Equivariant harmonic maps between manifolds with metrics of (p, q)-signature

by

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ABSTRACT. — The equivariant theory for harmonic maps is extended to manifolds with indefinite metrics. We apply it to produce many new examples of harmonic maps and global solutions for the associated Cauchy problems.

RÉSUMÉ. — La théorie équivariante pour les applications harmoniques est étendue aux variétées avec métriques indéfinies. Nous l'appliquons pour produire beaucoup de nouvelles applications harmoniques et solutions globales pour les problèmes de Cauchy associés.

Mots clés : Harmonic maps, Cauchy problem, hyperbolic systems, lorentzian manifolds.

INTRODUCTION

This paper deals with global existence problems for harmonic maps between manifolds with metrics of (p, q)-signature.

Existence problems for harmonic maps between *riemannian* manifolds have been studied since 1964 [ES] and global results of great interest in geometry and physics have been obtained by using many different methods (*see* [EL1], [EL2]). This paper deals with the *equivariant method:* roughly speaking, this method consists in exploiting the symmetries of certain maps in order to reduce the existence problem to the qualitative study of an *ordinary differential equation.*

We call a manifold endowed with a metric of signature of type (p, q) a (p, q)-*manifold.*

The class of (p, o)-manifolds coincides with the class of riemannian manifolds; $(p, 1)$-manifolds are called lorentzian manifolds. Because of their significance in mathematical physics, harmonic maps between (p, q)-manifolds are certainly a developping subject. However, at the present time, only a few global existence results have been obtained (*see* [CB], [G1], [G2], [GV], [HL]): this lack of general global existence results constitutes one of the main motivations for introducing the equivariant method in this context.

The paper is organized as follows: in section 1 we describe a general theoretical setting for equivariant theory on (p, q)-manifolds and establish a *Reduction theorem.*

In section 2 we describe many examples of equivariant maps: in particular, we produce two examples where the domain is the *Robertson-Walker space-time.*

In section 3 we prove a global existence theorem for the *Cauchy problem* for harmonic maps from Lorentzian into riemannian manifolds: this special case is particularly significant in mathematical physics (*see* [CB], [G2]).

Section 4 is devoted to maps between (p, q)-manifolds, $p, q \geqq 1$: we obtain existence and non-existence examples: in particular, we represent harmonically each homotopy class of maps from $T^{1, 1}$ to $S^{1, 1}$.

In section 5 we obtain qualitative results about the harmonic maps produced in sections 3 and 4.

The results of sections 3, 4 and 5 are based on the qualitative study of ordinary differential equations of physical interest.

1. (p, q)-EQUIVARIANT THEORY

Given a (p, q)-manifold W, its Chrystoffel symbols are defined analogously to the riemannian case: similarly, let $f: W_1 \to W_2$ be a differentiable map between two (p, q)-manifolds: thus the second fundamental form $\nabla(df)$ is defined and f is said to be harmonic if

$$\text{Trace } \nabla(df) = 0. \qquad (1.1)$$

If W_1 is a riemannian manifold, then the system (1.1) is elliptic; in the case where W_1 is a (p, q)-manifold, $p, q \geqq 1$, the system (1.1) has substantially different features from the riemannian case: in particular, the investigation of the existence of global solutions is much harder because this branch of the general theory for hyperbolic systems is much less advanced than the corresponding elliptic theory.

The equivariant method appears to be suitable to study global properties of (1.1) in quite many important cases, as we will see in sections 3, 4 and 5. Here we present a theoretical setting for this method.

As for equivariant theory for riemannian manifolds we refer to [Ba], [KW], [R2], [S2].

Before giving the general definitions we describe a simple but instructive example.

Let (M, g), (N, h) be riemannian manifolds of dimensions m, n. We consider warped products

$$\left. \begin{aligned} (W_1, g_1) &= (M \times \mathbb{R}, \, A^2(t) g - dt^2) \\ (W_2, g_2) &= (N \times \mathbb{R}, \, B^2(t) h + dt^2) \end{aligned} \right\} \qquad (1.2)$$

where $t \in \mathbb{R}$ and $A(t)$, $B(t)$ are positive functions defined on $\mathbb{R}$. Clearly W_1 and W_2 are respectively a lorentzian and a riemannian manifold.

We will be interested in maps $f : W_1 \to W_2$ of the following type:

$$\left. \begin{aligned} f : \quad & M \times \mathbb{R} \to N \times \mathbb{R} \\ & (x, t) \to (\Phi(x), \alpha(t)) \end{aligned} \right\} \qquad (1.3)$$

where $\Phi : M \to N$ is a *harmonic* map with *constant energy density* $e(\Phi)$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is a differentiable function.

The symmetries of f provide reduction of (1.1) to an O.D.E.: more precisely, the map f is harmonic if and only if the function $\alpha(t)$ satisfies

$$\alpha''(t) + m \frac{A'(t)}{A(t)} \alpha'(t) + \frac{2 e(\Phi) B(\alpha(t)) B'(\alpha(t))}{A^2(t)} = 0. \qquad (1.4)$$

Therefore in this case the existence of global solutions of the hyperbolic system (1.1) is reduced to the qualitative study of the ordinary differential equation (1.4).

In this order of ideas, we can proceed to the general definitions. Let (M, g) be an m-dimensional riemannian manifold on which there exist distributions S_j of dimension m_j, $j = 1 \ldots k$ such that:

(i) $\quad \left(\overset{k}{\underset{j=1}{\oplus}} S_j \right)_x = T_x M, \qquad \forall x \in M$

(ii) $\quad S_i \perp S_j, \qquad \forall i \neq j$

(iii) $\quad S_j$ is locally integrable, $\qquad j = 1 \ldots k$.

For example, M can be a product of riemannian manifolds; but a twisted product as well.

Let M_j be the integral submanifold associated to the distribution S_j and g_j be the induced metric on M_j (M_j, g_j in general make sense only locally).

Let i be the complex number such that $i^2 = -1$.

We consider the following two classes of functions:

$$v(t) \tag{1.5}$$

and

$$i \cdot v(t) \tag{1.6}$$

where in both cases $v(t)$ is a *positive* differentiable function defined on some open, not necessarily limited, real interval (a, b). We will develop (p, q)-equivariant theory on the following class of (p, q)-manifolds:

$$\left(M \times (a, b), \sum_{j=1}^{k} A_j^2(t) g_j + h^2(t) dt^2 \right) \tag{1.7}$$

where $A_j(t)$, $h(t)$ are functions either of type (1.5) or (1.6), and M and g_j are as above.

Let J be the set of indexes j such that A_j is of type (1.5) and assume for instance that $h(t)$ be of type (1.6): then the manifold in (1.7) is a (p, q)-manifold with

$$p = \sum_{j \in J} m_j, \qquad q = m + 1 - p \tag{1.8}$$

where we recall that $m_j = \dim S_j$ and $m = \dim M$.

