Ann. Inst. Henri Poincaré,

Vol. 6, n° 6, 1989, p. 437-480. *Analyse non linéaire*

An asymptotic expansion for the solution of the generalized Riemann problem. Part 2 : application to the equations of gas dynamics

by

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ABSTRACT. — We apply to the gas dynamics equations the general method of approximation for the generalized Riemann problem proposed by Le Floch-Raviart [6]. We both consider the equations of gas dynamics in Lagrangian coordinates in slab symmetry with a source term and the equations in Eulerian coordinates in plane, cylindrical or spherical symmetry. Explicit formulae are derived for the first order approximation of the solution of the generalized Riemann problem. Such an approximate solution may be easily used to construct a second order version of the numerical Godunov method.

Key words : Shock, nonlinear hyperbolic systems, asymptotic expansion, gas dynamics.

RÉSUMÉ. — Nous appliquons au système de la dynamique des gaz la méthode générale proposée par Le Floch-Raviart [6] pour l'approximation du problème de Riemann généralisé. Nous considérons à la fois les équations de la dynamique des gaz en coordonnées lagrangiennes en symétrie plane avec terme de source et les équations en coordonnées eulériennes en symétrie plane, cylindrique ou sphérique. Des formules explicites sont obtenues pour l'approximation du premier ordre de la solution du problème de Riemann généralisé. Ces résultats peuvent être facilement utilisés pour construire une version d'ordre 2 de la méthode de Godunov.

Classification A.M.S. : 35 L 65, 35 D 05, 35 C 05, 76 N 15, 35 L 67.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 6/89/06/437/44/$6.40/

1. INTRODUCTION

This paper is the second part of a work devoted to the approximation of the solution of the generalized Riemann problem. We study here the gas dynamics system in both Lagrangian and Eulerian coordinates and we apply the general method of approximation proposed in Le Floch-Raviart [6].

In the first part [6], we have considered systems of conservation laws of the form:

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\mathrm{U}+\frac{\partial}{\partial x}f(x, t, \mathrm{U})=g(x, t, \mathrm{U}),\\ &\mathrm{U}(x, t)\in\mathbb{R}^p, \qquad x\in\mathbb{R}, \qquad t>0,\end{aligned}\right\} \tag{1.1}$$

where f and g are smooth given functions. The *generalized Riemann problem* for (1.1) consists in solving the associated Cauchy problem with the initial condition:

$$\mathrm{U}(x, 0)=\mathrm{U}_{\mathrm{L}}(x) \quad \text{if } x<0, \qquad \mathrm{U}_{\mathrm{R}}(x) \quad \text{if } x>0, \tag{1.2}$$

where U_{L} and U_{R} are two smooth given functions. We assume that the system (1.1) is strictly hyperbolic; each characteristic field is supposed to be either genuinely nonlinear or linearly degenerate. First, we consider the following *classical Riemann problem*:

$$\frac{\partial}{\partial t}\mathrm{U}^0+\frac{\partial}{\partial x}f(\mathrm{U}^0)=0, \qquad \mathrm{U}^0(x, t)\in\mathbb{R}^p, \quad x\in\mathbb{R}, \quad t>0, \tag{1.3}$$

$$\mathrm{U}^0(x, 0)=\mathrm{U}_{\mathrm{L}}^0 \quad \text{if } x<0, \qquad \mathrm{U}_{\mathrm{R}}^0 \quad \text{if } x>0. \tag{1.4}$$

In (1.3)-(1.4), we have set:

$$f(\mathrm{U}^0)=f(0, 0, \mathrm{U}^0), \qquad \mathrm{U}_{\mathrm{L}}^0=\mathrm{U}_{\mathrm{L}}(0), \qquad \mathrm{U}_{\mathrm{R}}^0=\mathrm{U}_{\mathrm{R}}(0).$$

As it is well known (Smoller [9], for instance), for $|\mathrm{U}_{\mathrm{R}}^0-\mathrm{U}_{\mathrm{L}}^0|$ small enough, there exists a unique entropy solution of (1.3)-(1.4), depending only on the self-similarity variable $\xi=\dfrac{x}{t}$, which consists of $p+1$ constant states separated by p elementary waves (shock wave, contact discontibuity or rarefaction wave).

Then, let us recall that the problem (1.1)-(1.2) admits an entropy weak solution $\mathrm{U}=\mathrm{U}(x, t)$, locally defined in the neighbourhood of the origin $(x, t)=(0, 0)$, which has the same structure as that of U^0. For details, we refer to Li Tatsien-Yu Wenci [8] and Harabetian [5].

In [6], we looked for an *asymptotic expansion* for the function U in the form:

$$\mathrm{U}(x, t)=\sum_{k\in\mathbb{N}}\mathrm{U}^k(\xi).t^k, \qquad \xi=\frac{x}{t} \tag{1.5}$$

valid in each zone of smoothness of U. Each curve of discontinuity of U, say $x=\varphi(t)$, has also a Taylor expansion of the form:

$$\varphi(t)=\sum_{k \in \mathbb{N}} \sigma^k . t^{k+1}. \tag{1.6}$$

We have determined explicitely the functions $U^k: \mathbb{R} \to \mathbb{R}^p$ and each coefficient σ^k. Furthermore, we have given error bounds for the approximate solutions constructed from (1.5)-(1.6).

In this paper, we consider first the equations of gas dynamics in Lagrangian coordinates and in slab symmetry with a source term and a general equation of state. We study next the system of gas dynamics in Eulerian coordinates and in plane, cylindrical or spherical symmetry with an equation of state which is a generalization of that of a polytropic perfect gas. In both cases, we determine the first order approximation of the solution of (1.1)-(1.2), *i.e.* we derive explicit expressions for the function $U^1=U^1(\xi)$ of the expansion (1.5). These results are in fact identical with those of Ben Artzi-Falcovitz ([1], [2]) who have used a fairly different method.

Finally, let us recall that the approximate solution of (1.1)-(1.2) may be easily used to construct a second order version of the Godunov scheme. Such a numerical scheme appear to be very efficient in the numerical computation of very strong shocks ([1], Van Leer [10]).

An outline of the paper is as follows. Section 2 is devoted to the techniques of asymptotic expansion of Le Floch-Raviart [6] for a nonlinear hyperbolic system under a general nonconservative form. Then, in Section 3, we consider the system of gas dynamics in Lagrangian coordinates. In Section 4, the Eulerian system is treated.

2. OUTLINES OF THE GENERAL THEORY

Let us begin by recalling the techniques of asymptotic expansion of Le Floch-Raviart [6]. For the sake of simplicity, we shall restrict ourselves to systems of the form

$$\frac{\partial}{\partial t} U+\frac{\partial}{\partial x} f(U)=g(x, U). \tag{2.1}$$

In fact, it will be convenient in the applications to use nonconservative variables V such that

$$U=\Phi(V) \tag{2.2}$$

where Φ is a smooth diffeomorphism from a set $\mathcal{V} \subset \mathbb{R}^p$ into the set of states $\mathcal{U} \subset \mathbb{R}^p$. We first need to express the results of [6] in terms of the nonconservative variable V. In the domains of smoothness of the solution U of the generalized Riemann problem (2.1)-(1.2), the system (2.1) is equivalent to the *nonconservative system*:

$$\frac{\partial V}{\partial t}+B(V)\frac{\partial V}{\partial x}=h(x, V) \tag{2.3}$$

where the matrix B and the right hand side member h are given by

$$B(V)=D\Phi(V)^{-1}\,Df(\Phi(V)).D\Phi(V) \tag{2.4}$$

and

$$h(x, V)=D\Phi(V)^{-1}g(x, \Phi(V)). \tag{2.5}$$

Let us introduce some notations: the system (2.1) is assumed to be strictly hyperbolic, so that the eigenvalues of the matrix B(V) are real and distinct:

$$\lambda_1(V)<\lambda_2(V)<\ldots<\lambda_p(V), \qquad \forall\, V\in\mathscr{V}.$$

We denote by $\{r_i(V)\}_{1\leq i\leq p}$ a basis of corresponding right (column) eigenvectors and by $\{l_i(V)^T\}_{1\leq i\leq p}$ a basis of left (row) eigenvectors, *i. e.*

$$B(V)\,r_i(V)=\lambda_i(V)\,r_i(V), \qquad l_i(V)^T\,B(V)=\lambda_i(V)\,l_i(V)^T.$$

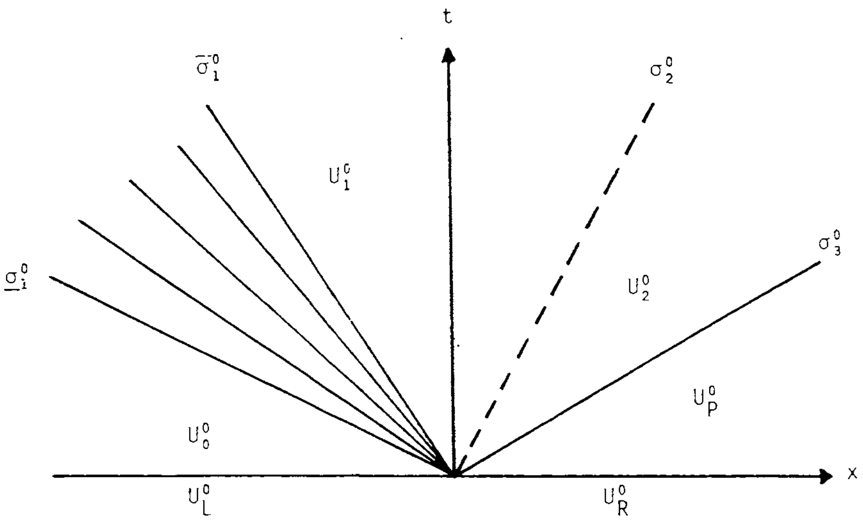

FIG. 2.1. — Classical Riemann problem.

Moreover, as usual we suppose that each i-characteristic field is either genuinely nonlinear

$$D\lambda_i(V)\,r_i(V)\neq 0, \qquad \forall\, V\in\mathscr{V}$$

or linearly degenerate

$$D\lambda_i(V)\, r_i(V)=0, \qquad \forall V \in \mathscr{V}.$$

Now, we look for an asymptotic expansion of the function $V=\Phi^{-1}(U)$ of the form

$$V(x, t)=\sum_{k \geq 0} t^k . V^k\left(\frac{x}{t}\right) \tag{2.6}$$

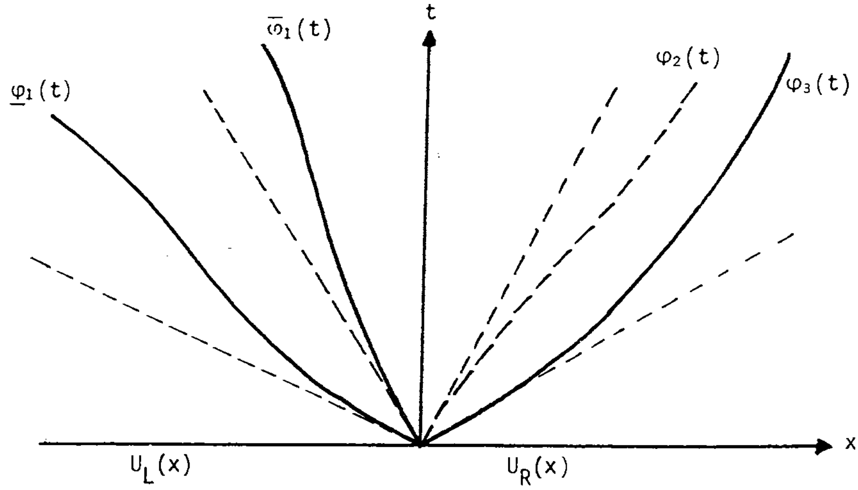

FIG. 2.2. – Generalized Riemann problem.

in the domains of smoothness of the solution U of (2.1)-(1.2). First, the zeroth order term $V^0=V^0(\xi)$ in (2.6) is given by

$$U^0=\Phi(V^0) \tag{2.7}$$

where $U^0(x, t)=U^0(\xi)\left(\xi=\frac{x}{t}\right)$ is the entropy weak solution of the *classical Riemann problem*

$$\left.\begin{aligned} &\frac{\partial U^0}{\partial t}+\frac{\partial}{\partial x} f(U^0)=0 \\ &U^0(x, t)=U_L^0 \quad \text{for } x<0, \qquad U_R^0 \quad \text{for } x>0. \end{aligned}\right\} \tag{2.8}$$

Thus, the function U^0 consists of $(p+1)$ constant states separated by rarefaction waves, shock waves or contact discontinuities. We denote by $\underline{\sigma}_i^0$ and $\bar{\sigma}_i^0$ the lower and upper bounds of the speeds of the i-wave

$(1 \leqq i \leqq p)$ and by U_i^0 $(0 \leqq i \leqq p)$ the $(p+1)$ constant states. If the i-wave is a shock wave or a contact discontinuity, we get

$$\underline{\sigma}_i^0 = \bar{\sigma}_i^0 \equiv \sigma_i^0.$$

For details concerning the classical Riemann problem, we refer for instance to [9].

Next, we note [8] that the function V has the same structure than the solution V^0 in a neighbourhood of the origin. The $(p+1)$ smoothness domains of V are separated either by a smooth curve denoted by $x = \varphi_i(t)$ such that

$$\varphi_i(0) = 0, \qquad \frac{d}{dt}\varphi_i(0) = \sigma_i^0$$

or by a rarefaction zone of the form

$$\{(x, t) \in \mathbb{R} \times \mathbb{R}_+;\ \underline{\varphi}_i(t) < x < \bar{\varphi}_i(t),\ 0 < t < \delta\}$$

where the curves $x = \underline{\varphi}_i(t)$ and $x = \bar{\varphi}_i(t)$ are smooth characteristic curves such that

$$\underline{\varphi}_i(0) = \bar{\varphi}_i(0) = 0, \qquad \frac{d}{dt}\underline{\varphi}_i(0) = \underline{\sigma}_i^0, \qquad \frac{d}{dt}\bar{\varphi}_i(0) = \bar{\sigma}_i^0.$$

We shall also look for an asymptotic expansion of the functions φ_i (and $\underline{\varphi}_i$, $\bar{\varphi}_i$) in the form

$$\varphi_i(t) = \sum_{b \geqq 0} \sigma_i^k . t^{k+1}. \tag{2.9}$$

If we compare the expansions (2.6) and

$$U(x, t) = \sum_{k \geqq 0} t^k U^k\left(\frac{x}{t}\right), \tag{2.10}$$

we obtain that

$$U^0 = \Phi(V^0), \qquad U^1 = D\Phi(V^0) V^1.$$

On the other hand, the higher order terms U^k, $k \geqq 2$, are related to the V^k in a more complicated way.

Replacing V by the expansion (2.6) in (2.3), we derive as in [6] the *ordinary differential equations* satisfied by the function V^k, $k \geqq 1$. We obtain:

$$(B(V^0) - \xi)\frac{d}{d\xi}V^k + (DB(V^0).V^k)\frac{d}{d\xi}V^0 + k.V^k = a^k, \tag{2.11}$$

where a^k is a function of ξ depending only on $V^0, \ldots, V^{k-1}$. It is a simple matter to check as in [6, Lemma 1], the following property: if V^1 is a polynomial of degree less than l for $0 \leqq l \leqq k-1$, then the function $\xi \mapsto a^k(\xi)$

is a polynomial of degree less than $k-1$. For $k=1$, we have

$$a^1=h(0,\ V^0)=D\Phi(V^0)^{-1}g(0,\ \Phi(V^0)).$$

In the sequel, it will be convenient to write (2.11) in the following way. Since for W_1, W_2 in $\mathbb{R}^p$, the function:

$$W \to (DB(W_1)\,W)\,W_2$$

is a linear mapping from $\mathbb{R}^p$ into $\mathbb{R}^p$, there exists a $p\times p$ matrix $C=C(W_1,\ W_2)$ such that

$$(DB(W_1)\,W)\,W_2=C(W_1,\ W_2)\,W. \tag{2.12}$$

Hence (2.11) becomes

$$(B(V^0)-\xi)\frac{dV^k}{d\xi}+\left(C\left(V^0,\ \frac{dV^0}{d\xi}\right)+k\right)V^k=a^k. \tag{2.13}$$

Note that the equation (2.13) is valid in each interval of smoothness of the function V^0.

