y)^{-1} \leqq Y, \end{cases}$$

and, in fact, either pair of relations (32) (i), (33) (i) or (32) (ii), (33) (ii) are equivalent to (12). We work below with parts (ii) of these relations — the other pair are entirely similar. Let us first note that (12) is equivalent (by definition) to

(34)
$$\langle X x, x \rangle - \langle Y y, y \rangle \leqq \| x - y \|^2, \qquad \forall x, \ y \in \mathbb{R}^n$$

and choosing successively $y = 0$, $x = 0$, $x = y$ we learn that

(35)
$$X \leqq I, \ Y \geqq -I \quad \text{and} \quad X \leqq Y.$$

Since $Y \geqq -I$, if (32) (ii) does not hold, then there is a unit vector z such that $Y z = -z$. Putting $x = z$, $y = \alpha z$ in (12) then quickly yields a contradiction.

The second inequality in (33) (ii) is true for scalars $y > -1$ and so for Y, and the first inequality in (33) (ii) follows from minimizing the right-hand side of

$$\langle X x, x \rangle \leqq \| x - y \|^2 + \langle Y y, y \rangle$$

over y (as been previously noted in [3] and is the reason for out notations $-X^\varepsilon$, Y_ε are just given by the appropriate versions of (20) with $\mathcal{O} = \mathbb{R}^n$) and so the asserted equivalence is evident. Next we assert that for $\varepsilon < 1$.

(36)
$$(1-\varepsilon) X \leqq ((1-\varepsilon) Y_\varepsilon)(1 + (1-\varepsilon) Y_\varepsilon)^{-1} = (1-\varepsilon) Y (1+Y)^{-1}.$$

Indeed, it is simple algebraic fact that the asserted identity holds and then the inequality is valid by (33) (ii). From the equivalence noted above, we conclude from (36) that

(37)
$$\begin{pmatrix} (1-\varepsilon) X & 0 \\ 0 & -(1-\varepsilon) Y_\varepsilon \end{pmatrix} \leqq \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Next we perform the analogous operation again, thid time replacing $(1-\varepsilon) X$ by $((1-\varepsilon) X)^\delta$ with $\delta = \dfrac{\varepsilon}{(1-\varepsilon)}$ and note the algebraic fact that

$(\nu X)^{\delta} = \nu X^{\nu \delta}$ to discover that

$$(38) \qquad \begin{pmatrix} X^{\varepsilon} & 0 \\ 0 & -Y_{\varepsilon} \end{pmatrix} \leqq \frac{1}{1-2\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

The first inequality of (15) is trivial from (35) and we have established (15). The relations $X \leqq X^{\varepsilon}$ and $Y_{\varepsilon} \leqq Y$ are obvious.

In order to produce Z as in the second assertion of Lemma 2, we consider more generally the problem of whether or not we can solve (16) given only the information

$$(39) \qquad\qquad X \leqq Y, \qquad X \leqq I, \qquad -I \leqq Y$$

which is a consequence of (12). First we claim that this possible if X, Y commute. Indeed, we may then assume that X and Y are both diagonal and the problem reduces to the scalar case. However, if X, $Y \in \mathbb{R}$, $X \leqq 1$, $-1 \leqq Y$, and $X \leqq Y$, then either $0 \in [X, Y]$ (in which case we choose $Z = 0$) or $Y \leqq 0$ (in which case we choose $Z = Y$) or $X \geqq 0$ (in which case we choose $Z = X$). In all cases $-1 \leqq Z \leqq 1$.

Next we recall (33) (ii) and that $Y(I+Y)^{-1} \leqq I$ and use the case just discussed to conclude that there is a Z satisfying

$$Y(I+Y)^{-1} \leqq Z \leqq Y, \qquad -I \leqq Z \leqq I$$

and we are done.

It is interesting that the commutativity assumption is essential in a very strong way as is shown be the next example. For small $\varepsilon > 0$ we define X_{ε}, Y_{ε} as

$$X_{\varepsilon} = \begin{pmatrix} -3 & \dfrac{2}{\sqrt{\varepsilon}} \\ \dfrac{2}{\sqrt{\varepsilon}} & \dfrac{-1}{\varepsilon} - 2 \end{pmatrix}, \qquad Y_{\varepsilon} = \begin{pmatrix} -1 & 0 \\ 0 & \dfrac{1}{\varepsilon} - 2 \end{pmatrix}.$$

It is straightforward to check that for $\varepsilon \leqq 1$ $X_{\varepsilon} \leqq Y_{\varepsilon}$, $X_{\varepsilon} \leqq I$, $Y_{\varepsilon} \geqq -1$ and that if $X_{\varepsilon} \leqq Z_{\varepsilon} \leqq Y_{\varepsilon}$, then Z_{ε} must be unbounded as $\varepsilon \downarrow 0$. (The reader will not appreciate the pain involved in producing this example.)