We call a manifold as in (1.7) an *equivariant* (p, q)-*manifold*.

In order to study maps between two equivariant (p, q)-manifolds, we use the above notations for the domain; for the range, we denote our equivariant (p, q)-manifold as follows:

$$\left(N \times (c, d), \sum_{r=1}^{l} B_r^2(t) h_r + K^2(t) dt^2 \right) \tag{1.9}$$

where N, (c, d), $B_r(t)$, $K(t)$ play respectively the role of M, (a, b), $A_j(t)$, $h(t)$ in (1.7) and h_r are associated to distributions T_r in an analogous way to g_j to S_j above.

Let $\Phi: M \to N$ and $\alpha: (a, b) \to (c, d)$ be smooth maps.

A map f of the form

$$\left. \begin{array}{l} f: \quad M \times (a, b) \to N \times (c, d) \\ \qquad (x, t) \to (\Phi(x), \alpha(t)) \end{array} \right\} \tag{1.10}$$

is said to be an *equivariant map* if $\Phi: M \to N$ satisfies

$$d\Phi(S_j) \subseteq T_{r_j} \quad \text{for some } r_j, \qquad j = 1 \ldots k. \tag{1.11}$$

and

$\Phi|_{S_j}$ is harmonic with constant energy density $e(\Phi)_j$, $j = 1 \ldots k$. (1.12)

More explicitly, condition (1.12) means: write $x \in M$ as $(x_1, \ldots, x_k)$, with $x_i \in M_i$, $i = 1 \ldots k$. This is possible locally because of assumption iii) and (1.12) is a local condition.

Let us fix $\bar{x}_i \in M_i$ for all $i \neq j$ and consider

$$x_j \to \Phi(\bar{x}_1, \ldots, x_j, \ldots, \bar{x}_k).$$

Condition (1.12) requires that such a map be harmonic with constant energy density $e(\Phi)_j$ which does not depend upon the choice of $\bar{x}_i \in M_i$, $i \neq j$.

A straightforward computation leads to

Reduction theorem ([R2], pp. 147-150)

Let $M \times (a, b)$, $N \times (c, d)$ *be equivariant* (p, q)-*manifolds as in* (1.7), (1.9); *and let* $f : M \times (a, b) \to N \times (c, d)$ *be an equivariant map as in* (1.10).

Then f *is harmonic if and only if the function* $\alpha(t)$ *satisfies*

$$\alpha''(t) + \left[\sum_{j=1}^{k} \frac{A'_j(t)}{A_j(t)} m_j - \frac{h'(t)}{h(t)} \right] \alpha'(t)$$

$$= \frac{2 h^2(t)}{K^2(\alpha(t))} \sum_{j=1}^{k} \frac{e(\Phi)_j B'_{r_j}(\alpha(t)) B_{r_j}(\alpha(t))}{A_j^2(t)}$$

$$- \frac{K'(\alpha(t))}{K(\alpha(t))} [\alpha'(t)]^2. \quad (1.13)$$

We notice that the example illustrated at the beginning of this section is a particular case of the Reduction theorem, with $(a, b) = (c, d) = \mathbb{R}$, $k = r = 1$, $h(t) \equiv i$, $K(t) \equiv 1$.

Remark (see [Ba], [KW], [R2]).

There are cases where (p, q)-manifolds as in (1.7), (1.9) are *open dense subsets* of other manifolds; each connected component of the complement of such an open set is usually called a *focal variety*.

In these cases global properties are usually achieved by imposing boundary conditions on $\alpha(t)$ and using regularity arguments.

2. EXAMPLES OF EQUIVARIANT MAPS BETWEEN (p, q)-MANIFOLDS

First of all we introduce the equivariant (p, q)-manifolds that we will be interested in.

Let S^p be the p-dimensional unit euclidean sphere and $\mathbb{R}^{p+1, q+1}$ the $(p+q+2)$-dimensional real vector space with inner product

$$\langle z, w \rangle = \sum_{i=1}^{p+1} z_i \cdot w_i - \sum_{i=p+2}^{p+q+2} z_i \cdot w_i. \tag{2.1}$$

We have

$$S^{p, q+1} \overset{\text{def}}{=} \{x \in \mathbb{R}^{p+1, q+1} : \langle x, x \rangle = 1\}. \tag{2.2}$$

We can parametrize $S^{p, q+1}$ in $\mathbb{R}^{p+1, q+1}$ by writing every point $z \in S^{p, q+1}$ as

$$z = \cosh tu + \sinh tv \tag{2.3}$$

where $u \in S^p$, $v \in S^q$ and $t \in [0, +\infty)$.

If $q > 0$, then it is easy to see that $S^{p, q+1}$ is isometric to

$$(S^p \times S^q \times [0, +\infty), h) \tag{2.4}$$

where the metric h is given by

$$\cosh^2 tg_1 - \sinh^2 t g_2 - dt^2 \tag{2.5}$$

with $t \in [0, +\infty)$, g_1, g_2 euclidean metrics of S^p, S^q.

The locus $t = 0$ is a focal variety homeomorphic to S^p.

It is clear from (2.4), (2.5) that $S^{p, q+1}$ minus the focal variety $t = 0$ is an equivariant $(p, q+1)$-manifold as in (1.7).

If $q = 0$, one sees easily that, up to an isometry,

$$S^{p, 1} = (S^p \times \mathbb{R}, \cosh^2 t - dt^2). \tag{2.6}$$

In this case there is no focal variety.

An important example of equivariant $(3,1)$-manifold is the *Robertson-Walker space-time* (see [SW]):

Let (M, g) be a simply connected 3-dimensional space form of curvature c ($c \in \{0, +1, -1\}$) and let $A(t)$ be a positive, strictly increasing, differentiable function defined on $(0, +\infty)$.

Then

$$(M \times (0, +\infty), A^2(t)g - dt^2) \tag{2.7}$$

is called *Robertson-Walker space-time* of c spatial curvature.

These space-times are important in general relativity because they are employed to construct cosmological models. Apparently, we have three different classes of Robertson-Walker space-times, according to the value of the constant c; but, when one actually uses these models to study cosmological phenomena, he finds that all three types make similar predictions for all observable effects.

Then nowadays the most common approach consists in studying all the Robertson-Walker space-times together as a one parameter set, the

parameter being

$$q \overset{\text{def}}{=} - \frac{A''(t) A(t)}{A'^2(t)}. \tag{2.8}$$

The parameter q is called *deceleration parameter:* the cases of physical interest are those corresponding to constant positive values of q, which can be obtained by choosing

$$A(t) = t^s, \qquad 0 < s < 1. \tag{2.9}$$

The case $q = 1/2$ [*i. e.* $s = 1/3$ in (2.9)] is particularly significant: in this case, the associated space-time is usually called *Einstein-De Sitter space-time*.