For integrating the differential system (2.13), we have to distinguish between the two types of intervals: $]\bar{\sigma}_i^0,\ \underline{\sigma}_{i+1}^0[$ and $]\underline{\sigma}_i^0,\ \bar{\sigma}_i^0[$. Let us first consider the case of an interval $]\bar{\sigma}_i^0,\ \underline{\sigma}_{i+1}^0[$ where the function V^0 takes the constant value V_i^0, with the convention that $\bar{\sigma}_0^0=-\infty$, $\underline{\sigma}_{p+1}^0=+\infty$. We have the analogous of [6, Lemma 2]:

LEMMA 2.1. — *Assume that ξ belongs to the interval $]\bar{\sigma}_i^0,\ \underline{\sigma}_{i+1}^0[$, $0\leqq i\leqq p$. Then, for all $k\geqq 1$, the general solution of the differential system* (2.13) *is given by*

$$V^k(\xi)=(\xi-B(V_i^0))^k.\,V_i^k+P_i^k(\xi) \tag{2.14}$$

where V_i^k is an arbitrary vector of $\mathbb{R}^p$ and $\xi \to P_i^k(\xi)$ is a polynomial function of degree $\leqq k-1$ *with values in $\mathbb{R}^p$ which depends only on $V^0, \ldots, V^{k-1}$. Moreover, for $k=1$, we have*

$$P_i^1=h(0,\ V_i^0). \tag{2.15}$$

Let us next consider the case of an interval $]\underline{\sigma}_i^0,\ \bar{\sigma}_i^0[$ which corresponds to an i-rarefaction wave for V^0. It is easy to verify that:

$$\lambda_i(V^0(\xi))=\xi$$

$$\frac{d}{d\xi}V^0 \text{ is proportional to } r_i(V^0).$$

Using the basis of eigenvectors $\{r_i(V^0(\xi))\}_{1\leqq i\leqq p}$, we set

$$V^k=\sum_{j=1}^{p}\alpha_j r_j(V^0),\qquad a^k=\sum_{j=1}^{p}\gamma_j r_j(V^0). \tag{2.16}$$

Then we have

$$(\mathrm{B}(\mathrm{V}^0)-\xi)\frac{d}{d\xi}\mathrm{V}^k=(\mathrm{B}(\mathrm{V}^0)-\xi)\sum_j\left\{\frac{d}{d\xi}\alpha_j r_j(\mathrm{V}^0)+\alpha_j\frac{d}{d\xi}r_j(\mathrm{V}^0)\right\}$$
$$=\sum_j(\lambda_j(\mathrm{V}^0)-\xi)\frac{d}{d\xi}\alpha_j.r_j(\mathrm{V}^0)+\sum_j\alpha_j(\mathrm{B}(\mathrm{V}^0)-\xi)\frac{d}{d\xi}r_j(\mathrm{V}^0).$$

Now, we introduce $\omega_{jl}=\omega_{jl}(\mathrm{V}^0)$ and $\beta_{jl}=\beta_{jl}(\mathrm{V}^0)$, $1\leqq j,\ l\leqq p$, defined by

$$\begin{gathered}\frac{d}{d\xi}r_j(\mathrm{V}^0)=\sum_l\omega_{jl}(\mathrm{V}^0)\,r_l(\mathrm{V}^0),\\ \mathrm{C}\left(\mathrm{V}^0,\frac{d}{d\xi}\mathrm{V}^0\right)r_j(\mathrm{V}^0)=\sum_l\beta_{jl}(\mathrm{V}^0)\,r_l(\mathrm{V}^0).\end{gathered}\tag{2.17}$$

Hence, we obtain

$$\sum_j\alpha_j(\mathrm{B}(\mathrm{V}^0)-\xi)\frac{d}{d\xi}r_j(\mathrm{V}^0)=\sum_{j,\,l}\alpha_j(\lambda_l(\mathrm{V}^0)-\xi)\,\omega_{jl}(\mathrm{V}^0)\,r_l(\mathrm{V}^0)$$

and

$$\mathrm{C}\left(\mathrm{V}^0,\frac{d\mathrm{V}^0}{d\xi}\right)\mathrm{V}^k=\sum_{l,\,j}\beta_{jl}(\mathrm{V}^0)\,\alpha_j r_l(\mathrm{V}^0).$$

Thus, the differential system (2.13) may be equivalently written as

$$(\lambda_j(\mathrm{V}^0)-\xi).\left\{\frac{d}{d\xi}\alpha_j+\sum_{l=1}^{p}\omega_{lj}(\mathrm{V}^0)\,\alpha_l\right\}$$
$$+k\,\alpha_j+\sum_{l=1}^{p}\beta_{lj}(\mathrm{V}^0)\,\alpha_l=\gamma_j,\qquad 1\leqq j\leqq p.\tag{2.18}$$

Since $\lambda_i(\mathrm{V}^0(\xi))=\xi$, we obtain for $j=i$ a *purely algebraic relation*

$$k\,\alpha_i+\sum_{l=1}^{p}\beta_{li}(\mathrm{V}^0)\,\alpha_l=\gamma_i\tag{2.19}$$

which determines the coefficient α_i in terms of α_l, $l\neq i$. In fact, let us check that in (2.19) β_{ii} is equal to 1. By differentiating the relation

$$\mathrm{B}(\mathrm{V}^0)\,r_i(\mathrm{V}^0)=\xi\,r_i(\mathrm{V}^0)$$

with respect to ξ, we get

$$\left(\mathrm{DB}(\mathrm{V}^0)\frac{d\mathrm{V}^0}{d\xi}\right)r_i(\mathrm{V}^0)+\mathrm{B}(\mathrm{V}^0)\frac{d}{d\xi}r_i(\mathrm{V}^0)=r_i(\mathrm{V}^0)+\xi\frac{d}{d\xi}r_i(\mathrm{V}^0).$$

Since $\frac{dV^0}{d\xi}$ is proportional to $r_i(V^0)$, we have

$$\left(DB(V^0)\frac{dV^0}{d\xi}\right)r_i(V^0)=(DB(V^0)\,r_i(V^0))\frac{dV^0}{d\xi}=C\left(V^0,\frac{dV^0}{d\xi}\right)r_i(V^0)$$

so that

$$C\left(V^0,\frac{dV^0}{d\xi}\right)r_i(V^0)=r_i(V^0)+(\lambda_i(V^0)-B(V^0))\frac{d}{d\xi}r_i(V^0)$$
$$=r_i(V^0)+\sum_{l=1}^{p}(\lambda_i(V^0)-\lambda_l(V^0))\,\omega_{il}(V^0)\,r_l(V^0).$$

Thus we find

$$\left.\begin{aligned}\beta_{ii}(V^0)&=1\\ \beta_{il}(V^0)&=(\xi-\lambda_l(V^0))\,\omega_{il}(V^0),\qquad l\neq i,\end{aligned}\right\}\tag{2.20}$$

and (2.19) becomes

$$(k+1)\,\alpha_i+\sum_{l\neq i}\beta_{li}(V^0)\,\alpha_l=\gamma_i.\tag{2.21}$$

LEMMA 2.2. — *Assume that* ξ *belongs to the interval* $]\underline{\sigma}_i^0,\bar{\sigma}_i^0[$. *then, the differential system* (2.13) *is equivalent to the algebraic equation* (2.21) *and the* $(p-1)$ *coupled differential equations* (2.18) *for* $j\neq i$.

In order to obtain the function V^k, we need to determine the vectors $V_i^k\in\mathbb{R}^p$, $0\leqq i\leqq p$, introduced in Lemma 2.1. Let us then consider the *jump relations* satisfied by a function V^k at a discontinuity of V^0. First, at a shock or a contact discontinuity, we may write the Rankine-Hugoniot jump relation corresponding to the hyperbolic system (2.1):

$$\varphi_i'(t).\,[U]=[f(U)]\tag{2.22}$$

where [.] denotes the jump at the discontinuity $x=\varphi_i(t)$. By means of the diffeomorphism Φ, we obtain

$$\varphi_i'(t)\,[\Phi(V)]=[f(\Phi(V))].$$

Then, by using the expansions (2.6) and (2.9), we can prove as in [6, Lemma 4]:

LEMMA 2.3. — *Assume that* V^0 *contains an i-shock wave or a i-contact discontinuity. Then for all* $k\geqq 1$, *there exists a vector* $q_i^k\in\mathbb{R}^p$ *which depends only on* $V^0,\ldots,V^{k-1}$ *and* $\sigma_i^0,\ldots,\sigma_i^{k-1}$ *such that*

$$D\Phi(V_i^0).\,(B(V_i^0)-\sigma_i^0)^{k+1}\,V_i^k=D\Phi(V_{i-1}^0).\,(B(V_{i-1}^0)-\sigma_i^0)^{k+1}\,V_{i-1}^k$$
$$+(-1)^k(k+1)\,\sigma_i^k.\,(\Phi(V_i^0)-\Phi(V_{i-1}^0))-q_i^k.\tag{2.23}$$

Moreover for $k=1$, we have

$$q_i^1 = D\Phi(V_i^0)(B(V_i^0)-\sigma_i^0)\,h(0, V_i^0) - D\Phi(V_{i-1}^0)(B(V_{i-1}^0)-\sigma_i^0)\,h(0, V_i^0). \quad (2.24)$$

Let us turn to the case where the i-wave of V^0 is a rarefaction wave. Recall that $x=\underline{\varphi}_i(t)$ and $x=\bar{\varphi}_i(t)$ are the smooth characteristic curves which bound the corresponding rarefaction zone of V. We set

$$\underline{\varphi}_i(t) = \sum_{k\geq 0} \underline{\sigma}_i^k . t^{k+1}, \qquad \bar{\varphi}_i(t) = \sum_{k\geq 0} \bar{\sigma}_i^k . t^{k+1}.$$

Using the continuity of V across the rarefaction zone, we get as in [6, Lemma 5]:

LEMMA 2.4. — *Assume that V^0 contains an i-rarefaction wave. Then, for all $k\geq 1$, there exists two vectors $\underline{q}_i^k = \underline{q}_i^k(\underline{\sigma}_i^0, \ldots, \underline{\sigma}_i^{k-1}, V^0, \ldots, V^{k-1})$ and $\bar{q}_i^k = \bar{q}_i^k(\bar{\sigma}_i^0, \ldots, \bar{\sigma}_i^{k-1}, V^0, \ldots, V^{k-1})$ such that*

$$V^k(\underline{\sigma}_i^0+0) + \underline{\sigma}_i^k \frac{r_i(V_{i-1}^0)}{\nabla\lambda_i(V_{i-1}^0).r_i(V_{i-1}^0)} = (\underline{\sigma}_i^0 - B(V_{i-1}^0))^k V_{i-1}^k + \underline{q}_i^k \quad (2.25)$$

and

$$(\bar{\sigma}_i^0 - B(V_i^0))^k V_i^k - \bar{\sigma}_i^k \frac{r_i(V_i^0)}{\nabla\lambda_i(V_i^0).r_i(V_i^0)} = V^k(\bar{\sigma}_i^0 - 0) + \bar{q}_i^k. \quad (2.26)$$

For $k=1$, we have

$$\underline{q}_i^k = h(0, V_{i-1}^0), \qquad \bar{q}_i^k = -h(0, V_i^0). \quad (2.27)$$

Finally, we need to determine the vectors V_0^k and V_p^k. Assuming that the initial data V_L, V_R defined by

$$U_L = \Phi(V_L), \qquad U_R = \Phi(V_R)$$

are written in the form

$$V_L(x) = \sum_{k\geq 0} x^k V_L^k, \qquad V_R(x) = \sum_{k\geq 0} x^k V_R^k$$

we obtain as in [6, Lemma 6]:

LEMMA 2.5. — *For all integer $k\geq 0$, we have*

$$V_0^k = V_L^k, \qquad V_p^k = V_R^k. \quad (2.28)$$

Let us end this section by recalling the local existence and uniqueness result:

THEOREM 2.1 (Le Floch-Raviart [6]). — *Let $k\geq 1$ be an integer and suppose thet the function $V^l = V^l(\xi)$ and the numbers σ_i^l or the pairs $(\underline{\sigma}_i^l, \bar{\sigma}_i^l)$, $1\leq i\leq p$, have been already determined for $l=1, \ldots, k-1$. Then, if $|V_R^0 - V_L^0|$ is small enough, there exists a unique function $V^k = V^k(\xi)$ and* *a unique set of numbers* σ_i^k *or pairs* $(\underline{\sigma}_i^k, \bar{\sigma}_i^k)$, $1 \leqq i \leqq p$, *solutions of the equations* (2.13), (2.14), (2.23), (2.25) and (2.26).

3. THE GAS DYNAMICS EQUATIONS IN LAGRANGIAN COORDINATES

We begin by considering the gas dynamics equations in Lagrangian coordinates and in slab symetry with a source term [4]

$$\frac{\partial \tau}{\partial t} - \frac{\partial u}{\partial x} = 0, \tag{3.1 a}$$

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = h, \tag{3.1 b}$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(pu) = hu. \tag{3.1 c}$$

In (3.1), $\tau > 0$ is the specific volume, u the velocity, $p > 0$ the pressure, ε the internal energy per unit mass, $e = \varepsilon + \frac{1}{2} u^2$ the total energy per unit mass, and the constant h represents a source term. We supplement the system (3.1) with an equation of state of the form

$$p = p(\tau, \varepsilon).$$

As usual, it is more convenient to use the set of nonconservative variables $\{\tau, u, S\}$ where the specific entropy S is defined by

$$T\, dS = d\varepsilon + p\, d\tau,$$

and T denotes the temperature. Then, for smooth solutions, the system (3.1) becomes:

$$\frac{\partial \tau}{\partial t} - \frac{\partial u}{\partial x} = 0, \tag{3.2 a}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \tau} p(\tau, S) \frac{\partial \tau}{\partial x} + \frac{\partial}{\partial S} p(\tau, S) \frac{\partial S}{\partial x} = h, \tag{3.2 b}$$

$$\frac{\partial S}{\partial t} = 0. \tag{3.2 c}$$

Let us define the functions $g = g(\tau, S)$ (g is the Lagrangian sound speed) and $q = q(\tau, S)$ by

$$g^2 = -\frac{\partial p}{\partial \tau}, \qquad q = -\frac{\partial p}{\partial \tau} \Big/ \frac{\partial p}{\partial S}. \tag{3.3}$$

Thus, setting

$$\mathrm{V}=\begin{pmatrix}\tau\\ u\\ \mathrm{S}\end{pmatrix},\qquad \mathrm{B}(\mathrm{V})=\begin{pmatrix}0 & -1 & 0\\ -g^2 & 0 & g^2/q\\ 0 & 0 & 0\end{pmatrix},\qquad h=\begin{pmatrix}0\\ h\\ 0\end{pmatrix},\tag{3.4}$$

we get a nonconservative form of the system (3.1)

$$\frac{\partial}{\partial t}\mathrm{V}+\mathrm{B}(\mathrm{V})\frac{\partial \mathrm{V}}{\partial x}=h.\tag{3.5}$$

The eigenvalues of the matrix B (V) are

$$\lambda_1(\mathrm{V})=-g<\lambda_2(\mathrm{V})=0<\lambda_3(\mathrm{V})=g$$

and one can choose the right eigenvectors of B (V)

$$r_1(\mathrm{V})=\begin{pmatrix}1\\ g\\ 0\end{pmatrix},\qquad r_2(\mathrm{V})=\begin{pmatrix}1\\ 0\\ q\end{pmatrix},\qquad r_3(\mathrm{V})=\begin{pmatrix}1\\ -g\\ 0\end{pmatrix}.$$

As it is well-known, the first and third characteristic fields are genuinely nonlinear and the second one is linearly degenerate.