We remark that there is nothing really finite dimensional in the discussion of Lemma 2 and the result is true in Hilbert spaces.

We also remark that the identity in (36) is a general (nonlinear) fact; indeed, it corresponds to part (x) of Step 3 in a slightly different technical setting ($\mathcal{C} = \mathbb{R}^m$). This observation corresponds to another proof of the first part of Theorem 1 parallel to the arguments leading to Lemma IV.1

of [3]. We sketch this path in a rather elegant manner which makes things quite transparent.

Let (1) hold with $\lambda = 1$ and assume, (without loss of generality, since the phenomenon under study is local) that Ω is a ball centered at $(\hat{x}, \hat{y})$. Replacing $u(x)$ by $u(x) - u(\hat{x}) - \langle \hat{x} - \hat{y}, x - \hat{x} \rangle$, $v(y)$ by $v(y) - v(\hat{y}) - \langle \hat{x} - \hat{y}, y - \hat{y} \rangle$ and then translating $(\hat{x}, \hat{y})$ to the origin, we achieve the following situation: u, $-v$ are bounded and upper-semicontinuous on Ω, which is a ball centered at the origin, and

$$(40) \quad u(0) = v(0) = 0, \qquad u(x) - v(y) \leq \frac{1}{2} \|x - y\|^2, \qquad \forall x, \; y \in \Omega.$$

We will simply say that $u \ll v$ if (40) holds. Then, with the obvious meaning, for $\frac{1}{2} > \varepsilon > 0$

$$(41) \qquad (1 - 2\varepsilon) u^{2\varepsilon} \ll (1 - 2\varepsilon) v_{2\varepsilon} \qquad \text{and} \qquad u \leq u^{2\varepsilon} \leq v_{2\varepsilon} \leq v.$$

To see this, note that (1) is equivalent to $u \leq v_2$ and then $(1 - \varepsilon) u \ll (1 - \varepsilon) v_{2\varepsilon}$ since $((1 - \varepsilon) v_{2\varepsilon})_2 = (1 - \varepsilon) v_2$ by (28) (or it's equivalent version for the subscripted approximations), and then continue as in the proof of Lemma 2 above to conclude quickly that (41) holds. Hence $u^{2\varepsilon}(x) - v_{2\varepsilon}(y) - \frac{1}{(1 - 2\varepsilon) 2} \|x - y\|^2$ has a maximum at the origin (0, 0). Using various parts of Step 3 of the proof of Theorem 1, one sees that there are X, Y $\in S^{n \times n}$ for which $(0, 0, X) \in \bar{\mathscr{D}}^{2, +} u^{2\varepsilon}(0)$, $(0, 0, Y) \in \bar{\mathscr{D}}^{2, -} v_{2\varepsilon}(0)$ and

$$(42) \qquad -\frac{1}{\varepsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{1}{(1 - 2\varepsilon)} \begin{pmatrix} I & -1 \\ -I & I \end{pmatrix}.$$

By Step 3, these points are also values of the closed semidifferentials of u, v at some points $(x_{\varepsilon, 0}, y_{\varepsilon, 0})$ such that

$$0 = u^{2\varepsilon}(0) = u(x_{\varepsilon, 0}) - \frac{\|x_{\varepsilon, 0}\|^2}{2\varepsilon}$$

$$0 = v_{2\varepsilon}(0) = v(y_{\varepsilon, 0}) - \frac{\|y_{\varepsilon, 0}\|^2}{2\varepsilon}.$$

Clearly $x_{\varepsilon, 0} = y_{\varepsilon, 0} = 0$ satisfy this relation. We can assume that it is the only such point upon replacing $u(x)$, $v(y)$ by $u(x) - \|x\|^4$ and $v(y) + \|y\|^4$ (which doesn't affect the semidifferentials at 0), and sketch is complete. We finally remark that we did not choose this (with appropariate expansion) as the primary presentation for several reasons. First, it does not seem to us to be the best way to reach the conclusion of Step 1 (the twice differentiable

case) and we regard this as an important orienting case. The "reason"
Step 1 holds is "really" Lemma 2. In a similar vein, one does not get the
insights provided by Lemma 2. Finally, we would need to introduce a
matrix lemma to obtain the conclusions involving Z in any case, and this
would decrease the unity of the presentation.