For our purposes it is convenient to employ the model of spatial curvature $+1$; that is to say $(M, g) = S^3$ in (2.7) and we write

$$R_s \overset{\text{def}}{=} (S^3 \times (0, +\infty), \ t^{2s} g - dt^2). \tag{2.10}$$

We will study cases where the target is a riemannian manifold: thus it is useful to recall some parametrizations of S^m and H^m, the m-dimensional hyperbolic space.

We write every point $z \in S^{r+1} \subset \mathbb{R}^{r+1} \times \mathbb{R}$ in the form

$$z = \sin \gamma . u + \cos \gamma \tag{2.11}$$

with $0 \leqq \gamma \leqq \pi$, $u \in S^r$.

This determines an identification between S^{r+1} minus the two poles and the warped product

$$(S^r \times (0, \pi), \ \sin^2 \gamma \, g + d\gamma^2) \tag{2.12}$$

where g is the canonical metric of S^r.

Similarly, we write every point $z \in S^{r+s+1} \subset \mathbb{R}^{r+1} \times \mathbb{R}^{s+1}$ as

$$z = \sin \gamma . u + \cos \gamma \, v \tag{2.13}$$

with $0 \leqq \gamma \leqq \pi/2$, $u \in S^r$, $v \in S^s$.

Then S^{r+s+1} minus the loci $\gamma = 0$, $\gamma = \pi/2$ is isometric to

$$(S^r \times S^s \times (0, \pi/2), \ \sin^2 \gamma \, g_1 + \cos^2 \gamma \, g_2 + d\gamma^2) \tag{2.14}$$

where g_1, g_2 are the canonical metrics of S^r, S^s.

The hyperbolic space H^{r+1} can be characterized in the following manner:

$$H^{r+1} = \{ x \in \mathbb{R}^{r+1, 1} : \langle x, x \rangle = -1 \}. \tag{2.15}$$

Therefore a parametrization of H^{r+1} in $\mathbb{R}^{r+1, 1}$ can be obtained by writing every point $z \in H^{r+1}$ as

$$z = \sinh \gamma \, u + \cosh \gamma \tag{2.16}$$

with $0 \leqq \gamma < +\infty$ and $u \in S^r$.

Thus H^{r+1} minus a point is isometric to

$$(S^r \times (0, +\infty), \sinh^2 \gamma \, g + d\gamma^2) \qquad (2.17)$$

where g is the canonical metric of S^r.

Now we are in the right position to give some examples of equivariant maps between (p, q)-manifolds and associated reduction equations.

In all examples below, the map $u \to \Phi(u)$ will denote a k-homogeneous harmonic polynomial map $\Phi: S^p \to S^r$; thus Φ is a harmonic map with constant energy density $e(\Phi) = \lambda/2$, $\lambda = k.(k+p-1)$. Important examples of this kind are the Hopf fibrations; other examples arise from orthogonal multiplications or as gradients of isoparametric functions. A discussion of these maps and their properties is given in sec. 8 of [EL1].

Example (a). $- f: S^{p,\,1} \to S^{r+1}$.

By using (2.6) and (2.11) we define

$$\left.\begin{array}{c} f: \quad S^{p,\,1} \to S^{r+1} \\ (u,\,t) \to \sin \alpha(t) \, \Phi(u) + \cos \alpha(t). \end{array}\right\} \qquad (2.18)$$

It is clear from (2.6), (2.12) that such a map is equivariant and as an application of the Reduction theorem we have that the condition of harmonicity is the following ordinary differential equation for $\alpha(t)$

$$\alpha''(t) + [p \tanh t] \, \alpha'(t) + \frac{\lambda}{\cosh^2 t} \sin \alpha(t) \cos \alpha(t) = 0 \qquad (2.19)$$

Example (b). $- f: R_s \to S^3$.

Let $H: S^3 \to S^2$ be the Hopf fibration: then H is a harmonic 2-homogeneous polynomial map with $\lambda = 8$.

By using (2.10), (2.11) with $r = 2$, we define

$$\left.\begin{array}{c} f: \quad R_s \to S^3 \\ (u,\,t) \to \sin \alpha(t) \, H(u) + \cos \alpha(t) \end{array}\right\} \qquad (2.20)$$

The Reduction theorem gives us the condition of harmonicity:

$$\alpha''(t) + \frac{3\,s}{t} \alpha'(t) + \frac{8}{t^{2\,s}} \sin \alpha(t) \cos \alpha(t) = 0. \qquad (2.21)$$

Example (c). $- f: S^{p,\,1} \to I^{r+1}$.

Let I^{r+1} be the hyperboloid

$$I^{r+1} \stackrel{\text{def}}{=} (S^r \times \mathbb{R}, \cosh^2 t g + dt^2) \qquad (2.22)$$

with g canonical metric of S^r.

By using (2.6) and (2.22) we define

$$\left.\begin{array}{c} f: \quad S^{p,\,1} \to I^{r+1} \\ (u,\,t) \to (\Phi(u), \, \alpha(t)) \end{array}\right\}. \qquad (2.23)$$

This map is harmonic if and only if

$$\alpha''(t) + [p \tanh t] \alpha'(t) + \frac{\lambda}{\cosh^2 t} \sinh \alpha(t) \cosh \alpha(t) = 0. \quad (2.24)$$

Example (d). $- f : S^{p,1} \to H^{r+1}$.
By using (2.6) and (2.16) we define

$$\left. \begin{array}{c} f : \quad S^{p,1} \to H^{r+1} \\ (u, t) \to \sinh \alpha(t) \Phi(u) + \cosh \alpha(t) \end{array} \right\}. \quad (2.25)$$

The condition of harmonicity is

$$\alpha''(t) + [p \tanh t] \alpha'(t) + \frac{\lambda}{\cosh^2 t} \sinh \alpha(t) \cosh \alpha(t) = 0. \quad (2.26)$$

Example (e). $- f : R_s \to H^3$.
By using (2.10) and (2.16) with $r = 2$, we define

$$\left. \begin{array}{c} f : \quad R_s \to H^3 \\ (u, t) \to \sinh \alpha(t) H(u) + \cosh \alpha(t) \end{array} \right\}. \quad (2.27)$$

The condition of harmonicity is

$$\alpha''(t) + \frac{3s}{t} \alpha'(t) + \frac{8}{t^{2s}} \sinh \alpha(t) \cosh \alpha(t) = 0. \quad (2.28)$$