As an example, we consider the following useful generalization of the equation of state for a polytropic perfect gas

$$p(\tau,\varepsilon)=(\gamma-1)\frac{\varepsilon}{\tau}+\frac{\bar{c}^2.(\bar{\tau}-\tau)}{\bar{\tau}.\tau}\tag{3.6}$$

where $\bar{c}>0$, $\bar{\tau}>0$ and $\gamma>1$ are given constants. Setting

$$\bar{p}=\frac{\bar{c}^2}{\gamma\bar{\tau}},$$

we have equivalently

$$p(\tau,\mathrm{S})=\frac{\mathrm{S}}{\tau^\gamma}-\bar{p}.\tag{3.6'}$$

For such a gas, we get

$$g^2=\frac{\gamma}{\tau}(p+\bar{p})=\frac{\gamma.S}{\tau^{\gamma+1}},\qquad q=\frac{\gamma.S}{\tau}=\gamma(p+\bar{p})\tau^{\gamma-1}.$$

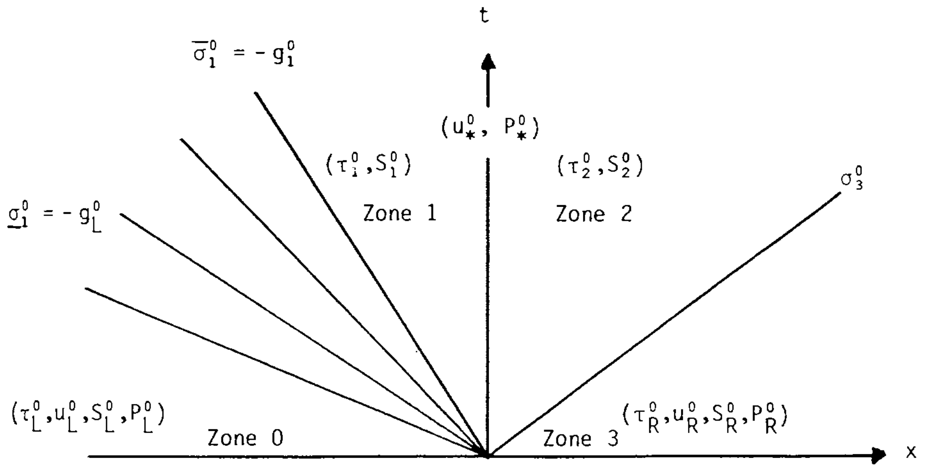

FIG. 3.1. – Zero order terms.

Now, for each state $V^0=\begin{pmatrix}\tau^0\\ u^0\\ S^0\end{pmatrix}$, the components $(\alpha_i)_{1\leqq i\leqq 3}$ of a state $V=\begin{pmatrix}\tau\\ u\\ S\end{pmatrix}$ on the basis $\{r_i(V^0)\}_{1\leqq i\leqq 3}$ are given by

$$\left.\begin{aligned}\alpha_1&=\frac{1}{2}\left(\frac{u}{g^0}+\tau-\frac{S}{q^0}\right),\\ \alpha_2&=\frac{S}{q^0},\\ \alpha_3&=\frac{1}{2}\left(-\frac{u}{g^0}+\tau-\frac{S}{q^0}\right),\end{aligned}\right\}\tag{3.7}$$

or equivalently

$$\left.\begin{aligned}\tau&=\alpha_1+\alpha_2+\alpha_3,\\ u&=g^0(\alpha_1-\alpha_3),\\ S&=q^0\alpha_2.\end{aligned}\right\}\tag{3.7'}$$

Furthermore for the source term, we have the decomposition

$$h(V)=\frac{h}{2g^0}(r_1(V^0)-r_3(V^0)).\tag{3.8}$$

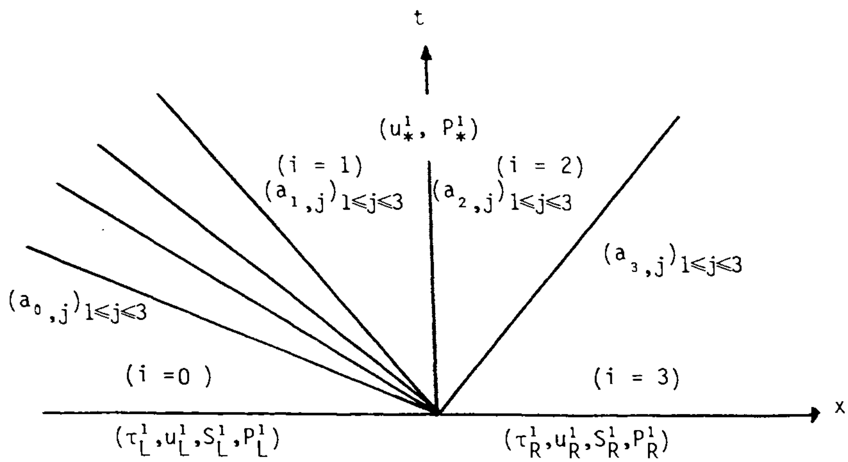

FIG. 3.2. — First order terms.

Let us apply the general theory recalled in Section 2. We now compute the first order approximation

$$V(x, t) = V^0\left(\frac{x}{t}\right) + t\,V^1\left(\frac{x}{t}\right) + \dots$$

of the solution V of the generalized Riemann problem for the system (3.1). Here $p=3$ and we denote by τ^0, u^0, p^0, c^0, g^0,... (respectively τ^1, u^1, p^1, c^1, g^1, ...) the quantities associated with the state V^0 (resp. V^1). Let us recall that the zones of smoothness of V^0 are numbered from $i=0$ to $i=3$. (See *Figs.* 3.1 and 3.2.)

Since V^0 is the entropy weak solution of the classical Riemann problem for (3.1), it consists of four constant states separated by i-waves, $1 \leqq i \leqq 3$. Recall that the 1- and 3-waves are rarefaction waves or shock waves, while the 2-wave is a stationnary contact discontinuity $\left(\frac{x}{t}=0\right)$. Moreover the velocity u^0 and the pressure p^0 are continuous at the contact discontinuity. Let us denote by u^0_* and p^0_* their respective values at $\xi=0$. In the sequel, we will assume that the functions V^0 has been determined explicitely: *see* for instance [4], [9].

Let us now compute the function $V^1 = V^1(\xi)$ of the expansion (2.6) associated with the system (3.1). As in Section 2, we set

$$V^1(\xi) = \sum_{1 \leqq j \leqq 3} \alpha_j(\xi)\, r_j(V^0(\xi)), \qquad \xi \in \mathbb{R}, \tag{3.9}$$

which defines the functions $\alpha_j=\alpha_j(\xi)$, $1\leqq j\leqq 3$. Then, using (3.8) and Lemma 2.1, we deduce the general form of the functions α_j in an interval where V^0 is constant.

LEMMA 3.1. — *In each interval* $]\bar{\sigma}_i^0, \underline{\sigma}_{i+1}^0[$, $i=0, 1, 2, 3$, *there exists three real constants* $a_{i,j}$, $j=1, 2, 3$, *such that*

$$\left.\begin{aligned}\alpha_1(\xi)&=(\xi+g_i^0)\,a_{i,1}+\frac{h}{2g_i^0},\\ \alpha_2(\xi)&=\xi\,.\,a_{i,2},\\ \alpha_3(\xi)&=(\xi-g_i^0)\,a_{i,3}-\frac{h}{2g_i^0}.\end{aligned}\right.\tag{3.10}$$

Next, we consider a 1-*rarefaction* zone $]\underline{\sigma}_1^0, \bar{\sigma}_1^0[=]-g_L^0, -g_1^0[$ or a 3-*rarefaction* zone $]\underline{\sigma}_3^0, \bar{\sigma}_3^0[=]g_2^0, g_R^0[$. Then using (2.18) and Lemma 2.2, we obtain the following expressions for the α_j's.

LEMMA 3.2. — *Assume that* V^0 *contains an i-rarefaction wave* ($i=1$ *or* 3). *Then, there exist two constants* $b_{1,2}$ *and* $b_{1,3}$ *for* $i=1$, *and* $b_{3,1}$ *and* $b_{3,2}$ *for* $1=3$, *such that for* $i=1$ *and* $\xi\in]-g_L^0, -g_1^0[$

$$\left.\begin{aligned}\alpha_1(\xi)&=-\frac{\xi}{4}\frac{d}{d\xi}(\operatorname{Log} q^0(\xi))\,\alpha_2(\xi)-\frac{1}{2}\alpha_3(\xi)+\frac{h}{4\xi},\\ \alpha_2(\xi)&=\xi\,.\,\frac{q_L^0}{q^0(\xi)}b_{1,2},\\ \alpha_3(\xi)&=\frac{-g_L^0}{4}\left(\frac{-g_L^0}{\xi}\right)^{1/2}\int_{-g_L^0}^{\xi}\left(\frac{y}{-g_L^0}\right)^{3/2}.\,\frac{d}{d\xi}\left(\frac{q_L^0}{q^0(y)}\right)dy\,.\,b_{1,2}\\ &\quad+-2g_L^0\,.\left(\frac{-g_L^0}{\xi}\right)^{1/2}b_{1,3}+\frac{h}{2\xi}\left(\left(\frac{-\xi}{g_L^0}\right)^{1/2}-1\right);\end{aligned}\right\}\tag{3.11}$$

and for $i=3$ *and* $\xi\in]g_2^0, g_R^0[$

$$\left.\begin{aligned}\alpha_1(\xi)&=-\frac{g_R^0}{4}\left(\frac{g_R^0}{\xi}\right)^{1/2}.\int_{\xi}^{g_R^0}\left(\frac{y}{g_R^0}\right)^{3/2}\frac{d}{d\xi}\left(\frac{q_R^0}{q^0(y)}\right)dy\,.\,b_{3,2}\\ &\quad+2g_R^0\left(\frac{g_R^0}{\xi}\right)^{1/2}.\,b_{3,1}+\frac{h}{2\xi}\left(\left(\frac{\xi}{g_R^0}\right)^{1/2}-1\right),\\ \alpha_2(\xi)&=\xi\frac{q_R^0}{q^0(\xi)}b_{3,2},\\ \alpha_3(\xi)&=-\frac{1}{2}\alpha_1(\xi)-\frac{\xi}{4}\frac{d}{d\xi}\operatorname{Log} q^0(\xi)\,.\,\alpha_2(\xi)+\frac{h}{4\xi}.\end{aligned}\right\}\tag{3.12}$$

In (3.11) *and* (3.12), $q^0=q^0(\xi)$ *is the function defined by* (3.3).

Proof. — Starting from (3.4), elementary calculations show that

$$(\mathrm{DB}(\mathrm{V}^0).\mathrm{V}^1)\frac{d\mathrm{V}^0}{d\xi}=\left(0,\ -2g^0\left(\left(\frac{\partial g}{\partial \tau}\right)^0\tau^1+\left(\frac{\partial g}{\partial \mathrm{S}}\right)^0\mathrm{S}^1\right)\frac{d}{d\xi}\tau^0,\ 0\right)^{\mathrm{T}},$$

or equivalently by the definition of C:

$$\mathrm{C}\left(\mathrm{V}^0,\frac{d\mathrm{V}^0}{d\xi}\right)\mathrm{V}^1=-\left\{\left(\frac{\partial g}{\partial \tau}\right)^0\tau^1+\left(\frac{\partial g}{\partial \mathrm{S}}\right)^0\mathrm{S}^1\right\}\frac{d}{d\xi}\tau^0.\ (r_1(\mathrm{V}^0)-r_3(\mathrm{V}^0)).$$

Hence, we have

$$\left.\begin{aligned}
\mathrm{C}\left(\mathrm{V}^0,\frac{d\mathrm{V}^0}{d\xi}\right)r_1(\mathrm{V}^0)&=-\left(\frac{\partial g}{\partial \tau}\right)^0\frac{d\tau^0}{d\xi}.\ (r_1(\mathrm{V}^0)-r_3(\mathrm{V}^0)),\\
\mathrm{C}\left(\mathrm{V}^0,\frac{d\mathrm{V}^0}{d\xi}\right)r_2(\mathrm{V}^0)&\\
&=-\left\{\left(\frac{\partial g}{\partial \tau}\right)^0+\left(\frac{\partial g}{\partial \mathrm{S}}\right)^0q^0\right\}\frac{d\tau}{d\xi}.(r_1(\mathrm{V}^0)-r_3(\mathrm{V}^0)),\\
\mathrm{C}\left(\mathrm{V}^0,\frac{d\mathrm{V}^0}{d\xi}\right)r_3(\mathrm{V}^0)&=-\left(\frac{\partial g}{\partial \tau}\right)^0\frac{d\tau^0}{d\xi}.\ (r_1(\mathrm{V}^0)-r_3(\mathrm{V}^0)).
\end{aligned}\right\}\quad(3.13)$$

Furthermore, we may write

$$\left.\begin{aligned}
\frac{d}{d\xi}r_1(\mathrm{V}^0)&=\left(0,\frac{dg^0}{d\xi},0\right)^{\mathrm{T}}=\frac{1}{2g^0}\frac{dg^0}{d\xi}.\ (r_1(\mathrm{V}^0)-r_3(\mathrm{V}^0))\\
\frac{d}{d\xi}r_2(\mathrm{V}^0)&=\left(0,\ 0,\ \frac{dq^0}{d\xi}\right)^{\mathrm{T}}\\
&=\frac{1}{q^0}\frac{dq^0}{d\xi}.\left(-\frac{1}{2}r_1(\mathrm{V}^0)+r_2(\mathrm{V}^0)-\frac{1}{2}r^3(\mathrm{V}^0)\right)\\
\frac{d}{d\xi}r_3(\mathrm{V}^0)&=\left(0,\ -\frac{dg^0}{d\xi},\ 0\right)=\frac{1}{2g^0}\frac{dg^0}{d\xi}.\ (-r_1(\mathrm{V}^0)+r_3(\mathrm{V}^0)).
\end{aligned}\right\}\quad(3.14)$$

Then, consider for instance the case of a 1-rarefaction zone, for which we know that

$$\frac{d}{d\xi}\mathrm{S}^0(\xi)=0,\qquad g^0(\xi)=-\xi.$$

Using (3.13) and (3.14), the differential system (2.18) for the functions $\alpha_j(\xi)$ becomes here:

$$\left.\begin{aligned}
&\alpha_1-\left(\frac{\partial g}{\partial \tau}\right)^0\{\alpha_1+\alpha_2+\alpha_3\}\frac{d\tau^0}{d\xi}-\left(\frac{\partial g}{\partial S}\right)^0\frac{d\tau^0}{d\xi}q^0\alpha_2=\frac{h}{2\xi},\\
&-\xi.\left(\frac{d\alpha_2}{d\xi}+\frac{1}{q^0}\frac{dq^0}{d\xi}.\alpha_2\right)+\alpha_2=0,\\
&-2\xi\left(\frac{d\alpha_3}{d\xi}+\frac{1}{2\xi}(-\alpha_1+\alpha_3)-\frac{1}{2q^0}\frac{dq^0}{d\xi}.\alpha_2\right)+\alpha_3\\
&\quad+\left(\frac{\partial g}{\partial \tau}\right)^0\frac{d\tau^0}{d\xi}\{\alpha_1+\alpha_2+\alpha_3\}+\left(\frac{\partial g}{\partial S}\right)^0\frac{d\tau^0}{d\xi}\alpha_2=-\frac{h}{2\xi}.
\end{aligned}\right\}$$

But an easy calculation gives

$$\left(\frac{\partial g}{\partial \tau}\right)^0\frac{d\tau^0}{d\xi}=-1,\qquad 1-\left(\frac{\partial g}{\partial S}\right)^0 q^0\frac{d\tau^0}{d\xi}=-\frac{\xi}{2q^0}\frac{d}{d\xi}q^0,$$

so that

$$2\alpha_1+\frac{\xi}{2q^0}\frac{dq^0}{d\xi}.\alpha_2+\alpha_3=\frac{h}{2\xi},\qquad (3.15\,a)$$

$$\frac{d}{d\xi}(q^0\alpha_2)=\frac{1}{\xi}q^0\alpha_2,\qquad (3.15\,b)$$

$$-2\xi\frac{d\alpha_3}{d\xi}+\frac{\xi}{2q^0}\frac{dq^0}{d\xi}.\alpha_2-\alpha_3=-\frac{h}{2\xi}.\qquad (3.15\,c)$$

Now, it is an easy matter to obtain the general solution of the system (3.15). In fact, (3.15 *b*) is an ODE for the function α_2, whose general solution is obvious. Next, knowing α_2, (3.15 *c*) gives an ODE for α_3 which is again easily solved. Finally, the algebraic equation (3.15 *a*) gives α_1. ■

In the particular case of the equation of state (3.6), we can obtain more explicit formulae.