We turn to a typical comparison result one may obtain from Theorem
1. Let F satisfy (9) and, in addition, assume that there is a $\gamma > 0$ such that

$$(43) \qquad \text{F}(x, r, p, \text{S}) - \text{F}(x, s, p, \text{S}) \geq \gamma (r - s)$$

for all $x \in \Omega$, r, $s \in \mathbb{R}$ and $\text{S} \in \text{S}^{n \times n}$ with $r \geq s$. Assume also that there is a
nonnegative function ω and a number $\theta \in (0, 2]$ such that $\omega(0+) = 0$ and

$$(44) \quad \text{F}(y, r, \lambda(x-y), \text{X}) - \text{F}(x, r, \lambda(x-y), \text{Y}) \leq \omega(\|x-y\| + \lambda \|x-y\|^\theta)$$

for all x, $y \in \Omega$, $\lambda > 1$, $r \in \mathbb{R}$ and X, $\text{Y} \in \text{S}^{n \times n}$ which satisfy (6). We will
prove, for completeness, the following result. It is the variant of [3]
Theorem IV.1 which arises from replacing their Lemma IV.1 with our
Theorem 1. The enormous scope will be evident upon perusing [3], as will
the way to proceed with other variants. It is worth pointing out a couple
of simple examples from [3]. The condition (44) holds with $\theta = 2$ if

$$\text{F}(x, r, p, \text{X}) = r - \text{trace}(\text{A}(x)\text{X})$$

provided either that $\text{A}(x): \bar{\Omega} \to \text{S}^{n \times n}$ is Hólder continuous with an
exponent $> \dfrac{1}{2}$ and is strictly positive on $\bar{\Omega}$ (a strictly elliptic case) or has
the form $\text{A}(x) = \Sigma(x) \Sigma(x)^*$ where $\Sigma: \bar{\Omega} \to \text{S}^{n \times m}$ is Lipschitz continuous (a
possibly degenerate case). Moreover, one may form $\inf_\alpha \sup_\beta \text{F}_{\alpha, \beta}$ given a
family $\text{F}_{\alpha, \beta}$ each of which satisfies (44) with the same ω and remain within
the class, producing highly nonlinear examples.

THEOREM 4. — *Let* F *satisfy* (9), (43), *and* (44) *with* $\theta = 2$. *Let* u, v *have
the continuity and boundedness properties of Theorem* 1, *and be, respectively,
a subsolution and a supersolution of* (10). *If* $u \leq v$ *on* $\partial\Omega$, *then* $u \leq v$ *in* Ω.
Moreover, if one of u, v *is locally Hólder continuous in* Ω *with exponent*
$\sigma \in (0, 1]$, *then it suffices to have* $\theta > 2 - \sigma$.

Proof. — Assume to the contrary of the assertion that $u(x) > v(x)$ for
some $x \in \Omega$; the reader may easily check that then for large λ we have (1)
for some point $(\hat{x}, \hat{y}) \in \Omega \times \Omega$ and, moreover,

$$(45) \qquad\qquad \lambda \|\hat{x} - \hat{y}\|^2 \to 0 \quad \text{as } \lambda \to \infty$$

— this is all analogous to some points in Step 3 above. We therefore
have the existence of X, Y satisfying (6) such that (11) holds. Small

manipulations using the conditions assumed on F yield

$$(46) \qquad \gamma(u(\hat{x}) - v(\hat{y})) \leqq \omega(\|\hat{x} - \hat{y}\| + \lambda \|\hat{x} - \hat{y}\|^\theta).$$

Since we clearly have $u(x) - v(x) \leqq u(\hat{y}) - v(\hat{y})$ for $x \in \Omega$, if $\theta = 2$ (45), (46) imply that $u \leqq v$ upon letting $\lambda \to \infty$, a contradiction. If, for example, u is Hölder continuous with exponent σ, the relation (1) with $x = y = \hat{y}$ shows that

$$(47) \qquad \lambda \|\hat{x} - \hat{y}\|^2 \leqq K \|\hat{x} - \hat{y}\|^\sigma$$

so $\lambda \|\hat{x} - \hat{y}\|^\theta \to 0$ as $\lambda \to \infty$ provided $\theta > 2 - \sigma$ and we are done. We make the standard remark that if $\sigma = 1$, then (47) provides a bound on $\lambda(\hat{x} - \hat{y})$ and we only need (44) to hold when this bound is satisfied by $\lambda(x - y)$.

We will not pursue further applications here, but let us remark that the first part of the proof of the strictly elliptic case in [3] can be eliminated using Theorem 1.