Example (f). $- f : S^{p,1} \to S^{r,1}$.
By using (2.6), we define

$$\left. \begin{array}{c} f : \quad S^{p,1} \to S^{r,1} \\ (u, t) \to (\Phi(u), \alpha(t)) \end{array} \right\}. \quad (2.29)$$

The condition of harmonicity is

$$\alpha''(t) + [p \tanh t] \alpha'(t) - \frac{\lambda}{\cosh^2 t} \sinh \alpha(t) \cosh \alpha(t) = 0. \quad (2.30)$$

Example (g). $- f : S^{p,q+1} \to S^{r,s+1}$.
Let $\Phi_1 : S^p \to S^r$ and $\Phi_2 : S^q \to S^s$ be two harmonic homogeneous polynomials with associated eigenvalues λ_1, λ_2.
By using (2.3) we define

$$\left. \begin{array}{c} f : \quad S^{p,q+1} \to S^{r,s+1} \\ (\cosh tu + \sinh tv) \to (\cosh \alpha(t) \Phi_1(u) + \sinh \alpha(t) \Phi_2(v)) \end{array} \right\} \quad (2.31)$$

where $\alpha(0) = 0$.

By using (2.4), (2.5) and the Reduction theorem, we obtain the condition of harmonicity

$$\alpha''(t) + \left[p \tanh t + \frac{q}{\tanh t} \right] \alpha'(t)$$
$$- \left[\frac{\lambda_1}{\cosh^2 t} + \frac{\lambda_2}{\sinh^2 t} \right] \sinh \alpha(t) \cosh \alpha(t) = 0. \quad (2.32)$$

Example (h): $f: S^{p,\, q+1} \to S^{r+s+1}$.

Let Φ_1, Φ_2 be as in example (g).

By using (2.3) and (2.13) we define

$$\left. \begin{array}{l} f: \quad S^{p,\, q+1} \to S^{r+s+1} \\ (\cosh tu + \sinh tv) \to (\sin \alpha(t)\, \Phi_1(u) + \cos \alpha(t)\, \Phi_2(v)) \end{array} \right\} \quad (2.23)$$

where $\alpha(0) = \pi/2$.

The condition of harmonicity is

$$\alpha''(t) + \left[p \tanh t + \frac{q}{\tanh t} \right] \alpha'(t)$$
$$+ \left[\frac{\lambda_1}{\cosh^2 t} + \frac{\lambda_2}{\sinh^2 t} \right] \sin \alpha(t) \cos \alpha(t) = 0. \quad (2.34)$$

Example (i). $- f: T^{1,\, 1} \to S^{1,\, 1}$.

Let $T^{1,\, 1}$ be

$$T^{1,\, 1} = (S^1 \times S^1,\ d\theta^2 - d\varphi^2). \quad (2.35)$$

We define

$$\left. \begin{array}{l} f: \quad T^{1,\, 1} \to S^{1,\, 1} \\ (\theta,\ \varphi) \to (m\,\theta + n\,\varphi,\ \alpha(\varphi)) \end{array} \right\}. \quad (2.36)$$

where $m,\ n \in \mathbb{Z}$

The condition of harmonicity is

$$\alpha''(\varphi) + (n^2 - m^2) \cosh \alpha(\varphi) \sinh \alpha(\varphi) = 0. \quad (2.37)$$

Further examples can be constructed, for instance by using different parametrizations of the space forms and $\mathbb{R}^{p+1,\, q+1}$ or by introducing suitable warped products.

3. THE CAUCHY PROBLEM FOR HARMONIC MAPS FROM LORENTZIAN INTO RIEMANNIAN MANIFOLDS

The Cauchy problem for harmonic maps from lorentzian into riemannian manifolds consists in the determination of a harmonic map from its value, and the value of its first derivative, on a space like submanifold of the source.

Global existence for this problem has been obtained in some particular cases: maps from $\mathbb{R}^{1,1}$ to complete manifolds (*see* [G2]); maps from $\mathbb{R}^{n,1}$, n odd, to complete manifolds provided that the Cauchy data are sufficiently small (*see* [CB]).

On the other hand, for large Cauchy data Shatah has constructed examples of maps from $\mathbb{R}^{3,1}$ to S^3 where global existence fails [EL2]. Here we consider the case of equivariant Cauchy data: for equivariant maps as in examples $(a)\ldots(e)$ of section 2, the Cauchy data are determined by the assignment of three real numbers $(\tilde{t}, \alpha_0, \alpha_1)$: more precisely, given $(\tilde{t}, \alpha_0, \alpha_1)$, does there exist a globally defined solution $\alpha(t)$ of the harmonicity equation such that

$$\alpha(\tilde{t}) = \alpha_0, \qquad \alpha'(\tilde{t}) = \alpha_1 ? \tag{3.1}$$

Remark. — In this equivariant context the existence of a *local solution* of the Cauchy problem follows from well-known results on ordinary differential equations; but in the general case the question whether there exist local solutions of the Cauchy problem is very delicate (*see* [CB]).

We prove the following

Global existence theorem:

Consider the Cauchy problem with equivariant data (3.1) *for the examples* $(a)\ldots(e)$ *of section 2.*

Then there always exists a unique global solution.

Proof. — The proof is a case by case study of examples $(a)\ldots(e)$. As for examples (a) and (b), our assertion follows immediately by writing the relevant equations (2.19) and (2.21) in vector form with $x = (\alpha, \alpha')$ and applying the following

LEMMA 3.1 (See [Ha]). — *Consider the system*

$$\dot{x}(t) = F(x(t), t) \tag{3.2}$$

where $(x, t) \in \mathbb{R}^2 \times (a, b)$ *and* $F : \mathbb{R}^2 \times (a, b) \to \mathbb{R}^2$ *is a differentiable function. Let* $\| \cdot \|$ *denote the euclidean norm of* $\mathbb{R}^2$.
Assume that

$$\| F(x, t) \| \leq \| x \| \cdot a(t) + b(t). \tag{3.3}$$

for some continuous function $a(t)$, $b(t)$.

Then every solution $x(t)$ *of* (3.2) *is defined for all* $t \in (a, b)$.

Now we occupy ourselves with examples (c), (d), (e).

It is convenient to make the substitution $t = e^u$, $u \in \mathbb{R}$, in equation (2.28) of example (e): we call $\beta(u) = \alpha(e^u)$, so that (2.28) becomes

$$\beta''(u) + (3s - 1)\beta'(u) + 8 e^{(2 - 2s)u} \sinh \beta(u) \cosh \beta(u) = 0, \qquad u \in \mathbb{R}. \tag{3.4}$$

Because (2.24) and (2.26) coincide, in order to end the proof of our theorem it is clearly sufficient to see that all solutions of (2.24) and (3.4) are globally defined.