LEMMA 3.3. — *Assume that* (3.6) *holds. Then, the functions* $(\alpha_j)_{1\leq j\leq 3}$ *of Lemma* 3.2 *are given for* $\xi\in]-g_L^0, g_1^0[$ *by*

$$\left.\begin{aligned}
&\alpha_1(\xi)=\frac{1}{2(\gamma+1)}\alpha_2(\xi)-\frac{1}{2}\alpha_3(\xi)+\frac{h}{4\xi},\\
&\alpha_2(\xi)=-\left(\frac{\xi}{-g_L^0}\right)^{(\gamma-1)/(\gamma+1)}.g_L^0.b_{1,2},\\
&\alpha_3(\xi)=\frac{g_L^0}{3\gamma-1}\left(\frac{-g_L^0}{\xi}\right)^{1/2}.\left\{-1+\left(\frac{\xi}{-g_L^0}\right)^{(3\gamma-1)/2(\gamma+1)}\right\}b_{1,2}\\
&\quad+-2g_L^0\left(\frac{-g_L^0}{\xi}\right)^{1/2}.b_{1,3}+\frac{h}{2\xi}\left(\left(\frac{-\xi}{g_L^0}\right)^{1/2}-1\right);
\end{aligned}\right\}\qquad (3.16)$$

and for $\xi \in]g_2^0, g_R^0[$ *by*

$$\left.\begin{aligned}\alpha_1(\xi) = \frac{+g_R^0}{3\gamma-1}\left(\frac{g_R^0}{\xi}\right)^{1/2}\left\{-\left(\frac{\xi}{g_R^0}\right)^{(3\gamma-1)/2(\gamma+1)}+1\right\}b_{3,2}\\ +2g_R^0\left(\frac{g_R^0}{\xi}\right)^{1/2}b_{3,1}+\frac{h}{2\xi}\left(\left(\frac{\xi}{g_R^0}\right)^{1/2}-1\right),\\ \alpha_2(\xi) = g_R^0 \cdot \left(\frac{\xi}{g_R^0}\right)^{(\gamma-1)/(\gamma+1)} \cdot b_{3,2},\\ \alpha_3(\xi) = -\frac{1}{2}\alpha_1(\xi)+\frac{1}{2(\gamma+1)}\alpha_2(\xi)+\frac{h}{4\xi}.\end{aligned}\right\} \tag{3.17}$$

Proof. — Consider for instance the case of 1-rarefaction wave. We know that

$$S^0(\xi) = \text{cte}, \qquad -g^0(\xi) = \xi.$$

Thus, it is a simple matter to show that

$$p^0(\xi) = \left(\frac{\xi}{-g_L^0}\right)^{2\gamma/(\gamma+1)} \cdot p_L^0 + \left(\left(\frac{\xi}{-g_L^0}\right)^{2\gamma/(\gamma+1)}-1\right)\frac{\overline{c}^2}{\gamma\overline{\tau}}$$

and

$$\tau^0(\xi) = \left(\frac{\xi}{-g_L^0}\right)^{-2/(\gamma+1)} \cdot \tau_L^0.$$

Therefore, we get the following expression for the function $q_0 = q_0(\xi)$:

$$\begin{aligned}q^0(\xi) = \frac{\gamma S_L^0}{\tau^0(\xi)} = \frac{\gamma S_L^0}{\tau_L^0}\left(\frac{-g_L^0}{\xi}\right)^{-2/(\gamma+1)}\\ = (\tau_L^0)^\gamma \cdot (g_L^0)^2 \cdot \left(\frac{-g_L^0}{\xi}\right)^{-2/(\gamma+1)},\end{aligned}$$

so that

$$q^0(\xi) = \left(\frac{\xi}{-g_L^0}\right)^{-2/(\gamma+1)} \cdot (g_L^0)^2 \cdot (\tau_L^0)^\gamma. \tag{3.18}$$

We obtain

$$\frac{d}{d\xi}\operatorname{Log} q^0(\xi) = -\frac{1}{-g_L^0} \cdot \frac{2}{\gamma+1}\left(\frac{-g_L^0}{\xi}\right) = \frac{-2}{(\gamma+1)\xi}$$

and

$$\int_{-g_L^0}^{\xi}\left(\frac{y}{-g_L^0}\right)^{3/2}\cdot\frac{d}{d\xi}\left(\frac{q_L^0}{q^0(y)}\right)dy=\int_{-g_L^0}^{\xi}\frac{-2}{\gamma+1}\cdot\left(\frac{y}{-g_L^0}\right)^{(3/2)-(2/(\gamma+1))-1}dy$$
$$=\frac{4}{3\gamma-1}\cdot\left(1-\left(\frac{\xi}{-g_L^0}\right)^{(3\gamma-1)/2(\gamma+1)}\right).$$

Hence the result follows from Lemma 3.2. ■

Let us now derive the boundary relations satisfied by the function V^1 at the discontinuities of V^0. Consider first rarefaction waves. From Lemmas 2.4 and 3.2, we have:

LEMMA 3.4. — *If* V^0 *contains a* 1-*rarefaction wave, the constants* $b_{1,2}$ *and* $b_{1,3}$ *of Lemma* 3.2 *satisfy*

$$b_{1,2}=a_{0,2},\qquad b_{1,3}=a_{0,3}. \tag{3.19}$$

Moreover, we have the continuity properties at $\xi=-g_1^0$

$$\alpha_j(-g_1^0-0)=\alpha_j(-g_1^0+0),\qquad j=2,3. \tag{3.20}$$

If V^0 *contains a* 3-*rarefaction wave, the constants* $b_{3,2}$ *and* $b_{3,2}$ *of Lemma* 3.2 *satisfy*

$$b_{3,1}=a_{3,1},\qquad b_{3,2}=a_{3,2}. \tag{3.21}$$

And we have the continuity properties at $\xi=g_2^0$

$$\alpha_j(g_2^0+0)=a_j(g_2^0-0),\qquad j=1,2. \tag{3.22}$$

Define the function $p^1=p^1(\xi)$ by $p^1=\left(\frac{\partial p}{\partial\tau}\right)^0\tau^1+\left(\frac{\partial p}{\partial S}\right)^0 S^1$. Then at the contact discontinuity, we get

LEMMA 3.5. — *The functions* $u^1=u^1(\xi)$ *and* $p^1=p^1(\xi)$ *are continuous at* $\xi=0$.

Proof. — At the contact discontinuity, the Rankine-Hugoniot jump relations for the system (3.1) become

$$[u]=[p]=0.$$

Following Lemma 2.3, we write

$$[u^0+tu^1+\ldots]=[p^0+tp^1+\ldots]=0$$

which yields

$$[u^0]=[p^0]=0$$

and

$$[u^1]=[p^1]=0.$$

Define the coefficients $a_{L,j}$ and $a_{R,j}$, $1 \leqq j \leqq 3$ by

$$V_L^1 = \sum_{j=1}^{3} a_{L,j} r_j(V_L^0), \qquad V_R^1 = \sum_{j=1}^{3} a_{R,j} r_j(V_R^0). \tag{3.23}$$

Then Lemma 2.5 enables us to determine the constants $(a_{i,j})_{1 \leqq j \leqq 3}$ of Lemma 3.1 for $i=0$ and $i=3$.

LEMMA 3.6. – *We have for* $1 \leqq j \leqq 3$

$$a_{0,j} = a_{L,j}, \qquad a_{3,j} = a_{R,j}. \tag{3.24}$$

Let us next derive the *jump relations* satisfied by the function V^1 at a shock discontinuity of V^0. Let us assume that a state (τ, u, p) is connected to a state (τ_a, u_a, p_a) by a shock discontinuity. Introducing the Hugoniot function $H_a(\tau, p)$

$$H_a(\tau, p) = \varepsilon(\tau, p) - \varepsilon(\tau_a, p_a) + \frac{1}{2}(p + p_a)(\tau - \tau_a), \tag{3.25}$$

we have [4]

$$H_a(\tau, p) = 0$$

which defines a function

$$\tau = G_a(p) = G(\tau_a, p_a, p). \tag{3.26}$$

Setting

$$\Phi_a(p) = \sqrt{(p - p_a)(\tau_a - G_a(p))} = \Phi(\tau_a, p_a, p),$$

it is a classical matter to check that

$$u = u_a + \varepsilon \Phi_a(p), \qquad \tau = G_a(p) \tag{3.27}$$

where $\varepsilon = -1$ for a 1-shock wave and $\varepsilon = 1$ for a 3-shock wave.

By writting

$$\tau^0 + t\tau^1 + \ldots = G(\tau_a^0 + t\tau_a^1 + \ldots, p_a^0 + t p_a^1 \ldots, p^0 + t p^1 + \ldots),$$

and

$$u^0 + t u^1 + \ldots = u_a^0 + t u_a^1 + \ldots + \varepsilon \Phi(\tau_a^0 + t\tau_a^1 + \ldots, p_a^0 + t p_a^1 + \ldots, p^0 + t p^1 + \ldots)$$

we obtain when V^0 contains a 1 or a 3 shock wave:

$$u^1 = u_a^1 + \varepsilon \left\{ \left(\frac{\partial \Phi}{\partial p_a}\right)^0 \tau_a^1 + \left(\frac{\partial \Phi}{\partial p_a}\right)^0 p_a^1 + \left(\frac{\partial \Phi}{\partial p}\right)^0 p^1 \right\}, \tag{3.28}$$

and

$$\tau^1 = \left(\frac{\partial G}{\partial \tau_a}\right)^0 \tau_a^1 + \left(\frac{\partial G}{\partial p_a}\right)^0 p_a^1 + \left(\frac{\partial G}{\partial p}\right)^0 p^1. \tag{3.29}$$

Here, for a 1-shock discontinuity we have set

$$\tau^1=\tau^1(\sigma_1^0+0),\qquad \tau_a^1=\tau^1(\sigma_1^0-0),\text{ etc.}$$

and for a 3-shock discontinuity:

$$\tau^1=\tau^1(\sigma_3^0-0),\qquad \tau_a^1=\tau^1(\sigma_1^0+0),\text{ etc.}$$

Now in the particular case of the equation of state (3.6), we can state

LEMMA 3.7. — *When the pressure p is given by* (3.6), *the relations* (3.28) *and* (3.29) *may be written:*

$$u^1-\left(\frac{\varepsilon}{W_a^0(p^0)}-Q_a^0\right)p^1 =u_a^1-\left(\frac{\varepsilon}{W_a^0(p^0)}+\mu^2.Q_a^0\right)p_a^1+(u^0-u_a^0)\frac{\tau_a^1}{2\tau_a^0},\tag{3.30}$$

and

$$\tau^1-\frac{(\mu^2\tau_a^0-\tau^0)Q_a^0}{u^0-u_a^0}p^1=\frac{\tau^0}{\tau_a^0}\tau_a^1+\frac{(\tau_a^0-\mu^2\tau^2)Q_a^0}{u^0-u_a^0}p_a^1\tag{3.31}$$

where μ, $W_a^0(p^0)$ *and* Q_a^0 *are defined by*

$$u^2=\frac{\gamma-1}{\gamma+1},$$

$$W_a^0(p)^2=\frac{p+\bar{p}+\mu^2(p_a+\bar{p})}{(1-\mu^2)\tau_a},\tag{3.32}$$

and

$$Q_a^0=\frac{u^0-u_a^0}{2(p^0+\bar{p}+\mu^2(p_a^0+\bar{p}))}.\tag{3.33}$$

Remark. — For our following derivations, we will use that we have (*see* [4], again) for a 1-shock wave

$$W_L^0(p_*^0)=\rho_L^0.(u_L^0-\sigma_1^0)=\rho_1^0.(u_*^0-\sigma_1^0)$$

and for a 3-shock wave:

$$W_R^0(p_*^0)=-\rho_R^0(u_R^0-\sigma_3^0)=-\rho_2^0.(u_*^0-\sigma_3^0).$$

Proof. — An easy calculation shows that

$$H_a(\tau,p)=\frac{1}{\gamma-1}\left\{\left(\frac{\gamma+1}{2}p+\gamma\bar{p}+\frac{\gamma-1}{2}p_a\right)\tau+-\left(\frac{\gamma+1}{2}p_a+\gamma\bar{p}+\frac{\gamma-1}{2}p\right)\tau_a\right\}$$

where $\bar{p}=\dfrac{\bar{c}^2}{\gamma\bar{\tau}}$. Hence, we have

$$\frac{2(\gamma-1)}{\gamma+1}H_a(\tau,p)=\left(p+\frac{2\gamma}{\gamma+1}\bar{p}+\mu^2p_a\right)\tau-\left(p_a+\frac{2\gamma\bar{p}}{\gamma+1}+\mu^2p\right)\tau_a$$

and

$$G(\tau_a, p_a, p)=\tau_a \frac{p_a+\mu^2 p+(2\gamma/(\gamma+1))\bar{p}}{p+\mu^2 p_a+(2\gamma/(\gamma+1))\bar{p}}$$

where μ is defined by

$$\mu^2=\frac{\gamma-1}{\gamma+1}.$$

Since

$$1+\mu^2=\frac{2\gamma}{\gamma+1},$$

we obtain

$$G(\tau_a, p_a, p)=\tau_a \frac{p_a+\bar{p}+\mu^2(p+\bar{p})}{p+\bar{p}+\mu^2(p_a+\bar{p})}. \tag{3.34}$$

Now, we have

$$\Phi(\tau_a, p_a, p)=(p-p_a)\sqrt{\frac{(1-\mu^2)\tau_a}{(p+\bar{p}+\mu^2(p_a+\bar{p}))}} \equiv \frac{p-p_a}{W_a(p)}. \tag{3.35}$$

Let us compute the partial derivatives of the function Φ. We find

$$\frac{\partial}{\partial \tau_a}\left(\frac{1}{W_a(p)}\right)=\frac{1}{2W_a(p)}\cdot\frac{1}{\tau_a},$$

$$\frac{\partial}{\partial p_a}\left(\frac{1}{W_a(p)}\right)=\frac{-\mu^2}{2W_a(p)}\frac{1}{((p+\bar{p})+\mu^2(p_a+\bar{p}))},$$

and

$$\frac{\partial}{\partial p}\left(\frac{1}{W_a(p)}\right)=-\frac{1}{2W_a(p)}\cdot\frac{1}{((p+\bar{p})+\mu^2(p_a+\bar{p}))}.$$

Hence, we have

$$\frac{\partial}{\partial \tau_a}\Phi_a(p)=\frac{\varepsilon(u-u_a)}{2\tau_a}, \tag{3.36 a}$$

$$\frac{\partial}{\partial p_a}\Phi_a(p)=-\frac{1}{W_a(p)}-\varepsilon\frac{\mu^2}{2}\cdot\frac{u-u_a}{(p+\bar{p}+\mu^2(p_a+\bar{p}))} \tag{3.36 b}$$

and similarly

$$\frac{\partial}{\partial p}\Phi_a(p)=\frac{1}{W_a(p)}-\frac{\varepsilon}{2}\cdot\frac{u-u_a}{(p+\bar{p}+\mu^2(p_a+\bar{p}))}. \tag{3.36 c}$$

Combining the above expressions with (3.28) yields (3.30).

In order to prove (3.31), we compute the first partial derivatives of the function G

$$\begin{aligned}\frac{\partial}{\partial \tau_a} G &= \frac{\tau}{\tau_a},\\ \frac{\partial G}{\partial p_a} &= \frac{\tau_a - \mu^2 \tau}{(p+\bar{p}+\mu^2(p_a+\bar{p}))},\\ \frac{\partial G}{\partial p} &= \frac{\mu^2 \tau_a - \tau}{(p+\bar{p}+\mu^2(p_a+\bar{p}))}.\end{aligned} \tag{3.37}$$

Together with (3.26), this gives

$$\tau^1 = \frac{\tau^0}{\tau_a^0} \tau_a^1 + \frac{(\tau_a^0 - \mu^2 \tau^0) p_a^1 + (\mu^2 \tau_a^0 - \tau^0) p^1}{((p^0+\bar{p}) + \mu^2 (p_a^0 + \bar{p}))},$$

and (3.31) follows. ■

We are now able to determine explicitely the function V^1. By Lemmas 3.1 and 3.2, we have only to compute the constants $a_{i,j}$, $1 \leqslant i, j \leqslant 3$, of Lemma 3.1. We have already noticed in Lemma 3.6 that $a_{0,j}$ and $a_{3,j}$, $1 \leqq j \leqq 3$, are known. Using Lemma 3.5, we may set

$$u_*^1 = u^1(0), \qquad p_*^1 = p^1(0). \tag{3.38}$$

We begin by deriving a 2×2 *algebraic system* in the two unknowns u_*^1 and p_*^1. Next, we will show how to compute the coefficients $a_{i,j}$ from u_*^1 and p_*^1. It is worthwile to notice that in the construction of the second order accurate version of Godunov method of approximation of (3.1), as developed in [1], we need only to compute the pair u_*^1, p_*^1 without constructing the whole function V^1.