3. THE PARABOLIC CASE

As a last topic, we briefly consider the case of parabolic equations and formulate a corresponding version of Theorem 1. If Ω is as above and $T > 0$, we put $\Omega_T = (0, T] \times \Omega$ and $\bar{\Omega}_T = [0, T] \times \bar{\Omega}$.

We consider a typical "parabolic equation" $F(t, x, u, u_t, Du, D^2 u) = 0$ in Ω_T where Du, $D^2 u$ refer to differentiations in the x-variables. Here

$$F: \quad \bar{\Omega}_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S^{n \times n} \to \mathbb{R}$$

and (degenerate) parabolicity corresponds to the condition

$(48) \quad F(t, x, r, a, p, X) \leqq F(t, x, s, b, p, Y)$ whenever $r \leqq s$, $a \leqq b$, $Y \leqq X$.

Let $u : \Omega_T \to \mathbb{R}$. We denote by $\mathscr{P}^{2,+}$, $\mathscr{P}^{2,-}$ the variant of the semidifferentials $\mathscr{D}^{2,+}$, $\mathscr{D}^{2,-}$ we will use in this parabolic situation and $(u(\hat{t}, \hat{x}), a, p, X) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times S^{n \times n}$ lies in $\mathscr{P}^{2,+} u(\hat{t}, \hat{x})$ if $(\hat{t}, \hat{x}) \in \bar{\Omega}_T$, $\hat{t} > 0$ and

$$(49) \quad u(t, x) \leqq u(\hat{t}, \hat{x}) + a(t - \hat{t}) + \langle p, x - \hat{x} \rangle$$
$$+ \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|t - \hat{t}| + \|x - \hat{x}\|^2)$$

for $t \leqq \hat{t}$ (etc.). We say that $u : \bar{\Omega}_T \to \mathbb{R}$ is a subsolution of $F = 0$ in Ω_T if u is bounded and upper-semicontinuous and

$$(50) \quad F(t, x, u(x, t), a, p, S) \leqq 0$$

$$\text{if} \quad (t, x) \in \Omega_T, \quad \text{and} \quad (u(x, t), a, p, S) \in \mathscr{P}^{2,+} u(t, x);$$

supersolutions are defined in the analogous way. We will also assume that F is coercive in u_t in the sense that

(51) $\lim\limits_{a \to \pm \infty} F(t, x, r, a, p, S) = \pm \infty$ uniformly for bounded t, x, r, p, S.

We have:

THEOREM 5. — *Let* u, $v : \Omega_T \to \mathbb{R}$ *be, respectively, sub and supersolutions of* F $= 0$ *in* Ω_T, $g \in C^1((0, T])$, $\hat{t}$, $\hat{x}$, $\hat{y} \in (0, T] \times \Omega \times \Omega$ *and*

(52) $u(t, x) - v(t, y) - g(t) - \dfrac{\lambda}{2} \| x - y \|^2 \leq u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) - g(\hat{t}) - \dfrac{\lambda}{2} \| \hat{x} - \hat{y} \|^2$

for x, $y \in \Omega$ *and* $0 < t \leq \hat{t}$. *Let* (51) *hold. Then there are* X, Y $\in S^{n \times n}$ *such that*

(53) $-4\lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 2\lambda \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$

and $a \in \mathbb{R}$ *such that*

(54) $(u(\hat{t}, \hat{x}), g'(\hat{t}) + a, \lambda(\hat{x} - \hat{y}), X) \in \bar{\mathscr{P}}^{2,+} u(\hat{t}, \hat{x})$,
            $(v(\hat{t}, \hat{y}), a, \lambda(\hat{x} - \hat{y}), Y) \in \bar{\mathscr{P}}^{2,-} v(\hat{t}, \hat{y})$.

Moreover, there are $a \in \mathbb{R}$ *and* Z $\in S^{n \times n}$ *such that* (54) *holds with* X $=$ Y $=$ Z *and*

(55) $-\lambda I \leq Z \leq \lambda I$.

The proof is a slight variation of the proof of Theorem 1 and will not be given. Observe, however the difference in the formulation — we assume here that u, v are sub and supersolutions of an equation for which (51) holds and no corresponding assumption is made in Theorem 1. The way this condition is used in the proof is just to obtain a bound on the t-component of the first order superdifferentials of (approximations of) $u(t, x) - v(s, y)$ at maximum points of

$$u(t, x) - v(s, y) - g(t) - M(t-s)^2 - \dfrac{\lambda}{2} \| x - y \|^2$$

as M $\to \infty$. It is a routine matter to establish the parabolic analogue of Theorem 3.

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