Both these equations are of the form

$$\alpha''(t) + D(t)\alpha'(t) + G(t)\sinh\alpha(t)\cosh\alpha(t) = 0 \qquad (3.5)$$

with $G(t)$, $D(t)$ differentiable functions, $D(t)$ bounded.

Equation (3.5) is of physical interest: in fact, let C be the curve in the cartesian 2-plane (x, y) parametrized by

$$x = \cosh\alpha, \qquad y = \sinh\alpha, \qquad \alpha \in \mathbb{R}. \qquad (3.6)$$

We denote by $\mathbf{T}_\alpha$ the unit tangent vector to C at α

$$\mathbf{T}_\alpha = \left(\frac{\sinh\alpha}{\sqrt{\sinh^2\alpha + \cosh^2\alpha}}, \frac{\cosh\alpha}{\sqrt{\sinh^2\alpha + \cosh^2\alpha}} \right). \qquad (3.7)$$

Consider the motion of a point of unit mass vinculated to C and write the Newton law

$$\mathbf{F} = \alpha''(t) \qquad (3.8)$$

under the assumption that the force $\mathbf{F}$ acting on the system be the sum of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ as follows: we require that $\mathbf{F}_1$ be a damping force

$$\mathbf{F}_1 = -D(t)\,\dot\alpha(t)\,\mathbf{T}_{\alpha(t)} \qquad (3.9)$$

and that $\mathbf{F}_2$ be a gravity force acting on the (x, y)-plane of the form

$$\mathbf{F}_2 = -[G(t).x.\sqrt{x^2 + y^2}]\,\mathbf{V}_{(x, y)} \qquad (3.10)$$

where $\mathbf{V}_{(x, y)}$ is a unit vector in $P = (x, y)$ parallel to the x-axis and oriented as in figure 1.

Then it is easy to see that (3.8) along the direction $\mathbf{T}_{\alpha(t)}$ takes the form (3.5).

Now the sign of $G(t)$ is crucial: we notice that in both equations (2.24) and (3.4) $G(t)$ is positive, so that the gravity force $\mathbf{F}_2$ is directed as in figure 1.

Standard theorems on ordinary differential equations tell us that a solution $\alpha(t)$ of equations of type (3.5) can cease to exist in some $\bar{t}$ if and only if $\lim\limits_{t \to \bar{t}} \alpha(t) = \infty$.

But it is physically clear that this cannot happen if $G(t)$ is positive, for in this case the gravity pushes the moving particle toward the position $\alpha = 0$: a detailed mathematical proof can be performed by comparing
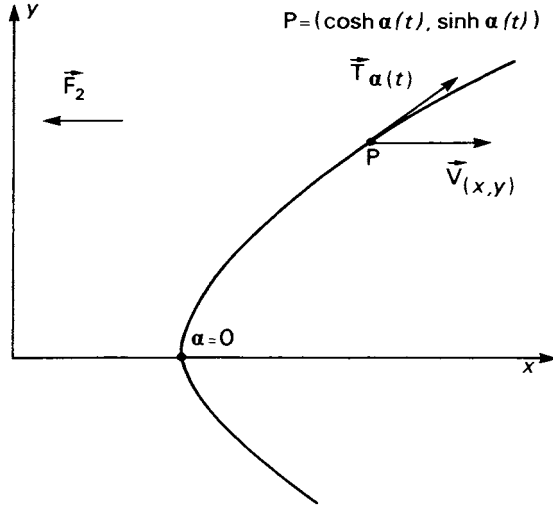
$P = (\cosh \alpha(t), \sinh \alpha(t))$

$\vec{T}_{\alpha(t)}$

$\vec{F}_2$

P

$\vec{V}_{(x,y)}$

$\alpha = 0$

FIG. 1.

equations (2.24) or (3.4) with the following

$$\alpha''(t) + K^2 \sinh \alpha(t) \cosh \alpha(t) = 0 \qquad (3.11)$$

where K^2 is a non-zero constant.

In fact, if one supposes that a solution $\alpha(t)$ of (2.24) or (3.4) blows up at a finite time $\bar{t}$, then, $D(t)$ being bounded, it is not difficult to show that also equation (3.11), with a suitable K^2, would have solutions which blow up at finite time. But this is not possible, because equation (3.11) has the following prime integral

$$\alpha'^2(t) = C_0^2 - K^2 \sinh^2 \alpha(t) \qquad (3.12)$$

where $C_0^2 = K^2 \sinh^2 \alpha(t_0) + \alpha'^2(t_0)$ for some $t_0 \in \mathbb{R}$.

Now it is clear that (3.12) forces a solution $\alpha(t)$ to remain for all t in some closed interval about the position $\alpha = 0$, contradicting $\lim_{t \to \bar{t}} \alpha(t) = \infty$.

Remark. — If $G(t)$ is negative in (3.5), then things drastically change: in fact it is well-known that equation

$$\alpha''(t) - K^2 \sinh \alpha(t) \cosh \alpha(t) = 0 \qquad (3.13)$$

does have solutions which blow up at finite time: this interesting phenomenon will lead us to cases where harmonic maps are not globally defined (*see* section 4).

4. HARMONIC MAPS BETWEEN (p, q)-MANIFOLDS, $p, q \geqq 1$

In this section we study examples (f), (g), (h), (i) of section 2. We start with

Example (i). $-f : \mathrm{T}^{1, 1} \to \mathrm{S}^{1, 1}$.

The set $[\mathrm{T}^{1, 1}, \mathrm{S}^{1, 1}]$ of homotopy classes of maps from $\mathrm{T}^{1, 1}$ to $\mathrm{S}^{1, 1}$ is $\mathbb{Z} \times \mathbb{Z}$.

We observe that the constant function $\alpha(\varphi) \equiv 0$ is a periodic solution of the harmonicity equation (2.37): its associated harmonic map of type (2.36) represents the homotopy class $\{m\} \times \{n\} \in \mathbb{Z} \times \mathbb{Z}$; this simple remark proves the following

PROPOSITION 4.1. — *Each element of* $[\mathrm{T}^{1, 1}, \mathrm{S}^{1, 1}] = \mathbb{Z} \times \mathbb{Z}$ *can be represented by a harmonic map.*

This proposition answers affirmatively to a question raised in [HL].

Remarks. — The constant solution $\alpha(\varphi) \equiv 0$ is the only *periodic* solution of equation (2.37) if and only if $m^2 \geqq n^2$.