THEOREM 3.1. — 1. *If the function* V^0 *contains a* 1-*rarefaction wave, we have*

$$u^1 + \frac{1}{g_1^0} \cdot p_*^1 = C_L + \left(\frac{g_1^0}{g_L^0}\right)^{1/2} \cdot \frac{h}{4}, \tag{3.39 a}$$

where

$$C_L = (g_1^0 g_L^0)^{1/2} \left\{ -u_L^1 + \frac{1}{g_L^0} p_L^1 + \frac{1}{4}\left(\frac{p_L^1}{g_L^0} + g_L^0 \tau_L^1\right) \cdot \int_{-g_L^0}^{-g_1^0} \left(\frac{y}{-g_L^0}\right)^{3/2} \cdot \frac{d}{d\xi}\left(\frac{q_L^0}{q^0(y)}\right) dy \right\}. \tag{3.39 b}$$

2. *If* V^0 *contains a* 3-*rarefaction wave, we have*

$$u_*^1 - \frac{1}{g_2^0} p_*^1 = C_R - \left(\frac{g_2^0}{g_R^0}\right)^{1/2} \cdot \frac{h}{4}, \tag{3.40 a}$$

where

$$C_R=(g_2^0 g_R^0)^{1/2}\left\{-u_R^1-\frac{1}{g_R^0}p_R^1 + \frac{1}{4}\left(\frac{p_R^1}{g_R^0}+g_R^0\tau_R^1\right).\int_{g_2^0}^{g_R^0}\left(\frac{y}{g_R^0}\right)^{3/2}\frac{d}{d\xi}\left(\frac{q_R^0}{q^0(y)}\right)dy\right\}. \quad (3.40\,b)$$

3. *If* V^0 *contains a* 1-*shock wave, we have*

$$\left\{1+\sigma_1^0\left(\frac{\partial}{\partial p}\Phi_L\right)_1^0\right\}u_*^1+\left\{\frac{-\sigma_1^0}{(g_1^0)^2}-\left(\frac{\partial}{\partial p}\Phi_L\right)_1^0\right\}p_*^1 = K_L+\left(1+\sigma_1^0.\left(\frac{\partial}{\partial p}\Phi_L\right)_1^0\right).h \quad (3.41\,a)$$

where

$$K_L=+\left(\frac{\partial}{\partial \tau_a}\Phi_L\right)_1^0.(\tau_L^1\sigma_1^0+u_L^1) + \left(\frac{\partial}{\partial p_a}\Phi_L\right)_1^0.(p_L^1\sigma_1^0-(g_L^0)^2u_L^1)+u_L^1\sigma_1^0-p_L^1. \quad (3.41\,b)$$

4. *If* V^0 *contains a* 3-*shock wave, we have*

$$\left\{1-\sigma_3^0\left(\frac{\partial}{\partial p}\Phi_R\right)_2^0\right\}u_*^1+\left\{\frac{-\sigma_3^0}{(g_2^0)^2}+\left(\frac{\partial}{\partial p}\Phi_R\right)_2^0\right\}p_*^1 = K_R+\left(1-\sigma_3^0.\left(\frac{\partial}{\partial p}\Phi_R\right)_2^0\right).h \quad (3.42\,a)$$

where

$$K_R=-\left(\frac{\partial}{\partial \tau_a}\Phi_R\right)_2^0.(\tau_R^1\sigma_3^0+u_R^1) - \left(\frac{\partial}{\partial p_a}\Phi_R\right)_2^0.\left(p_R^1\sigma_3^0-(g_R^0)^2u_R^1\right)+u_R^1\sigma_R^0-P_R^1. \quad (3.42\,b)$$

In (3.41) *and* (3.42), *we have set for* $a=L$ *or* R *and* $i=1,2$

$$\left(\frac{\partial}{\partial p}\Phi_a\right)_i^0=\frac{\partial}{\partial p}\Phi(\tau_a^0, p_a^0, p_i^0), \qquad etc...$$

Proof. — Consider first the case of a 1-rarefaction wave. By using (3.7) and Lemma 3.1, we have on one hand

$$\alpha_3(0-)=-g_1^0 a_{1,3}-\frac{h}{2g_1^0}$$

and, by

$$\tau_1^1-\frac{S_1^1}{q_1^0}=\left\{\left(\frac{\partial}{\partial\tau}p\right)_1^0\tau_1^1+\left(\frac{\partial}{\partial S}p\right)_1^0 S_1^1\right\}\Big/\left(\frac{\partial}{\partial\tau}p\right)_1^0=\frac{-p_*^1}{(g_1^0)^2},$$

we obtain on the other hand

$$\alpha_3(0-)=\frac{1}{2}\left(\frac{-u_*^1}{g_1^0}+\tau_1^1-\frac{S_1^1}{q_1^0}\right)=\frac{1}{2}\left(\frac{-u_*^1}{g_1^0}-\frac{p_*^1}{(g_1^0)^2}\right).$$

Hence

$$g_1^0 a_{1,3}+\frac{h}{2g_1^0}=\frac{1}{2}\cdot\left(\frac{u_*^1}{g_1^0}+\frac{p_*^1}{(g_1^0)^2}\right),$$

and therefore

$$u_*^1+\frac{1}{g_1^0}p_*^1=(g_1^0)^2 a_{1,3}+\frac{h}{2}. \qquad (3.43)$$

Next, the constant $a_{1,3}$ is determined as follows. By using Lemmae 3.1 and 3.4 we have

$$\alpha_3(-g_1^0+0)=\alpha_3(-g_1^0-0)=-2g_1^0 a_{1,3}-\frac{h}{2g_1^0}. \qquad (3.44)$$

It follows from Lemma 3.2 that

$$\alpha_3(-g_1^0)=\frac{-g_L^0}{4}\left(\frac{+g_L^0}{g_1^0}\right)^{1/2}\cdot\int_{-g_L^0}^{-g_1^0}\left(\frac{y}{-g_L^0}\right)^{3/2}\cdot\frac{d}{d\xi}\left(\frac{q_L^0}{q^0(y)}\right)dy\, a_{L,2}$$
$$+-2g_L^0.\left(\frac{g_L^0}{g_1^0}\right)^{1/2}.a_{L,3}+\frac{h}{2g_1^0}\left(1-\left(\frac{g_1^0}{g_L^0}\right)^{1/2}\right). \qquad (3.45)$$

By (3.6), (3.23) and (3.38), we have

$$a_{L,2}=\frac{1}{2}\left(\frac{u_*^1}{g_1^0}-\frac{p_*^1}{(g_1^0)^2}\right),\qquad a_{L,3}=\frac{1}{2}\left(\frac{-u_*^1}{g_1^0}-\frac{p_*^1}{(g_1^0)^2}\right). \qquad (3.46)$$

Hence, combining (3.44), (3.45) and (3.46) gives $a_{1,3}$. Then (3.43) yields (3.39). The case of a 3-rarefaction wave is similar.

Consider next, the case of a 3-shock wave (say!). We may write from (3.28):

$$u^1(\sigma_3^0-0)-\left(\frac{\partial\Phi}{\partial p}\right)_2^0 p^1(\sigma_3^0-0)$$
$$=u^1(\sigma_3^0+0)+\left(\frac{\partial\Phi}{\partial\tau_a}\right)_2^0\tau^1(\sigma_3^0+0)+\left(\frac{\partial\Phi}{\partial p_a}\right)_2^0 p^1(\sigma_3^0+0)\equiv K_R+h. \qquad (3.47)$$

On one hand, using (3.7), (3.10) and (3.24), we have

$$\begin{aligned} u^1(\sigma_3^0+0) &= g_R^0 . \{ \alpha_1(\sigma_3^0+0) - \alpha_3(\sigma_3^0+0) \} \\ &= g_R^0 . \left\{ (\sigma_3^0+g_R^0)\, a_{R,1} + (-\sigma_3^0+g_R^0)\, a_{R,3} + \frac{h}{g_R^0} \right\} \\ &= \sigma_3^0 u_R^1 - p_R^1 + h, \\ \tau^1(\sigma_3^0+0) &= (\alpha_1+\alpha_2+\alpha_3)(\sigma_3^0+0) \\ &= \sigma_3^0 . \tau_R^1 + u_R^1, \end{aligned}$$

and

$$\begin{aligned} p^1(\sigma_3^0+0) &= \left(\frac{\partial p}{\partial \tau}\right)_R^0 \tau^1(\sigma_3^0+0) + \left(\frac{\partial p}{\partial S}\right)_R^0 . S^1(\sigma_3^0+0) \\ &= -(g_R^0)^2 (\alpha_1(\sigma_3^0+0) + \alpha_3(\sigma_3^0+0)) \\ &= \sigma_3^0 p_R^1 - (g_R^0)^2 u_R^1. \end{aligned}$$

Hence the constant K_R defined in (3.47) is given by (3.42 *b*). On the other hand, we have in a similar way

$$\begin{aligned} u^1(\sigma_3^0-0) &= g_2^0 . \left\{ (\sigma_3^0+g_2^0)\, a_{2,1} + (-\sigma_3^0+g_2^0)\, a_{2,3} + \frac{h}{g_2^0} \right\}, \\ p^1(\sigma_3^0-0) &= -(g_2^0)^2 \left\{ (\sigma_3^0+g_2^0)\, a_{2,1} + (\sigma_3^0-g_2^0)\, a_{2,3} \right\}, \end{aligned}$$

and

$$\begin{aligned} u_*^1 &= g_2^0 . \left\{ g_2^0 a_{2,1} + g_2^0 a_{2,3} + \frac{h}{g_2^0} \right\} = (g_2^0)^2 . \left\{ a_{2,1} + a_{2,3} + \frac{h}{(g_2^0)^2} \right\}, \\ p_*^1 &= -(g_2^0)^2 . \left\{ g_2^0 a_{2,1} - g_2^0 a_{2,3} \right\} = -(g_2^0)^3 \{ a_{2,1} - a_{2,3} \}. \end{aligned}$$

Thus; the constants $a_{2,1}$ and $a_{2,3}$ are given by

$$\left. \begin{aligned} a_{2,1} &= \left(\frac{1}{g_2^0}\right)^2 \left\{ \frac{1}{2}\left(u_*^1 - \frac{p_*^1}{g_2^0}\right) - \frac{h}{2} \right\} \\ a_{2,3} &= \left(\frac{1}{g_2^0}\right)^2 \left\{ \frac{1}{2}\left(u_*^1 + \frac{p_*^1}{g_2^0}\right) - \frac{h}{2} \right\}. \end{aligned} \right\} \qquad (3.48)$$

Hence, the values of the functions u^1 and p^1 at $\xi=\sigma_3^0-0$ can be computed from their values at $\xi=0$:

$$u^1(\sigma_3^0-0)=\left(\frac{\sigma_3^0}{g_2^0}+1\right)\cdot\left\{\frac{1}{2}\left(u_*^1-\frac{p_*^1}{g_2^0}\right)-\frac{h}{2}\right\}+\left(\frac{-\sigma_3^0}{g_2^0}+1\right)\cdot\left\{\frac{1}{2}\left(u_*^1+\frac{p_*^1}{g_2^0}\right)-\frac{h}{2}\right\}+h$$
$$=u_*^1-\frac{\sigma_3^0}{(g_2^0)^2}\cdot p_*^1+2h,$$

and

$$p^1(\sigma_3^0-0)=-(\sigma_3^0+g_2^0)\left\{\frac{1}{2}\left(u_*^1-\frac{p_*^1}{g_2^0}\right)-\frac{h}{2}\right\}-(\sigma_3^0-g_2^0)\left\{\frac{1}{2}\left(u_*^1+\frac{p_*^1}{g_2^0}\right)-\frac{h}{2}\right\}$$
$$=-\sigma_3^0.u_*^1+p_*^1+\sigma_3^0.h.$$

Replacing, in (3.47), $u^1(\sigma_3^0-0)$ and $p^1(\sigma_3^0-0)$ by their above expressions yields (3.42 *a*). ■

We emphasize that the 2×2 system given by Theorem 3.1 is always numerically solvable. After having determined u_*^1 and p_*^1, we now compute the constants $a_{i,\,j}$.

THEOREM 3.2. — *For* $i=1, 2$, *we have*

$$a_{i,\,1}=\frac{1}{2(g_i^0)^2}\left\{u_*^1-\frac{p_*^1}{g_i^0}-h\right\} \tag{3.49}$$

and

$$a_{i,\,3}=\frac{1}{2(g_i^0)^2}\left\{u_*^1+\frac{p_*^1}{g_i^0}-h\right\}. \tag{3.50}$$

Moreover, if V^0 *contains a k-rarefaction wave, we obtain*

$$a_{1,\,2}=\frac{q_L^0}{q_1^0}a_{L,\,2},\qquad if\quad k=1 \tag{3.51}$$

and

$$a_{2,\,2}=\frac{q_R^0}{q_2^0}a_{R,\,2},\qquad if\quad k=3. \tag{3.52}$$

If V^0 *contains a k-shock wave, we find*

$$a_{1,\,2}=\frac{\sigma_1^0+g_1^0}{\sigma_1^0.(g_1^0)^2}\{-u_*^1+h\}+\left(\frac{\partial G}{\partial p}\right)_1^0\left\{-u_*^1+\frac{p_*^1}{\sigma_1^0}+h\right\}+\left(\frac{\partial G}{\partial \tau_a}\right)_L^0\left\{\tau_L^1+\frac{u_L^1}{\sigma_1^0}\right\}+\left(\frac{\partial G}{\partial p_a}\right)_L^0\left\{p_L^1-\frac{(g_L^0)^2}{\sigma_1^0}u_L^1\right\},\qquad if\quad k=1 \tag{3.53}$$

and

$$a_{2,2}=\frac{\sigma_3^0+g_2^0}{\sigma_3^0.(g_2^0)^2}.\{-u_*^1+h\}+\left(\frac{\partial G}{\partial p}\right)_2^0\left\{-u_*^1+\frac{p_*^1}{\sigma_3^0}+h\right\}$$
$$+\left(\frac{\partial G}{\partial \tau_a}\right)_R^0\left\{\tau_R^1+\frac{u_R^1}{\sigma_3^0}\right\}+\left(\frac{\partial G}{\partial p_a}\right)_R^0\left\{p_R^1-\frac{(g_R^0)^2}{\sigma_3^0}u_R^1\right\},\qquad \textit{if}\quad k=3. \qquad (3.54)$$

Proof. — The expressions (3.49) and (3.50) are derived as in (3.48). For $a_{1,2}$ or $a_{2,2}$, we first consider the case where for instance the function V^0 contains a 1-rarefaction wave. By Lemma 3.4, we have

$$\alpha_1(-g_1^0+0)=\alpha_1(-g_1^0-0),$$

where

$$\alpha_1(-g_1^0+0)=-g_1^0.a_{1,2},$$

and

$$\alpha_1(-g_1^0-0)=-g_1^0\frac{q_L^0}{q_1^0}d_{1,2}=-g_1^0\frac{q_L^0}{q_1^0}a_{0,2}$$
$$=-g_1^0\frac{q_L^0}{q_1^0}a_{L,2}.$$

This yields (3.51). The proof of (3.52) is similar. When the function V^0 contains a 3-shock wave, we use Lemma 3.7 and we write

$$\tau^1(\sigma_3^0-0)-\left(\frac{\partial G}{\partial p}\right)_2^0 p^1(\sigma_3^0-0)=\left(\frac{\partial G}{\partial \tau_a}\right)_R^0\tau^1(\sigma_3^0+0)+\left(\frac{\partial G}{\partial p_a}\right)_R^0 p^1(\sigma_3^0+0).$$

As in the proof of Theorem 3.1, we have

$$\tau^1(\sigma_3^0+0)=\sigma_3^0.\tau_R^1+u_R^1,$$
$$p^1(\sigma_3^0+0)=\sigma_3^0.p_R^1-(g_R^0)^2.u_R^1,$$

and

$$p^1(\sigma_3^0-0)=-\sigma_3^0u_*^1+p_*^1+\sigma_3^0.h.$$

Moreover, we get

$$\tau^1(\sigma_3^0-0)=(\alpha_1+\alpha_2+\alpha_3)(\sigma_3^0-0)$$
$$=\sigma_3^0\{a_{2,1}+a_{2,2}+a_{2,3}\}+g_2^0\{a_{2,1}+a_{2,3}\}$$
$$=\sigma_3^0a_{2,2}+\frac{\sigma_3^0+g_2^0}{(g_2^0)^2}\{u_*^1-h\},$$

which gives (3.54). The derivation of (3.53) is similar. ■

Finally, as a trivial consequence of Theorems 3.1 and 3.2, we obtain

COROLLARY. — *The values of the function* τ^1 *at* $\xi=0$ *are given by*

$$\tau^1(0-)=\frac{1}{g_1^0}\{u_*^1-h\}, \qquad (3.55)$$

and

$$\tau^1(0+)=\frac{1}{g_2^0}\{u_*^1-h\}. \qquad (3.56)$$

Let us close this section by computing the constants arising in Theorem 3.1 in the case of the equation of state (3.6). As a direct consequence of Lemmae 3.3 and 3.7, Theorem 3.1 and (3.36), (3.37) and (3.38), we obtain:

THEOREM 3.3. — 1. *If the function* V^0 *contains a* 1-*rarefaction wave, the constant* C_L *in* (3.39) *is given by*

$$C_L=(g_L^0 \cdot g^0)^{1/2}\left\{-u_L^1+\frac{p_L^1}{g_L^0}+\left(\frac{p_L^1}{g_L^0}+g_L^0\tau_L^1\right)\times\frac{1}{3\gamma-1}\left[\left(\frac{g_1^0}{g_L^0}\right)^{(3\gamma-1)/2(\gamma+1)}-1\right]\right\}. \qquad (3.57)$$

2. *If* V^0 *contains a* 3-*rarefaction wave, the constant* C_R *in* (3.40) *is given by*

$$C_R=-(g_R^0 \cdot g_2^0)^{1/2}\left\{+u_R^1+\frac{p_R^1}{g_R^0}+\left(\frac{p_R^1}{g_R^0}+g_R^0\tau_R^1\right)\times\frac{1}{3\gamma-1}\left[\left(\frac{g_2^0}{g_R^0}\right)^{(3\gamma-1)/2(\gamma+1)}-1\right]\right\}. \qquad (3.56)$$

3. *If* V^0 *contains a* 1-*shock wave, the coefficients of the equation* (3.41) *are explicitely given by*

$$\left(\frac{\partial}{\partial p}\Phi_L\right)_1^0=\frac{1}{W_L^0(p_*^0)}-\frac{1}{2}\cdot\frac{u_*^0-u_L^0}{(p_*^0+\bar{p})+\mu^2(p_L^0+\bar{p})} \qquad (3.57\,a)$$

$$\left(\frac{\partial}{\partial\tau_a}\Phi_L\right)_1^0=\frac{u_*^0-u_L^0}{2\tau_L^0}, \qquad (3.57\,b)$$

and

$$\left(\frac{\partial}{\partial p_a}\Phi_L\right)_1^0=\frac{-1}{W_L^0(p_*^0)}-\frac{\mu^2}{2}\cdot\frac{(u_*^0-u_L^0)}{p_L^0+\bar{p}+\mu^2(p_*^0+\bar{p})} \qquad (3.57\,c)$$

4. *If* V^0 *contains a* 3-*shock wave, the coefficients of the equation* (3.42) *are explicitely given by*

$$\left(\frac{\partial}{\partial p}\varphi_R\right)_2^0=\frac{1}{W_R^0(p_*^0)}+\frac{1}{2}\cdot\frac{u_*^0-u_R^0}{p_*^0+\bar{p}+\mu^2 p_R^0+\bar{p})}, \qquad (3.58\,a)$$

$$\left(\frac{\partial}{\partial\tau_a}\varphi_R\right)_2^0=-\frac{(u_*^0-u_R^0)}{2\tau_R^0} \qquad (3.58\,b)$$

and

$$\left(\frac{\partial}{\partial p_a}\varphi_R\right)_2^0 = \frac{-1}{W_R^0(p_*^0)} + \frac{\mu^2}{2}\cdot\frac{(u_*^0 - u_R^0)}{(p_R^0 + \bar{p} + \mu^2(p_*^0 + \bar{p})}. \qquad (3.58c)$$

4. THE GAS DYNAMICS EQUATIONS IN EULER COORDINATES

Let us now consider the gas dynamics system in Eulerian coordinates and in plane ($\nu=0$), cylindrical ($\nu=1$) or spherical ($\nu=2$) symetry [4]:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) + \frac{\nu}{x_0 + x}\rho u = 0, \qquad (4.1a)$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\nu}{x_0 + x}\rho u^2 = 0, \qquad (4.1b)$$

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}((\rho e + p) u) + \frac{\nu}{x_0 + x}(\rho e + p) u = 0. \qquad (4.1c)$$

In (4.1), x_0 is a positive number, $\rho = \frac{1}{v}$ is the density, while the other variables have the meaning defined in Section 3. For the sake of simplicity, we assume the equation of state (3.6).

Using the nonconservative variables (ρ, u, S), the system (4.1) becomes

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = -\frac{\nu}{x}\rho u, \qquad (4.2a)$$

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \frac{1}{\rho}\frac{\partial}{\partial x}p(\rho, S) = 0, \qquad (4.2b)$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0. \qquad (4.2c)$$

On the other hand, the equation of state can be written

$$p(\rho, S) = \rho^\gamma S - \frac{\bar{c}^2}{\gamma}.\bar{\rho}, \qquad (4.3)$$

with $\bar{\rho} = \frac{1}{\bar{v}}$. We get a nonconservative system of the form (2.3) where

$$V = \begin{pmatrix} \rho \\ u \\ S \end{pmatrix}, \quad B(V) = \begin{pmatrix} u & \rho & 0 \\ \frac{c^2}{\rho} & u & \rho^{\gamma-1} \\ 0 & 0 & u \end{pmatrix}, \quad h(x, V) = \begin{pmatrix} -\frac{\nu}{x}\rho u \\ 0 \\ 0 \end{pmatrix}, \qquad (4.4)$$

and the sound speed c is given by

$$c(\rho, S)^2 = \frac{\partial}{\partial \rho} p(\rho, S) = \frac{\gamma}{\rho}\left(p + \frac{\overline{c}^2}{\gamma}\overline{\rho}\right) \equiv \frac{\gamma}{\rho}(p+\overline{p}).$$

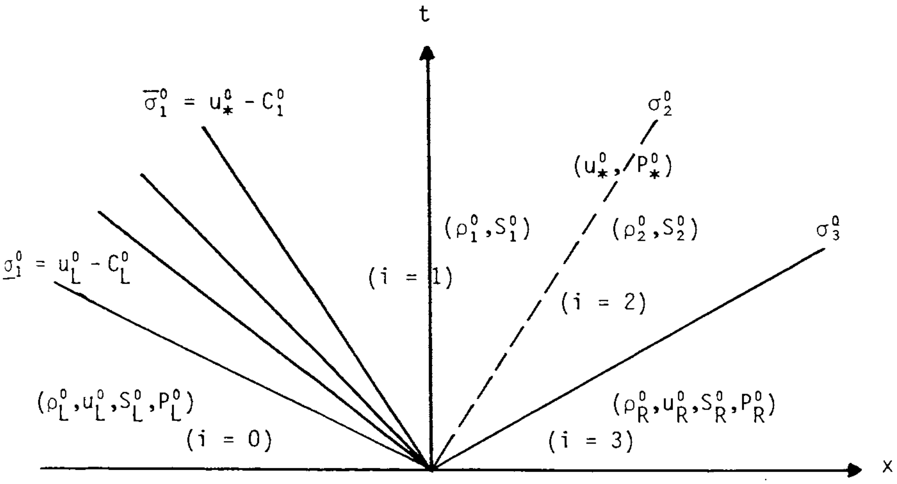

FIG. 4.1. — Zero order terms.

Now, the eigenvalues of the matrix $B(V)$ are

$$\lambda_1 = u - c < \lambda_2 = u < \lambda_3 = u + c,$$

and we take as a corresponding basis of eigenvectors

$$r_1(V) = \begin{pmatrix} \rho \\ -c \\ 0 \end{pmatrix}, \qquad r_2(V) = \begin{pmatrix} \rho \\ 0 \\ -\gamma S \end{pmatrix}, \qquad r_3(V) = \begin{pmatrix} \rho \\ c \\ 0 \end{pmatrix}. \tag{4.5}$$

A state $V^0 = \begin{pmatrix} \rho^0 \\ u^0 \\ S^0 \end{pmatrix}$ being fixed, the characteristic coordinates $(\alpha_i)_{1 \leq i \leq 3}$ of any arbitrary state $V = \begin{pmatrix} \rho \\ u \\ S \end{pmatrix}$, *i.e.*

$$V = \sum_{i=1}^{3} \alpha_i r_i(V^0),$$

are explicitely given by

$$\left.\begin{aligned} \alpha_1 &= \frac{1}{2}\left(\frac{\rho}{\rho_0} - \frac{u}{c^0} + \frac{S}{\gamma S^0}\right), \\ \alpha_2 &= -\frac{S}{\gamma S^0}, \\ \alpha_3 &= \frac{1}{2}\left(\frac{\rho}{\rho^0} + \frac{u}{c^0} + \frac{S}{\gamma S^0}\right), \end{aligned}\right\} \tag{4.6}$$

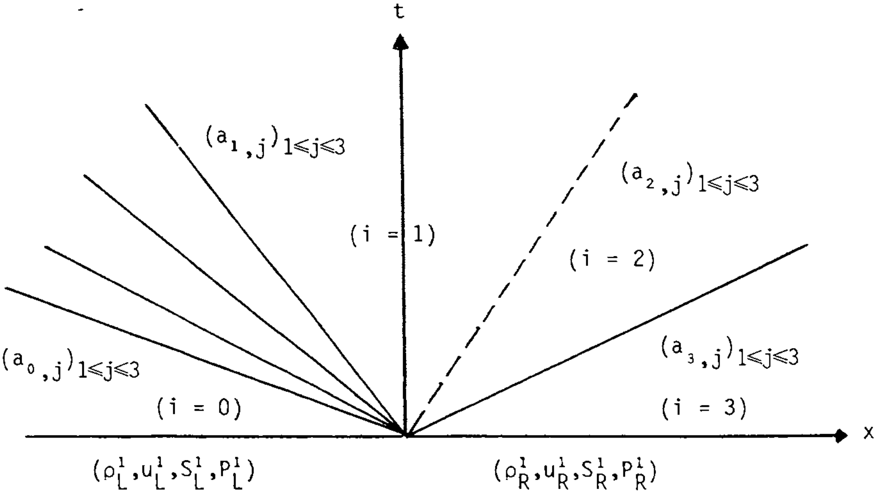

FIG. 4.2. – First order terms.

or equivalently by

$$\left.\begin{aligned} \rho &= \rho^0(\alpha_1+\alpha_2+\alpha_3), \\ u &= c^0(-\alpha_1+\alpha_3), \\ S &= -\gamma S^0 . \alpha_2. \end{aligned}\right\} \tag{4.7}$$

Moreover, we shall use the following decomposition of $h(x, V)$

$$h(x, V) = -\frac{v}{x}\rho u \frac{1}{2\rho_0}(r_1(V^0) + r_3(V^0)). \tag{4.8}$$

We apply again the theory of Section 2, and we derive the first order approximation

$$V(x, t) = V^0\left(\frac{x}{t}\right) + t\, V^1\left(\frac{x}{t}\right) + \dots$$

of the solution $V(x, t)$ of the generalized Riemann problem for the system (4.1). The notations are as shown in Figure 4.1 and 4.2. In order to determine the function $V^1 = V^1(\xi)$, we introduce as before its decomposition on the basis of eigenvectors $r_j(V^0)$

$$V^1(\xi) = \sum_{1 \leq j \leq 3} \alpha_j(\xi) r_j(V^0(\xi)), \qquad \xi \in \mathbb{R}.$$

We begin by deriving the general form of the functions $\alpha_j = \alpha_j(\xi)$ in an interval where the function V^0 remains constant:

LEMMA 4.1. — *In each interval* $]\bar{\sigma}_i^0, \underline{\sigma}_{i+1}^0[$, i=0, 1, 2, 3, *there exist three real constants* $a_{i,j}$, $j=1, 2, 3$, *such that*

$$\left.\begin{aligned} \alpha_1(\xi) &= (\xi - u_i^0 + c_i^0) a_{i,1} - \frac{\nu}{2x_0} u_i^0, \\ \alpha_2(\xi) &= (\xi - u_i^0) a_{i,2}, \\ \alpha_3(\xi) &= (\xi - u_i^0 - c_i^0) a_{i,3} - \frac{\nu}{2x_0} u_i^0. \end{aligned}\right\} \tag{4.9}$$

Next, we consider the case of an interval $]\underline{\sigma}_1^0, \bar{\sigma}_1^0[=]u_L^0 - c_L^0, u_1^0 - c_1^0[$ corresponding to a 1-*rarefaction wave* of V^0.

LEMMA 4.2. — *Assume that* V^0 *contains a* 1-*rarefaction wave. Then there exist two real constants* $b_{1,2}$ *and* $b_{1,3}$ *such that for all* $\xi \in]\underline{\sigma}_1^0, \bar{\sigma}_1^0[$:

$$\alpha_1(\xi) = \frac{a_2(\xi)}{2(\gamma+1)} - \frac{\gamma-3}{2(\gamma+1)}\alpha_3(\xi) - \frac{\nu}{4x_0}\left\{\frac{2\xi}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\left(u_L^0 + \frac{2c_L^0}{\gamma-1}\right)\right\}, \tag{4.10a}$$

$$\alpha_2(\xi) = b_{1,2}\, c^0(\xi)^{(\gamma+1)/(\gamma-1)}, \tag{4.10b}$$

$$\alpha_3(\xi) = \frac{b_{1,2}}{3\gamma-1} c^0(\xi)^{(\gamma+1)/(\gamma-1)} + b_{1,3}\, c^0(\xi)^{(3-\gamma)/2(\gamma-1)} + M^0(c^0(\xi))\, c^0(\xi))\, c^0(\xi) + N_L^0(c^0(\xi)). \tag{4.10c}$$

where $c^0 = c^0(\xi)$ *is the sound speed in a* 1-*rarefaction zone. The functions* $M^0 = M^0(c^0)$ *and* $N_L^0 = N_L^0(c^0)$ *are defined by*

$$\left.\begin{aligned} M^0(c^0) &= \frac{\nu}{x_0}\frac{\gamma+1}{(5-3\gamma)(\gamma-1)}, & &\text{if } \gamma \neq \frac{5}{3} \\ M^0(c^0) &= -\frac{\nu}{x_0}\frac{\gamma+1}{(\gamma-1)} \operatorname{Log} c^0, & &\text{if } \gamma = \frac{5}{3}, \end{aligned}\right\} \tag{4.11}$$

and

$$\left.\begin{aligned} N_L^0(c^0) &= \frac{\nu}{2x_0}\frac{\gamma+1}{\gamma-3}\left(u_L^0 + \frac{2c_L^0}{\gamma-1}\right), & &\text{if } \gamma \neq 3 \\ N_L^0(c^0) &= \frac{\nu}{2x_0}\frac{\gamma+1}{2(\gamma-1)}\left(u_L^0 + \frac{2c_L^0}{\gamma-1}\right). \operatorname{Log} c^0, & &\text{if } \gamma = 3. \end{aligned}\right\} \tag{4.12}$$