By contrast, if $n^2 > m^2$, then the condition of harmonicity is

$$\alpha''(\varphi) + (n^2 - m^2) \sinh \alpha(\varphi) \cosh \alpha(\varphi) = 0. \qquad (4.1)$$

$(n^2 - m^2)$ being positive, equation (4.1) has infinitely many solutions of period 2π which can be used to define interesting harmonic maps $f : \mathrm{T}^{1, 1} \to \mathrm{S}^{1, 1}$. All these solutions have the same image, which is a closed interval symmetric with respect to 0: it is easy to see that the image of these harmonic maps can be made arbitrarily large provided that $\mathrm{T}^{1, 1}$ is given a metric of the form

$$c^2 \, d\theta^2 - d\varphi^2 \qquad (4.2)$$

where c^2 is a suitable constant.

Reduction to equation (4.1) occurs also when one studies equivariant maps $f : \mathrm{T}^{1, 1} \to \mathrm{I}^2$, where I^2 is the hyperboloid defined in example (c) of section 2: therefore also in this case one can use non-costant periodic solutions of (4.1) to define an infinite family of harmonic maps.

We also remark that analogous results hold if one replaces the map $\theta \to m\theta$ in (2.36) (with $n = 0$) by any harmonic homogeneous polynomial map $\Phi : \mathrm{S}^p \to \mathrm{S}^r$.

Example (f). $- f : \mathrm{S}^{p, 1} \to \mathrm{S}^{r, 1}$.

We show that in this example there are solutions of the harmonicity equation which are globally defined and others which are not.

The relevant equation (2.30) is again of type (3.5): but the function $G(t) = -\lambda/\cosh^2 t$ is negative; thus, if one visualizes the situation as in Figure 1, he finds that now the force $\mathbf{F}_2$ is oriented in the direction of the x-axis: this fact, according to the Remark at the end of section 3, suggests that there may be solutions which blow up at finite time.

First we observe the following fact: suppose that $\lambda = p$ in equation (2.30): simple inspection shows that under this assumption the function $\bar{\alpha}(t) = t$ is a globally defined solution of (2.30) which solves the Cauchy problem

$$\alpha(0) = 0, \qquad \alpha'(0) = 1. \tag{4.3}$$

Now consider the solution $\alpha(t)$ of (2.30) determined by Cauchy data

$$\alpha(0) = 0, \qquad \alpha'(0) = \varepsilon, \qquad 0 \leqq \varepsilon < 1. \tag{4.4}$$

A comparison argument as in [R1] insures that the solution $\alpha(t)$ determined by (4.4) satisfies $|\alpha(t)| < |t| \ \forall \, t \neq 0$ and therefore it is globally defined: the physical meaning of this comparison is the following: take two particles P_1, P_2 which move on the curve C in (3.6) and assume that at a given time $\bar{t}$ the two particles are at the same position: if the speed of P_1 at time $\bar{t}$ is less than the speed of P_2 at time $\bar{t}$ and P_2 increases monotically toward $+\infty$, then P_1 will always be behind P_2.

Similarly, in the case where $\lambda \neq p$, the solution determined by (4.4) is globally defined provided that ε is sufficiently small. By contrast, now we exhibit initial conditions

$$\alpha(\tilde{t}) = \alpha_0, \qquad \alpha'(\tilde{t}) = \alpha_1, \qquad \alpha_0, \, \alpha_1 > 0 \tag{4.5}$$

in such a way that the solution of (2.30) distinguished by (4.5) blows up at finite time.

Our assertion follows if we produce a supersolution $F(t)$ of (2.30) in the following sense: we require that $F(t)$ satisfy

$$\left. \begin{array}{l} F(\tilde{t}) = \alpha_0, \qquad F'(\tilde{t}) = \alpha_1, \qquad \alpha_0, \, \alpha_1 > 0 \\ \lim_{t \to T} F(t) = +\infty \quad \text{for some } T > \tilde{t} \end{array} \right\} \tag{4.6}$$

and

$$F''(t) + [p \tanh t] \, F'(t)$$
$$- \frac{\lambda}{\cosh^2 t} \sinh F(t) \cosh F(t) \leqq 0, \qquad \forall \, t \in [\tilde{t}, T). \tag{4.7}$$

In fact, if both (4.6) and (4.7) hold, the comparison method of [PR], [R1] shows that the solution $\alpha(t)$ of (2.30) determined by (4.5) satisfies $\alpha(t) \geqq F(t) \ \forall \, t \in [\tilde{t}, T)$; thus the blowing up of $F(t)$ at T forces $\alpha(t)$ to blow up as well at some point $\tilde{T} \in (\tilde{t}, T]$.

In order to produce a function $F(t)$ which satisfies (4.6) and (4.7) we consider the following function

$$F(t) \overset{\text{def}}{=} \tanh^{-1}[1 - (1-t)^n] \tag{4.8}$$

where n is a fixed integer greater than 3.

If one substitutes (4.8) in the left-hand side of inequality (4.7) and uses the identity $\sinh x \cosh x = \tanh x \mid (1 - \tanh^2 x)$, he obtains

$$\frac{[1-t]^{-2}}{[2-(1-t)^n]} \left[-n(n-1) + np(\tanh t)(1-t) \right.$$
$$\left. + \frac{2[1-(1-t)^n]n^2}{[2-(1-t)^n]} - \frac{\lambda}{\cosh^2 t} \frac{1-(1-t)^n}{(1-t)^{n-2}} \right]. \quad (4.9)$$

Simple inspection shows that the expression in (4.9) is negative on $[1-\varepsilon, 1)$ for $\varepsilon > 0$ small.

Therefore, by construction, the function $F(t)$ defined by (4.8) satisfies both (4.6) and (4.7) with

$$\tilde{t} = 1 - \varepsilon, \qquad T = 1, \qquad \alpha_0 = F(1-\varepsilon), \qquad \alpha_1 = F'(1-\varepsilon).$$

In conclusion, we have proved

THEOREM 4.2. — *Consider equivariant maps* $f : S^{p,1} \to S^{r,1}$ *as in* (2.29) *and fix Cauchy data for the harmonicity equation* (2.30).

Then the existence of a global solution of this Cauchy problem depends upon the choice of the Cauchy data.

Similarly, for maps $f : S^{p,q+1} \to S^{r,s+1}$ as in example (g) we have globally defined harmonic maps, but also cases of blowing up of solutions at finite time. We conclude this section with some facts about

Example (h). — $f : S^{p,q+1} \to S^{r,s+1}$.

In this example, the harmonicity equation (2.34) describes the motion of a pendulum with variable gravity and damping: according to (2.33), we must look for solutions $\alpha(t)$ of (2.34) which are defined on $(0, +\infty)$ and also satisfy

$$\lim_{t \to 0} \alpha(t) = \pi/2. \qquad (4.10)$$

We observe that when $t \to 0$, then (2.34) behaves qualitatively as

$$\alpha''(t) + \frac{q}{t}\alpha'(t) + \frac{\lambda_2}{t^2}\sin\alpha(t)\cos\alpha(t) = 0. \qquad (4.11)$$

This, up to a substitution $t = e^u$, $u \in \mathbb{R}$, is the equation of a pendulum with constant gravity and damping: being $\lambda_2 > 0$, we can conclude that there exist solutions of (4.11) which satisfy condition (4.10); thus the same holds for equation (2.34). Finally, as an application of Lemma 3.1, we can conclude that all solutions of (2.34) are defined on the whole $(0, +\infty)$: therefore our construction (2.33) yields globally defined harmonic maps from $S^{p,q+1}$ into $S^{r,s+1}$.