Proof. — Starting with (4. 4), elementary calculations show that

$$\mathrm{DB}(\mathrm{V}^0)\,\mathrm{V}^1=\begin{pmatrix} u^1 & \rho^1 & 0 \\ (\gamma-2)\left(\dfrac{c^0}{\rho^0}\right)^2\rho^1+\gamma\rho_0^{\gamma-2}.\mathrm{S}^1 & u^1 & (\gamma-1)(\rho^0)^{\gamma-2}\rho^1 \\ 0 & 0 & u^1 \end{pmatrix},$$

so that the matrix $\mathrm{C}^0=\mathrm{C}\left(\mathrm{V}^0, \dfrac{d\mathrm{V}^0}{d\xi}\right)$, defined by (2. 12), is given by

$$\mathrm{C}^0=\begin{pmatrix} \dfrac{d}{d\xi}u^0 & \dfrac{d}{d\xi}\rho^0 & 0 \\ (\gamma-2)\left(\dfrac{c^0}{\rho_0}\right)^2\dfrac{d}{d\xi}\rho^0 & \dfrac{d}{d\xi}u^0 & \gamma\rho_0^{\gamma-2}\dfrac{d}{d\xi}\rho^0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next, in a 1-rarefaction wave, we know that the Riemann invariants S^0 and $u^0+\displaystyle\int\frac{1}{\rho}c(\rho,\mathrm{S})\,d\rho$ are constant and

$$u^0-c^0=\xi.$$

Thus, we have

$$\left.\begin{aligned} \frac{d}{d\xi}\rho^0=-\frac{2}{\gamma+1}\frac{\rho^0}{c^0}, \qquad \frac{d}{d\xi}u^0=\frac{2}{\gamma+1}, \\ \frac{d}{\delta\xi}c^0=-\frac{\gamma-1}{\gamma+1}, \qquad \frac{d}{d\xi}\mathrm{S}^0=0. \end{aligned}\right\} \tag{4.13}$$

Hence, the matric C^0 becomes

$$\mathrm{C}^0=\frac{2}{\gamma+1}\begin{pmatrix} 1 & -\rho^0/c^0 & 0 \\ (2-\gamma)\dfrac{c^0}{\rho^0} & 1 & -c^0/\mathrm{S}^0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, it is a simple matter to check that

$$\left.\begin{aligned} \mathrm{C}^0 r_1(\mathrm{V}^0)&=r_1(\mathrm{V}^0)+\frac{3-\gamma}{\gamma+1}r_3(\mathrm{V}^0), \\ \mathrm{C}^0 r_2(\mathrm{V}^0)&=-\frac{1}{\gamma+1}r_1(\mathrm{V}_0)+\frac{3}{\gamma+1}r_3(\mathrm{V}^0), \\ \mathrm{C}^0 r_3(\mathrm{V}^0)&=\frac{\gamma-3}{\gamma+1}r_1(\mathrm{V}^0)+\frac{3-\gamma}{\gamma+1}r_3(\mathrm{V}^0). \end{aligned}\right\} \tag{4.14}$$

On the other hand, we deduce the derivatives of $r_i(V^0)$ from (4.5) and (4.13)

$$\left.\begin{aligned} \frac{d}{d\xi}r_1(V^0)&=-\frac{1}{2c^0}r_1(V^0)+\frac{(\gamma-3)}{(2(\gamma+1)c^0}r_3(V^0),\\ \frac{d}{d\xi}r_2(V_0)&=-\frac{1}{(\gamma+1)c^0}(r_1(V^0)+r_3(V^0)),\\ \frac{d}{d\xi}r_3(V^0)&=\frac{(\gamma-3)}{2(\gamma+1)c^0}r_1(V^0)-\frac{1}{2c^0}r_3(V^0). \end{aligned}\right\} \quad (4.15)$$

Finally, using (4.14).(4.15), the ordinary differential equations (2.18) becomes

$$2\alpha_1-\frac{\alpha_2}{\gamma+1}+\frac{\gamma-3}{\gamma+1}\alpha_3=-\nu.\frac{u^0}{2x_0}, \quad (4.16a)$$

$$c^0\alpha_2'+\alpha_2=0, \quad (4.16b)$$

$$2c^0\alpha_3'+\frac{\alpha_2}{\gamma+1}+\frac{3-\gamma}{\gamma+1}\alpha_3=-\nu.\frac{u}{2x_0}. \quad (4.16c)$$

The integration of (4.16 *b*) is straightforward and gives (4.10*b*). Then, replacing α_2 by its expression (4.10*b*) in (4.16*c*), we obtain an O.D.E. for the function α_3, whose solution is given by (4.10*c*). Finally (4.10*a*) follows from the algebraic equation (4.16*a*). ■

Then, we treat similarly the case of an interval $]\underline{\sigma}_3^0, \bar{\sigma}_3^0[=]u_2^0+c_2^0, u_R^0+c_R^0[$ corresponding to a 3-*rarefaction wave* of V^0.

LEMMA 4.2 *bis*. — *Assume that* V^0 *contains a 3-rarefaction wave. Then, there exist two real constants* $b_{3,1}$ *and* $b_{3,2}$ *such that for all* $\xi\in]\underline{\sigma}_3^0, \bar{\sigma}_3^0[$ *we have*

$$\left.\begin{aligned} \alpha_1(\xi)&=b_{3,1}c^0(\xi)^{3-\gamma/2(\gamma-1)}\\ &\quad+\frac{b_{3,2}}{3\gamma-1}c^0(\xi)^{(\gamma+1)/(\gamma-1)}-M^0(c^0(\xi))c^0(\xi)+N_R^0(c^0(\xi)),\\ \alpha_2(\xi)&=b_{3,2}c^0(\xi)^{(\gamma+1)/(\gamma-1)},\\ \alpha_3(\xi)&=\frac{-(\gamma-3)}{2(\gamma+1)}\alpha_1(\xi)+\frac{\alpha_2(\xi)}{2(+1)}-\frac{\nu}{4x_0}\left\{\frac{2\xi}{\gamma+1}+\frac{\gamma-1}{\gamma+1}\left(u_R^0\right.\right.\\ &\qquad\left.\left.-\frac{2c_R^0}{\gamma-1}\right)\right\}, \end{aligned}\right\} \quad (4.17)$$

where $c^0 = c^0(\xi)$ is the sound speed in a 3-rarefaction zone. The functions $M^0 = M^0(c^0)$ *is defined by* (4.11), *and the function* $N^0_R = N^0_R(c^0)$ *is given by*

$$\left.\begin{aligned} N^0_R &= \frac{\nu}{2x_0}\frac{\gamma+1}{\gamma-3}\left(u^0_R - \frac{2c^0_R}{\gamma-1}\right), \quad \text{if} \quad \gamma \neq 3 \\ N^0_R &= \frac{\nu}{2x_0}\frac{\gamma+1}{2(\gamma-1)}\left(u^0_R - \frac{2c^0_R}{\gamma-1}\right)\operatorname{Log} c^0, \quad \text{if} \quad y=3. \end{aligned}\right\} \tag{4.18}$$

Let us now derive the boudary relations for the function V^1 at the discontinuities of V^0. We consider first *rarefaction waves*. As a direct consequence of Lemmas 2.4 and 4.2, we obtain:

LEMMA 4.3. – *Assume that* V^0 *contains a* 1-*rarefaction wave. Then the constants* $b_{1,2}$ *and* $b_{1,3}$ *of Lemma* 4.2 *satisfy*

$$b_{1,2} = -a_{0,2}\cdot(c^0_L)^{-2/(\gamma-1)} \tag{4.19}$$

and

$$b_{1,3} = (c^0_L)^{(\gamma-3)/2(\gamma-1)}\left\{-2c^0_L\cdot a_{0,3} - \frac{\nu u^0_L}{2x_0} + \frac{c^0_L}{3\gamma-1}a_{0,2} - M^0(c^0_L)\,c^0_L - N^0_L(c^0_L)\right\}, \tag{4.20}$$

while the relations at $\xi = u^0_* - c^0_1$ *are*

$$\alpha_j(u^0_* - c^0_1 - 0) = \alpha_j(u^0_* - c^0_1 + 0), \qquad j = 2 \text{ or } 3. \tag{4.21}$$

Assume that V^0 *contains a* 3-*rarefaction wave. Then, the constants* $b_{3,1}$ *and* $b_{3,2}$ *of Lemma* 4.2 *bis satisfy*

$$b_{3,2} = a_{3,2}(c^0_R)^{-2/(\gamma-1)}, \tag{4.22}$$

and

$$b_{3,1} = (c^0_R)^{(\gamma-3)/2(\gamma-1)}\left\{+2c^0_R a_{3,1} - \frac{\nu u^0_R}{2x_0} - \frac{c^0_R}{3\gamma-1}a_{3,2} + M^0(c^0_R)\,c^0_R - N^0_R(c^0_R)\right\}, \tag{4.23}$$

while the relations at $\xi = u^0_* + c^0_2$ *are*

$$\alpha_j(u^0_* + c^0_2 - 0) = \alpha_j(u^0_* + c^0_2 + 0), \qquad j = 1 \text{ or } 2. \tag{4.24}$$

Then, we emphasize that the jump relations for a *shock wave* have been already studied in the previous Section 3 (*see* Lemma 3.7). It remains to consider the *contact discontinuity* corresponding to $\xi = u^0_*\,(u^0_1 = u^0_2)$. As in

Section 3, we define the function p^1 by

$$p^1=\left(\frac{\partial p}{\partial \rho}\right)^0 \rho^1+\left(\frac{\partial p}{\partial S}\right)^0 S^1, \tag{4.25}$$

and get

LEMMA 4.4. — *The function u^1 and p^1 are continuous at the point $\xi=u_*^0$.*

Again, define the coefficients $a_{L,\,j}$ and $a_{R,\,j}$, $1\leqq j\leqq 3$, by

$$V_L^1=\sum_j a_{L,\,j}\,r_j(V_L^0),\qquad V_R^1=\sum_j a_{R,\,j}\,r_j(V_R^0).$$

Then, Lemma 2.5 enable us to determine the constants $(a_{i,\,j})_{1\leqq j\leqq 3}$ of Lemma 4.1 for $i=0$ and $i=3$:

LEMMA 4.5. — *For $j=1, 2, 3$, we have*

$$a_{0,\,j}=a_{L,\,j},\qquad a_{3,\,j}=a_{R,\,j}. \tag{4.26}$$

Remark. — Using (4.6) and (4.26), we easily get the following explicit formulae for the constants $a_{0,\,j}$:

$$a_{0,\,1}=a_{L,\,1}=\frac{1}{2}\left(-\frac{u_L^1}{c_L^0}+\frac{p_L^1}{\gamma(p_L^0+\bar{p})}\right),$$

$$a_{0,\,2}=a_{L,\,2}=\frac{-S_L^1}{\gamma S_L^0}=\frac{\rho_L^1}{\rho_L^0}-\frac{p_L^1}{\gamma(p_L^0+\bar{p})},$$

and

$$a_{0,\,3}=a_{L,\,3}=\frac{1}{2}\left(\frac{u_L^1}{c_L^0}+\frac{p_L^1}{\gamma(p_L^0+\bar{p})}\right).$$

Similar formulae may be derived for the $a_{3,\,j}$.

For obtaining the function V^1, it remains to determine the constants $a_{i,\,j}$ of Lemma 4.1 for $i=1$ and $i=2$. Thanks to Lemma 4.4, we can set

$$u_*^1=u^1(u_*^0),\qquad p_*^1=p^1(u_*^0). \tag{4.27}$$

As in Section 3, we begin by a 2×2 *algebraic system* in the unknown u_*^1 and p_*^1.

THEOREM 4.1. — 1. *If the function V^0 contains a 1-rarefaction wave, we have*

$$\frac{u_*^1}{c_1^0}+\frac{p_*^1}{\gamma(p_*^0+\bar{p})}=\frac{b_{1,\,2}}{3\gamma-1}(c_1^0)^{(\gamma+1)/(\gamma-1)} +b_{1,\,3}(c_1^0)^{(3-\gamma)/2(\gamma-1)}+M^0(c_1^0)\,c_1^0+N_L^0(c_1^0)-\frac{\nu u_*^0}{2x_0}, \tag{4.28}$$

where $b_{1,2}$ *and* $b_{1,3}$ *(respectively* M^0 *and* N_L^0*) are explicitely given by* (4.19) *and* (4.20) *(resp.* (4.11) *and* (4.12)).

2. *If the function* V^0 *contains a 3-rarefaction wave, we have*

$$-\frac{u_*^1}{c_2^0}+\frac{p_*^1}{\gamma(p_*^0+\bar{p})}=b_{3,1}(c_2^0)^{(3-\gamma)/2(\gamma-1)}+\frac{b_{3,2}}{(3\gamma-1)}(c_2^0)^{(\gamma+1)/(\gamma-1)}-M^0(c_2^0)c_2^0+N_R^0(c_2^0)-\frac{\nu u_*^0}{2x_0} \quad (4.29)$$

where $b_{3,1}$ *and* $b_{3,2}$ *(respectively* M^0 *and* N_R^0*) are explicitly given by* (4.22). (4.23) *(resp.* (4.11) *and* (4.18)).

3. *If the function* V^0 *contains a* 1*-shock wave, we have*

$$\{2-\rho_1^0.(\sigma_1^0-u_*^0).Q_L^0\}u_*^1+\left\{\frac{-(\sigma_1^0-u_*^0)}{\gamma(p_*^0+\bar{p})}+\frac{1}{W_L^0(p_*^0)}+Q_L^0\right\}p_*^1=(\sigma_1^0-u_*^0)\frac{\nu}{x_0}u_*^0+K_L^0, \quad (4.30)$$

where the constants $W_L^0(p_*^0)$, and K_L^0 *are given by* (3.32), (3.33) *and*

$$\left.\begin{aligned}K_L^0=(\sigma_1^0-u_L^0)u_L^1-\frac{p_L^1}{\rho_L^0}+\left(-\frac{1}{W_L^0(p_*^0)}+\mu^2Q_L^0\right)\Big\{-(\sigma_1^0-u_L^0)p_L^1\\+\gamma(p_L^0+\bar{p})\left(u_L^1+\frac{\nu}{x_0}u_L^0\right)\Big\}\\+\frac{(u_*^0-u_L^0)}{2.\rho_L^0}\left\{-(\sigma_1^0-u_L^0)\rho_L^1+\rho_L^0u_L^1+\frac{\nu}{x_0}\rho_L^0u_L^0\right\}\end{aligned}\right\} \quad (4.31)$$

4. *If the function* V^0 *contains a 3-shock wave, we have*

$$\{2-\rho_2^0(\sigma_3^0-u_*^0).Q_R^0\}u_*^1+\left\{\frac{-(\sigma_3^0-u_*^0)}{\gamma(p_*^0+\bar{p})}-\frac{1}{W_R^0(p_*^0)}+Q_R^0\right\}p_*^1=(\sigma_3^0-u_*^0)\frac{\nu}{x_0}u_*^0+K_R^0, \quad (4.32)$$

where the constants $W_R^0(p_*^0)$, Q_R^0 *and* K_R^0 *are given by* (3.32), (3.33) *and*

$$\left.\begin{aligned}K_R^0=(\sigma_3^0-u_R^0)u_R^1-\frac{p_R^1}{\rho_R^0}+\left(\frac{1}{W_R^0(p_*^0)}+\mu^2Q_R^0\right)\Big\{-(\sigma_3^0-u_R^0)p_R^1\\+\gamma(p_R^0+\bar{p})\left(u_R^1+\frac{\nu}{x_0}u_R^0\right)\Big\}+\frac{(u_*^0-u_R^0)}{2.\rho_R^0}\\\times\left\{-(\sigma_3^0-u_R^0)\rho_R^1+\rho_R^0u_R^1+\frac{\nu}{x_0}\rho_R^0u_R^0\right\}\end{aligned}\right\} \quad (4.33)$$

Proof. — Consider for instance the case where V^0 contains a 1-rarefaction wave (the derivation is similar in the case of a 3-rarefaction wave).

At $\xi = u_*^0 - 0$, we may write by Lemma 4.1:

$$\alpha_3(u^0 - 0) = -c_1^0 a_{1,3} - \frac{\nu}{2x_0} u_*^0,$$

and using (4.6) and (4.25), we have also

$$\alpha_3(u_*^0 - 0) = \frac{1}{2}\left(\frac{\rho_1^1}{\rho_1^0} + \frac{u_*^1}{c_1^0} + \frac{S_1^1}{\gamma S_1^0}\right) = \frac{1}{2}\left(\frac{u_*^1}{c_1^0} + \frac{p_*^1}{\gamma(p_*^0 + \bar{p})}\right).$$

Thus, we obtain

$$\frac{u_*^1}{c_1^0} + \frac{p_*^1}{\gamma(p_*^0 + \bar{p})} = 2\left\{-c_1^0 a_{1,3} - \frac{\nu}{2x_0} u_*^0\right\}. \tag{4.34}$$

Hence, to prove (4.28), it suffices to determine now the constant $a_{1,3}$.