5. SOME QUALITATIVE FEATURES
OF GLOBAL SOLUTIONS

An important feature of the equivariant method is that global solutions are always explicitly defined in terms of a function $\alpha(t)$ and homogeneous data on the cross section: thus the qualitative study of $\alpha(t)$ provides a complete description of the associated equivariant harmonic map.

Therefore in the equivariant case it is possible to give fearly complete answers to fundamental questions such as

(i) What is the image of a harmonic map?

(ii) How does the choice of Cauchy data determine global properties of solutions?

This is important because, at the present, questions such as (i), (ii) above are very poorly understood in the general case (1.1): in this order of ideas, now we present some qualitative results concerning examples (a) and (b) of section 2; all other examples of section 2 can be studied as well by using similar methods.

Example (a). $- f : S^{p, 1} \to S^{r+1}$.

In section 3 we showed global existence for any choice of Cauchy data: now we concentrate on two particularly interesting cases:

$$\alpha(0) = \pi/2, \qquad \alpha'(0) = \alpha_1, \qquad \alpha_1 \geqq 0 \qquad (5.1)$$

and

$$\alpha(0) = 0, \qquad \alpha'(0) = \alpha_1, \qquad \alpha_1 \geqq 0. \qquad (5.2)$$

Cauchy data (5.1) and (5.2) correspond to mapping the space like submanifold $S^p \times \{0\}$ respectively into the equator of S^{r+1} and into its North pole. We notice that, in both cases, if $\alpha_1 = 0$, then the associated global solution $\alpha(t)$ is constant. In the following two propositions we study how things vary when $\alpha_1 > 0$.

PROPOSITION 5.1. $-$ *Consider maps* $f : S^{p, 1} \to S^{r+1}$ *of type* (2.18) *with equivariant Cauchy data* (5.1).

Then there exists $b > 0$ *such that:*

(i) *if* $\alpha_1 > b$, *then the image of the associated solution* $\alpha(t)$ *contains the closed interval* $[0, \pi]$; *in particular, if the cross section map* $\Phi : S^p \to S^r$ *is surjective, then these global solutions are surjective as well.*

(ii) *if* $\alpha_1 = b$, *then the associated solution* $\alpha(t)$ *is strictly increasing and satisfies*

$$\lim_{t \to -\infty} \alpha(t) = 0, \qquad \lim_{t \to +\infty} \alpha(t) = \pi$$

in particular, if the cross section map $\Phi : S^p \to S^r$ *is surjective, then these global solutions cover* S^{r+1} *minus the two poles.*

(iii) *if $0 < \alpha_1 < b$, then the associated solution $\alpha(t)$ is strictly increasing and satisfies*

$$\lim_{t \to -\infty} \alpha(t) = \varepsilon, \qquad \lim_{t \to +\infty} \alpha(t) = \pi - \varepsilon \qquad (5.3)$$

where ε in (5.3) belongs to $[0, \pi/2)$ and may depend upon α_1.

PROPOSITION 5.2. — *Consider maps $f: S^{p, 1} \to S^{r+1}$ of type (2.18) with equivariant Cauchy data (5.2) and assume for simplicity $\Phi: S^p \to S^r$ surjective: then*

(i) *If α_1 is sufficiently large, then the associated global solution is surjective.*

(ii) *There exists $\bar{\alpha}_1$ such that the image of the associated global solution is exactly the Northern open hemisphere.*

Proofs. — The proof of these two propositions is rather long and technical: the method of proof is an adaptation of ideas introduced in [PR], [R1], [S2] and therefore we limit ourselves to an outline of proofs.

We start with proposition 5.1: it is convenient to make the substitution

$$\beta(t) = \alpha(t) - \pi/2 \qquad (5.4)$$

so that the harmonicity equation (2.19) and Cauchy data (5.1) become respectively

$$\beta''(t) + [p \tanh t] \beta'(t) - \frac{\lambda}{\cosh^2 t} \sin \beta(t) \cos \beta(t) = 0 \qquad (5.5)$$

and

$$\beta(0) = 0, \qquad \beta'(0) = \alpha_1. \qquad (5.6)$$

Because of the symmetries of equation (5.5), it is enough to perform our study on $[0, +\infty)$: the physical system described by (5.5) is a pendulum with damping $D(t) = p \tanh t$ and gravity $G(t) = -\lambda/\cosh^2 t$.

The function $D(t)$ is positive on $(0, +\infty)$: therefore the damping force reduces the speed of the motion on $(0, +\infty)$; the gravity force $G(t)$ acts in the direction indicated in Figure 2.

It is obvious that, if α_1 is sufficiently large, then the solution of the Cauchy problem (5.5), (5.6), which we denote by $\beta_1(t)$, reaches $\pi/2$ in finite time.

On the other hand, the fact that $\lim_{t \to +\infty} G(t) = 0$, together with the presence of the damping force, tell us that it makes sense to look for strictly increasing solutions $\beta(t)$ with image contained in $(-\pi/2, \pi/2)$.

It is worth noticing that the system in Figure 2 is qualitatively much different from the system arising from the study of harmonic maps between spheres. In fact, in that context, the damping force $D(t)$ is negative for t large, and thus it increases the speed of motion; but the gravity force

$\beta = \pi/2$

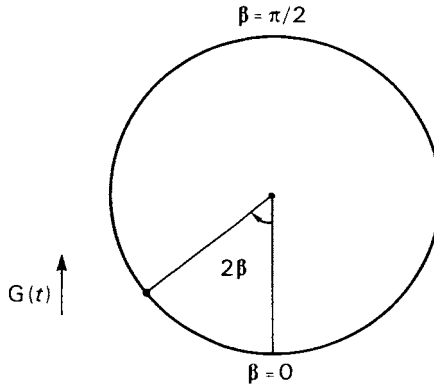$G(t)$

2β

$\beta = 0$

FIG. 2.

compensates for this by acting in the opposite direction with respect to Figure 2.