Thanks to (4.21), the function α_3 is continuous at $\xi = u_*^0 - c_1^0$. Therefore, using (4.9) and (4.10), we have

$$-2c_1^0 a_{1,3} - \frac{\nu}{2x_0} u_1^0 = \frac{b_{1,2}}{3\gamma - 1}(c_1^0)^{(\gamma+1)/(\gamma-1)} + b_{1,3}(c_1^0)^{(3-\gamma)/2(\gamma-1)} + M^0(c_1^0)c_1^0 + N_L^0(c_1^0)$$

which yields $a_{1,3}$.

Next, we consider the case of a 3-shock wave (again the derivation is similar in the case of a 1-shock wave). Lemma 3.7 gives the following relation between $u^1(\sigma_3^0 \pm 0)$, $p^1(\sigma_3^0 \pm 0)$ and $\rho^1(\sigma_3^0 + 0)$:

$$u^1(\sigma_3^0 - 0) - \left(\frac{1}{W_R^0(p_*^0)} - Q_R^0\right) p^1(\sigma_3^0 - 0) = u^1(\sigma_3^0 + 0) - \left(-\frac{1}{W_R^0(p_*^0)} + \mu^2 Q_R^0\right) p^1(\sigma_3^0 + 0) - (u_*^0 - u_R^0) \cdot \frac{\rho^1(\sigma_3^0 + 0)}{2 \cdot \rho_R^0}, \tag{4.35}$$

where the constants μ, $W_R^0(p_*^0)$ and Q_R^0 are defined by (3.32).(3.33).

We begin by computing the right hand side of (4.35). Using (4.7) and (4.9), we get

$$\begin{aligned} u^1(\sigma_3^0 + 0) &= c_R^0 \cdot (-\alpha_1 \sigma_3^0 +) + \alpha_3(\sigma_3^0 +)) \\ &= c_R^0 \cdot (-\sigma_3^0 + u_R^0 - c_R^0) a_{R,1} + (\sigma_3^0 - u_R^0 - c_R^0) a_{R,3}). \end{aligned}$$

Then because of (4.26) and (4.7), we have

$$\begin{aligned} u^1(\sigma_3^0 + 0) &= c_R^0 \left\{(\sigma_3^0 - u_R^0) \frac{u_R^1}{c_R^0} - c_R^0 \frac{p_R^1}{\gamma(p_R^0 + \bar{p})}\right\} \\ &= (\sigma_3^0 - u_R^0) u_R^1 - \frac{p_R^1}{\rho_R^0}. \end{aligned}$$

By the same arguments, we get also:

$$
\begin{aligned}
p^1(\sigma_3^0+0) &= \gamma(p_R^0+\bar{p}).\left\{(\sigma_3^0-u_R^0+c_R^0)\,Q_{R,\,1}+(\sigma_3^0-u_R^0-c_R^0)\,a_{R,\,3}-\frac{\nu}{x_0}\,u_R^0\right\}\\
&= \gamma(p_R^0+\bar{p}).\left\{(\sigma_3^0-u_R^0)\,\frac{p_R^1}{\gamma(p_R^0+\bar{p})}-c_R^0\,\frac{u_R^1}{c_R^0}-\frac{\nu}{x_0}\,u_R^0\right\}\\
&= (\sigma_3^0-u_R^0)\,p_R^1-\gamma(p_R^0+\bar{p}).\left(u_R^1+\frac{\nu}{x_0}\,u_R^0\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\rho^1(\sigma_3^0+0) &= \rho_R^0\left\{(\sigma_3^0-u_R^0)-\frac{\rho_R^1}{\rho_R^0}-c_R^0\,\frac{u_R^1}{c_R^0}-\frac{\nu}{x_0}\,u_R^0\right\}\\
&= (\sigma_3^0-u_R^0)\;\rho_R^1-\rho_R^0\,u_R^1-\frac{\nu}{x_0}\,\rho_R^0\,u_R^0.
\end{aligned}
$$

So that we have the value of the second member of the equation (4.35).

Now, we compute the left hand side of (4.35). Again using (4.7) and (4.9), we obtain

$$
\begin{aligned}
u^1(\sigma_3^0-0) &= c_2^0.\{-\alpha_1(\sigma_3^0-)+\alpha_3(\sigma_3^0-)\}\\
&= c_2^0.\{(-\sigma_3^0+u_2^0-c_2^0)\,a_{2,\,1}+(\sigma_3^0-u_2^0-c_2^0)\;a_{2,\,3}\}\\
&= c_2^0.\{(\sigma_3^0-u_2^0)(-a_{2,\,1}+a_{2,\,3})-c_2^0\,(a_{2,\,1}+a_{2,\,3})\},
\end{aligned}
$$

and

$$
\begin{aligned}
p^1(\sigma_3^0-0) &= \gamma(p_*^0+\bar{p}).\{\alpha_1(\sigma_3^0-)+\alpha_3(\sigma_3^0-)\}\\
&= \gamma(p_*^0+\bar{p}).\left\{(\sigma_3^0-u_3^0+c_2^0)\,a_{2,\,1}+(\sigma_3^0-u_2^0-c_2^0)\;a_{2,\,3}-\frac{\nu}{x_0}\,u_*^0\right\}\\
&= \gamma(p_*^0+\bar{p}).\left\{(\sigma_3^0-u_2^0)(a_{2,\,1}+a_{2,\,3})+c_2^0\,(a_{2,\,1}-a_{2,\,3})-\frac{\nu}{x_0}\,u_*^0\right\}.
\end{aligned}
$$

We need to express the constants $a_{2,\,1}$ and $a_{2,\,3}$ in terms of functions of u_*^1 and p_*^1. We may write by Lemma 4.1:

$$
\begin{aligned}
u_*^1 &= -(c_2^0)^2\,(a_{2,\,1}+a_{2,\,3}),\\
p_*^1 &= \gamma(p_*^0+\bar{p}).\left\{c_2^0\,(a_{2,\,1}-a_{2,\,3})-\frac{\nu}{x_0}\,u_*^0\right\},
\end{aligned}
$$

that is

$$a_{2,\,1}+a_{2,\,3}=-\frac{u_*^1}{(c_2^0)^2},\tag{4.36}$$
$$a_{2,\,1}-a_{2,\,3}=\frac{p_*^1}{\gamma(p_*^0+\bar{p})\,c_2^0}+\frac{\nu u_*^0}{x_0\,c_2^0}.$$

Hence, the values of u^1 and p^1 at $\xi=\sigma_3^0-0$ are:

$$u^1(\sigma_3^0-0)=-(\sigma_3^0-u_*^0)\frac{p^1}{\gamma(p_*^0+\bar{p})}+u_*^1-(\sigma_3^0-u_*^0)\frac{\nu u_*^0}{x_0},\tag{4.37}$$
$$p^1(\sigma_3^0-0)=-(\sigma_3^0-u_*^0).\,\rho_2^0.\,u_*^1+p_*^1.$$

Finally, noting that [4]

$$W_R^0(p_*^0)=\rho_2^0.(\sigma_3^0-u_*^0),$$

formula (4.32) results from (4.35). ■

We are now able to determine the whole of constants a_{ij}.

THEOREM 4.2. — *For $i=1$ and 2, the constants $a_{i,\,1}$ and $a_{i,\,3}$ are given by:*

$$a_{i,\,1}=\frac{1}{2c_i^0}.\left\{-\frac{u_*^1}{c_i^0}+\frac{p_*^1}{\gamma(p_*^0+\bar{p})}+\nu.\frac{u_*^0}{x_0}\right\},\tag{4.38}$$

$$a_{i,\,3}=\frac{-1}{2c_i^0}\left\{\frac{u_*^1}{c_i^0}+\frac{p_*^1}{\gamma(p_*^0+\bar{p})}+\nu.\frac{u_*^0}{x_0}\right\}\tag{4.39}$$

When V^0 *contains a k-rarefaction wave, we have*

$$a_{1,\,2}=\frac{c_L^0}{c_1^0+u_*^0}\left\{\frac{p_L^1}{\gamma(p_L^0+\bar{p})}-\frac{\rho_L^1}{\rho_L^0}\right\},\quad \text{if}\quad k=1,\tag{4.40}$$

and

$$a_{2,\,2}=\frac{c_R^0}{c_2^0-u_*^0}\left\{\frac{p_R^1}{\gamma(p_R^0+\bar{p})}-\frac{\rho_R^1}{\rho_R^0}\right\},\quad \text{if}\quad k=3.\tag{4.41}$$

When V^0 *contains a k-shock wave, we have*

$$\left.\begin{aligned} a_{1,2}=\frac{u_*^1}{(c_1^0)^2}-\frac{p_*^1}{(\sigma_1^0-u_*^0)\,\gamma\,(p_*^0+\bar{p})} &\\ +\frac{\rho_1^0}{\sigma_1^0-u_*^0}\Bigg[\left(\frac{\partial G}{\partial p}\right)_1^0((\sigma_1^0-u_*^0)\,\rho_2^0\,u_*^1-p_*^1) &\\ +\left(\frac{\partial G}{\partial \tau_a}\right)_L^0\left\{(\sigma_1^0-u_L^0)\,\frac{\rho_L^1}{(\rho_L^0)^2}-\frac{\nu}{x_0}\,\frac{u_L^0}{\rho_L^0}\right\} &\\ +\left(\frac{\partial G}{\partial p_a}\right)_L^0\left\{-(\sigma_1^0-u_L^0)\,p_L^1+\gamma\,(p_L^0+\bar{p})\left(u_L^1+\frac{\nu}{x_0}\,u_L^0\right)\right\}\Bigg] & \end{aligned}\right\}\tag{4.42}$$

and

$$\left.\begin{aligned} a_{2,2}=\frac{u_*^1}{(c_2^0)^2}-\frac{p_*^1}{(\sigma_3^0-u_*^0)\,\gamma\,(p_*^0+\bar{p})} &\\ +\frac{\rho_2^0}{(\sigma_3^0-u_*^0)}\Bigg[\left(\frac{\partial G}{\partial p}\right)_0^0((\sigma_3^0-u_*^0)\,\rho_2^0\,u_*^1-p_*^1) &\\ +\left(\frac{\partial G}{\partial \tau_a}\right)_R^0\left\{(\sigma_3^0-u_R^0)\,\frac{\rho_R^1}{(\rho_R^0)^2}-\frac{u_R^1}{\rho_R^0}-\frac{\nu}{x_0}\,\frac{u_R^0}{\rho_R^0}\right\} &\\ +\left(\frac{\partial G}{\partial p_a}\right)_R^0\left\{-(\sigma_3^0-u_R^0)\,p_R^1+\gamma\,(p_R^0+\bar{p})\left(u_R^1+\frac{\nu}{x_0}\,u_R\right)\right\}\Bigg] & \end{aligned}\right\}\tag{4.43}$$

where the derivatives of the function G *are given by* (3.37).

Proof. — Formulae (4.38) and (4.39) were previously demonstrated in the proof of Theorem 4.1. To prove (4.40) [it is similar for (4.41)], we have by (4.9)

$$\alpha_2(-c_1^0)=-(c_1^0+u_*^0)\,a_{1,2},$$

and by (4.10)

$$\alpha_2(-c_1^0)=b_{1,2}\,(c_1^0)^{(\gamma+1)/(\gamma-1)}.$$

But we deduce easily from (4.19), (4.26) and (4.6) the value of the constant $b_{1,2}$:

$$b_{1,2}=-a_{0,2}\,(c_L^0)^{-2/(\gamma-1)}=\frac{S_L^1}{\gamma\,S_L^0}\,(c_L^0)^{-2/(\gamma-1)}\left\{\frac{\rho_L^1}{\rho_L^0}-\frac{p_L^1}{\gamma\,(p_L^0+\bar{p})}\right\}.$$

Thus, we get

$$-(c_1^0+u_1^0)\,a_{1,2}=c_L^0\cdot\left\{\frac{\rho_L^1}{\rho_L^0}-\frac{p_L^1}{\gamma\,(p_L^0+\bar{p})}\right\},$$

which yields (4.40).

In the case where V^0 has a shock wave, say a 3-wave, (3.29) [or (3.31)] gives a relation between $\rho^1(\sigma_3^0 \pm 0)$ and $p^1(\sigma_3^0 \pm 0)$:

$$\frac{\rho^1(\sigma_3^0-0)}{(\rho_2^0)^2}+\left(\frac{\partial G}{\partial p}\right)_2^0 p^1(\sigma_3^0-0) = \left(\frac{\partial G}{\partial \tau_a}\right)_R^0 \frac{\rho^1(\sigma_3^0+0)}{(\rho_R^0)^2} - \left(\frac{\partial G}{\partial p_a}\right)_R^0 p^1(\sigma_3^0+0). \quad (4.44)$$

As in the proof of Theorem 4.1, we can compute $\rho^1(\sigma_3^0+0)$ and $p^1(\sigma_3^0 \pm 0)$ and get:

$$\rho^1(\sigma_3^0+0)=(\sigma_3^0-u_R^0)\,\rho_R^1-\rho_R^0 u_R^1-\frac{\nu}{x_0}\,\rho_R^0 u_R^0,$$

$$p^1(\sigma_3^0+0)=(\sigma_3^0-u_R^0)\,p_R^1-\gamma(p_R^0+\bar{p})\left(u_R^1+\frac{\nu}{x_0}\,u_R^0\right)$$

and

$$p^1(\sigma_3^0-0)=-(\sigma_3^0-u_*^0)\,\rho_2^0 u_*^1+p_*^1.$$

Using (4.7).(4.9), we also determine $\rho^1(\sigma_3^0-0)$:

$$\begin{aligned}\rho^1(\sigma_3^0-0)&=\rho_2^0.(\alpha_1+\alpha_2+\alpha_3)(\sigma_3^0-0)\\ &=\rho_2^0\left\{(\sigma_3^0-u_2^0)(a_{2,\,1}+a_{2,\,2}+a_{2,\,3})+c_2^0(a_{2,\,1}-a_{2,\,3})-\frac{\nu u_*^0}{x_0}\right\}\\ &=\rho_2^0(\sigma_3^0-u_2^0)\,a_{2,\,2}-(\sigma_3^0-u_2^0)\,\rho_2^0(c_2^0)^{-2}u_*^1+\frac{p_*^1}{(c_2^0)^2}.\end{aligned}$$

We introduce these expressions in (4.44):

$$\begin{aligned}&(\sigma_3^0-u_2^0)\,\frac{1}{\rho_2^0}\,a_{2,\,2}-(\sigma_3^0-u_2^0)\,\frac{u_*^1}{\rho_2^0(c_2^0)^2}+\frac{p_*^1}{(\rho_2^0)^2(c_2^0)^2}\\ &\quad+\left(\frac{\partial G}{\partial p}\right)_2^0.\left(-(\sigma_3^0-u_*^0)\,\rho_2^0 u_*^1+p_*^1\right)=\left(\frac{\partial G}{\partial \tau_a}\right)_R^0.\left((\sigma_3^0-u_R^0)\,\frac{\rho_R^1}{(\rho_R^0)^2}-\frac{u_R^1}{\rho_R^0}\right.\\ &\quad\left.+-\frac{\nu}{x_0}\,\frac{u_R^0}{\rho_R^0}\right)-\left(\frac{\partial G}{\partial p_a}\right)_R^0\left\{(\sigma_3^0-u_R^0)\,p_R^1-\gamma(p_R^0+\bar{p})\left(u_R^1+\frac{\nu}{x_0}\,u_R^0\right)\right\},\end{aligned}$$

which yields (4.42). ■

Finally, as a trivial consequence of Theorems 4.1 and 4.2, we get:

COROLLARY. — *The value of the function* ρ^1 *at* $\xi=u_*^0$ *are given by*

$$\rho^1(u_*^0-0)=\frac{\rho_1^0.p_*^1}{\gamma(p_*^0+\bar{p})}, \quad (4.45)$$

and

$$\rho^1(u_*^0+0)=\frac{\rho_2^0 p_*^1}{\gamma(p_*^0+\bar{p})}. \qquad (4.46)$$

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(*Manuscript received May* 16, 1988.)