We define a set $\beta^+(0)$ by

$$\beta^+(0) \stackrel{\text{def}}{=} \{\alpha_1 \in (0, +\infty): \beta_1(t) \text{ reaches } \pi/2 \text{ in finite time}\}$$

and let

$$b = \inf \beta^+(0). \tag{5.7}$$

The number b in (5.7) satisfies the requirements of Proposition 5.1: this can be achieved through the following steps

Step 1. — $\beta^+(0)$ is an open set and, if $b > 0$, then the solution $\beta_1(t)$ of the Cauchy problem with $\alpha_1 = b$ is increasing and satisfies $\lim_{t \to +\infty} \beta(t) = \pi/2$.

Step 2. — $b > 0$.

Step 3. — If $0 < \alpha_1 < b$, then $\beta_1(t)$ is increasing and

$$\beta_1(t) \in (0, \pi/2), \qquad \forall t \in (0, +\infty).$$

The proof of Step 1 is a straightforward adaptation of Lemma 3.2.6 of [S2].

Step 2 is more delicate: it can be proved by producing an explicit subsolution $F(t)$ in the sense of (4.6) and (4.7) with $\alpha_0 = 0$, $\tilde{t} = 0$, $T = +\infty$ and $\lim_{t \to +\infty} F(t) = \pi/2$ in (4.6); and $\leqq$ replaced by $\geqq$ in (4.7) (*see* [PR]).

Step 3 follows from a direct inspection of equation (5.5) together with a comparison argument as in [R1].

The proof of Proposition 5.2 uses similar methods and therefore its discussion is omitted.

Example (*b*). $- f : \mathbf{R}_s \to S^3$.

This example and its companion example (*e*) are important because they describe harmonic evolutions of a space time into a 3-dimensional space form.

Global existence of solutions of the Cauchy problem was obtained in section 3: here we will prove a stability result.

It is convenient to make the substitution $t = e^u$, $u \in \mathbb{R}$, in the harmonicity equation (2.21).

We call $\beta(u) = \alpha(e^u)$: in terms of $\beta(u)$, equation (2.21) becomes

$$\beta''(u) + (3s - 1) \beta'(u) + 8 e^{(2-2s)u} \sin \beta(u) \cos \beta(u) = 0. \qquad (5.8)$$

The constant function $\beta(u) \equiv 0$ is a solution of (5.8); its associated harmonic map is the constant map which sends the whole space-time $\mathbf{R}_s$ into the North pole of S^3.

In particular, the function $\beta(u) \equiv 0$ is the solution of the Cauchy problem

$$\beta(0) = 0, \qquad \beta'(0) = 0. \qquad (5.9)$$

We consider perturbations of (5.9) of type

$$\beta(0) = \alpha_0, \qquad \beta'(0) = \alpha_1 \qquad (5.10)$$

and call $\beta_1(u)$ the global solution of the Cauchy problem (5.10).

We have the following stability result

PROPOSITION 5.3. $-$ *Assume* $1/3 \leqq s < 1$; *and let* $\delta > 0$ *be an arbitrarily fixed number.*

Then there exists $\varepsilon > 0$ *such that if* $|\alpha_0|$, $|\alpha_1| \leqq \varepsilon$, *then* $\beta_1(u) \in [-\delta, \delta]$, $\forall u \geqq 0$.

Proof. $-$ It is helpful to visualize the situation on Figure 3 below by thinking of a particle moving about the position $\beta = 0$ under the influence of a gravity force $G(t)$ directed as in the figure.

To simplify the notations, we will write $\beta(u)$ instead of $\beta_1(u)$; also we assume both α_0 and α_1 positive: the other cases are similar.

It is obvious that, if the initial velocity α_1 is small, then $\beta(u)$ increases only until the time u reaches a certain value u_1, for which we have $\beta'(u_1) = 0$. Moreover, it is clear that if $\varepsilon > 0$ is chosen sufficiently small, then α_0, $\alpha_1 \leqq \varepsilon$ force $\beta(u_1) \leqq \delta$.

In order to end the proposition, it is sufficient to show that the amplitude of the oscillations around 0 decreases.

After the time u_1 the function $\beta(u)$ obviously decreases; suppose that $\beta(u)$ keep on decreasing until, at a certain time u_2, it reaches the position $\beta(u_2) = -\beta(u_1)$: we show that this is not acceptable.
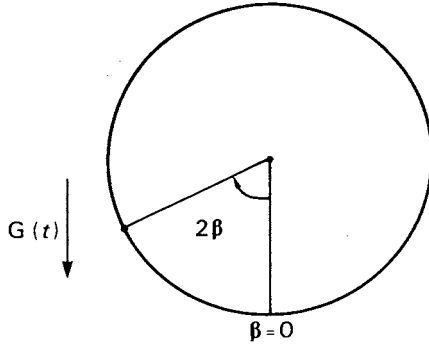
FIG. 3.

We multiply equation (5.8) by $2\beta'(u)$ and integrate between u_1 and u_2: we obtain

$$\beta'^2(u_2) + 2(3s-1) \int_{u_1}^{u_2} \beta'^2(u)\,du$$

$$+ 8 \int_{u_1}^{u_2} e^{(2-2s)u} \sin(2\beta(u))\,\beta'(u)\,du = 0. \quad (5.11)$$

Because of assumption $1/3 \leqq s$, the second term in the sum (5.11) is positive or zero.

After the substitution $\beta(u) = z$, the third term in (5.11) takes the form

$$-8 \int_{\beta(u_2)}^{\beta(u_1)} e^{(2-2s)\beta^{-1}(z)} \sin(2z)\,dz. \quad (5.12)$$

Being $s < 1$, and $\beta(u)$ decreasing on $[u_1, u_2]$, the function $e^{(2-2s)\beta^{-1}(z)}$ is decreasing on $[\beta(u_2), \beta(u_1)]$.

Thus also (5.12) is positive and this contradicts (5.11); therefore we must have $\beta'(\tilde{u}) = 0$ for some $\tilde{u}$ before the position $-\beta(u_1)$ is reached and this proves that the amplitude of oscillations decreases and so the proposition.

Remark. — A stability result analogous to Proposition 5.3 holds for example (e): this follows just by observing that, if $\beta(u)$ is small, then $\sinh\beta(u)\cosh\beta(u) \simeq \sin\beta(u)\cos\beta(u)$; therefore equations (5.8) and (3.4) have the same qualitative behavior for small values of $\beta(u)$.

As for equation (5.8), one can also investigate the stability of the constant solution $\beta(u) \equiv \pi/2$; this solution solves the Cauchy problem

$$\beta(0) = \pi/2, \qquad \beta'(0) = 0 \qquad (5.13)$$

and its associated harmonic map has image contained in the equator of S^3.

By contrast with proposition 5.3, it is not difficult to show that any arbitrarily small perturbation of (5.13) of type

$$\beta(0) = \pi/2 + \alpha_0, \qquad \beta'(0) = \alpha_1 \qquad\qquad (5.14)$$

gives rise to a harmonic map which covers the whole S^3.

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